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A Note on Quasi-Convex Functions

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We present an example of smooth quasi-convex functions in the positive octant of $\mathbb{R}^3$ which cannot be obtained as the images of convex smooth functions under a monotone smooth mapping of $\mathbb{R}$.

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1. Introduction

Quasi-convex functions play an important role in problems related to continuous optimization and mathematical programming such as generalizations of the von Neumann minimax theorem, the Kuhn-Tucker saddle-point theorem, and other optimization problems related to consumer demand and indirect utility function (see, for instance, [3], [5] and the references therein).

According to [3] a real-valued function $u(x)$ defined in a convex subset E of the Euclidean space $\mathbb{R}^d$ is called *quasi-convex* if

$$u(\lambda x + (1-\lambda)y) \le \max[u(x), u(y)]$$

as long as $\lambda \in [0,1]$, $x, y \in E$. Convex functions in [3] are those for which

$$f(\lambda x + (1-\lambda)y) \le \lambda f(x) + (1-\lambda) f(y).$$

Numerous properties of quasi-convex functions and their relation to convex functions are discussed, for instance, in [3], [6], and [1], in particular, that $F[f(x)]$ is quasi-convex if f is convex and F is nondecreasing. Somehow the following very natural question is left untouched: can any quasi-convex function u be represented as $F[f(x)]$ with convex f and nondecreasing F? This issue is also avoided in many other publications on the subject of quasi-convexity. The reason for that is probably because the answer to this question is negative and the corresponding counterexamples are given in [2]. In these examples, however, u is not smooth and for almost any point of E there exists a neighborhood such that in that neighborhood the above representation still holds.

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We want to present an example of a *smooth* function u, which is quasi-convex in a convex domain E such that there are no smooth and strictly monotone functions F such that $F[u(x)]$ is convex in a ball in E. Then, of course, u itself is not even locally an increasing smooth image of a smooth convex function. This directly contradicts the claim made in n. 1 of [2] that for any *smooth* quasi-convex function u one can find strictly increasing F such that $F[u(x)]$ is convex. Our arguments have much in common with Remark 5.14 of [4].

2. Main result

Theorem 2.1. *In $\mathbb{R}^3$ consider the domain $E := \{(x,y,z) : x,y,z > 0\}$, fix $\alpha > 0$, and introduce the function*

$$u(x,y,z) = \frac{z^\alpha(x^\alpha + y^\alpha)}{x^\alpha y^\alpha}.$$

Then E is convex, u is quasi-convex in E, and, if $\alpha \in (0,1]$, then for any $(x_0,y_0,z_0) \in E$ and any twice continuously differentiable function $F(t)$ on $\mathbb{R}$ such that $F'[u(x_0,y_0,z_0)] \neq 0$, the matrix of the second order derivatives of $F[u(x,y,z)]$ at (x_0,y_0,z_0) is neither nonnegative nor nonpositive. In consequence, $F[u(x,y,z)]$ is neither convex nor concave in any neighborhood of any point in E if $F' > 0$ or $F' < 0$ on $\mathbb{R}$.

Proof. First observe that the function

$$v(x,y) = \frac{x^\alpha + y^\alpha}{x^\alpha y^\alpha} = \frac{1}{x^\alpha} + \frac{1}{y^\alpha}$$

is convex in $(0,\infty)^2$. Then, in light of homogeneity of u for $(x_i,y_i,z_i) \in (0,\infty)^3$, $\lambda_i \in [0,1]$, $i = 1,2$, such that $\lambda_1 + \lambda_2 = 1$, we have

$$u\big(\lambda_1 x_1 + \lambda_2 x_2, \lambda_1 y_1 + \lambda_2 y_2, \lambda_1 z_1 + \lambda_2 z_2\big) =$$

$$= v\Big(\frac{\lambda_1 x_1 + \lambda_2 x_2}{\lambda_1 z_1 + \lambda_2 z_2}, \frac{\lambda_1 y_1 + \lambda_2 y_2}{\lambda_1 z_1 + \lambda_2 z_2}\Big) = v\Big(\mu_1 \frac{x_1}{z_1} + \mu_2 \frac{x_2}{z_2}, \mu_1 \frac{y_1}{z_1} + \mu_2 \frac{y_2}{z_2}\Big),$$

where $\mu_i = \lambda_i z_i/(\lambda_1 z_1 + \lambda_2 z_2)$. Since $\mu_i \geq 0$ and $\mu_1 + \mu_2 = 1$ and v is convex, the last expression above is less than

$$\mu_1 v\Big(\frac{x_1}{z_1}, \frac{y_1}{z_1}\Big) + \mu_2 v\Big(\frac{x_2}{z_{12}}, \frac{y_2}{z_2}\Big) = \mu_1 u(x_1,y_1,z_1) + \mu_2 u(x_2,y_2,z_2)$$

$$\leq \max\big[u(x_1,y_1,z_1), u(x_2,y_2,z_2)\big].$$

This shows that u is quasi-convex in E.

We now come to analyzing $F[u]$. Denote by Dv the column-vector gradient of v and by $D^2 v$ its matrix of the second-order derivatives. By a^* we mean the transpose of a matrix a and by $\langle a,b\rangle$ we mean the scalar product of $a,b\in\mathbb{R}^3$. We have

$$D\{F[u]\}=F'[u]Du,\quad D^2\{F[u]\}=F''[u]Du(Du)^*+F'[u]D^2u.$$

We fix $(x_0,y_0,z_0)\in E$ and take a column-vector $\xi=(\xi_1,\xi_2,\xi_3)\in\mathbb{R}^3$ (written in a common abuse of notation as a row vector) such that at (x_0,y_0,z_0) we have $\langle Du,\xi\rangle=0$, which means that

$$-\xi_1\alpha\frac{z_0^\alpha}{x_0^{\alpha+1}}-\xi_2\alpha\frac{z_0^\alpha}{y_0^{\alpha+1}}+\xi_3\alpha\frac{z_0^{\alpha-1}(x_0^\alpha+y_0^\alpha)}{x_0^\alpha y_0^\alpha}=0,$$

$$\xi_3=\frac{z_0x_0^\alpha y_0^\alpha}{x_0^\alpha+y_0^\alpha}\Big(\frac{\xi_1}{x_0^{\alpha+1}}+\frac{\xi_2}{y_0^{\alpha+1}}\Big). \tag{1}$$

Furthermore,

$$u_{xx}=\alpha(\alpha+1)\frac{z^\alpha}{x^{\alpha+2}},\quad u_{xy}=0,\quad u_{xz}=-\alpha^2\frac{z^{\alpha-1}}{x^{\alpha+1}},\quad u_{yz}=-\alpha^2\frac{z^{\alpha-1}}{y^{\alpha+1}},$$

$$u_{yy}=\alpha(\alpha+1)\frac{z^\alpha}{y^{\alpha+2}},\quad u_{zz}=\alpha(\alpha-1)\frac{z^{\alpha-2}(x^\alpha+y^\alpha)}{x^\alpha y^\alpha},$$

which implies that for ξ satisfying (1) we have at (x_0,y_0,z_0)

$$\begin{aligned}\big(F'[u]\big)^{-1}&\langle\xi,D^2\{F[u]\}\xi\rangle=\langle\xi,D^2u\xi\rangle=\\&=\alpha(\alpha+1)\frac{z_0^\alpha}{x_0^{\alpha+2}}\xi_1^2+\alpha(\alpha+1)\frac{z_0^\alpha}{y_0^{\alpha+2}}\xi_2^2-\\&\qquad-2\alpha^2\Big(\xi_1\frac{z_0^{\alpha-1}}{x_0^{\alpha+1}}+\xi_2\frac{z_0^{\alpha-1}}{y_0^{\alpha+1}}\Big)\frac{z_0x_0^\alpha y_0^\alpha}{x_0^\alpha+y_0^\alpha}\Big(\frac{\xi_1}{x_0^{\alpha+1}}+\frac{\xi_2}{y_0^{\alpha+1}}\Big)+\\&\qquad+\alpha(\alpha-1)\frac{z_0^{\alpha-2}(x_0^\alpha+y_0^\alpha)}{x_0^\alpha y_0^\alpha}\frac{z_0^2x_0^{2\alpha}y_0^{2\alpha}}{(x_0^\alpha+y_0^\alpha)^2}\Big(\frac{\xi_1}{x_0^{\alpha+1}}+\frac{\xi_2}{y_0^{\alpha+1}}\Big)^2.\end{aligned}$$

Here the sum of the last two terms equals

$$z_0^\alpha\frac{x_0^\alpha y_0^\alpha}{x_0^\alpha+y_0^\alpha}\Big(\frac{\xi_1}{x_0^{\alpha+1}}+\frac{\xi_2}{y_0^{\alpha+1}}\Big)^2\big[\alpha(\alpha-1)-2\alpha^2\big].$$

Therefore, $\big(F'[u]\big)^{-1}\langle\xi,D^2\{F[u]\}\xi\rangle=z_0^\alpha Q(\xi_1,\xi_2)),$ where

$$\begin{aligned}Q(\xi_1,\xi_2)&=\frac{1}{x_0^{\alpha+2}}\Big[\alpha(\alpha+1)-(\alpha^2+1)\frac{y_0^\alpha}{x_0^\alpha+y_0^\alpha}\Big]\xi_1^2+\\+\frac{1}{y_0^{\alpha+2}}&\Big[\alpha(\alpha+1)-(\alpha^2+1)\frac{x_0^\alpha}{x_0^\alpha+y_0^\alpha}\Big]\xi_1^2-2(\alpha^2+1)\frac{x_0^\alpha y_0^\alpha}{x_0^\alpha+y_0^\alpha}\frac{1}{x_0^{\alpha+1}}\frac{1}{y_0^{\alpha+1}}\xi_1\xi_2\,.\end{aligned}$$

It follows that to prove our claim it suffices to show that the quadratic form Q with respect to (ξ_1, ξ_2) is neither nonnegative nor nonpositive. For this to be true we need to show that the determinant of its matrix

$$\frac{1}{x_0^{\alpha+2}}\frac{1}{y_0^{\alpha+2}}\Big[\alpha(\alpha+1)-(\alpha^2+1)\frac{y_0^\alpha}{x_0^\alpha+y_0^\alpha}\Big]\Big[\alpha(\alpha+1)-(\alpha^2+1)\frac{x_0^\alpha}{x_0^\alpha+y_0^\alpha}\Big]-$$

$$-(\alpha^2+1)^2\frac{x_0^{2\alpha}y_0^{2\alpha}}{(x_0^\alpha+y_0^\alpha)^2}\frac{1}{x_0^{2\alpha+2}}\frac{1}{y_0^{2\alpha+2}}=:\frac{1}{x_0^{\alpha+2}}\frac{1}{y_0^{\alpha+2}}R(\xi_1,\xi_2)$$

is negative. One checks easily that

$$R(\xi_1,\xi_2)=\alpha^2(\alpha+1)^2-\alpha(\alpha+1)(\alpha^2+1)=\alpha(\alpha+1)(\alpha-1),$$

which is < 0 if $\alpha < 1$, and the theorem is proved in this case.

In case $\alpha = 1$, take ξ from above, introduce $\eta = \xi + \kappa Du(x_0, y_0, z_0)$, and note that

$$\big(F'[u]\big)^{-1}\langle\eta, D^2\{F[u]\}\eta\rangle = \langle\xi, D^2u\xi\rangle + 2\kappa\langle D^2u\xi, Du\rangle$$

$$+\Big[F''[u]\big(F'[u]\big)^{-1}|Du|^4+\langle D^2uDu, Du\rangle\Big]\kappa^2, \tag{2}$$

where at (x_0, y_0, z_0) as is easy to check

$$\langle\xi, D^2u\xi\rangle=\frac{2z_0}{x_0+y_0}\Big(\frac{\xi_1}{x_0}-\frac{\xi_2}{y_0}\Big)^2.$$

This quantity vanishes if $\xi_1 = tx_0, \xi_2 = ty_0$, $t \in \mathbb{R}$, and for the right-hand side of (2) not to change sign, say for $\kappa = 1$, when t runs through $\mathbb{R}$ it is necessary to have $\langle D^2uDu, \xi\rangle = 0$ for those ξ_1, ξ_2.

However, with this choice at (x_0, y_0, z_0) we have $\xi_3 = tz_0$ and

$$D^2u\xi = t\Big(\frac{z_0}{x_0^2}, \frac{z_0}{y_0^2}, -\frac{1}{x_0}-\frac{1}{y_0}\Big),$$

$$\langle D^2u\xi, Du\rangle = t\Big(-\frac{z_0^2}{x_0^4}-\frac{z_0^2}{y_0^4}-(\frac{1}{x_0}+\frac{1}{y_0})^2\Big).$$

This shows that for $\alpha = 1$ as well, the right-hand side of (2) has different signs for any fixed (x_0, y_0, z_0) if we vary ξ and κ and finishes the proof of the theorem.

Remark 2.2. One can show that, if $\alpha > 1$, then for any point $(x_0, y_0, z_0) \in E$ one can find large $\lambda > 0$ such that $\exp(\lambda u)$ is strictly convex in a neighborhood of (x_0, y_0, z_0). Of course, $\lambda \to \infty$ as $\alpha \downarrow 1$.

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