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A Guided Tour of Polyhedric Sets: Basic Properties, New Results on Intersections, and Applications

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The aim of this contribution is twofold. On the one hand, we give some new results concerning polyhedric sets. In particular, we show that sets with pointwise lower and upper bound are polyhedric in many important function spaces. Moreover, we show that the intersection of such a set with finitely many hyperplanes and half-spaces is polyhedric. We also provide counterexamples demonstrating that the intersection of polyhedric sets may fail to be polyhedric. On the other hand, we gather all important results from the literature concerning polyhedric sets in order to give a complete picture of the current knowledge. In particular, we illustrate the applications of polyhedricity.

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1. Introduction

The notion of polyhedricity of closed convex sets was first used in the seminal works by Mignot [22] and Haraux [14]. Haraux [14] also coined the term "polyhedricity". In these two works, the authors have shown that polyhedricity of a closed convex set in a Hilbert space implies that the metric projection onto this set (or equivalently, the solution map of a variational inequality) is directionally differentiable. Moreover, Mignot [22] verified a stationarity system for the optimal control of a variational inequality.

Since then, the notion of polyhedricity found some very important applications in infinite-dimensional optimization. In particular, polyhedricity helps to provide no-gap second-order optimality conditions in the infinite-dimensional case.

However, all the available results concerning polyhedricity are scattered in the literature and it is difficult to get an overview on the current knowledge. Hence, one important goal of the present paper is to bridge this gap. In particular, we

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collect sets which are known to be polyhedric, see Subsection 3.2, and we provide some known properties in Subsection 3.3.

On the other hand, we study the question whether the intersection of two polyhedric sets is again polyhedric and this is addressed in Section 4. We mention that we obtain completely new results, which generalize the results of [22, 14] concerning the polyhedricity of sets with bounds in vector lattices. We do not need a Dirichlet space setting as used by Mignot [22] and, thus, our assumptions are easier to verify. In Subsection 4.3 we give some counterexamples which demonstrate that intersections of polyhedric sets may fail to be polyhedric and we present a polyhedric set in finite dimensions which is not polyhedral.

Finally, we review the most important applications of polyhedricity, see Section 5. In particular, we prove the directional differentiability of the projection onto polyhedric sets, strong stationarity for optimal control of the projection, and provide no-gap second-order conditions in the polyhedric case. These results are presented in order to complete the picture of polyhedricity.

2. Notation

We denote by $\mathbb{N} = \{1, 2, \ldots\}$ and $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$ the natural numbers.

Let $A \subseteq X$ be a subset of the real Banach space X. We denote by $\operatorname{cl} A$, $\operatorname{conv} A$, $\operatorname{cone} A$, $\overline{\operatorname{conv}}\, A$, and $\operatorname{lin} A$ the *closure*, the *convex hull*, the (convex) *conic hull*, the *closed convex hull* and the *linear hull* of A, respectively. By $X^\star$ we denote the (topological) *dual space* of X with corresponding dual pairing $\langle \cdot, \cdot \rangle \colon X^\star \times X \to \mathbb{R}$. We define the *polar cone* and the *polar set* of A by

$$A^\circ := \{x^\star \in X^\star \mid \forall x \in A : \langle x^\star, x\rangle \le 0\}, \quad A^\square := \{x^\star \in X^\star \mid \forall x \in A : \langle x^\star, x\rangle \le 1\}.$$

Similarly, we use

$$B^\circ := \{x \in X \mid \forall x^\star \in B : \langle x^\star, x\rangle \le 0\}, \quad B^\square := \{x \in X \mid \forall x^\star \in B : \langle x^\star, x\rangle \le 1\}.$$

for any non-empty set $B \subseteq X^\star$. The *closure* of B w.r.t. the weak-$\star$ topology of $X^\star$ is denoted by $\operatorname{cl}_\star$. For a functional $\mu \in X^\star$ we denote the *annihilator* by

$$\mu^\perp := \{x \in X \mid \langle \mu, x\rangle = 0\}.$$

The *radial cone* and the *tangent cone* of a closed, convex set $K \subset X$ at $x \in K$ are given by

$$\mathcal{R}_K(x) := \operatorname{cone}(K - x) \quad \text{and} \quad \mathcal{T}_K(x) := \operatorname{cl} \mathcal{R}_K(x).$$

Finally, the *critical cone* of K w.r.t. $(x, \mu) \in K \times \mathcal{T}_K(x)^\circ$ is given by

$$\mathcal{K}_K(x, \mu) := \mathcal{T}_K(x) \cap \mu^\perp.$$

3. Definition, examples and basic properties

3.1. Definition

We start with the definition of polyhedricity and related concepts.

Definition 3.1. Let K be a closed, convex subset of the Banach space X.

(1) We say that K is *polyhedric* at (x, μ) with $x \in K$, $\mu \in \mathcal{T}_K(x)^\circ$, if

$$\mathcal{T}_K(x) \cap \mu^\perp = \operatorname{cl}\big(\mathcal{R}_K(x) \cap \mu^\perp\big). \tag{1}$$

We say that K is polyhedric at $x \in K$, if (1) holds for all $\mu \in \mathcal{T}_K(x)^\circ$. Finally, K is polyhedric, if it is polyhedric at all $x \in K$.

(2) The set K is called *polyhedral*, if it is the intersection of finitely many half-spaces, i.e., if there exist $n \in \mathbb{N}_0$, $\nu_1, \ldots, \nu_n \in X^\star$, and $c_1, \ldots, c_n \in \mathbb{R}$ such that

$$K = \big\{x \in X \mid \langle \nu_i,\, x_i\rangle \le c_i \;\forall i = 1, \ldots, n\big\}. \tag{2}$$

(3) We call K *co-polyhedral*, if it is the convex hull of finitely many points and rays, i.e., if there exist $n, m \in \mathbb{N}_0$, $x_1, \ldots, x_n \in X$, $r_1, \ldots, r_m \in X$ such that

$$K = \operatorname{conv}\{x_1, \ldots, x_n\} + \operatorname{cone}\{r_1, \ldots, r_m\}.$$

(4) Finally, K is said to be *generalized polyhedral*, if it is the intersection of a polyhedral set with a closed, affine subspace.

It is well-known that the notions of polyhedrality, co-polyhedrality and generalized polyhedrality coincide in finite-dimensional spaces, see, e.g., [17, Theorem 2.12] or [40, Theorem 1.2].

In infinite dimensions, the concepts of polyhedrality and co-polyhedrality are dual. Indeed, if K satisfies (2), then we have

$$K^\square = \operatorname{conv}\left(\{0\} \cup \left\{\frac{\nu_i}{c_i}\right\}_{\{i \mid c_i > 0\}}\right) + \operatorname{cone}\{\nu_i\}_{\{i \mid c_i \le 0\}}.$$

Hence, $K^\square$ is co-polyhedral. Similarly, the polar set of a co-polyhedral set is polyhedral.

Now we discuss the notion of polyhedricity, which was introduced by Mignot [22] and Haraux [14]. From $\mathcal{R}_K(x) \subset \mathcal{T}_K(x)$ it is clear that the left-hand side of (1) always contains the right-hand side. Moreover, (1) follows trivially if $\mathcal{R}_K(x) = \mathcal{T}_K(x)$ and it is straightforward to check that this is the case for polyhedral, co-polyhedral and generalized polyhedral sets.

We will see in Subsection 3.2 that there are many polyhedric sets that are not generalized polyhedral if the space X is infinite-dimensional. In finite dimensions, one is tempted to conjecture that polyhedric sets are polyhedral. However, one has to restrict this question to bounded polyhedric sets or broaden the notion of polyhedrality, since polyhedricity is a local property whereas polyhedrality is global. Indeed, the set

$$\operatorname{conv}\{(s, s^2)^\top\}_{s \in \mathbb{Z}} \subset \mathbb{R}^2$$

is polyhedric, but has countably infinitely many vertices. Thus, it is not polyhedral. In Example 4.24 we will present a compact, convex and polyhedric set in $\mathbb{R}^3$ which fails to be a polyhedron. To our knowledge, such an example was not known previously.

3.2. Examples

Now, we give the most important examples for polyhedric sets in infinite-dimensional spaces, which can be found in the literature.

A broad class of examples can be found in Banach spaces which possess additionally a lattice structure with strongly-weakly continuous lattice operations, see Subsection 4.2 for references and further details. In fact, this yields the polyhedricity of the sets

$$\begin{aligned}
&\{u \in L^p(\Omega) \mid u_a \le u \le u_b \text{ a.e. in } \Omega\} && \text{in } L^p(\Omega),\ p \in [1,\infty],\ u_a, u_b \in L^p(\Omega),\\
&\{u \in W_0^{1,p}(\Omega) \mid u_a \le u \le u_b \text{ a.e. in } \Omega\} && \text{in } W_0^{1,p}(\Omega),\ p \in [1,\infty),\ u_a, u_b \in W_0^{1,p}(\Omega),\\
&\{u \in W^{1,p}(\Omega) \mid u_a \le u \le u_b \text{ a.e. in } \Omega\} && \text{in } W^{1,p}(\Omega),\ p \in [1,\infty),\ u_a, u_b \in W^{1,p}(\Omega),\\
&\{u \in W^{1,p}(\Omega) \mid u_a \le u \le u_b \text{ a.e. on } \partial\Omega\} && \text{in } W^{1,p}(\Omega),\ p \in [1,\infty),\ u_a, u_b \in W^{1,p}(\Omega).
\end{aligned}$$

In all cases, $\Omega \subset \mathbb{R}^d$ is an open, bounded set. In the last case, we have to assume additionally that Ω is a Lipschitz domain in order to get a well-defined trace and the notion “a.e.” refers to the surface measure on the boundary $\partial\Omega$. Such sets where already studied in the works by Mignot [22] and Haraux [14], at least in the Hilbert space case $p = 2$. The regularity requirements on u_a, u_b can be significantly reduced. In particular, it is sufficient that the bounds u_a and u_b are measurable functions. We refer to Subsection 4.2 for details and to Example 4.21 for more polyhedric sets defined by (pointwise) bounds.

The pointwise ordering on $H_0^2(\Omega)$ does not induce a lattice structure. Nevertheless, it is shown in [27] that subsets $K \subset H_0^2(\Omega)$ which are defined by pointwise constraints are polyhedric at some $u \in K$, but not at all $u \in K$. Similarly, $H^{-1/2}(\Omega)$ does not posses a lattice structure, but [33, Lemma 1] shows the polyhedricity of the set

$$\{\mu \in H^{-1/2}(\partial\Omega) \mid \mu \in L^\infty(\partial\Omega) \text{ and } -1 \le \mu \le 1 \text{ a.e. on } \partial\Omega\}$$

in $H^{-1/2}(\partial\Omega)$ at (μ, u) for smooth Ω and under some regularity assumptions on (μ, u).

Finally, we mention that there are certain results in the vector-valued case. Using the proof of [5, Proposition 4.3], one can show that

$$\{u \in L^p(\Omega)^n \mid u(x) \in K(x) \text{ for almost all } x \in \Omega\}$$

is polyhedric in $L^p(\Omega)^n$, $1 \le p < \infty$, if the set-valued mapping $x \mapsto K(x)$ is measurable and if $K(x)$ is a polyhedral set for almost all $x \in \Omega$, see Lemma 4.20.

A similar result for vector-valued Sobolev spaces was recently obtained in [39]: Let $\{a_i\}_{i=1,\ldots,N} \subset \mathbb{R}^n$ and $\{b_i\}_{i=1,\ldots,N} \subset \mathbb{R}$ with $N \in \mathbb{N}$ be given, such that the polyhedral set $C := \{x \in \mathbb{R}^n \mid a_i^\top x \le b_i \; \forall i = 1, \ldots, N\}$ satisfies $0 \in \operatorname{int}(C)$ and the linear-independence constraint qualification (in the sense of nonlinear optimization). That is, the family $\{a_i\}_{i \in I(x)}$ is assumed to be linearly independent for all $x \in C$, where $I(x) = \big\{i \in \{1, \ldots, N\} \mid a_i^\top x = b_i\big\}$ is the set of active indices. Then, the set

$$\{u \in H_0^1(\Omega)^n \mid u(x) \in C \text{ for almost all } x \in \Omega\}$$

is polyhedric in $H_0^1(\Omega)^n$. The technique of proof is much more involved as in the $L^p(\Omega)^n$-case due to the spatial coupling of the $H_0^1(\Omega)^n$-norm. It is not clear whether the result can be generalized to a constraint set C which depends on $x \in \Omega$, and whether the assumption $0 \in \operatorname{int}(C)$ or the linear-independence constraint qualification can be dropped.

Finally, by using a surjectivity argument, see also Lemma 3.3 below, one can show the polyhedricity of

$$\{u \in H^1_\Gamma(\Omega; \mathbb{R}^d) \mid u^\top \nu \le b \text{ a.e. on } \Gamma_C\}$$

in $H^1_\Gamma(\Omega; \mathbb{R}^d) := \{u \in H^1(\Omega; \mathbb{R}^d) \mid u = 0 \text{ a.e. on } \Gamma\}$, where Ω is a Lipschitz domain, $\Gamma, \Gamma_C \subset \partial\Omega$ are closed and measurable (w.r.t. the surface measure) and have positive distance, and $\nu\colon \bar{\Omega} \to \mathbb{R}^d$ is Lipschitz with $|\nu|_{\mathbb{R}^d} \ge 1$ on $\bar{\Omega}$, cf. [34, Section 4.6], [4, Section 3.2.2], and [24, Corollary 4.16] for similar results.

3.3. Basic properties

In this section, we review two basic properties of polyhedric sets. First, we recall a characterization of polyhedricity for cones. Second, we show a stability result of polyhedric sets under linear mappings.

Lemma 3.2. *Let $K \subset X$ be a closed, convex cone in the reflexive Banach space X. For $x \in K$, $\mu \in \mathcal{T}_K(x)^\circ = K^\circ \cap x^\perp$ the following conditions are equivalent:*

(1) *K is polyhedric at (x, μ).*

(2) *K° is polyhedric at (μ, x).*

(3) *$\mathcal{K}_K(x, \mu)^\circ = \mathcal{K}_{K^\circ}(\mu, x)$.*

(4) *$\mathcal{K}_{K^\circ}(\mu, x)^\circ = \mathcal{K}_K(x, \mu)$.*

The relation $\mathcal{T}_K(x)^\circ = K^\circ \cap x^\perp$ for closed, convex cones K can be found in [6, (2.110)] and we refer to [37, Lemma 5.2] for a proof of Lemma 3.2. To our knowledge, there is no similar result which characterizes the polyhedricity of a closed, convex set K with $0 \in K$ via its polar set $K^\square$. In principle, this should be possible since $K = K^{\square\square}$ is determined by $K^\square$. However, it is easily checked that the set

$$K := \overline{\operatorname{conv}}\{(1, 1 - 1/n)^\top\}_{n \in \mathbb{N}} \subset \mathbb{R}^2$$

is polyhedric, but its polar set is not polyhedric at $((0,1),(1,0))$.

The next lemma shows that the preimage of a polyhedric set under a linear mapping is again polyhedric if a certain condition is satisfied.

Lemma 3.3. *Let X, Y be Banach spaces, $K_Y \subset Y$ be a closed, convex set and $S\colon X \to Y$ be a bounded and linear operator. Set $K_X := S^{-1}(K_Y)$ and let $x \in K_X$, $\mu \in \mathcal{T}_{K_X}(x)^\circ$ be given. Assume that*

$$S\,X - \mathcal{R}_{K_Y}(S\,x) = Y \tag{3}$$

is satisfied. Then, there is $\lambda \in \mathcal{T}_{K_Y}(S\,x)^\circ$ with $\mu = S^\star\lambda$.

Additionally, we suppose that

$$S\,X - \big(\mathcal{R}_{K_Y}(S\,x) \cap \lambda^\perp\big) = Y \tag{4}$$

holds and that K_Y is polyhedric at $(S\,x, \lambda)$. Then, K_X is polyhedric at (x, μ).

On the other hand, suppose that

$$S\,X - \big(\mathcal{R}_{K_Y}(S\,x) \cap [-\mathcal{R}_{K_Y}(S\,x)]\big) = Y \tag{5}$$

is satisfied and that K_X is polyhedric at (x, μ). Then, K_Y is polyhedric at $(S\,x, \lambda)$.

Proof. In the case that (3) is satisfied, we have

$$\mathcal{T}_{K_X}(x) = \mathcal{T}_{S^{-1}(K_Y)}(x) = S^{-1}(\mathcal{T}_{K_Y}(S\,x)), \tag{6}$$

see [6, Corollary 2.91]. Consequently,

$$\mathcal{T}_{K_X}(x)^\circ = \big(S^{-1}(\mathcal{T}_{K_Y}(S\,x))\big)^\circ = S^\star\,\mathcal{T}_{K_Y}(S\,x)^\circ,$$

see [18, Theorem 2.1]. This shows the first assertion.

Now, assert (4) and that K_Y is polyhedric at $(S\,x, \lambda)$. Let $h \in \mathcal{T}_{K_X}(x) \cap \mu^\perp$ be given. This implies $S\,h \in \mathcal{T}_{K_Y}(S\,x)$, see (6), and

$$\langle \lambda,\, S\,h\rangle_{Y^\star,Y} = \langle S^\star\lambda,\, h\rangle_{X^\star,X} = \langle \mu,\, h\rangle_{X^\star,X} = 0.$$

By the polyhedricity of K_Y, we get a sequence $\{\hat h_k\}_{k\in\mathbb{N}} \subset \mathcal{R}_{K_Y}(S\,x) \cap \lambda^\perp$ with $\hat h_k \to S\,h$. Due to the assumption (4) we can apply the generalization of the open mapping theorem [42, Theorem 2.1] we find a constant $M > 0$ and sequences $\{h_k\}_{k\in\mathbb{N}} \subset X$, $\{\tilde h_k\}_{k\in\mathbb{N}} \subset \mathcal{R}_{K_Y}(S\,x)\cap\lambda^\perp$ with $\|h_k\|_X + \|\tilde h_k\|_Y \le M\,\|\hat h_k - S\,h\|_Y \to 0$ and

$$\hat h_k - S\,h = S\,h_k - \tilde h_k.$$

Hence, $h + h_k \to h$ in X and $S\,(h + h_k) = \hat h_k + \tilde h_k \in \mathcal{R}_{K_Y}(S\,x) \cap \lambda^\perp$. Hence, $h + h_k \in \mathcal{R}_{K_X}(x) \cap \mu^\perp$ and this demonstrates the polyhedricity of K_X at (x, μ).

Now, suppose that (5) holds and that K_X is polyhedric at (x, μ). Let $v \in \mathcal{T}_{K_Y}(S\,x) \cap \lambda^\perp$ be given. Due to (5), there is $h \in X$, $z \in \mathcal{R}_{K_Y}(S\,x) \cap (-\mathcal{R}_{K_Y}(S\,x))$ such that $v = S\,h - z$. Moreover, $\pm z \in \mathcal{R}_{K_Y}(S\,x)$ and $\lambda \in \mathcal{T}_{K_Y}(S\,x)^\circ$ implies $z \in \lambda^\perp$. Hence, $S\,h = v + z \in \mathcal{T}_{K_Y}(S\,x) \cap \lambda^\perp$. By (6) we find $h \in \mathcal{T}_{K_X}(x) \cap \mu^\perp$. From the polyhedricity of K_X we get a sequence $\{h_n\}_{n\in\mathbb{N}} \subset \mathcal{R}_{K_X}(x) \cap \mu^\perp$ with $h_n \to h$ in X. Now, it is easy to see that $S\,h_n - z \in \mathcal{R}_{K_Y}(x) \cap \lambda^\perp$ and that this sequence converges to v. This shows the polyhedricity of K_Y at $(S\,x, \lambda)$. □

The condition (3) is the constraint qualification of Robinson, Zowe and Kurcyusz. Conditions (4) and (5) are stricter variants, which also imply the uniqueness of the multiplier λ, see [32, Theorem 2.2]. Finally, we mention that the result of Lemma 3.3 was previously only known in the case that S is surjective, see [34, 208, 209], [6, Proposition 3.54] and [24, Lemma 4.11]. Note that the surjectivity of S implies all of the conditions (3)–(5).

4. Polyhedricity of intersections

In this section, we are interested in the polyhedricity of intersections of (polyhedric) sets. In particular, we are going to study conditions which ensure that the intersection is again polyhedric.

First, we give some general remarks concerning the intersection of a polyhedric set with a polyhedral set in Subsection 4.1. Then, we restrict our attention to sets with bounds in Banach spaces featuring a lattice structure in Subsection 4.2. In particular, we extend the classical results by Mignot [22] and Haraux [14] by showing that intersections of sets with bounds with polyhedral sets are polyhedric. Finally, we give certain counterexamples in Subsection 4.3.

4.1. Intersections of a polyhedric set with a polyhedral set

In this section, we study the intersection of a polyhedric set with a polyhedral set. In general, this intersection may fail to be polyhedric, see Example 4.24, and we can only give some partial results.

We start with a simple observation.

Lemma 4.1. *Let $K \subset X$ be polyhedric at x. Then,*

$$\mathcal{T}_K(x) \cap \mu^\perp = \operatorname{cl}\left(\mathcal{R}_K(x) \cap \mu^\perp\right) \qquad \forall \mu \in X^\star.$$

Proof. If $\mu \in \mathcal{T}_K(x)^\circ$ or $\mu \in -\mathcal{T}_K(x)^\circ$, the claim follows from the definition of polyhedricity. On the other hand, if $\mu \notin \mathcal{T}_K(x)^\circ$ and $\mu \notin -\mathcal{T}_K(x)^\circ$, then there are $v^+, v^- \in \mathcal{R}_K(x)$ such that

$$\langle \mu, v^+ \rangle > 0 \quad \text{and} \quad \langle \mu, v^- \rangle < 0.$$

Now, let $v \in \mathcal{T}_K(x) \cap \mu^\perp$ be given. By definition of $\mathcal{T}_K(x)$, there is a sequence $\{v_k\}_{k\in\mathbb{N}} \subset \mathcal{R}_K(x)$, such that $\|v - v_k\|_X \to 0$. Now, it is easy to see that there are non-negative null-sequences $\{\lambda_k^+\}_{k\in\mathbb{N}}$, $\{\lambda_k^-\}_{k\in\mathbb{N}}$, such that

$$\langle \mu, v_k + \lambda_k^+ v^+ + \lambda_k^- v^- \rangle = 0.$$

This shows

$$v_k + \lambda_k^+ v^+ + \lambda_k^- v^- \in \mathcal{R}_K(x) \cap \mu^\perp \quad \text{and} \quad v_k + \lambda_k^+ v^+ + \lambda_k^- v^- \to v.$$

Hence, $v \in \operatorname{cl}\left(\mathcal{R}_K(x) \cap \mu^\perp\right)$. □

By using this lemma, we get a formula for the tangent cone of the intersection of a polyhedric set and a hyperplane or half-space.

Lemma 4.2. *Let K be polyhedric at x and let $\mu \in X^\star$ be given. We set*

$$K_\mu := \{y \in K \mid \langle \mu, y - x\rangle = 0\} \quad \text{and} \quad \hat{K}_\mu := \{y \in K \mid \langle \mu, y - x\rangle \le 0\}$$

Then,

$$\begin{aligned} \mathcal{R}_{K_\mu}(x) &= \mathcal{R}_K(x) \cap \mu^\perp, & \mathcal{R}_{\hat{K}_\mu}(x) &= \mathcal{R}_K(x) \cap \mu^\circ, \\ \mathcal{T}_{K_\mu}(x) &= \mathcal{T}_K(x) \cap \mu^\perp, & \mathcal{T}_{\hat{K}_\mu}(x) &= \mathcal{T}_K(x) \cap \mu^\circ, \\ \mathcal{T}_{K_\mu}(x)^\circ &= \operatorname{cl}_\star\big(\mathcal{T}_K(x)^\circ + \operatorname{lin}\{\mu\}\big), & \mathcal{T}_{\hat{K}_\mu}(x)^\circ &= \operatorname{cl}_\star\big(\mathcal{T}_K(x)^\circ + \operatorname{cone}\{\mu\}\big). \end{aligned}$$

Proof. We have

$$\mathcal{R}_K(x) \cap \mu^\perp = \bigcup_{\lambda>0} \lambda\,(K - x) \cap \mu^\perp = \bigcup_{\lambda>0} \lambda\,(K_\mu - x) = \mathcal{R}_{K_\mu}(x).$$

By using the previous lemma, we find

$$\mathcal{T}_K(x) \cap \mu^\perp = \operatorname{cl}\big(\mathcal{R}_K(x) \cap \mu^\perp\big) = \operatorname{cl}\Big(\mathcal{R}_{K_\mu}(x)\Big) = \mathcal{T}_{K_\mu}(x).$$

The formula for $\mathcal{T}_{K_\mu}(x)^\circ$ follows from taking polars. The formulas for $\hat{K}_\mu$ follow from similar considerations. □

In order to study intersections with more than one hyperplane, we introduce the concept of higher-order polyhedricity.

Definition 4.3. Let $K \subset X$ be a closed, convex subset of the Banach space X and let $n \in \mathbb{N}_0$ be given. We call K n-polyhedric at $x \in K$, if

$$\mathcal{T}_K(x) \cap \bigcap_{i=1}^n \mu_i^\perp = \operatorname{cl}\Big(\mathcal{R}_K(x) \cap \bigcap_{i=1}^n \mu_i^\perp\Big) \qquad \forall \mu_1, \ldots, \mu_n \in X^\star \tag{7}$$

holds.

Note that polyhedric sets are 1-polyhedric (by Lemma 4.1) and all closed, convex sets are 0-polyhedric. Similar to Lemma 4.2 we can prove the following result.

Lemma 4.4. *We fix a closed, convex set $K \subset X$ and a polyhedral set*

$$P := \{x \in X \mid \langle \nu_i, x\rangle = c_i, 1 \le i \le n;\ \langle \mu_j, x\rangle \le d_j, 1 \le j \le m\}$$

such that $K \cap P$ is not empty and we fix $x \in K \cap P$. Here, $n, m \in \mathbb{N}_0$, $\nu_i, \mu_j \in X^\star$, $c_i, d_j \in \mathbb{R}$ for $1 \le i \le n$ and $1 \le j \le m$. Moreover, we set $N = m + n$.

To simplify the presentation, we assume that all inequality constraints are active at x, i.e., $\langle \mu_j, x\rangle = d_j$ holds for all $1 \le j \le m$.

Then, $\mathcal{R}_{K\cap P}(x) = \mathcal{R}_K(x) \cap \bigcap_{i=1}^n \nu_i^\perp \cap \bigcap_{j=1}^m \mu_j^\circ$.

In the case that K is N-polyhedric at x, we have

$$\mathcal{T}_{K\cap P}(x) = \mathcal{T}_K(x) \cap \bigcap_{i=1}^n \nu_i^\perp \cap \bigcap_{j=1}^m \mu_j^\circ,$$

$$\mathcal{T}_{K\cap P}(x)^\circ = \operatorname{cl}_\star\big(\mathcal{T}_K(x)^\circ + \operatorname{lin}\{\nu_1, \ldots, \nu_n\} + \operatorname{cone}\{\mu_1, \ldots, \mu_m\}\big).$$

If, additionally, K is $(N+q)$-polyhedric at x for some $q \in \mathbb{N}_0$, then $K \cap P$ is q-polyhedric at x.

Proof. The verification of the formula for $\mathcal{R}_{K\cap P}(x)$ is straightforward.

The inclusion "$\subset$" in the formula for the tangent cone follows from the formula for the radial cone.

Now, let $v \in \mathcal{T}_K(x) \cap \bigcap_{i=1}^n \nu_i^\perp \cap \bigcap_{j=1}^m \mu_j^\circ$ be given. W.l.o.g., we assume that there is $\hat{m} \in \mathbb{N}_0$ such that $\langle \mu_j, v\rangle = 0$ for $1 \le j \le \hat{m}$ and $\langle \mu_j, v\rangle < 0$ for $\hat{m} < j \le m$. Since K is $(n+\hat{m})$-polyhedric, there is a sequence $\{v_k\}_{k\in\mathbb{N}} \subset \mathcal{R}_K(x) \cap \bigcap_{i=1}^n \nu_i^\perp \cap \bigcap_{j=1}^{\hat{m}} \mu_j^\perp$ with $v_k \to v$ in X. Hence, $v_k \in \mathcal{R}_K(x) \cap \bigcap_{i=1}^n \nu_i^\perp \cap \bigcap_{j=1}^m \mu_j^\circ$ for m large enough. Hence, $v \in \mathcal{T}_{K\cap P}(x)$.

The formula for the normal cone follows by taking polars.

The assertion concerning polyhedricity of $K \cap P$ is straightforward to check. $\square$

The following lemma generalizes Lemma 4.1 to higher-order polyhedricity.

Lemma 4.5. *Let $n \ge 1$ be given and assume that $K \subset X$ is $(n-1)$-polyhedric, but not n-polyhedric. Then, there exist $\mu_1, \ldots, \mu_n \in X^\star$ such that*

$$\mu_i \in \operatorname{cl}_\star\big(\mathcal{T}_K(x)^\circ + \operatorname{lin}\{\mu_1, \ldots, \mu_{i-1}\}\big) \qquad \forall i \in \{1, \ldots, n\}$$

and

$$\mathcal{T}_K(x) \cap \bigcap_{i=1}^n \mu_i^\perp \ne \operatorname{cl}\Big(\mathcal{R}_K(x) \cap \bigcap_{i=1}^n \mu_i^\perp\Big).$$

Proof. We prove the result by induction over n.

The case $n = 1$ follows from Lemma 4.1.

Let $n > 1$ be given and assume that the assertion holds for all sets $\tilde{K} \subset X$ which are $(n-2)$-polyhedric, but not $(n-1)$-polyhedric.

Since K is not n-polyhedric, there exist $\lambda_1, \ldots, \lambda_n \subset X^\star$ such that

$$\mathcal{T}_K(x) \cap \bigcap_{i=1}^n \lambda_i^\perp \ne \operatorname{cl}\Big(\mathcal{R}_K(x) \cap \bigcap_{i=1}^n \lambda_i^\perp\Big).$$

Hence, there exist $v \in \mathcal{T}_K(x) \cap \bigcap_{i=1}^n \lambda_i^\perp$ and $\varepsilon > 0$, such that

$$B_\varepsilon(v) \cap \mathcal{R}_K(x) \cap \bigcap_{i=1}^n \lambda_i^\perp = \emptyset. \tag{8}$$

We define the bounded, linear map $P\colon X \to \mathbb{R}^n$, $x \mapsto (\langle \lambda_1, x\rangle, \ldots, \langle \lambda_n, x\rangle)$. Next, we define the convex sets

$$S := B_\varepsilon(v) \cap \mathcal{R}_K(x) \subset X \quad \text{and} \quad M := PS \subset \mathbb{R}^n.$$

By (8) we have $0 \notin M$ and it is clear that the set M is convex. Hence, we can separate M and 0, i.e., there exists $t \in \mathbb{R}^n \setminus \{0\}$, such that $M \subset t^\circ$, see [6, Thm. 2.17]. Since $t \neq 0$, we can find a regular matrix $R \in \mathbb{R}^{n\times n}$, such that $R_{1i} = t_i$ for all $i = 1, \ldots, n$. We define

$$\nu_i := \sum_{j=1}^n R_{ij}\lambda_j \qquad \forall i \in \{1, \ldots, n\}$$

and by the regularity of R we get $\operatorname{lin}\{\lambda_1, \ldots, \lambda_n\} = \operatorname{lin}\{\nu_1, \ldots, \nu_n\}$ and therefore $\bigcap_{i=1}^n \lambda_i^\perp = \bigcap_{i=1}^n \nu_i^\perp$.

As a next step, we show that $\nu_1 \in \mathcal{T}_K(x)^\circ$. To the contrary, assume that $\nu_1 \notin \mathcal{T}_K(x)^\circ$. Then, there exists $w \in \mathcal{R}_K(x)$ with $\langle \nu_1, w\rangle > 0$. We set $h = (\langle \nu_i, w\rangle)_{i=1}^n \in \mathbb{R}^n$. Since $h \neq 0$, it is possible to choose vectors $g_1, \ldots, g_{n-1} \in \mathbb{R}^n$, such that

$$\operatorname{lin}\{h\} = \bigcap_{i=1}^{n-1} g_i^\perp.$$

Since $v \in \mathcal{T}_K(x) \cap \bigcap_{i=1}^n \lambda_i^\perp = \mathcal{T}_K(x) \cap \bigcap_{i=1}^n \nu_i^\perp \subset \mathcal{T}_K(x) \cap \bigcap_{i=1}^{n-1} \Big(\sum_{j=1}^n (g_i)_j \nu_j\Big)^\perp$, and since K is $(n-1)$-polyhedric, there exists a sequence

$$\{v_k\} \subset \mathcal{R}_K(x) \cap \bigcap_{i=1}^{n-1} \Big(\sum_{j=1}^n (g_i)_j \nu_j\Big)^\perp,$$

with $v_k \to v$. Then, we have for all $i \in \{1, \ldots, n-1\}$

$$\big(g_i,\, (\langle \nu_j, v_k\rangle)_{j=1}^n\big)_{\mathbb{R}^n} = \sum_{j=1}^n (g_i)_j \,\langle \nu_j, v_k\rangle = \Big\langle \sum_{j=1}^n (g_i)_j\, \nu_j,\, v_k \Big\rangle = 0.$$

Hence, $(\langle \nu_j, v_k\rangle)_{j=1}^n$ belongs to the linear hull of h. Thus, there exists a sequence $\{\alpha_k\} \subset \mathbb{R}$ such that

$$\langle \nu_i, v_k\rangle = \alpha_k \,\langle \nu_i, w\rangle \qquad \forall i = 1, \ldots, n. \tag{9}$$

For $i = 1$, we find $\langle \nu_1, v_k \rangle = \alpha_k \langle \nu_1, w \rangle$. Since $v_k \in S$ (it holds for k large enough and w.l.o.g. we can drop the terms in the sequence for which $v_k \notin S$), we have

$$\langle \nu_1, v_k \rangle = \sum_{j=1}^{n} R_{1j} \langle \lambda_j, v_k \rangle = \sum_{j=1}^{n} t_j \langle \lambda_j, v_k \rangle = (t, P v_k)_{\mathbb{R}^m} \leq 0,$$

since $P v_k \in PS = M \subset t^\circ$. Hence, $\alpha_k = \langle \nu_1, v_k \rangle / \langle \nu_1, w \rangle \leq 0$ and $\alpha_k \to 0$, since $\langle \nu_1, v \rangle = 0$ and $v_k \to v$. Then, the sequence $\{v_k - \alpha_k w\}_{k \in \mathbb{N}}$ belongs to $\mathcal{R}_K(x)$ and converges towards v. By (9) we have $\langle \nu_i, v_k - \alpha_k w \rangle = 0$ for all $i = 1, \ldots, n$ and $k \in \mathbb{N}$. Hence,

$$v_k - \alpha_k w \in \mathcal{R}_K(x) \cap \bigcap_{i=1}^{n} \nu_i^\perp = \mathcal{R}_K(x) \cap \bigcap_{i=1}^{n} \lambda_i^\perp$$

and this is a contradiction to (8). Hence, we have $\nu_1 \in \mathcal{T}_K(x)^\circ$.

Now, we set $\mu_1 := \nu_1$. From Lemma 4.4, we find that the set $K_{\mu_1} := \{y \in K \mid \langle x - y, \mu_1 \rangle = 0\}$ is $(n-2)$-polyhedric and

$$v \in \mathcal{T}_K(x) \cap \bigcap_{i=1}^{n} \nu_i = \mathcal{T}_{K_{\mu_1}}(x) \cap \bigcap_{i=2}^{n} \nu_i$$
$$v \notin \operatorname{cl}\big(\mathcal{R}_K(x) \cap \bigcap_{i=1}^{n} \nu_i\big) = \operatorname{cl}\big(\mathcal{R}_{K_{\mu_1}}(x) \cap \bigcap_{i=2}^{n} \nu_i\big),$$

see Lemma 4.2. Hence, K_{μ_1} is not $(n-1)$-polyhedric. By the induction hypothesis there exist $\mu_2, \ldots, \mu_n \in X$, such that

$$\mu_i \in \operatorname{cl}_\star \big(\mathcal{T}_{K_{\mu_1}}(x)^\circ + \operatorname{lin}\{\mu_2, \ldots, \mu_{i-1}\}\big) = \operatorname{cl}_\star \big(\mathcal{T}_K(x)^\circ + \operatorname{lin}\{\mu_1, \ldots, \mu_{i-1}\}\big)$$

holds for all $n = 2, \ldots, n$ and

$$\mathcal{T}_K(x) \cap \bigcap_{i=1}^{n} \mu_i^\perp = \mathcal{T}_{K_{\mu_1}}(x) \cap \bigcap_{i=1}^{n} \mu_i^\perp \neq \operatorname{cl}\big(\mathcal{R}_{K_{\mu_1}}(x) \cap \bigcap_{i=1}^{n} \mu_i^\perp\big) = \operatorname{cl}\big(\mathcal{R}_K(x) \cap \bigcap_{i=1}^{n} \mu_i^\perp\big).$$

This finishes the induction step. □

Hence, in order to check 2-polyhedricity of a polyhedric set, it is sufficient to show

$$\mathcal{T}_K(x) \cap \mu_1^\perp \cap \mu_2^\perp = \operatorname{cl}\big(\mathcal{R}_K(x) \cap \mu_1^\perp \cap \mu_2^\perp\big)$$

for $\mu_1 \in \mathcal{T}_K(\bar{x})^\circ$ and $\mu_2 \in \operatorname{cl}_\star \big(\mathcal{T}_K(\bar{x})^\circ + \operatorname{lin}\{\mu_1\}\big)$. However, it is not possible to verify this condition in the case of a general polyhedric set. Indeed, we will show the existence of polyhedric sets which are not 2-polyhedric in Example 4.24.

4.2. Intersections of a set with bounds with a polyhedral set

In this section, we study sets which are compatible to the lattice structure of a Banach space. The precise assumption on this lattice structure is given in the following definition and Assumption 4.7.

Definition 4.6. Let X be a (real) Banach space and $C \subset X$ be a closed convex cone which is pointed, i.e., which satisfies $C \cap -C = \{0\}$. For $x, y \in X$ we say $x \geq y$ if and only if $x - y \in C$.

We say that $z \in X$ is the supremum of $x, y \in X$ if $z \geq x$, $z \geq y$ and

$$(w \geq x \text{ and } w \geq y) \quad \Rightarrow \quad w \geq z \qquad \forall w \in X.$$

It is easy to see that the supremum (if it exists) is unique and it will be denoted by $\max(x, y)$. If the supremum of all $x, y \in X$ exists, we say that X is a *vector lattice*.

It is easy to check that the relation $\geq$ on X is antisymmetric ($x \geq y$ and $y \geq x$ implies $x = y$), transitive ($x \geq y$ and $y \geq z$ implies $x \geq z$) and reflexive ($x \geq x$ for all $x \in X$). Further, it is compatible with the linear structure on X, i.e.,

$$\begin{aligned} x \geq y \quad &\Longrightarrow \quad x + z \geq y + z \quad && \forall x, y, z \in X, \\ x \geq y \quad &\Longrightarrow \quad \alpha\, x \geq \alpha\, y && \forall x, y \in X,\ \alpha \geq 0, \\ x \geq y \quad &\Longrightarrow \quad \alpha\, y \geq \alpha\, x && \forall x, y \in X,\ \alpha \leq 0. \end{aligned}$$

For convenience, we will use the natural notations

$$\min(x, y) := -\max(-x, -y) \quad \text{and} \quad |x| := \max(x, 0) - \min(x, 0),$$

for $x, y \in X$. Note that

$$x = x + \max(0, -x) - \max(-x, 0) = \max(x, 0) + \min(x, 0).$$

This readily implies

$$\begin{aligned} x + y = x - y + 2y &= \max(x - y, 0) + \min(x - y, 0) + 2y \\ &= \max(x, y) + \min(x, y). \end{aligned} \tag{10}$$

For more information on vector lattices, we refer to [31, Chapter II].

In what follows, we need a slight assumption on the continuity of $\max(\cdot, \cdot)$ and this will be a standing assumption in this section.

Assumption 4.7. *We assume that X is a vector lattice (induced by the closed, convex cone C). Moreover, we assume that for all sequences $\{x_n\}_{n \in \mathbb{N}} \subset X$ with $x_n \to x$ in X we have*

$$\max(0, x_n) \rightharpoonup \max(0, x) \quad \textit{in } X.$$

That is, $\max(0, \cdot)$ *is assumed to be strongly-weakly sequentially continuous.*

In many important cases, the mapping $\max(0,\cdot)$ is even strongly-strongly continuous, see Example 4.21. Further, we note that Assumption 4.7 even implies (strong-weak) continuity in both arguments, since

$$\max(x_n, y_n) = \max(x_n - y_n, 0) + y_n \rightharpoonup \max(x - y, 0) + y = \max(x, y)$$

for $x_n \to x$, $y_n \to y$.

We emphasize that we do not require the stronger property that X is a Banach lattice, which would amount to

$$|x| \le |y| \quad \Rightarrow \quad \||x|\|_X \le \||y|\|_X \qquad \forall x, y \in X,$$

see [31, Section II.5]. Indeed, the important space $X = H_0^1(\Omega)$, where $\Omega \subset \mathbb{R}^d$ is a bounded, open set, equipped with its natural, pointwise ordering satisfies Assumption 4.7, but it is *not* a Banach lattice.

A convenient result to show the satisfaction of Assumption 4.7 is the following results, which is essentially [14, Theorem 3]. We give the proof for the convenience of the reader.

Lemma 4.8. *Let X be a reflexive Banach space which is additionally a vector lattice. Suppose that $x \mapsto \max(x, 0)$ is bounded, i.e., there exists $M > 0$ with $\|\max(x, 0)\|_X \le M \|x\|_X$ for all $x \in X$. Then, Assumption 4.7 is satisfied.*

Proof. Let $\{x_n\}_{n\in\mathbb{N}} \subset X$ be a sequence with $x_n \to x$ in X. The boundedness of $x \mapsto \max(x, 0)$ implies the boundedness of $\{\max(x_n, 0)\}_{n\in\mathbb{N}}$. Due to the reflexivity of X, there is a subsequence (without relabeling) such that $\max(x_n, 0) \rightharpoonup y$ in X for some $y \in X$. It remains to show $y = \max(x, 0)$. By the definition of the maximum, we have

$$\max(x_n, 0) + \max(x - x_n, 0) \ge x_n + (x - x_n) = x.$$

Together with

$$\max(x_n, 0) + \max(x - x_n, 0) \ge 0 + 0 = 0$$

we infer

$$\max(x_n, 0) + \max(x - x_n, 0) \ge \max(x, 0).$$

Since C is closed and convex, it is weakly closed and we can pass to the weak limit to obtain $y + 0 \ge \max(x, 0)$, since $\|\max(x - x_n, 0)\|_X \le M \|x - x_n\|_X \to 0$. Similarly, we show

$$\max(x, 0) + \max(x_n - x, 0) \ge \max(x_n, 0)$$

and $n \to \infty$ leads to $\max(x, 0) + 0 \ge y$. In consequence, $y = \max(x, 0)$. The uniqueness of the weak limit ensures the weak convergence of the entire sequence $\{\max(x_n, 0)\}_{n\in\mathbb{N}}$. $\square$

Should X be additionally a Hilbert space, the boundedness of $x \mapsto \max(x, 0)$ follows from $\big(\max(x, 0), \min(x, 0)\big)_X \ge 0$ for all $x \in X$, see [14, Corollary 1].

In what follows, we will consider sets with bounds in the sense of the following definition.

Definition 4.9. We assume that the Banach space X satisfies Assumption 4.7. Let $\underline{K} \subset X$ be a closed convex set. We say that $\underline{K}$ is a *set with lower bound*, if $x, y \in \underline{K}$ implies $\min(x, y) \in \underline{K}$ and if we have $\underline{K} + C \subset \underline{K}$. Similarly, a closed convex set $\overline{K} \subset X$ is called a *set with upper bound*, if $-\overline{K}$ is a set with lower bound. Finally, $K \subset X$ is said to be a *set with bounds*, if it can be written as the intersection of a set with upper bound and a set with lower bound.

We note that Mignot uses the same definition in the special case of *Dirichlet spaces*, cf. [22, Définition 3.2].

We give some examples illustrating Definition 4.9. For all $x \in X$, the set

$$x + C = \{y \in X \mid y \geq x\}$$

is a set with lower bound. Similarly, X itself is a set with lower bound.

It is also possible to consider the case that the lower bound does not belong to X. To illustrate this situation, we use the Lebesgue space $X := L^2(0, 1)$ with the natural, pointwise ordering, and pick a measurable function $x\colon (0, 1) \to \mathbb{R} \cup \{-\infty\}$. Then,

$$\{y \in X \mid y \geq x \text{ a.e. in } (0, 1)\}$$

is a set with lower bound, even in case that $x \notin L^2(0, 1)$.

It is easy to see that if $\underline{K}$ is a set with lower bound, $x, y \in \underline{K}$ implies $\max(x, y) \in \underline{K}$, since $\max(x, y) = x + \max(0, y - x) \in x + C \subset \underline{K}$. Hence, $\underline{K}$ is closed under taking maximums and minimums.

The next assumption is a standing assumption in this section.

Assumption 4.10. *We suppose that Assumption 4.7 is satisfied. The sets $\underline{K}$ and $\overline{K}$ denote sets with lower and upper bound, respectively. We set $K := \underline{K} \cap \overline{K}$ and fix some $x \in K$.*

Now, we give some straightforward results concerning the structure of the sets $\underline{K}$, $\overline{K}$ and K.

Lemma 4.11. *For $v \in \mathcal{R}_{\underline{K}}(x)$, we have* $\max(v, 0), \min(v, 0) \in \mathcal{R}_{\underline{K}}(x)$.
For $v \in \mathcal{R}_{\overline{K}}(x)$, we have $\max(v, 0), \min(v, 0) \in \mathcal{R}_{\overline{K}}(x)$.
For $v \in \mathcal{R}_{K}(x)$, we have $\max(v, 0), \min(v, 0) \in \mathcal{R}_{K}(x)$.

Proof. Due to $\mathcal{R}_{\overline{K}}(x) = -\mathcal{R}_{-\overline{K}}(x)$ and $\mathcal{R}_K(x) = \mathcal{R}_{\underline{K}}(x) \cap \mathcal{R}_{\overline{K}}(x)$, it is sufficient to prove the claim for $v \in \mathcal{R}_{\underline{K}}(x)$.

For a given $v \in \mathcal{R}_{\underline{K}}(x)$, there is $t > 0$, such that $x + t\,v \in \underline{K}$. Together with $x \in \underline{K}$, we find $x + t \max(v, 0) = \max(x, x + t\,v) \in \underline{K}$ and $x + t \min(v, 0) = \min(x, x + t\,v) \in \underline{K}$. This shows $\max(v, 0), \min(v, 0) \in \mathcal{R}_{\underline{K}}(x)$. □

In the following, we will often use the well-known fact that closed, convex sets are weakly sequentially closed. That is, if the set $G \subset X$ is closed and convex, and if the sequence $\{v_n\} \subset G$ satisfies $v_n \rightharpoonup v$, then $v \in G$.

Lemma 4.12. *For* $v \in \mathcal{T}_{\underline{K}}(x)$*, we have* $\max(v, 0), \min(v, 0) \in \mathcal{T}_{\underline{K}}(x)$*.*
For $v \in \mathcal{T}_{\overline{K}}(x)$*, we have* $\max(v, 0), \min(v, 0) \in \mathcal{T}_{\overline{K}}(x)$*.*

Proof. Similar to the proof of Lemma 4.11, it is sufficient to consider the case $v \in \mathcal{T}_{\underline{K}}(x)$. By definition, there is a sequence $\{v_n\} \subset \mathcal{R}_{\underline{K}}(x)$, such that $v_n \to v$ as $n \to \infty$. Using Assumption 4.7, we get

$$\max(v_n, 0) \rightharpoonup \max(v, 0) \quad \text{and} \quad \min(v_n, 0) \rightharpoonup \min(v, 0).$$

By Lemma 4.11, we have $\max(v_n, 0), \min(v_n, 0) \in \mathcal{R}_{\underline{K}}(x) \subset \mathcal{T}_{\underline{K}}(x)$. Closedness and convexity of $\mathcal{T}_{\underline{K}}(x)$ yield $\max(v, 0), \min(v, 0) \in \mathcal{T}_{\underline{K}}(x)$. □

The following result generalizes [22, Lemme 3.4(ii)]. Note that, in general, the tangent cone to an intersection is (strictly) smaller than the intersection of the tangent cones.

Lemma 4.13. *We have* $\mathcal{R}_K(x) = \mathcal{R}_{\underline{K}}(x) \cap \mathcal{R}_{\overline{K}}(x)$ *and* $\mathcal{T}_K(x) = \mathcal{T}_{\underline{K}}(x) \cap \mathcal{T}_{\overline{K}}(x)$*. In particular,* $v \in \mathcal{T}_K(x)$ *implies* $\max(v, 0), \min(v, 0) \in \mathcal{T}_K(x)$*.*

Proof. Since $K = \underline{K} \cap \overline{K}$, we have

$$\mathcal{R}_K(x) = \mathcal{R}_{\underline{K}}(x) \cap \mathcal{R}_{\overline{K}}(x) \quad \text{and} \quad \mathcal{T}_K(x) \subset \mathcal{T}_{\underline{K}}(x) \cap \mathcal{T}_{\overline{K}}(x).$$

To show the reverse inclusion, let $v \in \mathcal{T}_{\underline{K}}(x) \cap \mathcal{T}_{\overline{K}}(x)$ be given. Then, there exist sequences $\{v_n\} \subset \mathcal{R}_{\underline{K}}(x)$ and $\{w_n\} \subset \mathcal{R}_{\overline{K}}(x)$, such that $v_n \to v$ and $w_n \to v$. In particular, $\min(v_n, 0) + \max(w_n, 0) \rightharpoonup v$. By Lemma 4.11, we have $\min(v_n, 0) \in \mathcal{R}_{\underline{K}}(x)$ and together with $\min(v_n, 0) \in -C$, we have $\min(v_n, 0) \in \mathcal{R}_{\underline{K}}(x) \cap \mathcal{R}_{\overline{K}}(x) = \mathcal{R}_K(x)$. Similarly, $\max(w_n, 0) \in \mathcal{R}_K(x)$ follows. Hence, $\min(v_n, 0) + \max(w_n, 0) \in \mathcal{R}_K(x)$. This shows $v \in \mathcal{T}_K(x)$.

Together with Lemma 4.12, we get $\max(v, 0), \min(v, 0) \in \mathcal{T}_K(x)$ for $v \in \mathcal{T}_K(x)$. □

The next result shows that the radial cone of K is compatible with the order structure of X.

Lemma 4.14. *Let* $u, v, w \in X$ *with* $u \ge v \ge w$ *and* $u, w \in \mathcal{R}_K(x)$ *be given. Then,* $v \in \mathcal{R}_K(x)$*.*

Proof. For $u, w \in \mathcal{R}_K(x)$, there is $t > 0$ with $x + t\,u, x + t\,w \in K$. From $x + t\,u \ge x + t\,v \ge x + t\,w$ we find $v \in \mathcal{R}_K(x)$. □

The next lemma demonstrates a possibility to define a projection onto the set $\{v \in X \mid u \ge v \ge w\}$ for $v \ge w$.

Lemma 4.15. *Let* $u, v, w \in X$ *be given such that* $u \ge w$*. Then,* $u \ge \min(u, v) + \max(w, v) - v \ge w$*.*

Proof. We set $\tilde{v} := \min(u, v) + \max(w, v) - v$. From $u \ge w$, we get $\max(u, v) \ge \max(w, v)$. Using (10) gives $u \ge \min(u, v) + \max(w, v) - v = \tilde{v}$. Similarly, $u \ge w$, implies $\min(u, v) \ge \min(w, v)$ and, consequently, $\tilde{v} = \min(u, v) + \max(v, w) - v \ge w$. Thus, we have shown $u \ge \tilde{v} \ge w$. □

Note that the mapping $v \mapsto \min(u, v) + \max(w, v) - v$ is strongly-weakly continuous due to Assumption 4.7.

The following lemma is the analogue to Lemma 4.14 for the tangent cone. It provides some further insight into the structure of the set K.

Lemma 4.16. *Let $u, v, w \in X$ with $u \ge v \ge w$ and $u, w \in \mathcal{T}_K(x)$ be given. Then, $v \in \mathcal{T}_K(x)$.*

Proof. We start with sequences $\{\tilde u_n\}_{n\in\mathbb{N}}, \{\tilde w_n\}_{n\in\mathbb{N}} \subset \mathcal{R}_K(x)$ with $\tilde u_n \to u$ and $\tilde w_n \to w$. By Assumption 4.7, we have $\min(\tilde w_n, \tilde u_n) \rightharpoonup w$. By applying Mazur's lemma to the sequence $\{(\tilde u_n, \min(\tilde w_n, \tilde u_n))\}_{n\in\mathbb{N}}$, we get the sequences $\{u_n\}_{n\in\mathbb{N}}$, $\{w_n\}_{n\in\mathbb{N}} \subset \mathcal{R}_K(x)$ with $u_n \to u$, $w_n \to w$ and $u_n \ge w_n$.

Now we set $v_n := \min(u_n, v) + \max(w_n, v) - v$. Using Lemma 4.15 we infer $u_n \ge v_n \ge w_n$ and from Lemma 4.14 we get $v_n \in \mathcal{R}_K(x) \subset \mathcal{T}_K(x)$.

Finally, Assumption 4.7 implies $v_n \rightharpoonup \min(u, v) + \max(w, v) - v = v \in \mathcal{T}_K(x)$. $\square$

Now we provide an auxiliary lemma which will be the crucial ingredient in the proof of Theorem 4.18 below. It contains all the structure of K from Assumption 4.10 in the sense that Assumption 4.10 is not explicitly used in Theorem 4.18 below. It is inspired by the proof of [22, Théorème 3.2].

Lemma 4.17. *Let $v \in \mathcal{T}_K(x)$ be given. Then there exists a sequence $\{v_k\}_{k\in\mathbb{N}} \subset \mathcal{R}_K(x) \cap \big(v - \mathcal{T}_K(x)\big)$ with $v_k \to v$. That is,*

$$v \in \operatorname{cl}\big(\mathcal{R}_K(x) \cap \big(v - \mathcal{T}_K(x)\big)\big)$$

holds for all $v \in \mathcal{T}_K(x)$.

Proof. We take a sequence $\{w_k\} \subset \mathcal{R}_K(x)$, such that $w_k \to v$. Then we define $v_k^+ := \min(\max(v, 0), w_k) + \max(0, w_k) - w_k$. By Assumption 4.7, we get $v_k^+ \rightharpoonup \min\big(\max(v, 0), \max(v, 0)\big) = \max(v, 0)$ and $\{v_k^+\} \subset \mathcal{R}_K(x)$ follows from $\max(w_k, 0) \ge v_k^+ \ge 0$ and $\max(w_k, 0) \in \mathcal{R}_K(x)$, see Lemmas 4.14, 4.15 and 4.11.

Next, we show $\max(v, 0) - v_k^+ \in \mathcal{T}_K(x)$. By Lemma 4.15, we find $\max(v, 0) \ge v_k^+ \ge 0$ and this implies $\max(v, 0) \ge \max(v, 0) - v_k^+ \ge 0$. Hence, $\max(v, 0) - v_k^+ \in \mathcal{T}_K(x)$ by Lemmas 4.13 and 4.16.

To summarize, the sequence $\{v_k^+\}$ satisfies $v_k^+ \rightharpoonup \max(v, 0)$, $v_k^+ \in \mathcal{R}_K(x)$ as well as $\max(v, 0) - v_k^+ \in \mathcal{T}_K(x)$ for all $k \in \mathbb{N}$.

Similarly, we can show that $v_k^- := \min(0, w_k) + \max(\min(v, 0), w_k) - w_k$ satisfies $v_k^- \rightharpoonup \min(v, 0)$, $v_k^- \in \mathcal{R}_K(x)$ and $\min(v, 0) - v_k^- \in \mathcal{T}_K(x)$. Now, we define $v_k := v_k^+ + v_k^-$ and it is easy to check that $v_k \rightharpoonup v$, $v_k \in \mathcal{R}_K(x)$ and $v - v_k \in \mathcal{T}_K(x)$, i.e., $v_k \in v - \mathcal{T}_K(x)$, hold for all $k \in \mathbb{N}$.

Thus, we have established $v_k \rightharpoonup v$ and $v_k \in \operatorname{cl}\big(\mathcal{R}_K(x) \cap \big(v - \mathcal{T}_K(x)\big)\big)$. Since this set is closed and convex, we infer $v \in \operatorname{cl}\big(\mathcal{R}_K(x) \cap \big(v - \mathcal{T}_K(x)\big)\big)$. $\square$

Now we are able to prove the main theorem of this section.

Theorem 4.18. *Let Assumption* 4.10 *be satisfied by K. Then, the set K is n-polyhedric for all $n \in \mathbb{N}_0$.*

Proof. By induction over $n \in \mathbb{N}_0$, we prove that K is n-polyhedric at all $x \in K$.

Assume that K is n-polyhedric for some $n \in \mathbb{N}_0$. We proceed by contradiction and assume that K is not $(n+1)$-polyhedric. Hence, we can invoke Lemma 4.5 and get $\mu_1, \ldots, \mu_{n+1} \in X^\star$ such that $\mu_i \in \operatorname{cl}_\star\big(\mathcal{T}_K(x)^\circ + \operatorname{lin}\{\mu_1, \ldots, \mu_{i-1}\}\big)$ for all $i \in \{1, \ldots, n+1\}$ and

$$\mathcal{T}_K(x) \cap \bigcap_{i=1}^{n+1} \mu_i^\perp \neq \operatorname{cl}\Big(\mathcal{R}_K(x) \cap \bigcap_{i=1}^{n+1} \mu_i^\perp\Big). \tag{11}$$

Let $v \in \mathcal{T}_K(x) \cap \bigcap_{i=1}^{n+1} \mu_i^\perp$ be arbitrary. Using Lemma 4.17, we get a sequence $\{v_k\} \subset \mathcal{T}_K(x)$ such that $v_k \in \mathcal{R}_K(x)$, $v - v_k \in \mathcal{T}_K(x)$ and $v_k \to v$.

Now, we use induction over j to show that

$$v_k \in \bigcap_{i=1}^{j} \mu_i^\perp \qquad \forall k \in \mathbb{N} \tag{12}$$

holds for all $j \in \{0, \ldots, n+1\}$. The case $j = 0$ is trivially satisfied.

Now, suppose that (12) holds for some $j \in \{0, \ldots, n\}$. Since

$$\mu_{j+1} \in \operatorname{cl}_\star\big(\mathcal{T}_K(x)^\circ + \operatorname{lin}\{\mu_1, \ldots, \mu_j\}\big) = \Big(\mathcal{T}_K(x) \cap \bigcap_{i=1}^{j} \mu_j^\perp\Big)^\circ, \tag{13}$$

we have $0 \geq \langle \mu_{j+1}, v - v_k\rangle = \langle \mu_{j+1}, -v_k\rangle \geq 0$, where we have used (12), (13), $v \in \bigcap_{i=1}^{j+1} \mu_i^\perp$ and $v_k, v - v_k \in \mathcal{T}_K(x)$. Hence, $v_k \in \mu_{j+1}^\perp$. By induction, (12) holds for all $j \in \{0, \ldots, n+1\}$.

Hence, the sequence $\{v_k\}$ belongs to $\mathcal{R}_K(x) \cap \bigcap_{i=1}^{n+1} \mu_i^\perp$ and converges towards v. Thus, $v \in \operatorname{cl}\big(\mathcal{R}_K(x) \cap \bigcap_{i=1}^{n+1} \mu_i^\perp\big)$ and this is a contradiction to (11). Therefore, K is $(n+1)$-polyhedric and this finishes the induction over n. $\square$

Before we give some remarks, we mention an easy corollary which follows together with Lemma 4.4.

Corollary 4.19. *Let $P \subset X$ be a polyhedral set with $x \in P$. Then, $K \cap P$ is polyhedric and*

$$\mathcal{T}_{K\cap P}(x) = \mathcal{T}_K(x) \cap \mathcal{T}_P(x), \quad \textit{and} \quad \mathcal{T}_{K\cap P}(x)^\circ = \operatorname{cl}_\star\big(\mathcal{T}_K(x)^\circ \cap \mathcal{T}_P(x)^\circ\big).$$

Now we consider some remarks concerning Theorem 4.18.

First, we mention that Theorem 4.18 extends previously known results significantly:

1. Mignot [22, Théorème 3.2] shows the polyhedricity of sets with bounds in Dirichlet spaces. In the proof, he uses a pointwise characterization of the tangent cone $\mathcal{T}_K(x)$, which is not available in our general setting.
2. Haraux [14, Corollary 2] shows the polyhedricity of the cone of non-negative vectors $C \subset X$ under Assumption 4.7 in the case that X is a Hilbert space. This situation is much easier to handle, since there is only a lower bound and this set C is even a cone.

Moreover, both results show only the polyhedricity of the set under consideration, whereas our result also enables us to show n-polyhedricity, which, in turn, provides additionally the polyhedricity of the intersection with a polyhedral set, see Corollary 4.19. To our knowledge, even in special cases, such a result was not available.

Second, by inspecting the proof of Theorem 4.18, we see that we only invoked Lemma 4.5, which holds for arbitrary closed, convex sets K, and Lemma 4.17. In particular, the lattice structure of the underlying space X was not explicitly used. Hence, the proof of Theorem 4.18 provides the n-polyhedricity at $x \in K$ of all closed, convex sets K satisfying

$$v \in \operatorname{cl}\big(\mathcal{R}_K(x) \cap \big(v - \mathcal{T}_K(x)\big)\big) \qquad \forall v \in \mathcal{T}_K(x). \tag{14}$$

We emphasize that no further assumptions on K are necessary. This enables us to show polyhedricity also in situations in which a lattice structure is not available. We demonstrate this possibility in the following lemma, which significantly relaxes the requirements of [5, Proposition 4.3], and also shows the n-polyhedricity of K.

Lemma 4.20. *Let $(\Omega, \Sigma, \mathfrak{m})$ be a complete, σ-finite measure space. Further, let $Q\colon \Omega \rightrightarrows \mathbb{R}^d$ be a measurable (in the sense of* [3, *Definition* 8.1.1]*), set-valued map such that $Q(\omega) \subset \mathbb{R}^d$ is a polyhedron for a.a. $\omega \in \Omega$. Then, for $1 \le p < \infty$, the set*

$$K = \{u \in L^p(\mathfrak{m}; \mathbb{R}^d) \mid u(\omega) \in Q(\omega) \text{ for a.a. } \omega \in \Omega\}$$

is n-polyhedric at all $u \in K$ for all $n \in \mathbb{N}_0$.

Proof. We follow the proof of [5, Proposition 4.3]. It is easily checked that K is closed in $L^p(\mathfrak{m}; \mathbb{R}^d)$ and convex. Let $v \in \mathcal{T}_K(u)$ be given. For $k \in \mathbb{N}$, we define

$$v_k(\omega) := \begin{cases} v(\omega) & \text{if } u(\omega) + \frac{1}{k} v(\omega) \in Q(\omega), \\ 0 & \text{else.} \end{cases}$$

From the measurability of Q, we get the measurability of v_k. Moreover, $v_k \in \mathcal{R}_K(u)$ by construction. Owing to [3, Corollary 8.5.2], we easily find $v - v_k \in \mathcal{T}_K(u)$. Since $Q(\omega)$ is a polyhedron and $v(\omega) \in \mathcal{T}_{Q(\omega)}(u(\omega)) = \mathcal{R}_{Q(\omega)}(u(\omega))$, we obtain the pointwise convergence $v_k(\omega) \to v(\omega)$ for a.a. $\omega \in \Omega$. Now, $v_k \to v$ in $L^p(\mathfrak{m}; \mathbb{R}^d)$ follows from the dominated convergence theorem, since $|v_k(\omega)|_{\mathbb{R}^d} \le |v(\omega)|_{\mathbb{R}^d}$. Hence, condition (14) is satisfied by the set K and the n-polyhedricity follows as in the proof of Theorem 4.18. □

We mention that the polyhedrality assertion on $Q(\omega)$ can be relaxed to $\mathcal{R}_{Q(\omega)}(u) = \mathcal{T}_{Q(\omega)}(u)$ for all $u \in Q(\omega)$ and a.e. $\omega \in \Omega$. Note that, in contrast to the proofs of Lemma 4.17 and Theorem 4.18, we did use the actual characterization of $\mathcal{T}_K(u)$ in the above proof. We do not know if the result of Lemma 4.20 holds for $p = \infty$. Moreover, we mention that the above proof does not generalize to the case of $W^{1,p}(\Omega;\mathbb{R}^d)$, $p \in [1,\infty)$.

We close this section by giving some examples to which Theorem 4.18 can be applied.

Example 4.21. (1) Let $(\Omega, \Sigma, \mathfrak{m})$ be a complete, σ-finite measure space, $a, b : \Omega \to \mathbb{R} \cup \{\pm\infty\}$ measurable and $p \in [1,\infty]$. Then,

$$K = \{u \in L^p(\mathfrak{m}) \mid a \le u \le b \text{ a.e.}\}$$

is n-polyhedric for all $n \in \mathbb{N}_0$. Here, the lattice property is evident and Assumption 4.7 follows from the fact that $\max(\cdot, 0)$ is globally Lipschitz on $\mathbb{R}$.

(2) Let $\Omega \subset \mathbb{R}^d$ be an open, bounded set (equipped with the Lebesgue measure), $p \in [1,\infty)$ and $a, b \colon \Omega \to \mathbb{R} \cup \{\pm\infty\}$ measurable. Then,

$$K = \{u \in W^{1,p}_0(\Omega) \mid a \le u \le b \text{ a.e.}\}$$

is n-polyhedric for all $n \in \mathbb{N}_0$. The lattice property follows from Stampacchia's lemma and Assumption 4.7 (with strong-strong continuity) from Lebesgue's dominated convergence theorem, cf. [6, Proposition 6.45] and [2, Theorem 5.8.2].

(3) Let $\Omega \subset \mathbb{R}^d$ be an open, bounded set (equipped with the Lebesgue measure), $p \in [1,\infty)$ and $a, b \colon \Omega \to \mathbb{R} \cup \{\pm\infty\}$ measurable. Then,

$$K = \{u \in W^{1,p}(\Omega) \mid a \le u \le b \text{ a.e.}\}$$

is n-polyhedric for all $n \in \mathbb{N}_0$. As in the case of $W^{1,p}_0(\Omega)$, the lattice property follows from Stampacchia's lemma and Assumption 4.7 (with strong-strong continuity) from Lebesgue's dominated convergence theorem.

(4) We consider the Besov space $B^s_{p,q}(\mathbb{R}^d)$ with $p, q \in (1,\infty)$ and $s \in (0, 1 + 1/p)$. Then, we know from [8, Théorème 2] and [26, Theorem 1] that $u \mapsto \max(u, 0)$ is a bounded operator from $B^s_{p,q}(\mathbb{R}^d)$ into itself. Now, since $B^s_{p,q}(\mathbb{R}^d)$ is reflexive, we can use Lemma 4.8 to ensure the satisfaction of Assumption 4.7. Hence, we obtain the n-polyhedricity of

$$K = \{u \in B^s_{p,q}(\mathbb{R}^d) \mid a \le u \le b \text{ a.e.}\},$$

where $a, b \colon \mathbb{R}^d \to \mathbb{R} \cup \{\pm\infty\}$ are measurable.
If $\Omega \subset \mathbb{R}^d$ is a bounded Lipschitz domain, we can use the extension operator from [30] to get the boundedness of $u \mapsto \max(u, 0)$ from $B^s_{p,q}(\Omega)$ into itself. Hence, we also get n-polyhedricity of sets with pointwise bounds in $B^s_{p,q}(\Omega)$. The same reasoning can be used in the Triebel-Lizorkin spaces $F^s_{p,q}(\mathbb{R}^d)$ and $F^s_{p,q}(\Omega)$, see [29, Theorem 5.4.1].

Finally, we mention that in most of these cases, in particular if $s \in (1, 1+1/p)$, the strong-strong continuity of $u \mapsto \max(u, 0)$ is an open problem, see [9, discussion after Remark 25] and [25, statement before Proposition 1].

(5) Suppose that $\Omega \subset \mathbb{R}^d$ is a bounded Lipschitz domain. We choose $a, b \colon \partial\Omega \to \mathbb{R} \cup \{\pm\infty\}$, measurable w.r.t. the surface measure on $\partial\Omega$ and denote by $T \colon W^{1,p}(\Omega) \to L^p(\partial\Omega)$ the trace operator, cf. [41, 190f]. Then, it is easy to check that the set

$$K = \{u \in W^{1,p}(\Omega) \mid a \le T(u) \le b \text{ a.e. on } \partial\Omega\}$$

is a set with bounds under the natural ordering of $W^{1,p}(\Omega)$ and, thus, n-polyhedric at every $u \in W^{1,p}(\Omega)$ for all $n \in \mathbb{N}_0$.

(6) We use the same setting as in the previous example and, additionally, a vector field $\nu \colon \bar{\Omega} \to \mathbb{R}^d$ such that ν and $\nu/|\nu|^2_{\mathbb{R}^d}$ are Lipschitz continuous on $\bar{\Omega}$. This follows, e.g., if $\nu \in W^{1,\infty}(\Omega)$ is uniformly positive. Then, the operator $R \colon W^{1,p}(\Omega)^d \to W^{1,p}(\Omega)$ defined via $R(v) = v^\top \nu$ is surjective, since $R(w\,\nu/|\nu|^2_{\mathbb{R}^d}) = w$ for all $w \in W^{1,p}(\Omega)$.
Hence, Lemma 3.3 together with the previous example shows that the set

$$K = \{u \in W^{1,p}(\Omega; \mathbb{R}^d) \mid a \le T(u)^\top \nu \le b \text{ a.e. on } \partial\Omega\}$$

is polyhedric.

(7) Let X be a locally compact Hausdorff space. By the Riesz-Markov theorem [28, Theorem 6.19], we know that the dual space of the Banach lattice $C_0(X)$ (equipped with the natural, pointwise ordering) can be identified with the space $\mathcal{M}(X)$ of regular, countably additive and signed Borel measures equipped with the total variation norm. Owing to [31, Proposition 5.5], $\mathcal{M}(X)$ is also a Banach lattice and, thus, satisfies Assumption 4.7. Hence, for all measures $\mu_a, \mu_b \in \mathcal{M}(X)$, the set

$$K = \{\mu \in \mathcal{M}(X) \mid \mu_a \le \mu \le \mu_b\}$$

is n-polyhedric for all $n \in \mathbb{N}_0$. Here, $\mu_a \le \mu$ is to be understood with the ordering in $\mathcal{M}(X)$ induced by the ordering of $C_0(X)$, i.e., $\int_X f \,\mathrm{d}\mu_a \le \int_X f \,\mathrm{d}\mu$ for all $f \in C_0(X)$ satisfying $f(x) \ge 0$ for all $x \in X$.

(8) We denote by $BV_0[0,1]$ the space of all functions $f \colon [0,1] \to \mathbb{R}$ of bounded variation with $f(0) = 0$. This set becomes a Banach lattice if it is equipped with the bounded variation norm and the non-negative cone

$$C = \{f \in BV_0[0,1] \mid f \text{ is monotonically non-decreasing}\},$$

see [1, Section 9.8, Theorem 9.51]. Thus, for every $a, b \in BV_0[0,1]$, the set

$$K = \{f \in BV_0[0,1] \mid f - a, b - f \text{ are monotonically non-decreasing}\}$$

is n-polyhedric for all $n \in \mathbb{N}_0$.

4.3. Counterexamples

In this section, we give some counterexamples. To our knowledge, these are the first available counterexamples which show that the intersection of polyhedric sets may fail to be polyhedric.

The first counterexample limits possible generalizations of Corollary 4.19. In particular, the intersection of a set with bounds with a co-polyhedral set can, in general, fail to be polyhedric.

Example 4.22. We consider the unit interval $(0,1)$ equipped with the Lebesgue measure. We define $b \in L^2(0,1)$ via $b(x) := \big(x^2 + (1-x)^2\big)^{1/2}$ and we set

$$K := \big\{u \in L^2(0,1) \mid 0 \le u(x) \le b(x) \quad \text{f.a.a. } x \in (0,1)\big\}.$$

Moreover, we denote by $P := \operatorname{lin}\{e_1, e_2\}$ the two-dimensional subspace spanned by $e_1(x) := 1 - x$ and $e_2(x) := x$. Then, it is easy to check that

$$K \cap P = \{\alpha_1 \, e_1 + \alpha_2 \, e_2 \mid \alpha_1, \alpha_2 \ge 0 \text{ and } \alpha_1^2 + \alpha_2^2 \le 1\}.$$

Indeed, we have $\alpha_1 \, e_1 + \alpha_2 \, e_2 \in K$ if and only if

$$0 \le \alpha_1 \, (1-x) + \alpha_2 \, x \le \big(x^2 + (1-x)^2\big)^{1/2}$$

for all $x \in [0,1]$. Now, the above identity follows from the fact that equality in the Cauchy-Schwarz inequality

$$\alpha_1 \, (1-x) + \alpha_2 \, x \le (\alpha_1^2 + \alpha_2^2)^{1/2} \, \big((1-x)^2 + x^2\big)^{1/2}$$

holds if and only if the vectors (α_1, α_2), $(1-x, x)$ are colinear.

Let $u = \alpha_1 \, e_1 + \alpha_2 \, e_2 \in K \cap P$ be given. Then the set $\{x \in (0,1) \mid u(x) = b(x)\}$ has measure zero. Therefore we get $\mathcal{T}_K(u) = L^2(0,1)$. This shows that, in general, $\mathcal{T}_{K \cap P}(u) \ne \mathcal{T}_K(u) \cap \mathcal{T}_P(u)$.

Moreover, it is easy to check that $K \cap P$ is not polyhedric, in particular, it is not polyhedric at $(e_1, e_1 - 2\, e_2)$. Here, we identified $L^2(0,1)^\star$ with $L^2(0,1)$.

The same negative results are obtained by intersecting K with the compact co-polyhedral set $\operatorname{conv}\{0, 2\, e_1, 2\, e_2\}$.

The next example shows that sets which are defined by pointwise bounds may fail to be polyhedric if the underlying Banach space does not posses a lattice structure.

Example 4.23. Let $\Omega \subset \mathbb{R}^d$ be an open, bounded set. We set $C := \{v \in H_0^1(\Omega) \mid v \ge 0\}$. Then, the cone $-C^\circ \subset H^{-1}(\Omega) := (H_0^1(\Omega))^\star$ induces an order on $H^{-1}(\Omega)$. It is known that $H^{-1}(\Omega)$ equipped with this order fails to be a vector lattice, see, e.g., [39, Appendix B]. However, from Lemma 3.2 we infer that $-C^\circ$ is polyhedric.

Now, we define $\Lambda := \big\{\lambda \in H^{-1}(\Omega) \mid 1 \ge \lambda \ge -1\big\}$, where $1 \in H^{-1}(\Omega)$ is the functional $v \mapsto \int_\Omega v \, \mathrm{d}x$ and "$\ge$" is the ordering induced by $-C^\circ$. Note that Λ is the intersection of the polyhedric sets

$$-C^\circ - 1 = \{\lambda \in H^{-1}(\Omega) \mid \lambda \ge -1\} \quad \text{and} \quad C^\circ + 1 = \{\lambda \in H^{-1}(\Omega) \mid 1 \ge \lambda\}.$$

However, we will show that Λ fails to be polyhedric at some points. Without loss of generality, we assume that the closed unit cube $[-1,1]^d$ belongs to Ω. Then, there exists $u \in C$ such that $u(x) = |x_1|$ for all $x \in [-1,1]^d$ and that $\{x \in \Omega \mid u(x) = 0\}$ has Lebesgue measure zero. We set $\lambda := 1$. It is clear that $\lambda \in \Lambda$. Moreover, it is easy to check that $\Lambda \subset L^\infty(\Omega)$ and

$$\mathcal{R}_\Lambda(\lambda) = \{\mu \in L^\infty(\Omega) \mid \mu(x) \le 0 \text{ for a.a. } x \in \Omega\}.$$

Now, $u \in \mathcal{R}_\Lambda(\lambda)^\circ = \mathcal{T}_\Lambda(\lambda)^\circ$ follows. Moreover, $\mathcal{R}_\Lambda(\lambda) \cap u^\perp = \{0\}$.

We define the surface measure $\mu \in H^{-1}(\Omega)$ via

$$\langle \mu,\, v\rangle := -\int_S v \,\mathrm{d}s,$$

where $S = \{0\} \times (-1,1)^{d-1}$ is equipped with the $(d-1)$-dimensional Lebesgue measure. Now, we define the functions $\mu_n := -\alpha_n\, \chi_{A_n} \in \mathcal{R}_\Lambda(\lambda)$ with $A_n := (-1/n, 1/n) \times (-1,1)^{d-1}$ and $\alpha_n := n/2$. We show that $\mu_n \rightharpoonup \mu$ in $H^{-1}(\Omega)$. Indeed, for all $\varphi \in C_0^\infty(\Omega)$ we have $\langle \mu_n,\, \varphi\rangle \to \langle \mu,\, \varphi\rangle$ and, thus, it is sufficient to show that the sequence $\{\mu_n\}_{n\in\mathbb{N}}$ is bounded in $H^{-1}(\Omega)$. For $v \in C_0^\infty(\Omega)$ we have

$$v(x) = \int_{-\infty}^{x_1} \frac{\partial v}{\partial x_1}(t, x_2, x_3, \ldots)\,\mathrm{d}t \le c\,\|v(\cdot, x_2, x_3, \ldots)\|_{L^2(\mathbb{R})} \qquad \forall x \in \Omega.$$

Hence,

$$\int_{-1/n}^{1/n} \frac{n}{2}\, v(x)\,\mathrm{d}x_1 \le c\,\|v(\cdot, x_2, x_3, \ldots)\|_{L^2(\mathbb{R})} \qquad \forall x_2, \ldots, x_d \in (-1,1).$$

Using Fubini's theorem, we find

$$\langle \mu_n,\, v\rangle = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \int_{-1/n}^{1/n} \frac{n}{2}\, v(x)\,\mathrm{d}x_1\,\mathrm{d}x_2 \,\ldots\, \mathrm{d}x_d \le \tilde{c}\,\|v\|_{H_0^1(\Omega)}\,.$$

Since $\mu_n \in H^{-1}(\Omega)$ and $C_0^\infty(\Omega)$ is dense in $H_0^1(\Omega)$, we find $\|\mu_n\|_{H^{-1}(\Omega)} \le \tilde{c}$ for all $n \in \mathbb{N}$. Hence, we have shown $\mu_n \rightharpoonup \mu$ in $H^{-1}(\Omega)$. Using Mazur's lemma, we find $\mu \in \mathcal{T}_\Lambda(\lambda)$. Further, it is clear that $\mu \in u^\perp$. Hence,

$$\mu \in \mathcal{T}_\Lambda(\lambda) \cap u^\perp \ne \operatorname{cl}\big(\mathcal{R}_\Lambda(\lambda) \cap u^\perp\big) = \{0\}$$

and this shows that Λ is not polyhedric at (λ, u).

The above example is crucial for studying variational inequalities of the second kind in $H_0^1(\Omega)$, cf. the technique for a similar problem in $H^{1/2}(\partial\Omega)$ in [34, Section 4.5], and also [12, Section 2]. In [12], a construction similar to Example 4.22 is carried out for sets with pointwise bounds in dual spaces and this yields the non-polyhedricity in many similar situations. In [11], the authors provided a directional differentiability result for a variational inequality of second kind in

$H_0^1(\Omega)$ and additional terms appear in the variational inequality for the derivative. This cannot happen if $\Lambda \subset H^{-1}(\Omega)$ would be polyhedric. However, the results of [13] suggest that Λ may be polyhedric at some (λ, u) possessing certain regularity properties.

Finally, we present a counterexample in $\mathbb{R}^3$ which possesses some very bizarre properties. It is a compact polyhedric set, but fails to be a polyhedron. Moreover, it is not 2-polyhedric and the intersection with a hyperplane (which is polyhedral and co-polyhedral) is not polyhedric.

Example 4.24. In $\mathbb{R}^3$ we consider the points $O := (0,0,0)$, and for all $n \in \mathbb{N}$, $P_n := (n^{-2}, n^{-1}, -n^{-3})$ and $Q_n := (-(n/\gamma)^{-2}, (n/\gamma)^{-1}, 0)$, where $\gamma = (2+\sqrt{7})/3$. We set

$$K := \operatorname{conv}\big(\{O\} \cup \{P_n, Q_n\}_{n\in\mathbb{N}}\big),$$

see also Figure 4.1. Since the sequences $\{P_n\}$ and $\{Q_n\}$ converge towards O, the set K is closed. In what follows, we show that K is polyhedric, but not polyhedral.

First, we show that K is polyhedric at O. We have

$$\begin{aligned}\mathcal{R}_K(O) &= \operatorname{cone} K = \operatorname{cone}\{P_n, Q_n\}_{n\in\mathbb{N}} = \operatorname{cone}\{(n^{-1}, 1, -n^{-2}), (-\gamma\, n^{-1}, 1, 0)\}_{n\in\mathbb{N}} \\ &= \operatorname{cone}\big(\{(n^{-1}, 1, -n^{-2})\}_{n\in\mathbb{N}} \cup \{(-\gamma, 1, 0), (0,1,0)\}\big) \setminus \operatorname{cone}\{(0,1,0)\}.\end{aligned}$$

By taking the closure, we obtain

$$\mathcal{T}_K(O) = \operatorname{cone}\big(\{(n^{-1}, 1, -n^{-2})\}_{n\in\mathbb{N}} \cup \{(-\gamma, 1, 0), (0,1,0)\}\big).$$

The ray $R := \operatorname{cone}\{(0,1,0)\}$ is not an exposed ray of $\mathcal{T}_K(O)$, cf. Figure 4.1. In particular, if an intersection of $\mathcal{T}_K(O)$ with a hyperplane $\mu^\perp$, $\mu \in \mathbb{R}^3$, contains $(0,1,0)$, we have $\mu_2 = 0$. Moreover, this implies

$$\begin{aligned}\operatorname{cone}\{(-\gamma, 1, 0), (0,1,0)\} \setminus R \subset \mathcal{R}_K(O) \cap \mu^\perp \qquad & \text{if } \mu_1 = 0, \\ \operatorname{cone}\{(1-\lambda\,\gamma, 1+\lambda, -1), (0,1,0)\} \setminus R \subset \mathcal{R}_K(O) \cap \mu^\perp \qquad & \text{if } \mu_3/\mu_1 \le 1, \\ \operatorname{cone}\Big\{\Big(1+\tfrac{\kappa}{k}, 1+\kappa, -1-\tfrac{\kappa}{k^2}\Big), (0,1,0)\Big\} \setminus R \subset \mathcal{R}_K(O) \cap \mu^\perp \qquad & \text{if } \mu_3/\mu_1 > 1.\end{aligned}$$

Here, $\lambda = \gamma^{-1}\,(1 - \mu_3\,\mu_1^{-1}) \ge 0$ and $k \in \mathbb{N}$ is chosen large enough such that $\kappa = (\mu_3\,\mu_1^{-1} - 1)/(k^{-1} - k^{-2}\,\mu_3\,\mu_1^{-1}) > 0$. In any case, we can assure $\operatorname{cone}\{(0,1,0)\} \subset \operatorname{cl}\big(\mathcal{R}_K(O) \cap \mu^\perp\big)$. This shows $\mathcal{T}_K(O) \cap \mu^\perp = \operatorname{cl}\big(\mathcal{R}_K(O) \cap \mu^\perp\big)$ and K is polyhedric at O.

Next, it is a little bit tedious to check that K is the intersection of the half-spaces which are defined by the following inequalities and that the points on the right-hand side are exactly those points of O, P_n, Q_n which lie on the boundary of the half-spaces:

$$\begin{aligned}x^\top(0,0,1) &\le 0, && O, Q_n \quad \forall n \in \mathbb{N}, \\ x^\top(1, \gamma, 1+\gamma) &\ge 0, && O, Q_1, P_1, \\ x^\top(2\,k+1, -1, k\,(k+1)) &\le 0, && O, P_k, P_{k+1},\end{aligned}$$

$$x^\top(a_k^{(1)}, b_k^{(1)}, c_k^{(1)}) \le \gamma^2, \qquad Q_k, Q_{k+1}, P_k,$$
$$x^\top(a_k^{(2)}, b_k^{(2)}, c_k^{(2)}) \le 1, \qquad P_k, P_{k+1}, Q_{k+1},$$

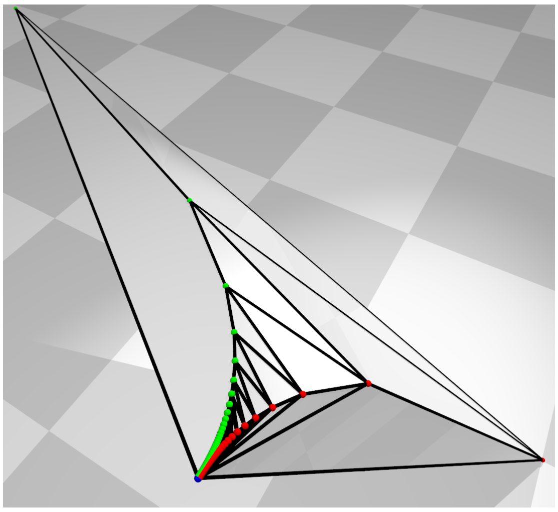
where $k \in \mathbb{N}$. In the last two lines, we have used the coefficients

$$a_k^{(1)} := k\,(k+1), \qquad b_k^{(2)} := \frac{\gamma^2\,(3\,k^2 + 3\,k + 1) + (k+1)^2}{(2\,k+1)\,\gamma^2 + (k+1)\,\gamma},$$
$$b_k^{(1)} := \gamma\,(2\,k+1), \qquad a_k^{(2)} := 3\,k^2 + 3\,k + 1 - (2\,k+1)\,b_k^{(2)},$$
$$c_k^{(1)} := k\,a_k^{(1)} + k^2\,b_k^{(1)} - \gamma^2\,k^3, \qquad c_k^{(2)} := k\,a_k^{(2)} + k^2\,b_k^{(2)} - k^3.$$

From this representation of K, we learn two things. First, all P_k, Q_k are extreme points of K and, thus, K is not polyhedral. Second, the intersection $K \cap \{x \in \mathbb{R}^3 \mid x^\top(1,1,1) \ge \varepsilon\}$ is a polyhedron for all $\varepsilon > 0$, since it can be written as a finite intersection of half-spaces. Thus, $\mathcal{R}_K(x)$ is closed for all $x \in K \setminus \{O\}$.

Hence, we have shown that K is polyhedric, but not polyhedral. Moreover, it is easy to check that the intersection of K with the hyperplane $x^\top(0,0,1) = 0$ is not polyhedric. Hence, K is not 2-polyhedric, cf. Lemma 4.4.

Finally, we mention that the example can be lifted to $\mathbb{R}^n$, $n > 3$, by considering $K \times \mathbb{R}^{n-3}$.



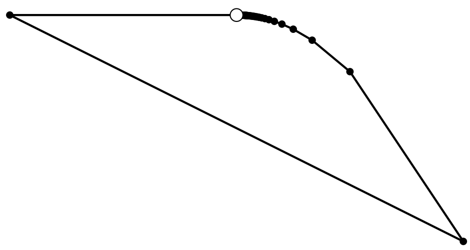


Figure 4.1: The polyhedric, non-polyhedral set K from Example 4.24 is shown left. The points Q_n and P_n are shown on the left and the right side, respectively. The right figure shows the intersection of $\mathcal{R}_K(O)$ with the two-dimensional hyperplane $x^\top(0,1,0) = 1$. Note that the hollow dot (corresponding to the non-exposed ray in direction $(0,1,0)$ of $\mathcal{T}_K(O)$) does not belong to this intersection.

However, it can be shown that all bounded, $(n-1)$-polyhedric sets in $\mathbb{R}^n$ are polyhedral. Indeed, if K is bounded and $(n-1)$-polyhedric, intersections of K with two-dimensional subspaces are bounded and polyhedric by Lemma 4.4. In $\mathbb{R}^2$, bounded and polyhedric sets are precisely the bounded polyhedral sets and this is easy to prove. Now, we can invoke [17, Theorem 4.7] to obtain the polyhedrality of K. Moreover, the same reasoning shows that $(n-1)$-polyhedric sets in $\mathbb{R}^n$ are boundedly polyhedral in the sense of Klee [17], that is, they have polyhedral intersections with all bounded polyhedral sets.

4.4. Summary

We briefly summarize the findings of Section 4. We provided results concerning the polyhedricity of the intersection of sets. These results are collected in Table 4.1. The reader should bear in mind that "with bounds" refer to a set with bounds in the sense of Definition 4.9 in a Banach space with a lattice structure satisfying Assumption 4.7, see also Example 4.23. Moreover, for the polyhedricity of the intersection of two sets with bounds we require that both sets satisfy Assumption 4.10 w.r.t. the same ordering in X. Then, the intersection is again a set with bounds and, thus, polyhedric.The other positive results are straightforward (lower right corner) or follow from Corollary 4.19. Moreover, Examples 4.22 to 4.24 show the failure of polyhedricity for all negative cases.

	polyhedric	with bounds	polyhedral	co-polyhedral
polyhedric	✗	✗	✗	✗
with bounds	✗	✓	✓	✗
polyhedral	✗	✓	✓	✓
co-polyhedral	✗	✗	✓	✓

Table 4.1: A summary concerning the polyhedricity of intersections. The notions of the sets are given in Definitions 3.1 and 4.9. A check mark (✓) indicates intersections which are always polyhedric, whereas a cross (✗) signifies that this type of intersection may fail to be polyhedric.

5. Applications of polyhedricity

In order to complete the picture of polyhedricity, we illustrate its important applications. For the convenience of the reader, we included the rather short proofs.

5.1. Directional differentiability of projections

The very first application of polyhedricity was a result on the differentiability of (metric) projections onto closed, convex sets. In fact, these differentiability results motivated the introduction of the notion of polyhedricity in [22, 14].

We state the results in the generality of Mignot [22, Section 2], but use the proofs of Haraux [14, Section 1]. Another difference to both works is that we do not use the identification $H \cong H^\star$, but we will explicitly work with the dual space $H^\star$.

In this section, we use the following standing assumption.

Assumption 5.1. *The space H is a real Hilbert space, $A\colon H \to H^\star$ is a bounded, linear operator which is coercive, i.e., there exists $\alpha > 0$ such that*

$$\langle A\,x,\, x\rangle \ge \alpha\,\|x\|^2 \quad \forall x \in H,$$

Finally, $K \subset H$ is a closed, convex set.

Note that it is also possible to work with the bounded, coercive and bilinear form a defined via $a(x,y) = \langle A\,x,\, y\rangle$.

We further point out that the coercivity of A already implies that H is isomorphic to a Hilbert space since $a_s(x,y) := (a(x,y) + a(y,x))$ is a scalar product on H and the induced norm is equivalent to the original norm of H.

For a given functional $\ell \in H^\star$, we consider the variational inequality

$$\text{Find} \quad y \in K \qquad \text{such that} \quad \langle A\,y - \ell,\, v - y\rangle \ge 0 \quad \forall v \in K. \tag{15}$$

In the case that A is symmetric, i.e., $\langle A\,x,\, y\rangle = \langle A\,y,\, x\rangle$ for all $x, y \in H$, this problem is equivalent to a projection problem. In particular, y is the projection of the Riesz representative of ℓ (w.r.t. the a-scalar product) onto K w.r.t. the norm $\|x\|_A^2 := \langle A\,x,\, x\rangle = a(x,x)$.

It is well known that for all $\ell \in H^\star$ there is a unique solution $y := S(\ell)$ of (15), see [35]. Moreover, this solution operator $S\colon H^\star \to H$ is globally Lipschitz continuous. From (15) we immediately find $\ell - A\,y \in \mathcal{T}_K(y)^\circ$. We recall the definition of the critical cone $\mathcal{K}_K(y,\lambda) := \mathcal{T}_K(y) \cap \lambda^\perp$.

Theorem 5.2. *Let $\ell \in H^\star$ be given and let $y := S(\ell)$, $\lambda := \ell - A\,y$. Moreover, assume that K is polyhedric at (y,λ). Then, S is directionally differentiable at ℓ and the directional derivative $\delta y := S'(\ell; h)$ in direction $h \in H^\star$ solves the variational inequality*

$$\delta y \in \mathcal{K}_K(y,\lambda) \qquad \textit{and} \qquad \langle A\,\delta y - h,\, v - \delta y\rangle \ge 0 \quad \forall v \in \mathcal{K}_K(y,\lambda). \tag{16}$$

Proof. Let a direction $h \in H^\star$ be given. For $t > 0$, we define $y_t := S(\ell + t\,h)$ and $\lambda_t := \ell + t\,h - A\,y_t$. We find $y_t \in K$ and $\lambda_t \in \mathcal{T}_K(y_t)^\circ$, see (15). Using the Lipschitz continuity of S we find

$$\Big\|\frac{y_t - y}{t}\Big\|_H \le C \qquad \text{and} \qquad \Big\|\frac{\lambda_t - \lambda}{t}\Big\|_{H^\star} \le C.$$

Since H and $H^\star$ are reflexive, we find a subsequence (without relabeling) and $\delta y \in H$, $\delta\lambda \in H^\star$ such that

$$\frac{y_t - y}{t} \rightharpoonup \delta y \qquad \text{and} \qquad \frac{\lambda_t - \lambda}{t} \rightharpoonup \delta\lambda.$$

From $y_t \in K$ we immediately obtain $\delta y \in \mathcal{T}_K(y)$.

Now, we consider an arbitrary $x \in K$ with $\langle \lambda, x - y \rangle = 0$. We obtain

$$\langle \delta\lambda, x - y \rangle \leftarrow \Big\langle \frac{\lambda_t - \lambda}{t}, x - y_t \Big\rangle \le \Big\langle \frac{-\lambda}{t}, x - y_t \Big\rangle = \Big\langle \frac{-\lambda}{t}, y - y_t \Big\rangle \to \langle \lambda, \delta y \rangle \le 0.$$

This shows

$$\delta\lambda \in \{z \in K - y \mid \langle \lambda, z \rangle = 0\}^\circ = \big(\mathcal{R}_K(y) \cap \lambda^\perp\big)^\circ$$

and by using $x = y$ we find $\delta y \in \lambda^\perp$. The definition $\lambda_t = \ell + t\,h - A\,y_t$ and polyhedricity of K implies immediately $\delta\lambda = h - A\,\delta y \in \mathcal{K}_K(y, \lambda)^\circ$. Hence, it remains to show the convergence of the entire sequence and the strong convergence $(y_t - y)/t \to \delta y$. By testing the variational inequality (15) for y with $v = y_t$ and vice versa, we obtain

$$\Big\langle A\,\frac{y_t - y}{t} - h, \frac{y_t - y}{t} \Big\rangle \le 0.$$

Using the weak convergence of the difference quotient and the coercivity of A, we get

$$\begin{aligned} 0 \le -\langle \delta\lambda, \delta y \rangle = \langle A\,\delta y - h, \delta y \rangle &\le \liminf_{t \searrow 0} \Big\langle A\,\frac{y_t - y}{t} - h, \frac{y_t - y}{t} \Big\rangle \\ &\le \limsup_{t \searrow 0} \Big\langle A\,\frac{y_t - y}{t} - h, \frac{y_t - y}{t} \Big\rangle \le 0. \end{aligned}$$

On the one hand, this shows $\langle \delta\lambda, \delta y \rangle = 0$ and, thus, $\delta y \in \mathcal{T}_K(y) \cap \lambda^\perp = \mathcal{K}_K(y, \lambda)$ solves the VI (16). Thus, the weak limit δy is independent of the chosen subsequence and a standard argument shows the convergence of the entire sequence.

On the other hand, this shows the convergence

$$\langle A\,\delta y, \delta y \rangle = \lim_{t \searrow 0} \Big\langle A\,\frac{y_t - y}{t}, \frac{y_t - y}{t} \Big\rangle$$

and, by coercivity of A, the strong convergence $(y_t - y)/t \to \delta y$ follows. □

We mention some extensions of Theorem 5.2. First of all, the result [14, Theorem 1] also allows for non-polyhedric sets K by including an operator L, which takes into account the curvature of the boundary of K. [20, Theorem 1.1] and [6, Theorem 5.5] treat nonlinear variational inequalities.

Theorem 5.2 can be applied to show the shape differentiability of the solution mapping to variational inequalities, see [34, Chapter 4], and to compute the graphical derivative of the normal cone mapping to K, see [23, Theorem 5.2], [38, Lemma 3.10].

5.2. Optimal control of VIs: strong stationarity

A second application of polyhedricity is the derivation of optimality conditions for the optimal control of variational inequalities, and was first considered in [22, Proposition 4.1]. This is strongly linked to the differentiability result Theorem 5.2. The same technique can be used to derive optimality conditions for control problems involving other non-smooth maps, if the derivative can be written as the solution of a variational inequality.

In addition to Assumption 5.1, we use the following setting: $f \in H^\star$, U is a real Banach space, $B \in \mathcal{L}(U, H^\star)$ has a dense range, and $J\colon H \times U \to \mathbb{R}$ is Fréchet differentiable.

We consider the optimal control problem

$$\begin{cases} \text{Minimize} & J(y,u) \\ \text{such that} & y \in K \text{ and } \langle A\,y - B\,u - f,\, v - y\rangle_{H^\star,H} \ge 0 \quad \forall v \in K. \end{cases} \tag{17}$$

Theorem 5.3. *Suppose that $(\bar y, \bar u) \in H \times U$ is locally optimal for (17). We set $\bar\lambda := B\,\bar u + f - A\,\bar y$ and assume that K is polyhedric at $(\bar y, \bar\lambda)$. Then, there exist $p \in H$, $\mu \in H^\star$ satisfying the optimality system*

$$J_y(\cdot) + A^\star\, p + \mu = 0, \tag{18a}$$

$$J_u(\cdot) - B^\star\, p = 0, \tag{18b}$$

$$p \in -\mathcal{K}_K(\bar y, \bar\lambda), \tag{18c}$$

$$\mu \in \mathcal{K}_K(\bar y, \bar\lambda)^\circ. \tag{18d}$$

Here, $J_y(\cdot) \in H^\star$ and $J_u(\cdot) \in U^\star$ are the partial Fréchet derivatives of J at $(\bar u, \bar y)$.

Proof. We follow the proof of [22, Proposition 4.1].

As in Subsection 5.1, we denote by S the solution operator of the variational inequality and use the reduced problem

$$\text{Minimize} \quad J(S(B\,u + f), u).$$

Since $\bar u$ is a local minimizer and using Theorem 5.2, we have

$$\langle J_y(\cdot),\, S'(B\,\bar u + f; B\,h)\rangle_{H^\star,H} + \langle J_u(\cdot),\, h\rangle_{U^\star,U} \ge 0 \qquad \forall h \in U.$$

Since A is coercive, we get $\|S'(B\,\bar u + f; B\,h)\|_H \le C\,\|B\,h\|_{H^\star}$ for all $h \in U$, and this yields $|\langle J_u(\cdot),\, h\rangle_{U^\star,U}| \le C\,\|B\,h\|_{H^\star}$ for all $h \in U$.

Hence, there is $p \in H^{\star\star} \cong H$ (by defining it as in the next line on the dense subspace image$(B) \subset H^\star$ and extending it by continuity on the whole space $H^\star$) such that

$$\langle J_u(\cdot),\, h\rangle_{U^\star,U} = \langle p,\, B\,h\rangle_{H,H^\star} \qquad \forall h \in U.$$

This yields (18b) and

$$\langle J_y(\cdot),\, S'(B\,\bar u + f; B\,h)\rangle_{H^\star,H} + \langle p,\, B\,h\rangle_{H,H^\star} \ge 0 \qquad \forall h \in U.$$

Using the density of image(B) in $H^\star$ we get

$$\langle J_y(\cdot),\, S'(B\,\bar u + f; h)\rangle_{H^\star,H} + \langle p,\, h\rangle_{H,H^\star} \ge 0 \qquad \forall h \in H^\star. \tag{19}$$

In what follows, we set $\mathcal{K} := \mathcal{K}_K(\bar{y}, \bar{\lambda})$ for convenience. We choose $h \in \mathcal{K}^\circ$ in (19). Then, (16) shows $S'(\bar{u} + f; h) = 0$ and, thus,

$$\langle p, h \rangle_{H,H^\star} \geq 0 \qquad \forall h \in \mathcal{K}^\circ,$$

i.e., $p \in -\mathcal{K}$, which shows (18c).

Now, we choose $h \in A\,\mathcal{K}$ in (19), hence $S'(B\,\bar{u} + f; h) = A^{-1}\,h$ by (16). We get

$$\langle J_y(\cdot), A^{-1}\,h \rangle_{H^\star,H} + \langle p, h \rangle_{H,H^\star} \geq 0 \qquad \forall h \in A\,\mathcal{K}.$$

Rearranging terms yields

$$\langle J_y(\cdot) + A^\star\,p, v \rangle_{H^\star,H} \geq 0 \qquad \forall v \in \mathcal{K}.$$

Now, we define $\mu := -J_y(\cdot) - A^\star\,p$ and get $\mu \in \mathcal{K}^\circ$. This shows (18a) and (18d). □

The crucial ingredients in the proof of Theorem 5.3 are the density of the image of B and the absence of control constraints on u. In the presence of control constraints, system (18) is no longer necessary without further assumptions. We refer to [36, Section 6] for counterexamples.

Using terminology from finite-dimensional optimization, the optimality system (18) is of strongly stationary type. By using methods from variational analysis (in particular, [20, Theorem 3.1]), the same result is shown in [16, Theorem 4.6].

For more general optimization problems with complementarity constraints in Banach spaces a notion of strong stationarity was introduced in [37]. In the polyhedric case, this notion is of reasonable strength and it is a necessary optimality condition under some constraint qualification, see [37, Section 5].

5.3. No-gap second-order conditions

As a last application of polyhedricity, we consider second-order optimality conditions. In particular, polyhedricity is one possibility to obtain no gap between the necessary and sufficient optimality conditions of second order.

In this section, we consider the optimization problem

$$\text{minimize } f(x), \quad \text{subject to } \; g(x) \in K. \tag{20}$$

Here, X, Y are Banach spaces and the mappings $f\colon X \to \mathbb{R}$, $g\colon X \to Y$ are assumed to be twice Fréchet differentiable at $\bar{x}$, where $\bar{x} \in X$ satisfies $g(\bar{x}) \in K$. The set $K \subset Y$ is closed and convex.

The Lagrangian associated to problem (20) is

$$L(x, \lambda) := f(x) + \langle \lambda, g(x) \rangle$$

and $\bar{\lambda} \in Y^\star$ is said to be a *Lagrange multiplier* at $\bar{x}$ if $\bar{\lambda} \in \mathcal{T}_K(g(\bar{x}))^\circ$ and $0 = L'(\bar{x}, \bar{\lambda}) = f'(\bar{x}) + g'(\bar{x})^\star\,\bar{\lambda}$. Here, L' is the partial Fréchet derivative of L with respect to x. The second partial Fréchet derivative of L w.r.t. x is denoted by L'' and we write $L''(\bar{x}, \bar{\lambda})\, h^2 := L''(\bar{x}, \bar{\lambda})[h, h]$.

The *critical cone* $C(\bar{x})$ at $\bar{x}$ is defined as

$$C(\bar{x}) := \{h \in X \mid g'(\bar{x})\, h \in \mathcal{T}_K(g(\bar{x})) \text{ and } f'(\bar{x})\, h \leq 0\}.$$

If $\bar{\lambda}$ is a Lagrange multiplier, it is easy to check that

$$C(\bar{x}) = \{h \in X \mid g'(\bar{x})\, h \in \mathcal{T}_K(g(\bar{x})) \cap \bar{\lambda}^\perp\} = g'(\bar{x})^{-1}\, \mathcal{K}_K(g(\bar{x}), \bar{\lambda}).$$

Let $\bar{\lambda}$ be a Lagrange multiplier. To derive second-order necessary optimality conditions we apply the constraint qualification of Robinson, Zowe and Kurcyusz to the constraint

$$g(x) - g(\bar{x}) \in (K - g(\bar{x})) \cap \bar{\lambda}^\perp =: K_{\bar{\lambda}}.$$

Using $\mathcal{R}_{K_{\bar{\lambda}}}(0) = \mathcal{R}_K(g(\bar{x})) \cap \bar{\lambda}^\perp$, the condition of Robinson, Zowe and Kurcyusz becomes

$$Y = g'(\bar{x})\, X - \big(\mathcal{R}_K(g(\bar{x})) \cap \bar{\lambda}^\perp\big). \tag{21}$$

From [32, Theorem 2.2], we obtain that this condition implies that the given multiplier $\bar{\lambda}$ is the unique Lagrange multiplier for (20). In finite dimensions, such a condition was introduced in [19] and coined "strict Mangasarian-Fromovitz condition". Therefore, we call (21) "strict Robinson-Zowe-Kurcyusz condition". In particular, (21) is satisfied if $g'(\bar{x})$ is surjective.

We start with a result on second-order necessary conditions. The first part of the following theorem was given in [21, Theorem 3.3].

Theorem 5.4. *We assume that $\bar{x}$ is a local minimizer of* (20) *and let $\bar{\lambda} \in \mathcal{T}_K(g(\bar{x}))^\circ$ denote the unique corresponding Lagrange multiplier, i.e., $f'(\bar{x}) + g'(\bar{x})^\star\, \bar{\lambda} = 0$ such that the strict Robinson-Zowe-Kurcyusz condition* (21) *is satisfied. Then, we have*

$$L''(\bar{x}, \bar{\lambda})\, h^2 \geq 0 \qquad \forall h \in X : g'(\bar{x})\, h \in \mathcal{R}_K(\bar{x}), f'(\bar{x})\, h = 0. \tag{22}$$

If K is polyhedric at $(g(\bar{x}), \bar{\lambda})$, we get

$$L''(\bar{x}, \bar{\lambda})\, h^2 \geq 0 \qquad \forall h \in C(\bar{x}). \tag{23}$$

The proof of (22) is straightforward by using

$$\mathcal{T}_F(\bar{x}) = g'(\bar{x})^{-1} \Big(\operatorname{cl} \big(\mathcal{R}_K(g(\bar{x})) \cap \bar{\lambda}^\perp\big)\Big)$$

due to (21), see [6, Corollary 2.91]. Here,

$$F := \{x \in X \mid g(x) \in K \text{ and } g(x) - g(\bar{x}) \in \bar{\lambda}^\perp\}.$$

Moreover, (23) follows from (22) by using the following lemma which is just a translation of Lemma 3.3 to the critical cone $C(\bar{x})$.

Lemma 5.5. *Let us assume that $\bar\lambda \in \mathcal{T}_K(g(\bar x))^\circ$ satisfies $f'(\bar x) + g'(\bar x)^\star \bar\lambda = 0$ and the strict Robinson-Zowe-Kurcyusz condition* (21). *Moreover, we suppose that K is polyhedric at $(g(\bar x), \bar\lambda)$. Then, problem* (20) *satisfies the strong extended polyhedricity condition at $\bar x$ in the sense of* [6, *Definition* 3.52]. *That is,*

$$C_R(\bar x) := \{h \in X \mid g'(\bar x)\, h \in \mathcal{R}_K(\bar x) \text{ and } f'(\bar x)\, h = 0\}$$

is dense in the critical cone $C(\bar x)$.

Proof. We apply Lemma 3.3 with the setting

$$S = g'(\bar x), \qquad K_y = K - g(\bar x), \qquad \mu = g'(\bar x)\, \bar\lambda.$$

Recall that (21) implies the uniqueness of the multiplier $\bar\lambda$, see [32], hence, we get $\lambda = \bar\lambda$ in Lemma 3.3. An easy calculation using (6) shows

$$C(\bar x) = \{h \in X \mid g'(\bar x)\, h \in \mathcal{T}_{K-g(\bar x)}(0) \cap \bar\lambda^\perp\} = \mathcal{T}_{g'(\bar x)^{-1}(K-g(\bar x))}(0) \cap \mu^\perp.$$

Hence, Lemma 3.3 implies that this set is the closure of

$$\mathcal{R}_{g'(\bar x)^{-1}(K-g(\bar x))}(0) \cap \mu^\perp = g'(\bar x)^{-1}\, \mathcal{R}_K(g(\bar x)) \cap \mu^\perp = C_R(\bar x). \qquad \square$$

Another route to second-order necessary conditions is the utilization of so-called second-order tangent sets, see [6, Section 3.2]. Then, the necessary condition is similar to (23), but contains a term corresponding to the curvature of the set K, see [6, Theorem 3.45]. Under the "extended polyhedricity condition", one arrives at (23). Indeed, using Lemma 5.5, Theorem 5.4 is also an easy corollary of [6, Proposition 3.53].

Next, we provide a second-order sufficient condition.

Theorem 5.6. *We assume that $\bar\lambda$ is a Lagrange multiplier at $\bar x$. Further, we assume that X is finite-dimensional or that the constraint qualification*

$$Y = g'(\bar x)\, X - \mathcal{T}_K(g(\bar x)) \tag{24}$$

is satisfied. Suppose that

$$L''(\bar x, \bar\lambda)\, h^2 \ge \alpha\, \|h\|_X^2 \quad \forall h \in X : g'(\bar x)\, h \in \mathcal{T}_K(g(\bar x)) \text{ and } f'(\bar x)\, h \le \eta\, \|h\|_X \tag{25}$$

holds for some $\alpha > 0$, $\eta > 0$. Then, for all $\tilde\alpha \in (0, \alpha/2)$ there is $\varepsilon > 0$, such that

$$f(x) \ge f(\bar x) + \tilde\alpha\, \|x - \bar x\|_X^2 \quad \forall x \in F \cap B_\varepsilon(\bar x) \tag{26}$$

holds.

Proof. We verify (26) by contradiction. Assume that (26) does not hold. Then, there is a sequence $\{x_n\}$ with $g(x_n) \in K$ and $x_n \ne \bar x$ such that $x_n \to x$ and $f(x_n) < f(\bar x) + \tilde\alpha\, \|x_n - \bar x\|_X^2$. Next, we construct a sequence $\{h_n\}_{n\in\mathbb{N}} \subset X$, such that $\|h_n\|_X = o(\|x_n - \bar x\|_X)$ and $g'(\bar x)\,(x_n - \bar x + h_n) \in \mathcal{T}_K(g(\bar x))$ for all $n \in \mathbb{N}$.

1. In the case that X is finite-dimensional, we have w.l.o.g. $(x_n - \bar{x}) / \|x_n - \bar{x}\|_X \to h$, thus $g'(\bar{x})\, h \in \mathcal{T}_K(\bar{x})$. We set $h_n := \|x_n - \bar{x}\|_X\, h - (x_n - \bar{x})$ and have $h_n = o(\|x_n - \bar{x}\|_X)$ and $x_n - \bar{x} + h_n = \|x_n - \bar{x}\|_X\, h \in g'(\bar{x})^{-1}\, \mathcal{T}_K(g(\bar{x}))$.
2. Now, suppose that the condition (24) is satisfied. Since g is Fréchet-differentiable, we have $r_n := g(\bar{x}) + g'(\bar{x})\,(x_n - \bar{x}) - g(x_n) = o(\|x_n - \bar{x}\|_X)$. Due to the constraint qualification (24), we can apply the generalized open mapping theorem [42, Theorem 2.1] and obtain the existence of $h_n \in X$, $v_n \in \mathcal{T}_K(g(\bar{x}))$ satisfying
$$-r_n = g'(\bar{x})\, h_n - v_n$$
and $h_n = \mathcal{O}(\|r_n\|_X) = o(\|x_n - \bar{x}\|_X)$. In particular, we have
$$g'(\bar{x})\,(x_n - \bar{x} + h_n) = g(x_n) - g(\bar{x}) + v_n \in \mathcal{T}_K(g(\bar{x})).$$
Thus, $x_n - \bar{x} + h_n \in g'(\bar{x})^{-1}\, \mathcal{T}_K(g(\bar{x}))$.

Since
$$\begin{aligned} f'(\bar{x})\,(x_n - \bar{x}) &= f(x_n) - f(\bar{x}) + o(\|x_n - \bar{x}\|_X) \\ &\le \tilde{\alpha}\, \|x_n - \bar{x}\|_X^2 + o(\|x_n - \bar{x}\|_X) = o(\|x_n - \bar{x}\|_X) \end{aligned}$$
we conclude $f'(\bar{x})\,(x_n - \bar{x} + h_n) = o(\|x_n - \bar{x}\|_X) = o(\|x_n - \bar{x} + h_n\|_X)$. Hence, $x_n - \bar{x} + h_n$ belongs to the set on the right-hand side of (25) for n large enough.

Finally, we have for n large enough
$$\begin{aligned} \tilde{\alpha}\, \|x_n - \bar{x}\|^2 \ge f(x_n) - f(\bar{x}) &\ge L(x_n, \bar{\lambda}) - L(\bar{x}, \bar{\lambda}) \\ &= L'(\bar{x}, \bar{\lambda})\,(x_n - \bar{x}) + \frac{1}{2}\, L''(\bar{x}, \bar{\lambda})\,(x_n - \bar{x})^2 + o(\|x_n - \bar{x}\|_X^2) \\ &= \frac{1}{2}\, L''(\bar{x}, \bar{\lambda})\,(x_n - \bar{x} + h_n)^2 - L''(\bar{x}, \bar{\lambda})[x_n - \bar{x}, h_n] - \\ &\qquad - \frac{1}{2}\, L''(\bar{x}, \bar{\lambda})\, h_n^2 + o(\|x_n - \bar{x}\|_X^2) \\ &= \frac{1}{2}\, L''(\bar{x}, \bar{\lambda})\,(x_n - \bar{x} + h_n)^2 + o(\|x_n - \bar{x}\|_X^2) \\ &\ge \frac{\alpha}{2}\, \|x_n - \bar{x}\|^2 + o(\|x_n - \bar{x}\|_X^2) \end{aligned}$$
and this is a contradiction to $\tilde{\alpha} < \alpha/2$. □

Without the constraint qualification (24) we have to work with a larger critical cone, cf. [6, Lemma 3.65 and Remark 3.68]. In case that the Lagrange multiplier $\bar{\lambda}$ is not unique, it is easy to see that the left-hand side in (25) can be replaced by $\sup_\lambda L''(\bar{x}, \lambda)\, h^2$, where the sup ranges over a bounded set of Lagrange multipliers λ, by using essentially the same proof, see also [6, Section 3.3.1]. Recall that the set of Lagrange multipliers is already bounded if the constraint qualification (24) is satisfied, cf. [42, Theorem 4.1].

In the case that $L''(\bar{x}, \bar{\lambda})$ is a Legendre form, see [6, Definition 3.73], we can apply [6, Lemma 3.75] and get the equivalence of (25) with

$$L''(\bar{x}, \bar{\lambda})\, h^2 > 0 \qquad \forall h \in C(\bar{x}) \setminus \{0\},$$

in the case that X is reflexive.

Putting everything together, we get the following second-order conditions with no gap.

Theorem 5.7. *We assume that $\bar{\lambda}$ is a Lagrange multiplier at $\bar{x}$ and*

(1) *K is polyhedric at $(g(\bar{x}), \bar{\lambda})$,*

(2) *the strict Robinson-Zowe-Kurcyusz condition* (21) *holds (implying the uniqueness of $\bar{\lambda}$),*

(3) *$L''(\bar{x}, \bar{\mu})$ is a Legendre form and X is reflexive.*

Then, the following hold:

(i) *If $\bar{x}$ is locally optimal, we have $L''(\bar{x}, \bar{\lambda})\, h^2 \geq 0$ for all $h \in C(\bar{x})$.*

(ii) *The second-order condition $L''(\bar{x}, \bar{\lambda})\, h^2 > 0$ for all $h \in C(\bar{x}) \setminus \{0\}$ is equivalent to the quadratic growth condition* (26). *In particular, $\bar{x}$ is a strict local minimizer.*

This result provides an indication that polyhedricity plays a crucial role for deriving second-order conditions with no (or a very small) gap. We refer exemplarily to the contributions [5, 7, 10] for similar results by using polyhedricity and further extensions.

Finally, we mention that the existence of a Legendre form on a reflexive Banach space X implies that the norm of X is equivalent to a Hilbert space norm, i.e., X is isomorphic to a Hilbert space, see [15].

6. Conclusions

We have given an overview of polyhedric sets. Moreover, we have expanded the classical polyhedricity results from Mignot [22] and Haraux [14] and thus we are able to show the polyhedricity of intersections of sets with bounds with polyhedral sets. On the other hand, we have given some counterexamples which show that intersections of polyhedric sets might be non-polyhedric in various situations. From this point of view, it would be interesting to have further results concerning the intersections of polyhedric sets.

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