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Strong Convergence Theorems by Hybrid Methods for New Demimetric Mappings in Banach Spaces

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Using a new nonlinear mapping called generalized demimetric and the C-Q method, we first prove a strong convergence theorem for finding a fixed point for the mapping in a Banach space which generalizes simultaneously the results by Nakajo and Takahashi [12], and Solodov and Svaiter [14] in a Hilbert space. Furthermore, using the mapping and the shrinking projection method, we prove another strong convergence theorem in a Banach space. We apply these results to obtain new strong convergence theorems in a Hilbert space and a Banach space.

Keywords: Fixed point, demimetric mapping, maximal monotone operator, metric resolvent, metric projection, hybrid method, shrinking projection method, duality mapping.

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1. Introduction

Let H be a real Hilbert space and let C be a nonempty, closed and convex subset of H. For a mapping $U\colon C \to H$, we denote by $F(U)$ the *set of fixed points* of U. Let t be a real number with $0 \le t < 1$. A mapping $U\colon C \to H$ is called a *t-strict pseudo-contraction* [3] if

$$\|Ux - Uy\|^2 \le \|x-y\|^2 + t\|x - Ux - (y - Uy)\|^2$$

for all $x, y \in C$. If U is a t-strict pseudo-contraction and $F(U) \ne \emptyset$, then we have that, for $x \in C$ and $q \in F(U)$,

$$\|Ux - q\|^2 \le \|x-q\|^2 + t\|x - Ux\|^2$$

and hence

$$\|Ux - x\|^2 + \|x-q\|^2 + 2\langle Ux - x, x - q\rangle \le \|x-q\|^2 + t\|x - Ux\|^2.$$

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Therefore, we have that

$$\frac{2}{1-t}\langle x - Ux, x - q\rangle \geq \|x - Ux\|^2 \tag{1}$$

for all $x \in C$ and $q \in F(U)$. A mapping $U\colon C \to H$ is called *generalized hybrid* [7] if there exist $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha\|Ux - Uy\|^2 + (1-\alpha)\|x - Uy\|^2 \leq \beta\|Ux - y\|^2 + (1-\beta)\|x - y\|^2$$

for all $x, y \in C$. Such a mapping U is called *(α, β)-generalized hybrid.* Notice that the class of generalized hybrid mappings covers several well-known mappings. For example, a (1,0)-generalized hybrid mapping is *nonexpansive*, i.e.,

$$\|Ux - Uy\| \leq \|x - y\|, \quad \forall x, y \in C.$$

It is *nonspreading* [8, 9] for $\alpha = 2$ and $\beta = 1$, i.e.,

$$2\|Ux - Uy\|^2 \leq \|Ux - y\|^2 + \|Uy - x\|^2, \quad \forall x, y \in C.$$

It is also *hybrid* [19] for $\alpha = \frac{3}{2}$ and $\beta = \frac{1}{2}$, i.e.,

$$3\|Ux - Uy\|^2 \leq \|x - y\|^2 + \|Ux - y\|^2 + \|Uy - x\|^2, \quad \forall x, y \in C.$$

In general, nonspreading and hybrid mappings are not continuous; see [5]. If U is generalized hybrid and $F(U) \neq \emptyset$, then we have that, for $x \in C$ and $q \in F(U)$,

$$\alpha\|q - Ux\|^2 + (1-\alpha)\|q - Ux\|^2 \leq \beta\|q - x\|^2 + (1-\beta)\|q - x\|^2$$

and hence $\|Ux - q\|^2 \leq \|x - q\|^2$. From

$$\|Ux - x\|^2 + \|x - q\|^2 + 2\langle x - q, Ux - x\rangle \leq \|x - q\|^2,$$

we have that

$$2\langle x - q, x - Ux\rangle \geq \|x - Ux\|^2. \tag{2}$$

We also know that there exists such a mapping in a Banach space. Let E be a smooth Banach space and let B be a maximal monotone operator with $B^{-1}0 \neq \emptyset$. Then, for the metric resolvent J_λ of B for $\lambda > 0$, we have from [17] that, for any $x \in E$ and $q \in B^{-1}0$,

$$\langle J_\lambda x - q, J(x - J_\lambda x)\rangle \geq 0.$$

Then we get $\langle J_\lambda x - x + x - q, J(x - J_\lambda x)\rangle \geq 0$ and hence

$$\langle x - q, J(x - J_\lambda x)\rangle \geq \|x - J_\lambda x\|^2. \tag{3}$$

Motivated by (1), (2) and (3), Kawasaki and Takahashi [6] introduced a new nonlinear mapping as follows: Let E be a smooth Banach space, let C be a nonempty, closed and convex subset of E and let θ be a real number with $\theta \neq 0$.

A mapping $U\colon C \to E$ with $F(U) \neq \emptyset$ is called *generalized demimetric* [6] if there exists $\theta \in \mathbb{R}$ with $\theta \neq 0$ such that, for any $x \in C$ and $q \in F(U)$,

$$\theta\langle x - q, J(x - Ux)\rangle \geq \|x - Ux\|^2. \tag{4}$$

Such a mapping U is called *θ-generalized demimetric.* According to the definition, we get that a t-strict pseudo-contraction U with $F(U) \neq \emptyset$ is $\frac{2}{1-t}$-generalized demimetric, an $(\alpha,\ \beta)$-generalized hybrid mapping U with $F(U) \neq \emptyset$ is 2-generalized demimetric and the metric resolvent J_λ with $B^{-1}0 \neq \emptyset$ is 1-generalized demimetric.

On the other hand, we know the C-Q method introduced by Solodov and Svaiter [14] for finding a solution of an optimization problem; see also [12, 13]. Furthermore, we know the shrinking projection method introduced by Takahashi, Takeuchi and Kubota [21] for finding a fixed point of a nonexpansive mapping.

In this paper, using this new nonlinear mapping called generalized demimetric and the C-Q method, we first prove a strong convergence theorem for finding a fixed point for the mapping in a Banach space. Furthermore, using the mapping and the shrinking projection method, we prove another strong convergence theorem in a Banach space. We apply these results to obtain new strong convergence theorems in a Hilbert space and a Banach space. In particular, one of them extends simultaneously the results by Nakajo and Takahashi [12], and Solodov and Svaiter [14] in a Hilbert space to that in a Banach space.

2. Preliminaries

Let E be a real Banach space with norm $\|\cdot\|$ and let E^* be the dual space of E. We denote the value of $y^* \in E^*$ at $x \in E$ by $\langle x, y^*\rangle$. When $\{x_n\}$ is a sequence in E, we denote the *strong convergence* of $\{x_n\}$ to $x \in E$ by $x_n \to x$ and the *weak convergence* by $x_n \rightharpoonup x$. The *modulus δ of convexity* of E is defined by

$$\delta(\epsilon) = \inf\left\{1 - \frac{\|x + y\|}{2} : \|x\| \leq 1, \|y\| \leq 1, \|x - y\| \geq \epsilon\right\}$$

for every ϵ with $0 \leq \epsilon \leq 2$. A Banach space E is said to be *uniformly convex* if $\delta(\epsilon) > 0$ for every $\epsilon > 0$. It is known that a Banach space E is uniformly convex if and only if, for any two sequences $\{x_n\}$ and $\{y_n\}$ in E such that

$$\lim_{n\to\infty} \|x_n\| = \lim_{n\to\infty} \|y_n\| = 1 \text{ and } \lim_{n\to\infty} \|x_n + y_n\| = 2,$$

$\lim_{n\to\infty} \|x_n - y_n\| = 0$ holds. A uniformly convex Banach space is *strictly convex* and *reflexive.* We also know that a uniformly convex Banach space has the Kadec-Klee property, i.e., $x_n \rightharpoonup u$ and $\|x_n\| \to \|u\|$ imply $x_n \to u$; see [4].

The *duality mapping* J from E into 2^{E^*} is defined by

$$Jx = \{x^* \in E^* : \langle x, x^*\rangle = \|x\|^2 = \|x^*\|^2\}$$

for every $x \in E$. Let $U = \{x \in E : \|x\| = 1\}$. The norm of E is said to be *Gâteaux differentiable* if for each $x, y \in U$, the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{5}$$

exists. In this case, E is called *smooth*. We know that E is smooth if and only if J is a single-valued mapping of E into E^*. We also know that E is reflexive if and only if J is surjective, and E is strictly convex if and only if J is one-to-one. Therefore, if E is a smooth, strictly convex and reflexive Banach space, then J is a single-valued bijection and in this case, the inverse mapping J^{-1} coincides with the duality mapping J_* on E^*. For more details, see [16] and [17]. We know the following result:

Lemma 2.1. ([16]) *Let E be a smooth Banach space and let J be the duality mapping on E. Then, $\langle x - y, Jx - Jy \rangle \geq 0$ for all $x, y \in E$. Furthermore, if E is strictly convex and $\langle x - y, Jx - Jy \rangle = 0$, then $x = y$.*

Let C be a nonempty, closed and convex subset of a strictly convex and reflexive Banach space E. Then we know that for any $x \in E$, there exists a unique element $z \in C$ such that $\|x - z\| \leq \|x - y\|$ for all $y \in C$. Putting $z = P_C x$, we call such a mapping P_C the *metric projection* of E onto C.

Lemma 2.2. ([16]) *Let E be a smooth, strictly convex and reflexive Banach space. Let C be a nonempty, closed and convex subset of E and let $x_1 \in E$ and $z \in C$. Then, the following conditions are equivalent:*

(1) $z = P_C x_1$;

(2) $\langle z - y, J(x_1 - z) \rangle \geq 0, \quad \forall y \in C$.

Let E be a Banach space and let A be a mapping of E into 2^{E^*}. The *effective domain* of A is denoted by $\mathrm{dom}(A)$, that is, $\mathrm{dom}(A) = \{x \in E : Ax \neq \emptyset\}$. A multi-valued mapping A on E is said to be *monotone* if $\langle x - y, u^* - v^* \rangle \geq 0$ for all $x, y \in \mathrm{dom}(A)$, $u^* \in Ax$, and $v^* \in Ay$. A monotone operator A on E is said to be *maximal* if its graph is not properly contained in the graph of any other monotone operator on E. The following theorem is due to Browder [2]; see also [17, Theorem 3.5.4].

Theorem 2.3. ([2]) *Let E be a uniformly convex and smooth Banach space and let J be the duality mapping of E into E^*. Let A be a monotone operator of E into 2^{E^*}. Then A is maximal if and only if for any $r > 0$, $R(J + rA) = E^*$, where $R(J + rA)$ is the range of $J + rA$.*

Let E be a uniformly convex Banach space with a Gâteaux differentiable norm and let A be a maximal monotone operator of E into 2^{E^*}. For all $x \in E$ and $r > 0$, we consider the following equation

$$0 \in J(x_r - x) + rAx_r.$$

This equation has a unique solution x_r; see [17]. We define J_r by $x_r = J_r x$. Such J_r, $r > 0$ are called the *metric resolvents* of A. The set of null points of A is defined by $A^{-1}0 = \{z \in E : 0 \in Az\}$. We know that $A^{-1}0$ is closed and convex; see [17].

For a sequence $\{C_n\}$ of nonempty, closed and convex subsets of a Banach space E, define $\text{s-Li}_n C_n$ and $\text{w-Ls}_n C_n$ as follows: $x \in \text{s-Li}_n C_n$ if and only if there exists $\{x_n\} \subset E$ such that $\{x_n\}$ converges strongly to x and $x_n \in C_n$ for all $n \in \mathbb{N}$. Similarly, $y \in \text{w-Ls}_n C_n$ if and only if there exist a subsequence $\{C_{n_i}\}$ of $\{C_n\}$ and a sequence $\{y_i\} \subset E$ such that $\{y_i\}$ converges weakly to y and $y_i \in C_{n_i}$ for all $i \in \mathbb{N}$. If C_0 satisfies

$$C_0 = \operatorname*{s-Li}_n C_n = \operatorname*{w-Ls}_n C_n, \tag{6}$$

it is said that $\{C_n\}$ converges to C_0 in the sense of Mosco [11] and we write $C_0 = \text{M-lim}_{n\to\infty} C_n$. It is easy to show that if $\{C_n\}$ is nonincreasing with respect to inclusion, then $\{C_n\}$ converges to $\bigcap_{n=1}^{\infty} C_n$ in the sense of Mosco. For more details, see [11]. The following lemma was proved by Tsukada [24].

Lemma 2.4 ([24]). *Let E be a uniformly convex Banach space. Let $\{C_n\}$ be a sequence of nonempty, closed and convex subsets of E. If $C_0 = M\text{-lim}_{n\to\infty} C_n$ exists and nonempty, then for each $x \in E$, $\{P_{C_n}x\}$ converges strongly to $P_{C_0}x$, where P_{C_n} and P_{C_0} are the mertic projections of E onto C_n and C_0, respectively.*

Let E be a smooth Banach space, let C be a nonempty, closed and convex subset of E and let θ be a real number with $\theta \neq 0$. Then a mapping $U\colon C \to E$ with $F(U) \neq \emptyset$ is called *generalized demimetric* if it satisfies (4), that is, for any $x \in C$ and $q \in F(U)$,

$$\theta\langle x - q, J(x - Ux)\rangle \geq \|x - Ux\|^2.$$

Such a mapping U is called *θ-generalized demimetric*. This generalized the concept of demimetric mappings in the sense of Takahashi [20]. Let η be a real number with $\eta \in (-\infty, 1)$. A mapping $U\colon C \to E$ with $F(U) \neq \emptyset$ is called *η-demimetric* [20] if, for any $x \in C$ and $q \in F(U)$,

$$2\langle x - q, J(x - Ux)\rangle \geq (1 - \eta)\|x - Ux\|^2.$$

If a mapping U is θ-generalized demimetric and $\theta > 0$, then U is $\left(1 - \frac{2}{\theta}\right)$-demimetric. The concept of generalized demimetric mappings is new if $\theta < 0$. We know examples of generalized demimetric mappings with $\theta < 0$; see [6].

Example 2.5. ([6]). Let H be a Hilbert space, let C be a nonempty, closed and convex subset of H and let T be a mapping from C into H. Suppose that T is Lipschitzian, that is, there exists $L > 0$ such that

$$\|Tx - Ty\| \leq L\|x - y\|$$

for all $x, y \in C$. Let $S = (L + 1)I - T$. If $F(\frac{T}{L}) \neq \emptyset$, then S is $(-2L)$-generalized demimetric. □

Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H. For $\alpha > 0$, a mapping $B\colon C \to H$ is called *α-inverse strongly monotone* if

$$\langle x - y, Bx - By\rangle \geq \alpha\|Bx - By\|^2, \quad \forall x, y \in C.$$

Example 2.6. ([6]). Let H be a real Hilbert space, let C be a nonempty, closed and convex subset of H and let $\alpha > 0$. If B be an α-inverse strongly monotone mapping from C into H with $B^{-1}0 \neq \emptyset$, then $T = I + B$ is $\left(-\frac{1}{\alpha}\right)$-generalized demimetric. $\square$

3. Main results

Let E be a Banach space and let C be a nonempty, closed and convex subset of E. A mapping $U\colon C \to E$ is called demiclosed if, for a sequence $\{x_n\}$ in C such that $x_n \rightharpoonup p$ and $x_n - Ux_n \to 0$, $p = Up$ holds. In this section, using the C-Q method, we first prove a strong convergence theorem for finding a fixed point of a generalized demimetric mapping in a Banach space which generalizes simultaneously the results by Nakajo and Takahashi [12] and Solodov and Svaiter [14] in a Hilbert space. Before proving the result, we need two important lemmas for generalized demimetric mappings in a smooth, strictly convex and reflexive Banach space.

Lemma 3.1. ([6]) *Let E be a smooth, strictly convex and reflexive Banach space and let C be a nonempty, closed and convex subset of E. Let θ be a real number with $\theta \neq 0$. Let T be a θ-generalized demimetric mapping of C into E. Then $F(T)$ is closed and convex.*

Lemma 3.2. ([6]) *Let E be a smooth Banach space, let C be a nonempty, closed and convex subset of E and let θ be a real number with $\theta \neq 0$. Let T be a θ-generalized demimetric mapping from C into E and let $k \in \mathbb{R}$ with $k \neq 0$. Then $(1-k)I + kT$ is θk-generalized demimetric from C into E.*

Using Lemmas 3.1 and 3.2, we obtain the following strong convergence theorem by the hybrid method [12] in a Banach space.

Theorem 3.3. *Let E be a uniformly convex and smooth Banach space and let J_E be the duality mapping on E. Let C be a nonempty, closed and convex subset of E and let θ be a real number with $\theta \neq 0$. Let $T\colon C \to E$ be a θ-generalized demimetric and demiclosed mapping with $F(T) \neq \emptyset$. Let k be a real number with $\theta k > 0$. Let $x_1 \in C$ and let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} z_n = (1-\alpha_n)x_n + \alpha_n((1-k)I + kT)x_n, \\ C_n = \{z \in C : \theta k\langle x_n - z, J_E(x_n - z_n)\rangle \geq \|x_n - z_n\|^2\}, \\ Q_n = \{z \in C : \langle x_n - z, J_E(x_1 - x_n)\rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_1, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $0 < a \leq \alpha_n \leq 1$ for some $a \in \mathbb{R}$. Then $\{x_n\}$ converges strongly to a point $z_0 \in F(T)$, where $z_0 = P_{F(T)}x_1$.

Proof. It is obvious that $C_n \cap Q_n$ is closed and convex for all $n \in \mathbb{N}$. Let us show that $F(T) \subset C_n$ for all $n \in \mathbb{N}$. Putting $U = (1-k)I + kT$, we have that $F(T) = F(U)$ and U is θk-generalized demimetric from Lemma 3.2. We have that for any $z \in F(U) = F(T)$ and $n \in \mathbb{N}$,

$$\begin{aligned}\theta k\langle x_n - z,\ J_E(x_n - z_n)\rangle &= \theta k\langle x_n - z, \alpha_n J_E(x_n - Ux_n)\rangle \\ &= \theta k\alpha_n\langle x_n - z, J_E(x_n - Ux_n)\rangle \geq \alpha_n\|x_n - Ux_n\|^2 \qquad (7)\\ &= \frac{\alpha_n^2}{\alpha_n}\|x_n - Ux_n\|^2 = \frac{1}{\alpha_n}\|x_n - z_n\|^2 \geq \|x_n - z_n\|^2.\end{aligned}$$

Then we have that $F(U) = F(T) \subset C_n$ for all $n \in \mathbb{N}$. We show that $F(U) = F(T) \subset Q_n$ for all $n \in \mathbb{N}$. Since $Q_1 = \{z \in C : \langle x_1 - z, J_E(x_1 - x_1)\rangle \geq 0\} = C$, it is obvious that $F(U) = F(T) \subset Q_1$. Suppose that $F(U) = F(T) \subset Q_j$ for some $j \in \mathbb{N}$. Then $F(U) = F(T) \subset C_j \cap Q_j$. By $x_{j+1} = P_{C_j \cap Q_j}x_1$, we have that

$$\langle x_{j+1} - z, J_E(x_1 - x_{j+1})\rangle \geq 0, \quad \forall z \in C_j \cap Q_j$$

and hence

$$\langle x_{j+1} - z, J_E(x_1 - x_{j+1})\rangle \geq 0, \quad \forall z \in F(U) = F(T).$$

Then, $F(U) = F(T) \subset Q_{j+1}$. We have by mathematical induction that $F(U) = F(T) \subset Q_n$ for all $n \in \mathbb{N}$. Thus, we have that $F(U) = F(T) \subset C_n \cap Q_n$ for all $n \in \mathbb{N}$. This implies that $\{x_n\}$ is well defined.

Since $F(T)$ is nonempty, closed and convex from Lemma 3.1, there exists $z_0 \in F(T)$ such that $z_0 = P_{F(T)}x_1$. By $x_{n+1} = P_{C_n \cap Q_n}x_1$, we have that

$$\|x_1 - x_{n+1}\| \leq \|x_1 - y\|, \quad \forall y \in C_n \cap Q_n.$$

Since $z_0 \in F(T) \subset C_n \cap Q_n$, we have that

$$\|x_1 - x_{n+1}\| \leq \|x_1 - z_0\|. \tag{8}$$

This means that $\{x_n\}$ is bounded.

Next, we show that $\lim_{n\to\infty}\|x_n - x_{n+1}\| = 0$. From the definition of Q_n, we have that $x_n = P_{Q_n}x_1$. From $x_{n+1} = P_{C_n \cap Q_n}x_1$ we have that $x_{n+1} \in Q_n$. Thus

$$\|x_n - x_1\| \leq \|x_{n+1} - x_1\|, \quad \forall n \in \mathbb{N}.$$

This implies that $\{\|x_1 - x_n\|\}$ is bounded and nondecreasing. Then there exists the limit of $\{\|x_1 - x_n\|\}$. Put $\lim_{n\to\infty}\|x_n - x_1\| = c$. If $c = 0$, then $\lim_{n\to\infty}\|x_n - x_{n+1}\| = 0$. Assume that $c > 0$. Since $x_n = P_{Q_n}x_1$, $x_{n+1} \in Q_n$ and $\frac{x_n + x_{n+1}}{2} \in Q_n$, we have that

$$\|x_1 - x_n\| \leq \|x_1 - \frac{x_n + x_{n+1}}{2}\| \leq \frac{1}{2}(\|x_1 - x_n\| + \|x_1 - x_{n+1}\|)$$

and hence

$$\lim_{n\to\infty}\|x_1 - \frac{x_n + x_{n+1}}{2}\| = c.$$

Since E is uniformly convex, we get that $\lim_{n\to\infty}\|x_n - x_{n+1}\| = 0$.

We have from $x_{n+1} \in C_n$ that

$$\|x_n - z_n\|^2 \leq \theta k \langle x_n - x_{n+1}, J_E(x_n - z_n)\rangle. \tag{9}$$

We also have from $\theta k > 0$ that for $z \in F(T)$ and $n \in \mathbb{N}$,

$$\|x_n - z_n\|^2 \leq \theta k \langle x_n - z, J_E(x_n - z_n)\rangle \leq \theta k \|x_n - z\| \|x_n - z_n\|$$

and hence $\|x_n - z_n\| \leq \theta k \|x_n - z\|$.

Then, $\{z_n\}$ is bounded. Since $\lim_{n\to\infty} \|x_n - x_{n+1}\| = 0$ and $\{z_n\}$ is bounded, we have from (9) that $\lim_{n\to\infty} \|x_n - z_n\| = 0$. Since $\theta k > 0$ and

$$\|x_n - z_n\|^2 = \alpha_n^2 \|x_n - Ux_n\|^2 \geq a^2 k^2 \|x_n - Tx_n\|^2,$$

we get that

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0. \tag{10}$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ converging weakly to w. Since T is demiclosed, we have that $w \in F(T)$.

From $z_0 = P_{F(T)} x_1$ and $w \in F(T)$, we have from (8) that

$$\begin{aligned} \|x_1 - z_0\| \leq \|x_1 - w\| &\leq \liminf_{i\to\infty} \|x_1 - x_{n_i}\| \\ &\leq \limsup_{i\to\infty} \|x_1 - x_{n_i}\| \leq \|x_1 - z_0\|. \end{aligned}$$

Then we get that

$$\lim_{i\to\infty} \|x_1 - x_{n_i}\| = \|x_1 - w\| = \|x_1 - z_0\|.$$

From the Kadec-Klee property of E, we have that $x_1 - x_{n_i} \to x_1 - w$ and hence

$$x_{n_i} \to w = z_0.$$

Therefore, we have $x_n \to z_0$. This completes the proof. □

Next, using the shrinking projection method [21], we prove a strong convergence theorem for finding a fixed point of a generalized demimetric mapping in a Banach space.

Theorem 3.4. *Let E be a uniformly convex and smooth Banach space and let J_E be the duality mapping on E. Let C be a nonempty, closed and convex subset of E and let θ be a real number with $\theta \neq 0$. Let $T\colon C \to E$ be a θ-generalized demimetric and demiclosed mapping with $F(T) \neq \emptyset$. Let k be a real number with $\theta k > 0$. Let $x_1 \in C$ and let $C_1 = C$. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} z_n = (1-\alpha_n)x_n + \alpha_n((1-k)I + kT)x_n, \\ C_{n+1} = \{z \in C_n : \theta k \langle x_n - z, J_E(x_n - z_n)\rangle \geq \|x_n - z_n\|^2\}, \\ x_{n+1} = P_{C_{n+1}} x_1, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $0 < a \leq \alpha_n \leq 1$ for some $a \in \mathbb{R}$. Then $\{x_n\}$ converges strongly to a point $z_0 \in F(T)$, where $z_0 = P_{F(T)} x_1$.

Proof. It is obvious that C_n are closed and convex for all $n \in \mathbb{N}$. We show that $F(T) = F(U) \subset C_n$ for all $n \in \mathbb{N}$. It is obvious that $F(T) \subset C = C_1$. Suppose that $F(T) \subset C_j$ for some $j \in \mathbb{N}$. As in the proof of Theorem 3.3, we have that for all $z \in F(T)$,

$$\begin{aligned} \theta k\langle x_j - z,\ J_E(x_j - z_j)\rangle &= \theta k\alpha_j\langle x_j - z, J_E(x_j - Ux_j)\rangle \\ &\geq \alpha_j\|x_j - Ux_j\|^2 = \frac{1}{\alpha_j}\|x_j - Ux_j\|^2 \geq \|x_j - Ux_j\|^2. \end{aligned} \tag{11}$$

Then, $F(T) \subset C_{j+1}$. We have by mathematical induction that $F(T) \subset C_n$ for all $n \in \mathbb{N}$. This implies that $\{x_n\}$ is well defined.

Since $F(T)$ is nonempty, closed and convex from Lemma 3.1, there exists $z_0 \in F(T)$ such that $z_0 = P_{F(T)}x_1$. From $x_n = P_{C_n}x_1$, we have that

$$\|x_1 - x_n\| \leq \|x_1 - y\|$$

for all $y \in C_n$. Since $z_0 \in F(T) \subset C_n$, we have that

$$\|x_1 - x_n\| \leq \|x_1 - z_0\|. \tag{12}$$

Let $C_0 = \bigcap_{n=1}^{\infty} C_n$. Since $C_0 \supset F(U) \neq \emptyset$, we have that C_0 is nonempty. Since $C_0 =$ M-$\lim_{n\to\infty} C_n$ and $x_n = P_{C_n}x_1$ for every $n \in \mathbb{N}$, by Lemma 2.4 we have

$$x_n \to x_0 = P_{C_0}x_1.$$

From this, we have that $\|x_n - x_{n+1}\| \to 0$. From $x_{n+1} = P_{C_{n+1}}x_1$, we have $x_{n+1} \in C_{n+1}$. As in the proof of Theorem 3.3, we get that $x_n - z_n \to 0$. Since

$$\|x_n - z_n\|^2 = \alpha_n^2\|x_n - Ux_n\|^2 \geq a^2k^2\|x_n - Tx_n\|^2,$$

we get that

$$\lim_{n\to\infty}\|x_n - Tx_n\| = 0. \tag{13}$$

Since $\{x_n\}$ converges strongly to x_0, $\{x_n\}$ converges weakly to x_0. Since T is demiclosed, we have that $x_0 \in F(T)$.

From $z_0 = P_{F(T)}x_1$, $x_0 \in F(T)$, $x_n \to x_0$ and (12), we have that

$$\|x_1 - z_0\| \leq \|x_1 - x_0\| = \lim_{n\to\infty}\|x_1 - x_n\| \leq \|x_1 - z_0\|.$$

Then we get that $x_0 = z_0$. Therefore, we have $x_n \to x_0 = z_0$. This completes the proof. $\square$

4. Applications

In this section, using Theorems 3.3 and 3.4, we get new strong convergence theorems in a Hilbert space and a Banach space. We know the following lemma obtained by Marino and Xu [10]; see also [22].

Lemma 4.1. ([10, 22]) *Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H. Let t be a real number with $0 \leq t < 1$ and let $U\colon C \to H$ be a t-strict pseudo-contraction. If $x_n \rightharpoonup z$ and $x_n - Ux_n \to 0$, then $z \in F(U)$.*

We also know the following lemma from Kocourek, Takahashi and Yao [7]; see also [23].

Lemma 4.2. ([7, 23]) *Let H be a Hilbert space, let C be a nonempty, closed and convex subset of H and let $U\colon C \to H$ be generalized hybrid. If $x_n \rightharpoonup z$ and $x_n - Ux_n \to 0$, then $z \in F(U)$.*

Let H be a Hilbert space and let C be a nonempty subset of H. Suppose that $T\colon C \to H$ is a Lipschitzian mapping, that is, there exists $L > 0$ such that

$$\|Tx - Ty\| \leq L\|x - y\|$$

for all $x, y \in C$. Let $S = (L+1)I - T$ and suppose $F(\frac{T}{L}) \neq \emptyset$. Then we know from Example 2.5 that S is $(-2L)$-generalized demimetric. In fact, we have that $z \in F(\frac{T}{L})$ if and only if $z \in F(S)$. Furthermore, if $F(S) \neq \emptyset$, then we have that for any $x \in C$ and $z \in F(S)$,

$$\begin{aligned}
0 &\geq L^2\Big(\Big\|\frac{T}{L}x - z\Big\|^2 - \|x - z\|^2\Big) \\
&= L^2\Big(\Big\|\frac{T}{L}x - x\Big\|^2 + \|x - z\|^2 + 2\Big\langle\frac{T}{L}x - x, x - z\Big\rangle - \|x - z\|^2\Big) \\
&= \|Tx - Lx\|^2 + 2L\langle Tx - Lx, x - z\rangle = \|x - Sx\|^2 + 2L\langle x - Sx, x - z\rangle
\end{aligned}$$

and hence

$$-2L\langle x - z, x - Sx\rangle \geq \|x - Sx\|^2.$$

Then we have that S is $(-2L)$-generalized demimetric.

Theorem 4.3. *Let H be Hilbert spaces and let C be a nonempty, closed and convex subset of H. Let $L > 0$. Assume that $T\colon C \to H$ is L-Lipschitzian and $F(\frac{T}{L}) \neq \emptyset$. Let $x_1 \in C$ and let $\{x_n\}$ be a sequence generated by*

$$\begin{cases}
z_n = ((1-L)I + T)x_n, \\
C_n = \{z \in C : 2L\langle x_n - z, x_n - z_n\rangle \geq \|x_n - z_n\|^2\}, \\
Q_n = \{z \in C : \langle x_n - z, x_1 - x_n\rangle \geq 0\}, \\
x_{n+1} = P_{C_n \cap Q_n} x_1, \quad \forall n \in \mathbb{N}.
\end{cases}$$

Then $\{x_n\}$ converges strongly to a point $z_0 \in F(\frac{T}{L})$, where $z_0 = P_{F(\frac{T}{L})}x_1$.

Proof. Since T is L-Lipschitzian and $F(\frac{T}{L}) \neq \emptyset$, $S = (L+1)I - T$ is $(-2L)$-generalized demimetric. Take $k = -1$ and $\alpha_n = 1$ in Theorem 3.3. Then we have that $\theta k = 2L$ and

$$z_n = (1-\alpha_n)x_n + \alpha_n((1-k)I + kS)x_n = (2I - ((L+1)I - T))x_n = ((1-L)+T)x_n.$$

Furthermore, from Lemma 4.1, S is demiclosed. In fact, if $x_n \rightharpoonup z$ and $x_n - Sx_n \to 0$, then

$$\frac{1}{L}(x_n - Sx_n) = \frac{1}{L}(Tx_n - Lx_n)x_n = \frac{T}{L}x_n - x_n \to 0.$$

Since $\frac{T}{L}$ is nonexpansive and hence demiclosed, we have $z \in F(\frac{T}{L}) = F(S)$. Therefore, we have the desired result from Theorem 3.3. $\square$

Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H. Let $\alpha > 0$ and let $A\colon C \to H$ be an α-inverse strongly monotone mapping. Putting $U = I + A$, we have from Example 2.6 that $U\colon C \to H$ is a $\frac{-1}{\alpha}$-generalized demimetric mapping. In fact, since $A\colon C \to H$ is α-inverse strongly monotone, we have that for any $x \in C$ and $z \in A^{-1}0 = F(U)$,

$$\langle x - z, Ax \rangle \geq \alpha \|Ax\|^2$$

and hence $\langle x - z, Ux - x \rangle \geq \alpha \|Ux - x\|^2$. Then, we have

$$\frac{-1}{\alpha}\langle x - z, x - Ux \rangle \geq \|x - Ux\|^2. \tag{14}$$

Theorem 4.4. *Let H be Hilbert spaces and let C be a nonempty, closed and convex subset of H. Let $T\colon C \to H$ be a nonexpansive mapping such that $F(T) \neq \emptyset$. Let $x_1 \in C$ and let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} z_n = 2Tx_n - x_n, \\ C_n = \{z \in C : 4\langle x_n - z, x_n - z_n \rangle \geq \|x_n - z_n\|^2\}, \\ Q_n = \{z \in C : \langle x_n - z, x_1 - x_n \rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_1, \quad \forall n \in \mathbb{N}. \end{cases}$$

Then $\{x_n\}$ converges strongly to a point $z_0 \in F(T)$, where $z_0 = P_{F(T)}x_1$.

Proof. Since T is a nonexpansive mapping of C into H such that $F(T) \neq \emptyset$, $I - T$ is $\frac{1}{2}$-inverse strongly monotone. From (14), $U = 2I - T$ is (-2)-generalized demimetric. Take $k = -2$ and $\alpha_n = 1$ in Theorem 3.3. Then we have that $\theta k = 4$ and

$$z_n = (1 - \alpha_n)x_n + \alpha_n((1 - k)I + kU)x_n = (3I - 2(2I - T))x_n = (2T - I)x_n.$$

Furthermore, from Lemma 4.1, U is demiclosed. In fact, if $x_n \rightharpoonup z$ and $x_n - Ux_n \to 0$, then

$$Tx_n - x_n = x_n - (2x_n - Tx_n) = x_n - Ux_n \to 0.$$

Since T is nonexpansive and hence it is demiclosed, we have $z \in F(T) = F(U)$. Therefore, we have the desired result from Theorem 3.3. $\square$

Theorem 4.5. *Let H be Hilbert spaces and let C be a nonempty, closed and convex subset of H. Let t be a real number with $t \in [0,1)$. Let $U\colon C \to H$ be a t-strict pseudo-contraction such that $F(U) \neq \emptyset$. Let $x_1 \in C$ and let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} z_n = (1-\alpha_n)x_n + \alpha_n U x_n, \\ C_n = \{z \in C : \frac{2}{1-t}\langle x_n - z, x_n - z_n\rangle \geq \|x_n - z_n\|^2\}, \\ Q_n = \{z \in C : \langle x_n - z, x_1 - x_n\rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_1, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $0 < a \leq \alpha_n \leq 1$ for some $a \in \mathbb{R}$. Then $\{x_n\}$ converges strongly to a point $z_0 \in F(U)$, where $z_0 = P_{F(U)}x_1$.

Proof. Since U be a t-strict pseudo-contraction of C into H such that $F(U) \neq \emptyset$, from (1), U is $\frac{2}{1-t}$-generalized demimetric. Furthermore, from Lemma 4.1, U is demiclosed. Therefore, we have the desired result from Theorem 3.3. □

Theorem 4.6. *Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H. Let $U\colon C \to H$ be a generalized hybrid mapping with $F(U) \neq \emptyset$. Let $x_1 \in C$ and let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} z_n = (1-\alpha_n)x_n + \alpha_n U x_n, \\ C_n = \{z \in C : \|z - z_n\| \leq \|z - x_n\|\}, \\ Q_n = \{z \in C : \langle x_n - z, x_1 - x_n\rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_1, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $0 < a \leq \alpha_n \leq 1$ for some $a \in \mathbb{R}$. Then $\{x_n\}$ converges strongly to a point $z_0 \in F(U)$, where $z_0 = P_{F(U)}x_1$.

Proof. Since U be a generalized hybrid mapping of C into H such that $F(U) \neq \emptyset$, from (2), U is 2-generalized demimetric. Thus, the inequality in Theorem 3.3 is

$$2\langle x_n - z, x_n - z_n\rangle \geq \|x_n - z_n\|^2.$$

Using this inequality and

$$2\langle x_n - z, x_n - z_n\rangle = \|x_n - z_n\|^2 + \|z - x_n\|^2 - \|z - z_n\|^2,$$

we have $\|z - z_n\|^2 \leq \|z - x_n\|^2$, that is, $\|z - z_n\| \leq \|z - x_n\|$ in Theorem 4.6. Furthermore, from Lemma 4.2, U is demiclosed. Therefore, we have the desired result from Theorem 3.3. □

In the case when U is nonexpansive, we have the following result by Nakajo and Takahashi [12].

Theorem 4.7. ([12]) *Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H. Let $U\colon C \to H$ be a nonexpansive mapping with $F(U) \neq \emptyset$. Let $x_1 \in C$ and let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} z_n = (1-\alpha_n)x_n + \alpha_n U x_n, \\ C_n = \{z \in C : \|z - z_n\| \leq \|z - x_n\|\}, \\ Q_n = \{z \in C : \langle x_n - z, x_1 - x_n \rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_1, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $0 < a \leq \alpha_n \leq 1$ for some $a \in \mathbb{R}$. Then $\{x_n\}$ converges strongly to a point $z_0 \in F(U)$, where $z_0 = P_{F(U)} x_1$.

Using Theorem 3.3, we also have the following strong convergence theorem for finding a zero point of a maximal monotone operator in a Banach space. This theorem extends the result of Solodov and Svaiter [14] in a Hilbert space to that in a Banach space.

Theorem 4.8. *Let E be a uniformly convex and smooth Banach space. Let J_E be the duality mapping on E. Let A be a maximal monotone operator of E into E^* and let J_λ be the metric resolvent of A for $\lambda > 0$. Let $x_1 \in E$ and let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} z_n = (1-\alpha_n)x_n + \alpha_n J_\lambda x_n, \\ C_n = \{z \in E : \langle z_n - z, J_E(x_n - z_n) \rangle \geq 0\}, \\ Q_n = \{z \in E : \langle x_n - z, J_E(x_1 - x_n) \rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_1, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $0 < a \leq \alpha_n \leq 1$ for some $a \in \mathbb{R}$. Then $\{x_n\}$ converges strongly to a point $z_0 \in A^{-1}0$, where $z_0 = P_{A^{-1}0} x_1$.

Proof. Since J_λ is the metric resolvent of A, from (3), J_λ is 1-generalized demimetric. Thus, the inequality in Theorem 3.3 is

$$\langle x_n - z, J_E(x_n - z_n) \rangle \geq \|x_n - z_n\|^2.$$

Using this inequality and $\|x_n - z_n\|^2 = \langle x_n - z_n, J_E(x_n - z_n) \rangle$, we have the inequality

$$\langle z_n - z, J_E(x_n - z_n) \rangle \geq 0$$

in Theorem 4.8. We also have that if $\{x_n\}$ is a sequence in E such that $x_n \rightharpoonup p$ and $x_n - J_\lambda x_n \to 0$, then $p = J_\lambda p$. In fact, assume that $x_n \rightharpoonup p$ and $x_n - J_\lambda x_n \to 0$. It is clear that $J_\lambda x_n \rightharpoonup p$. We also know that $\|J_F(x_n - J_\lambda x_n)\| = \|x_n - J_\lambda x_n\| \to 0$. Since J_λ is the metric resolvent of A, we have from [1] that

$$\langle J_\lambda x_n - J_\lambda p, J_E(x_n - J_\lambda x_n) - J_E(p - J_\lambda p) \rangle \geq 0.$$

This implies $\langle p - J_\lambda p, -J_E(p - J_\lambda p) \rangle \geq 0$ and hence $-\|p - J_\lambda p\|^2 \geq 0$. Then we have $p = J_\lambda p$. Therefore, we have the desired result from Theorem 3.3. $\square$

Similarly, using Theorem 3.4, we have the following results.

Theorem 4.9. *Let H be Hilbert spaces and let C be a nonempty, closed and convex subset of H. Let $L > 0$. Assume that $T\colon C \to H$ is L-Lipschitzian and $F(\frac{T}{L}) \neq \emptyset$. For $x_1 \in C$ and $C_1 = C$, let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} z_n = ((1-L)I + T)x_n, \\ C_{n+1} = \{z \in C_n : 2L\langle x_n - z, x_n - z_n\rangle \geq \|x_n - z_n\|^2\}, \\ x_{n+1} = P_{C_{n+1}}x_1, \quad \forall n \in \mathbb{N}. \end{cases}$$

Then $\{x_n\}$ converges strongly to a point $z_0 \in F(\frac{T}{L})$, where $z_0 = P_{F(\frac{T}{L})}x_1$.

Theorem 4.10. *Let H be Hilbert spaces and let C be a nonempty, closed and convex subset of H. Let $T\colon C \to H$ be a nonexpansive mapping such that $F(T) \neq \emptyset$. For $x_1 \in C$ and $C_1 = C$, let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} z_n = 2Tx_n - x_n, \\ C_{n+1} = \{z \in C_n : 4\langle x_n - z, x_n - z_n\rangle \geq \|x_n - z_n\|^2\}, \\ x_{n+1} = P_{C_{n+1}}x_1, \quad \forall n \in \mathbb{N}. \end{cases}$$

Then $\{x_n\}$ converges strongly to a point $z_0 \in F(T)$, where $z_0 = P_{F(T)}x_1$.

Theorem 4.11. *Let H be a Hilbert space. Let t be a real number with $t \in [0,1)$. Let C be a nonempty, closed and convex subset of H and let $U\colon C \to H$ be a t-strict pseudo-contraction such that $F(U) \neq \emptyset$. For $x_1 \in C$ and $C_1 = C$, let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} z_n = (1-\alpha_n)x_n + \alpha_n U x_n, \\ C_{n+1} = \{z \in C_n : \frac{2}{1-t}\langle x_n - z, x_n - z_n\rangle \geq \|x_n - z_n\|^2\}, \\ x_{n+1} = P_{C_{n+1}}x_1, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $0 < a \leq \alpha_n \leq 1$ for some $a \in \mathbb{R}$. Then $\{x_n\}$ converges strongly to $z_0 \in F(U)$, where $z_0 = P_{F(U)}x_1$.

Theorem 4.12. *Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H. Let $U\colon C \to H$ be a generalized hybrid mapping with $F(U) \neq \emptyset$. For $x_1 \in C$ and $C_1 = C$, let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} z_n = (1-\alpha_n)x_n + \alpha_n U x_n, \\ C_{n+1} = \{z \in C_n : \|z - z_n\| \leq \|z - x_n\|\}, \\ x_{n+1} = P_{C_{n+1}}x_1, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $0 < a \leq \alpha_n \leq 1$ for some $a \in \mathbb{R}$. Then $\{x_n\}$ converges strongly to $z_0 \in F(U)$, where $z_0 = P_{F(U)}x_1$.

In the case when U is nonexpansive, Theorem 4.12 is a result by Takahashi, Takeuchi and Kubota [21]. Using Theorem 3.4, we also have the following strong convergence theorem for finding a zero point of a maximal monotone operator in a Banach space.

Theorem 4.13. *Let E be a uniformly convex and smooth Banach space. Let J_E be the duality mapping on E. Let A be a maximal monotone operator of E. Let J_λ be the metric resolvent of A for $\lambda > 0$. Suppose that $A^{-1}0 \neq \emptyset$. For $x_1 \in C$ and $C_1 = C$, let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} z_n = (1-\alpha_n)x_n + \alpha_n J_\lambda x_n, \\ C_{n+1} = \{z \in C_n : \langle z_n - z, J_E(x_n - z_n)\rangle \geq 0\}, \\ x_{n+1} = P_{C_{n+1}} x_1, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $0 < a \leq \alpha_n \leq 1$ for some $a \in \mathbb{R}$. Then $\{x_n\}$ converges strongly to a point $z_0 \in A^{-1}0$, where $z_0 = P_{A^{-1}0} x_1$.

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