

Fixed Points of Legendre-Fenchel Type Transforms

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A recent result characterizes the fully order reversing operators acting on the class of lower semicontinuous proper convex functions in a real Banach space as certain linear deformations of the Legendre-Fenchel transform. Motivated by the Hilbert space version of this result and by the well-known result saying that this convex conjugation transform has a unique fixed point (namely, the normalized energy function), we investigate the fixed point equation in which the involved operator is fully order reversing and acts on the above-mentioned class of functions. It turns out that this nonlinear equation is very sensitive to the involved parameters and can have no solution, a unique solution, or several (possibly infinitely many) ones. Our analysis yields a few by-products, such as results related to positive definite operators, and to functional equations and inclusions involving monotone operators.

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1. Introduction

1.1. Background

Let X be a real Hilbert space. Our goal is to solve the fixed point equation

$$f(x) = \tau f^*(Ex + c) + \langle w, x \rangle + \beta, \quad x \in X, \quad (1)$$

where $\tau > 0$, $c \in X$, $w \in X$ and $\beta \in \mathbb{R}$ are given, $E: X \rightarrow X$ is a given continuous linear invertible operator, and $f: X \rightarrow [-\infty, \infty]$ is the unknown function. Here

$$f^*(x^*) := \sup\{\langle x^*, x \rangle - f(x) : x \in X\}, \quad x^* \in X, \quad (2)$$

is the *Legendre-Fenchel transform* of the function f . This transform (which has many other names such as the *Legendre transformation*, or the *convex conjugation*, and sometimes has a form which is slightly different from (2)) plays a central role in classical mechanics [2], thermodynamics [29], convex analysis [22], nonlinear analysis [12] and optimization [14].

The motivation for discussing (1) stems from several known results. First, it is a more general version of the equation

$$f = f^*, \quad (3)$$

the solutions of which describe all the self-conjugate functions. It is well known (see [8, Proposition 13.19, p. 225], [22, p. 106]) that (3) has a unique solution, namely $f(x) = \frac{1}{2}\|x\|^2$, $x \in X$ (the “normalized energy function”). This fact was mentioned briefly and without proof already in the pioneering work of Fenchel [13, p. 73] for $X = \mathbb{R}^n$, and later it was extended (with a proof) by Moreau [18, Proposition 9.a] to general real Hilbert spaces.

Second, (1) is related to a relatively recent development in convex analysis. As shown in the pioneering work of Artstein-Avidan and Milman [4, Theorem 7] when $X = \mathbb{R}^n$, the right-hand side of (1) is closely related to the characterization of fully order reversing operators T acting on the class $\mathcal{C}(X)$ of all proper, convex, and lower semi-continuous convex functions from X to $\mathbb{R} \cup \{+\infty\}$. More precisely, if T is fully order reversing, in the sense that T is invertible and it reverses the point-wise order between functions in $\mathcal{C}(X)$ (that is, if $f(x) \leq g(x)$ for all $x \in X$, then $(Tf)(u) \geq (Tg)(u)$ for all $u \in X$), and also T^{-1} reverses the order between functions, then T must have the form $T = T[E, c, w, \tau, \beta]$, where

$$T[E, c, w, \tau, \beta](f)(x) := \tau f^*(Ex + c) + \langle w, x \rangle + \beta, \quad x \in X \quad (4)$$

for some $\tau > 0$, $c \in X$, $w \in X$, $\beta \in \mathbb{R}$ and an invertible linear operator $E: X \rightarrow X$. We think of T as being a Legendre-Fenchel type transform since, up to certain inner and outer linear deformations, it indeed is this transform. The converse statement holds too: as can be verified directly, any operator of the form $T = T[E, c, w, \tau, \beta]$ is a fully order reversing acting on $\mathcal{C}(X)$. See [3, 5, 6, 7, 9, 25] for variations of this result regarding other classes of functions and geometric objects, still in a finite-dimensional setting.

As shown in [15, Theorem 2], the above-mentioned characterization of fully order reversing operators holds also in the case of arbitrary real Banach spaces. Here one considers fully order reversing operators T acting between $\mathcal{C}(X)$ and the class of $\mathcal{C}_{w^*}(X^*)$ of all weak* lower semicontinuous proper and convex functions from the dual X^* to $\mathbb{R} \cup \{+\infty\}$. Now f^* is defined on X^* as in (2) (where now $x^* \in X^*$ and $\langle x^*, x \rangle$ denotes $x^*(x)$ for all $x \in X$), the linear operator $E: X^* \rightarrow X^*$ is continuous on X^* , c is a vector in X^* , w is a vector in the canonical embedding of X in X^{**} , τ is positive, and β is real.

Since such an operator T acts between two different sets, one cannot speak of its possible fixed points. However, in the specific case where X is a Hilbert space

the well-known strong correspondence between X and X^* enables us to identify them. This fact, when combined with the fact that for convex functions strong and weak lower semicontinuity coincide (and the same holds for the weak and weak* topologies in $X^* \cong X$), leads us to take $E: X \rightarrow X$, $c, w \in X$, and $T: \mathcal{C}(X) \rightarrow \mathcal{C}(X)$, and to conclude that (1) describes the form of the most general fixed point equation of fully order reversing operators acting on $\mathcal{C}(X)$.

We mention now some works related to fixed point theory in the context of conjugates and convex analysis. One type of works we have already mentioned earlier, namely works which completely solve (3) (for instance, [8, Proposition 13.19, p. 225], [13, p. 73], [18, Proposition 9.a] and [22, p. 106]). A second type of relevant works are [17] and [27]. More precisely, [17] discusses (3) in a rather general setting: X is a nonempty set and the conjugation f^* is abstract. It was proved in [17, Theorem 1.1] that in this case there exists at least one solution f to (3). Variations of this theorem for more concrete settings (with the stronger result that now the fixed point must belong to the Fitzpatrick family of a maximal monotone operator) appeared earlier in [27, Theorem 2.4] and later in [17, Theorem 4.4]. Neither (1) nor the questions of uniqueness of solutions (to (3)) and their classification have been considered in these papers.

Another work which is somewhat related to our context is that of Rotem [23] in which self-polar functions on the positive ray were investigated. More precisely, the considered equation was $f = f^\circ$, where here f° is the polarity transform of f , a transform which was introduced in [22, pp. 136–139] and was extensively investigated in [6]. A complete characterization of all the self-polar functions on the ray was presented in [23, Theorem 5] and an example presented later [23, p. 838] shows that this characterization fails already in a two-dimensional setting. Anyway, (1) was not investigated in [23] (but it is an interesting open problem to investigate a version of (1) in which f^* is replaced by f°).

1.2. Contributions

The main theorem of this paper is Theorem 3.1 below which shows that the solution set of the nonlinear equation (1) is very sensitive to the various parameters which appear in it, and classifies the possible solutions in many cases. In a nutshell, (1) can have no solution, a unique solution, or many (possibly infinitely many) solutions. More precisely, the governing parameter seems to be the invertible linear operator E . If E is positive definite, then there always exists a solution to (1), and this solution is quadratic and strictly convex. Sometimes uniqueness can also be established, and this existence and uniqueness result generalizes the well-known result mentioned in Subsection 1.1 that $f = \frac{1}{2}\|\cdot\|^2$ is the unique solution to (3). On the other hand, if E is not positive definite, then there can be several (perhaps infinitely many) solutions to (1) or no solution at all, depending on the values of the other parameters which appear in (1). Moreover, in some cases there exist non-quadratic solutions.

To the best of our knowledge, (1) has not been considered in the literature. Its analysis is somewhat technical and requires separation into several cases, according to the relevant parameters which appear in (1).

Along the way we obtain a number of by-products which seem to be of independent interest. Among them, we mention Lemmas 6.1–6.2 below, concerning certain functional inclusions and equations (see also Remark 6.5), and Corollary 6.3 below regarding the uniqueness of square roots of the identity operator in the class of positive semidefinite linear operators.

1.3. Paper layout

After some preliminaries given in Section 2, we formulate the main classification theorem (Theorem 3.1) in Section 3. The tools needed in the proof of this theorem are developed in Sections 4 to 13, and the proof itself is presented in Section 14. We finish the paper with Section 15 which contains several concluding remarks and open problems.

2. Preliminaries

We work with a real Hilbert space $X \neq \{0\}$ endowed with an inner product $\langle \cdot, \cdot \rangle$ and an induced norm $\|\cdot\|$. A function $f: X \rightarrow [-\infty, \infty]$ is called *proper* whenever $f(x) > -\infty$ for all $x \in X$ and, in addition, $f(x) \neq \infty$ for at least one point $x \in X$. The *effective domain* of $f: X \rightarrow [-\infty, \infty]$ is the set $\text{dom}(f) := \{x \in X : f(x) \in \mathbb{R}\}$. The *Fenchel-Legendre transform* of $f: X \rightarrow [-\infty, \infty]$ is the function $f^*: X \rightarrow [-\infty, \infty]$ which is defined in (2).

It is well known that f^* is always convex and lower semicontinuous in the norm topology of X , and, in addition, that $f^* \equiv -\infty$ if and only if $f \equiv \infty$ (see, for instance, [8, 28] for the proofs of many known facts from convex analysis which are mentioned here without proofs). The *biconjugate* of f is the function $f^{**}: X \rightarrow [-\infty, \infty]$ defined by $f^{**} = (f^*)^*$. A well-known result, sometimes called the Fenchel-Moreau theorem [11, Theorem 1.11, p.13], says that $f = f^{**}$ whenever $f \in \mathcal{C}(X)$. Here $\mathcal{C}(X)$ denotes the set of lower semicontinuous proper convex functions $f: X \rightarrow \mathbb{R} \cup \{+\infty\}$.

We consider the pointwise order between functions, that is, we write $f \leq g$ whenever $f(x) \leq g(x)$ for all $x \in X$. It is well known and easily follows from (2) that $f \leq g$ if and only if $f^* \geq g^*$. The *subdifferential* of f at $x \in \text{dom}(f)$ is the set $(\partial f)(x)$ defined by $(\partial f)(x) := \{x^* \in X : f(x) + \langle x^*, y - x \rangle \leq f(y), \quad \forall y \in X\}$. In general, $(\partial f)(x)$ is not necessarily a singleton and it can be empty. If, however, f is continuous (and, hence, finite everywhere), then $(\partial f)(x)$ is nonempty for all $x \in X$. We say that $f: X \rightarrow (-\infty, \infty]$ is strictly convex if for all $x, y \in \text{dom}(f)$ satisfying $x \neq y$ and for all $\lambda \in (0, 1)$ we have $f(\lambda x + (1-\lambda)y) < \lambda f(x) + (1-\lambda)f(y)$.

Given a linear and continuous operator $E: X \rightarrow X$, the *adjoint* of E is the operator $E^*: X \rightarrow X$ defined by the equation $\langle E^*a, b \rangle = \langle a, Eb \rangle$ for all $(a, b) \in X^2$. It is well known that E^* is continuous, and $\|E^*\| = \|E\|$.

A *self-adjoint operator* is a continuous linear operator $E: X \rightarrow X$ satisfying $E = E^*$. Such an operator is also called *symmetric*. A self-adjoint operator $E: X \rightarrow X$ satisfying $\langle Ex, x \rangle \geq 0$ for all $x \in X$ is called *positive semidefinite*. A self-adjoint operator $E: X \rightarrow X$ satisfying $\langle Ex, x \rangle > 0$ for all $0 \neq x \in X$ is

called *positive definite* or simply *positive*. We denote by $I: X \rightarrow X$ the *identity operator*.

A set-valued operator on X is a mapping $A: X \rightarrow 2^X$, where 2^X is the set of all subsets of X . Such a mapping is frequently identified with the graph of A , that is, with the set $G(A) := \{(x, y) \in X^2 : y \in Ax\}$. We say that $A: X \rightarrow 2^X$ is single-valued if $A(x)$ is a singleton for all $x \in X$. In this case we regard A as an ordinary function from X to X . We say that $A: X \rightarrow 2^X$ is contained in $B: X \rightarrow 2^X$ whenever $G(A) \subseteq G(B)$, or, equivalently, when $Ax \subseteq Bx$ for each $x \in X$. For set-valued operators $A: X \rightarrow 2^X$ and $B: X \rightarrow 2^X$ and $x \in X$ we define $(A + B)(x) := \{a + b : a \in Ax, b \in Bx\}$ if $Ax \neq \emptyset$ and $Bx \neq \emptyset$ and $(A + B)(x) := \emptyset$ otherwise. The composition BA (also denoted by $B \circ A$) is the operator from X to 2^X defined by $(BA)(x) := \bigcup_{x' \in Ax} Bx'$, $x \in X$. This is an associative operation. The *inverse* of $A: X \rightarrow 2^X$, denoted by A^{-1} , is the set-valued operator the graph of which is $G(A^{-1}) = \{(y, x) : y \in Ax\}$. We call A *monotone* whenever

$$\langle y_2 - y_1, x_2 - x_1 \rangle \geq 0 \quad \forall (x_1, x_2) \in X^2, \forall y_i \in Ax_i, i = 1, 2. \quad (5)$$

The set-valued operator A is called *strictly monotone* if there is a strict inequality in (5) whenever $x_1 \neq x_2$. We say that A is *maximal monotone* whenever it is monotone and there exists no monotone operator $B: X \rightarrow 2^X$ such that $A \neq B$ and A is contained in B . It is well known and straightforward to check that a set-valued operator is maximal monotone if and only if its inverse is maximal monotone. In the sequel we make use of the well-known facts that the subdifferential of a proper function is monotone and any positive semidefinite linear operator $A: X \rightarrow X$ is maximal monotone (if A is positive definite, then it is even strictly monotone). More information regarding the theory of set-valued (monotone) operators can be found in [8, 10, 12, 26].

A function $f: X \rightarrow (-\infty, \infty]$ is called *at most quadratic* whenever

$$f(x) = \frac{1}{2} \langle Ax, x \rangle + \langle b, x \rangle + \gamma, \quad x \in X, \quad (6)$$

for some self-adjoint operator $A: X \rightarrow X$, a vector $b \in X$, and a real number $\gamma \in \mathbb{R}$. The operator A is *the leading coefficient* of f . We say that f is *quadratic* when it has the form (6) with $A^* = A \neq 0$. A well-known fact (that follows from [1, p. 29]) which we use in Lemma 10.1 below is that if f'' exists at each point and is constant, then f is at most quadratic.

A function $f: X \rightarrow (-\infty, \infty]$ is (Fréchet) *differentiable* at some $x \in X$ if $x \in \text{dom}(f)$ and there exists a continuous linear functional $f'(x): X \rightarrow \mathbb{R}$ such that for every $h \in X$ sufficiently small,

$$f(x + h) = f(x) + f'(x)(h) + o(\|h\|).$$

The identification between X and its dual X^* allows us to write $\langle f'(x), h \rangle$ instead of $f'(x)(h)$. The function f is twice differentiable at $x \in X$ if $f': X \rightarrow X^*$ exists in a neighborhood of x and is differentiable at x . The second derivative $f''(x)$ can be identified with a continuous and symmetric bilinear form acting from X^2 to $\mathbb{R}$. It is well known that any symmetric bilinear form $B: X^2 \rightarrow \mathbb{R}$ can be written as $B(a, b) = \frac{1}{2}\langle Aa, b \rangle$, $(a, b) \in X^2$, where $A: X \rightarrow X$ is a continuous and symmetric linear operator, and hence we identify $f''(x)$ with the operator A associated with it. If f'' exists and is continuous in a neighborhood of x , then f has a second order Taylor expansion about x :

$$f(x+h) = f(x) + \langle f'(x), h \rangle + \frac{1}{2}\langle f''(x)h, h \rangle + o(\|h\|^2), \quad h \in X. \quad (7)$$

According to a well-known fact, if f is convex and differentiable at x , then $(\partial f)(x) = \{f'(x)\}$, in which case we write $(\partial f)(x) = f'(x)$.

3. The classification theorem

The main result of this paper is the following classification theorem which analyzes the set of solutions of (1) under various assumptions on the relevant parameters and on the class of allowed solutions.

Theorem 3.1. *Let X be a real Hilbert space. Take $\tau > 0$, $c \in X$, $w \in X$ and $\beta \in \mathbb{R}$. Let $E: X \rightarrow X$ be an invertible and continuous linear operator. Consider the fixed point equation (1). Then the following statements hold:*

- (a) *Any solution $f: X \rightarrow [-\infty, \infty]$ of (1) must be proper, convex and lower semicontinuous.*
- (b) *If E is positive definite, then there exists a strictly convex quadratic solution f to (1), namely $f: X \rightarrow \mathbb{R}$ and it has the form (6). Its coefficients satisfy the following relations:*

$$A = \sqrt{\tau}E, \quad b = \frac{w + \sqrt{\tau}c}{1 + \sqrt{\tau}}, \quad \gamma = \frac{\beta(1 + \sqrt{\tau})^2 + \frac{1}{2}\sqrt{\tau}\langle c - w, E^{-1}(c - w) \rangle}{(1 + \sqrt{\tau})^2(\tau + 1)}. \quad (8)$$

This solution is unique in the class of quadratic functions having a leading coefficient which is invertible.

- (c) *Suppose that E is positive definite and (i) or (ii) holds:*
 - (i) $\tau = 1$ and $c = w$,
 - (ii) X is finite dimensional, $\tau \neq 1$, and f belongs to the class of functions from X to $\mathbb{R}$ which are twice differentiable and their second derivative is continuous at the point

$$x_0 := \frac{1}{1 - \tau} (E^{-1}w - E^{-1}c). \quad (9)$$

Then there exists a unique solution f to (1) (in the first case the uniqueness is in the class of all functions from X to $[-\infty, \infty]$, and in the second case in the class of functions mentioned in Part (c), (ii) above). In fact, this solution is quadratic and strictly convex and its coefficients satisfy (8).

- (d) If E is not positive definite, then there are cases (which depend on E and on the other parameters which appear in (1)) in which (1) does not have any solution, cases in which it has at least one solution, and cases in which it has several solutions (possibly infinitely many) and some of these solutions are not quadratic.

The proof of Theorem 3.1 is given in Section 14 below. It is quite long and technical, and is based on several results presented in Sections 4 to 13.

4. A few simple claims

Here we recall without proofs known elementary facts which are needed later.

Lemma 4.1. *Let X be a real Hilbert space. Assume that function $f: X \rightarrow \mathbb{R}$ is at most quadratic. Then f is convex if and only if its leading coefficient A from (6) is positive semidefinite; f is strictly convex if and only if A is positive definite.*

Lemma 4.2. *Let X be a real Hilbert space. Assume that $h: X \rightarrow \mathbb{R}$ has the form*

$$h(x) = \frac{1}{2} \langle Ax, x \rangle, \quad x \in X, \quad (10)$$

for some positive semidefinite invertible operator $A: X \rightarrow X$. Then

$$h^*(x^*) = \frac{1}{2} \langle A^{-1}x^*, x^* \rangle, \quad x^* \in X. \quad (11)$$

Lemma 4.3. *Let X be a real Hilbert space and let $E: X \rightarrow X$ be a continuous invertible linear operator, $c \in X, w \in X, \beta \in \mathbb{R}$, and $\tau > 0$ be given. Let $h: X \rightarrow [-\infty, \infty]$ and let $g: X \rightarrow [-\infty, \infty]$ be defined by*

$$g(x) := \tau h(Ex + c) + \langle w, x \rangle + \beta, \quad x \in X.$$

Then

$$g^*(x^*) = \tau h^*(Hx^* + v) + \langle z, x^* \rangle + \rho, \quad x^* \in X, \quad (12)$$

where

$$\begin{cases} H := \tau^{-1}(E^{-1})^*, & v := -\tau^{-1}(E^{-1})^*w, \\ z := -\tau^{-1}E^{-1}c, & \rho := \tau^{-1}(\langle w, E^{-1}c \rangle - \beta). \end{cases} \quad (13)$$

5. General properties of solutions to (1)

The results presented in this section describe some properties that any solution f to (1) must satisfy.

Lemma 5.1. *If $f: X \rightarrow [-\infty, \infty]$ solves (1), then $f \in \mathcal{C}(X)$.*

Proof. It is well known that f^* is always convex and lower semicontinuous. Hence the right-hand side of (1) is convex and lower semicontinuous, and therefore so is the left-hand side, namely f . It remains to be shown that f is proper. If $f(x) = -\infty$ at some $x \in X$, then $f^*(Ex + c) = -\infty$ from (1). It is well known

and easy to see that if f^* is equal to $-\infty$ at some point, then $f \equiv \infty$ and $f^* \equiv -\infty$ leading to a contradiction, in view of (1). By the same token if $f \equiv \infty$, then $f^* \equiv -\infty$ and again (1) leads to a contradiction. Therefore f is proper. $\square$

Lemma 5.2. *Any solution $f: X \rightarrow [-\infty, \infty]$ to (1) satisfies the following functional equations:*

$$f(x) = \tau^2 f(\tau^{-1}(E^{-1})^*Ex + \tau^{-1}(E^{-1})^*c - \tau^{-1}(E^{-1})^*w) + \langle w - E^*E^{-1}c, x \rangle + \langle w, E^{-1}c \rangle - \langle E^{-1}c, c \rangle, \quad x \in X, \quad (14)$$

and

$$f(\tau E^{-1}E^*x + E^{-1}E^*E^{-1}w - E^{-1}c) = \tau^2 f(x) + \langle w, \tau E^{-1}E^*x + E^{-1}E^*E^{-1}w - E^{-1}c \rangle - \langle \tau^3 c, x \rangle + \beta(1 - \tau^2), \quad \forall x \in X. \quad (15)$$

In addition, if E is self-adjoint, then

$$f(\tau x + E^{-1}w - E^{-1}c) = \tau^2 f(x) + \langle w, \tau x + E^{-1}w - E^{-1}c \rangle - \langle \tau^3 c, x \rangle + \beta(1 - \tau^2), \quad \forall x \in X. \quad (16)$$

Proof. We first prove (14). By applying $T = T[E, c, w, \tau, \beta]$ on both sides of (1), we have $f = Tf = T^2f$. From (1), Lemma 4.3, the equality $f = f^{**}$ (which holds since $f \in \mathcal{C}(X)$ according to Lemma 5.1) and elementary calculations, we have

$$\begin{aligned} (T^2f)(x) &= \tau(Tf)^*(Ex + c) + \langle w, x \rangle + \beta \\ &= \langle w, x \rangle + \langle w, E^{-1}c \rangle - \langle E^{-1}c, c \rangle - \langle E^*E^{-1}c, x \rangle + \\ &\quad + \tau^2 f(\tau^{-1}(E^{-1})^*Ex + \tau^{-1}(E^{-1})^*c - \tau^{-1}(E^{-1})^*w) \end{aligned} \quad (17)$$

and this implies (14). Now we prove (15). The equality $f = f^{**}$, equation (1), Lemma 4.3 and the change of variables $x \mapsto Ex + c$ imply that

$$\begin{aligned} f(x) &= (f^*)^*(x) = ((1/\tau)f(E^{-1}(\cdot) - E^{-1}c)) + \langle -E^{-1}w/\tau, \cdot - c \rangle - \beta/\tau)^*(x) \\ &= (1/\tau)f^*(\tau E^*x + E^*Ew) + \langle \tau c, x \rangle + \beta. \end{aligned} \quad (18)$$

From (1) and elementary calculations it follows that

$$\begin{aligned} f^*(\tau E^*x + E^*Ew) &= f^*(c + E(\tau E^{-1}E^*x + E^{-1}E^*E^{-1}w - E^{-1}c)) \\ &= (1/\tau)(f(\tau E^{-1}E^*x + E^{-1}E^*E^{-1}w - E^{-1}c) - \\ &\quad - \langle w, \tau E^{-1}E^*x + E^{-1}E^*E^{-1}w - E^{-1}c \rangle - \beta). \end{aligned}$$

This equality and (18) imply (15), and since $E = E^*$, we get (16) from (15). $\square$

Lemma 5.3. *Let $f: X \rightarrow [-\infty, \infty]$ be a solution to (1). Then there exist a linear operator $Q: X \rightarrow X$, a vector $q \in X$, and a real number θ such that*

$$\frac{1}{2}\langle Qx, x \rangle + \langle q, x \rangle + \theta \leq f(x), \quad \forall x \in X. \quad (19)$$

$$\text{In fact} \quad Q = \frac{2\tau}{\tau+1}E, \quad q = \frac{\tau}{\tau+1} \left(\frac{1}{\tau}w + c \right), \quad \theta = \frac{\beta}{\tau+1}. \quad (20)$$

Proof. Lemma 5.1 implies that $f \in \mathcal{C}(X)$. Hence it satisfies the Fenchel-Young inequality

$$f^*(x^*) + f(x) \geq \langle x^*, x \rangle, \quad \forall x, x^* \in X. \quad (21)$$

In particular, this inequality holds for an arbitrary $x \in X$ and for $x^* := Ex + c$. Since f satisfies (1) it follows that

$$f^*(x^*) = \frac{1}{\tau} f(x) - \left\langle \frac{1}{\tau} w, x \right\rangle - \frac{\beta}{\tau}.$$

This equality and (21) imply that for all $x \in X$

$$\begin{aligned} f(x) &\geq \frac{\tau}{\tau+1} \left(\left\langle \frac{1}{\tau} w + Ex + c, x \right\rangle + \frac{\beta}{\tau} \right) \\ &= \frac{1}{2} \left\langle \frac{2\tau}{\tau+1} Ex, x \right\rangle + \left\langle \frac{\tau}{\tau+1} \left(\frac{1}{\tau} w + c \right), x \right\rangle + \frac{\beta}{\tau+1}. \end{aligned} \quad (22)$$

This inequality implies (19) and (20). $\square$

6. Two general lemmas and additional results

In this section we present two general lemmas and a by-product of possible independent interest.

Lemma 6.1. *Let X be a real Hilbert space. Assume that $L: X \rightarrow 2^X$ is single-valued, invertible, strictly monotone and maximal monotone. Let $Q: X \rightarrow 2^X$ be a monotone operator. If $I \subseteq QLQL$ or $I \subseteq LQLQ$, then $Q = L^{-1}$, and hence Q is actually single-valued, invertible, strictly monotone and maximal monotone.*

Proof. Suppose first that $I \subseteq QLQL$. Hence, for all $x \in X$ there exists $z \in (QL)(x)$ such that $x \in (QL)(z)$. Since L is single-valued, we have $z \in Q(L(x))$ and $x \in Q(L(z))$. Therefore, if $x \neq z$, then, by the monotonicity of Q and the strict monotonicity of L we have

$$0 \leq \langle z - x, L(x) - L(z) \rangle = -\langle x - z, L(x) - L(z) \rangle < 0.$$

This contradiction implies that $x = z$. Since $x \in Q(L(z))$, it follows that $x \in Q(L(x))$. Therefore $L^{-1}(y) \in Q(y)$ for all $y \in X$, and hence $L^{-1} \subset Q$. Since L^{-1} is maximal monotone (a fact which follows directly from the assumption that L is by maximal monotone), we conclude that $Q = L^{-1}$.

Assume now that $I \subseteq LQLQ$. We claim that this inclusion implies that $I \subseteq LQ$. Indeed, suppose to the contrary that for some $x \in X$ we have $x \notin (LQ)(x)$. Since $x \in (LQLQ)(x)$, there exists $z \in (LQ)(x)$ such that $x \in (LQ)(z)$. We have $x \neq z$ because $x \notin (LQ)(x)$. Since $z \in (LQ)(x)$ and $x \in (LQ)(z)$ and L is single-valued, there exist $x' \in Q(x)$ and $z' \in Q(z)$ such that $z = L(x')$ and $x = L(z')$, so that

$$x - z = L(z') - L(x') = -(L(x') - L(z')). \quad (23)$$

Since $z \neq x$, we get from (23) that $x' \neq z'$. Since $x' \in Q(x)$ and $z' \in Q(z)$, we get from the monotonicity of Q , (23), and the strict monotonicity of L ,

$$0 \leq \langle x' - z', x - z \rangle = -\langle x' - z', L(x') - L(z') \rangle < 0.$$

This is a contradiction, and so $I \subseteq LQ$. Thus, $x \in (LQ)(x)$ for all $x \in X$, namely $I \subset LQ$, so that $L^{-1} \subset Q$. Hence $L^{-1} = Q$ because L^{-1} is maximal monotone and Q is monotone. $\square$

Lemma 6.2. *Let X be a real Hilbert space. Assume that $L: X \rightarrow 2^X$ is single-valued, invertible, strictly monotone and maximal monotone. If $Q: X \rightarrow 2^X$ is monotone, satisfies $Q(x) \neq \emptyset$ for all $x \in X$ and*

$$LQL = Q^{-1}, \quad (24)$$

then $Q = L^{-1}$. In other words, (24) has a unique solution in the set

$$\Omega(X) := \{Q: X \rightarrow 2^X : Q(x) \neq \emptyset \ \forall x \in X \text{ and } Q \text{ is monotone}\}, \quad (25)$$

and this solution is $Q = L^{-1}$.

Proof. Take $x \in X$ and $y \in Q(x)$. Hence $x \in Q^{-1}y \subseteq (Q^{-1}Q)(x)$. In view of (24) we get $x \in (Q^{-1}Q)(x) = (LQLQ)(x)$ for all $x \in X$, namely $I \subseteq LQLQ$. We conclude from Lemma 6.1 that $Q = L^{-1}$. In other words, any solution $Q \in \Omega(X)$ to (24) must coincide with L^{-1} . Finally, it is immediate to see that L^{-1} (which belongs to $\Omega(X)$) does solve (24). $\square$

Corollary 6.3. *If X is a real Hilbert space and $Q: X \rightarrow X$ is a positive semidefinite linear operator satisfying $Q^2 = I$, then $Q = I$.*

Proof. It follows from Lemma 6.2 with $L := I$. $\square$

Remark 6.4. When $\dim(X) = n \in \mathbb{N}$, Corollary 6.3 is just a simple consequence of the fact that Q , being self-adjoint, can be diagonalized. In other words, there exists a linear operator $U: X \rightarrow X$ satisfying $U^*U = I = UU^*$ such that $D := U^*QU$ is a diagonal operator. Thus, $I = U^*U = U^*Q^2U = U^*QUU^*QU = (U^*QU)^2 = D^2$. Since Q is positive semidefinite, so is D . Thus $D = I$ and therefore $Q = UDU^* = I$.

Remark 6.5. To the best of our knowledge, the functional equation (24) has not been investigated so far. However, interestingly, versions of (24) can be found in several places in the literature. We mentioned next two of them. The first one appears in [20, Equation (10), p. 1440], where the considered equation is

$$Q = P + \sum_{k=1}^m A_k^*(Q - C)^{-1}A_k, \quad (26)$$

where m and n are natural numbers, $A_1, \dots, A_m$ are arbitrary $n \times n$ matrices (not necessarily invertible), P is an $n \times n$ positive definite matrix, C is positive

semidefinite, and the unknown Q is an $n \times n$ matrix such that $Q - C$ is positive definite. This equation is inspired by the closely related matrix equation [24, Equation 7.1.30. p.95] which appears in the study of extremal interpolation problems. Related matrix-type equations appear in [19, Equation (5.1), p.416] and [20, Equation (11), p.1440]. Although (24) and (26) have similarities, there are important differences between them (the classes in which the unknown Q is sought and other differences).

A second version of (24) is simply the involution equation

$$h^2 = I, \quad (27)$$

with unknown function h . It is equivalent to (24) if we assume that the unknown Q in (24) is single-valued and make the change of variables $h = LQ$. There is a vast literature on involutions in various settings. For instance, [3, 4, 6, 15] discuss involutions in the context of operators acting on $\mathcal{C}(X)$ or on closely related classes of functions and convex sets, and [16, Chapter 11] discusses involutions of functions from an interval to itself. However, we are not aware of works which investigate the equation $(LQ)^2 = I$ in the context of Lemma 6.2.

7. Properties of the solutions to (1): the positive semidefinite and quadratic cases

This section presents properties of the solutions to (1) under additional assumptions on E and/or f .

Lemma 7.1. *Assume that $f: X \rightarrow \mathbb{R}$ has the form (6), where A is invertible and positive semidefinite. If f satisfies (1), then*

$$\begin{cases} A = \tau E^* A^{-1} E, & (\tau E^* A^{-1} + I)b = w + \tau E^* A^{-1} c, \\ \gamma = \frac{\beta + \langle \tau(c - b), \frac{1}{2} A^{-1}(c - b) \rangle}{\tau + 1}. \end{cases} \quad (28)$$

On the other hand, if the coefficients of f satisfy (28), where $\tau > 0$ and $c, w \in X$ are given and $E: X \rightarrow X$ is a given self-adjoint invertible linear operator, then f solves (1) and we also have

$$(\sqrt{\tau} A^{-1} E)^2 = I = \left(\frac{1}{\sqrt{\tau}} E^{-1} A \right)^2. \quad (29)$$

Proof. We can write $f(x) = h(x) + \langle b, x \rangle + \gamma$ where $h(x) := \frac{1}{2} \langle Ax, x \rangle$ for all $x \in X$. By using Lemma 4.2 and Lemma 4.3 we see that

$$f^*(x^*) = h^*(x^* - b) - \gamma = \langle x^* - b, \frac{1}{2} A^{-1}(x^* - b) \rangle - \gamma, \quad \forall x^* \in X. \quad (30)$$

Fix $x \in X$ and denote $x^* := Ex + c$. Suppose first that f solves (1). This equation, (30), the facts that A and hence A^{-1} are self-adjoint, all imply that

$$\begin{aligned} f(x) &= \tau(\langle Ex + c - b, \frac{1}{2} A^{-1}(Ex + c - b) \rangle - \gamma) + \langle w, x \rangle + \beta = \\ &= \langle x, \frac{1}{2} \tau E^* A^{-1} E x \rangle + \langle x, \tau E^* A^{-1}(c - b) + w \rangle + \beta + \langle \tau(c - b), \frac{1}{2} A^{-1}(c - b) \rangle - \tau \gamma. \end{aligned}$$

Since the left-most and right-most sides of this equation are quadratic functions and their leading coefficients are self-adjoint (A by assumption, hence so is $\tau E^* A^{-1} E$), we can equate the coefficients of both functions and after doing this we obtain (28). On the other hand, suppose that f satisfies (28) and $E = E^*$. These assumptions and (30) imply that

$$\begin{aligned}
& \tau f^*(Ex + c) + \langle w, x \rangle + \beta = \\
& = \tau(\langle Ex + c - b, \frac{1}{2}A^{-1}(Ex + c - b) \rangle - \gamma) + \langle w, x \rangle + \beta \\
& = \tau\langle Ex + c - b, \frac{1}{2}\tau^{-1}E^{-1}AE^{-1}(c - b) + \frac{1}{2}\tau^{-1}E^{-1}AE^{-1}Ex \rangle - \tau\gamma + \langle w, x \rangle + \beta \\
& = \frac{1}{2}\langle x, AE^{-1}(c - b) \rangle + \frac{1}{2}\langle Ax, x \rangle + \tau\langle c - b, \frac{1}{2}A^{-1}(c - b) \rangle + \langle c - b, \frac{1}{2}E^{-1}Ax \rangle - \\
& \quad - \tau\gamma + \langle \tau EA^{-1}(b - c) + b, x \rangle + (\tau\gamma + \gamma - \tau\langle c - b, \frac{1}{2}A^{-1}(c - b) \rangle) \\
& = \frac{1}{2}\langle Ax, x \rangle + \langle AE^{-1}(c - b) + \tau EA^{-1}(b - c), x \rangle + \langle b, x \rangle + \gamma \\
& = \frac{1}{2}\langle Ax, x \rangle + \langle b, x \rangle + \gamma = f(x), \tag{31}
\end{aligned}$$

using the fact that (28) implies the equality $AE^{-1} = \tau EA^{-1}$ in the last but one equation. Therefore f satisfies (1), as required. Finally, since $E = E^*$, it follows from the first equality in (28) that the leftmost equality in (29) holds, from which the right equality in (29) also follows by taking inverses. $\square$

Lemma 7.2. *Assume that E is positive definite and that $f: X \rightarrow \mathbb{R}$ has the form (6), where A is positive semidefinite and invertible. If f satisfies (1), then its coefficients satisfy (8). In particular, A is actually positive definite.*

Proof. Since A is positive semidefinite and since f solves (1) and has the form (6), Lemma 7.1 implies (28). Denote $Q := A$ and $L := (\sqrt{\tau}E)^{-1}$. With this notation and the fact that E is self-adjoint, we see that the first equation in (28) is equivalent to the equation $LQL = Q^{-1}$. Since E is continuous and invertible, L is continuous and invertible. Since E is positive definite and hence strictly monotone and maximal monotone, so is L . Since A is positive semidefinite, Q is monotone. Thus Lemma 6.2 implies that $Q = L^{-1}$, namely $A = \sqrt{\tau}E$. Hence A is positive definite. By substituting the previous expressions in the second equation of (28) we see that $b = (w + \sqrt{\tau}c)/(1 + \sqrt{\tau})$. Therefore $c - b = (c - w)/(1 + \sqrt{\tau})$ and the expression for γ in (8) follows. $\square$

Lemma 7.3. *Let $f: X \rightarrow [-\infty, \infty]$ be a solution to (1). If E is invertible and positive semidefinite, then there exist a positive semidefinite invertible linear operator $Q': X \rightarrow X$, a vector $q' \in X$, and a real number θ' such that*

$$f(x) \leq \frac{1}{2}\langle Q'x, x \rangle + \langle q', x \rangle + \theta', \quad \forall x \in X. \tag{32}$$

$$\text{In fact: } \begin{cases} Q' = \frac{1}{2}(\tau + 1)E^{-1}, & q' = \frac{1}{2}(c + w), \\ \theta' = \frac{\beta}{\tau + 1} + \left\langle \frac{1-\tau}{4}E^{-1}c - \frac{1}{2}E^{-1}w, c \right\rangle + \frac{\langle E^{-1}(w + \tau c), w + \tau c \rangle}{4(\tau + 1)}. \end{cases} \quad (33)$$

Proof. Since the conjugation reverses the order, by using Lemma 5.3 and taking conjugates on both sides of (22), it follows from Lemmas 4.3 and Lemma 4.2 that for all $x^* \in X$,

$$\begin{aligned} f^*(x^*) &\leq \frac{1}{2} \left\langle \frac{\tau+1}{2\tau} E^{-1} \left(x^* - \frac{\tau}{\tau+1} \left(\frac{1}{\tau} w + c \right) \right), x^* - \frac{\tau}{\tau+1} \left(\frac{1}{\tau} w + c \right) \right\rangle - \frac{\beta}{\tau+1} \\ &= \frac{1}{2} \left\langle \frac{\tau+1}{2\tau} E^{-1} x^*, x^* \right\rangle - \frac{1}{2} \left\langle E^{-1} \left(\frac{1}{\tau} w + c \right), x^* \right\rangle + \\ &\quad + \frac{1}{4} \left\langle \frac{\tau}{\tau+1} E^{-1} \left(\frac{1}{\tau} w + c \right), \frac{1}{\tau} w + c \right\rangle - \frac{\beta}{\tau+1}. \end{aligned} \quad (34)$$

Note that (1) implies that $f^*(x^*) = (1/\tau)f(x) - \langle (1/\tau)w, x \rangle - (1/\tau)\beta$ for every $x \in X$ and for $x^* = Ex + c$. Combining this fact with (34), we obtain (32) and (33) after some algebra. $\square$

Corollary 7.4. *Let $f: X \rightarrow [-\infty, \infty]$ be a solution to (1), where the invertible linear operator $E: X \rightarrow X$ is assumed to be positive semidefinite. Then f is finite and locally Lipschitz continuous everywhere.*

Proof. Lemma 5.3 and Lemma 7.3 imply that f is finite everywhere (hence proper), that is, its effective domain is X . Since, in addition, f is convex and lower semicontinuous (Lemma 5.1) and since X is a Banach space, it follows from a well-known result that f is continuous on X . As a matter of fact, either the continuity of f or (32) imply that f is locally bounded above everywhere and hence, by another well-known result in convex analysis, f is locally Lipschitz continuous everywhere. $\square$

Lemma 7.5. *Suppose that the invertible linear operator $E: X \rightarrow X$ from (1) is positive semidefinite. Then any solution $f: X \rightarrow \mathbb{R}$ to (1) which is at most quadratic must be strictly convex and, in particular, quadratic.*

Proof. Lemma 5.1 ensures that f is convex. Hence Lemma 4.1 implies that the leading coefficient A of f must be positive semidefinite, that is, $\langle Ax, x \rangle \geq 0$ for all $x \in X$. Assume to the contrary that $\langle Ay, y \rangle = 0$ for some nonzero vector $y \in X$. This assumption, the fact that f satisfies (6), and the definition of f^* (in (2)), all imply that for each $t \in \mathbb{R}$,

$$f^*(y + b) \geq \langle y + b, ty \rangle - \left(\frac{1}{2} \langle A(ty), ty \rangle + \langle b, ty \rangle + \gamma \right) = t\|y\|^2 - \gamma. \quad (35)$$

By taking the limit $t \rightarrow \infty$ in (35) and using the assumption that $y \neq 0$ we find that $f^*(y + b) = \infty$. From (1) with $x := E^{-1}(y + b - c)$ and $x^* := y + b$ it follows

that $f(x) = \infty$. This contradicts Corollary 7.4 which ensures that f must be finite everywhere. Hence A is positive definite. This fact and Lemma 4.1 imply that f is strictly convex and quadratic. $\square$

Corollary 7.6. *Suppose that X is finite dimensional and that $f: X \rightarrow \mathbb{R}$ is a solution to (1) which is at most quadratic. If E is positive semidefinite, then the leading coefficient A of f must be invertible.*

Proof. Lemma 7.5 ensures that f is strictly convex. Thus (Lemma 4.1) A is positive definite. Since $\dim(X) < \infty$, we conclude that A is invertible. $\square$

Lemma 7.7. *Suppose that f and p solve (1), where the invertible linear operator $E: X \rightarrow X$ is assumed to be positive semidefinite. Then there exists a continuous function $g_{f,p}: X \rightarrow \mathbb{R}$ satisfying*

$$f(x) = p(x) + g_{f,p}(x), \quad x \in X, \quad (36)$$

and

$$g_{f,p}(\tau x + E^{-1}w - E^{-1}c) = \tau^2 g_{f,p}(x), \quad x \in X. \quad (37)$$

Proof. From Corollary 7.4 we know that both f and p are finite and continuous everywhere. Thus, if we define $g_{f,p}: X \rightarrow [-\infty, \infty]$ by $g_{f,p}(x) := f(x) - p(x)$ for each $x \in X$, then $g_{f,p}$ is well defined, finite, and continuous everywhere. Lemma 5.2 implies that both f and p satisfy (16). By considering the version of (16) with f , subtracting from it the version of (16) with p , and substituting $g_{f,p}$ in the corresponding places, we obtain (37). $\square$

8. E is positive definite: existence and partial uniqueness

The following proposition shows the existence of a solution to (1) when E is positive definite. This solution is unique in the class of quadratic functions having a leading coefficient which is invertible.

Proposition 8.1. *If the invertible linear operator $E: X \rightarrow X$ in (1) is positive definite, then there exists a solution $p: X \rightarrow [-\infty, \infty]$ to (1). This solution is quadratic and strictly convex, and its coefficients are defined by (8). Furthermore, p is the unique function which solves (1) in the class of quadratic functions having a leading coefficient which is invertible.*

Proof. Let p be the function defined by (6) and having coefficients defined by (8). Since E is invertible, we have $A = \sqrt{\tau}E \neq 0$. Thus p is quadratic. Direct calculations show that (28) is satisfied. Since E is positive definite, so is A . Therefore Lemma 7.1 implies that p satisfies (1). Moreover, since A is positive definite, Lemma 4.1 ensures that p is strictly convex.

Suppose now that f is a quadratic function which solves (1) and its leading coefficient A (from (6)) is invertible. Lemma 5.1 implies that f is convex. Hence Lemma 4.1 ensures that A is positive semidefinite and thus Lemma 7.2 implies

that the coefficients of f satisfy (8) (and A is actually positive definite). Therefore f coincides with p , namely there exists a unique solution to (1) in the class of quadratic functions having a leading coefficient which is invertible. $\square$

9. E is positive definite: existence and uniqueness when $\tau = 1$ and $w = c$

In Proposition 9.1 below we establish the uniqueness of solutions to (1) when E is positive definite, $\tau = 1$ and $w = c$. An immediate consequence of this proposition is the classical fact mentioned in Subsection 1.1 that the normalized energy function is the unique solution to (3).

Proposition 9.1. *Consider (1) under the assumptions that $\tau = 1$ and $w = c$ (in particular, when $w = c = 0$), namely*

$$f(x) = f^*(Ex + c) + \langle c, x \rangle + \beta, \quad x \in X. \quad (38)$$

If, in addition, E is positive definite, then there exists a unique solution $f: X \rightarrow [-\infty, \infty]$ to (1). This solution coincides with the strictly convex quadratic function p from Proposition 8.1.

Proof. Existence follows from Proposition 8.1. As for uniqueness, suppose that some function $f: X \rightarrow [-\infty, \infty]$ solves (1). Lemma 5.1 implies that $f \in \mathcal{C}(X)$. From Corollary 7.4 it follows that f is continuous (and finite). As is well known, this fact implies that $(\partial f)(x) \neq \emptyset$ for each $x \in X$. The change of variables $x \mapsto Ex + c$, elementary calculations and (1) lead to

$$f(Bx + d) = f^*(x) + \langle B^*c, x \rangle + \langle c, d \rangle + \beta, \quad x \in X, \quad (39)$$

where $B := E^{-1}$ and $d := -Bc$. Now we apply the subdifferential operator to both sides of (39) and we use the following known facts:

- (i) $\partial f^* = (\partial f)^{-1}$;
- (ii) $B^* = B$ (because E is positive definite);
- (iii) $(\partial f_d)(z) = (\partial f)(z + d)$ for all $z \in X$ where $f_d(z) := f(z + d)$ for all $z \in X$;
- (iv) the subdifferential of the sum of two lower semicontinuous proper convex functions is equal to the sum of the subdifferentials when the effective domain of one of the functions is the whole space;
- (v) if g is convex and differentiable, then $\partial g(x) = \{g'(x)\}$ for all $x \in X$;
- (vi) $\partial(g \circ B) = B^* \circ (\partial g) \circ B$ for all $g \in \mathcal{C}(X)$.

We conclude from (i)–(vi) that $BQL = Q^{-1} + Bc$, where $Q: X \rightarrow 2^X$ and $L: X \rightarrow X$ are the operators defined by $Q(x) := (\partial f)(x)$ and $L(x) := Bx + d$ for all $x \in X$. Therefore, by adding d to both sides of this equation and recalling that $d = -Bc$, we arrive at the equation $LQL = Q^{-1}$. Since E is positive definite and invertible, so is B . Thus L is the translation by a vector of an invertible, strictly monotone and maximal monotone operator, and so the same holds for L . Since Q is clearly monotone, Lemma 6.2 can be used to conclude that $Q = L^{-1}$. Hence $(\partial f)(x) = E(x - d)$ for each $x \in X$, and so ∂f is single-valued and continuous.

We conclude that $f'(x) = (\partial f)(x)$ and, as a result, $f'(x) = E(x - d) = Ex + c$ for all $x \in X$.

Let $x \in X$ be fixed and let $g: \mathbb{R} \rightarrow \mathbb{R}$ be the function defined by $g(t) := f(tx)$ for all $t \in \mathbb{R}$. Then g is differentiable and for all $t \in \mathbb{R}$

$$g'(t) = \langle x, f'(tx) \rangle = \langle x, E(tx) + c \rangle = \langle Ex, x \rangle t + \langle c, x \rangle.$$

Thus for every $x \in X$, we have

$$f(x) = g(1) = g(0) + \int_0^1 g'(t) dt = \frac{1}{2} \langle Ex, x \rangle + \langle c, x \rangle + f(0). \quad (40)$$

Therefore f is quadratic and its leading coefficient is E , which is an invertible and positive definite operator. Since f solves (1), Proposition 8.1 implies that f coincides with the strictly convex quadratic function p defined there, as claimed. (Note: from (40) the linear coefficient of f is c , but from (6) and (8) it should be b ; there is no contradiction since (8) and $w = c$ imply that $b = c$.) $\square$

10. E is positive definite: existence and uniqueness when both $\tau \neq 1$ and f'' exists and is continuous at a point

In this section we show that if E is positive definite, X is finite-dimensional and $\tau \neq 1$, then there exists a unique solution to (1) in the class of functions having a second derivative which is continuous at a certain point (the finite-dimensionality of X is only needed in Proposition 10.2, but not in Lemma 10.1).

Lemma 10.1. *Suppose that $\tau \neq 1$ and that the invertible operator $E: X \rightarrow X$ is positive definite. Assume that $f: X \rightarrow [-\infty, \infty]$ solves (1). If f is twice differentiable on X and its second derivative is continuous at the point $x_0 := (1/(1 - \tau))(E^{-1}w - E^{-1}c)$, then f must be strictly convex and quadratic.*

Proof. Let $p: X \rightarrow \mathbb{R}$ be the strictly convex and quadratic solution to (1) from Proposition 8.1. Lemma 7.7 implies the existence of a function $g_{f,p}: X \rightarrow \mathbb{R}$ such that (36) and (37) hold. Since p is quadratic, it has a continuous second derivative. Thus $g_{f,p} := f - p$ is twice differentiable and its second derivative is continuous at the point x_0 .

Let $x_1 := E^{-1}w - E^{-1}c$. Then $x_1 = (1 - \tau)x_0$. By differentiating (37) we conclude that $\tau g'_{f,p}(\tau x + x_1) = \tau^2 g'_{f,p}(x)$ for each $x \in X$. Since $\tau \neq 0$, it follows that $g'_{f,p}(\tau x + x_1) = \tau g'_{f,p}(x)$ for each $x \in X$. A second differentiation and a division by τ yields

$$g''_{f,p}(\tau x + x_1) = g''_{f,p}(x) \quad \forall x \in X. \quad (41)$$

Assume first that $\tau \in (0, 1)$. We fix x and use (41) iteratively to obtain

$$\begin{aligned} g''_{f,p}(x) &= g''_{f,p}(\tau x + x_1) = g''_{f,p}(\tau(\tau x + x_1) + x_1) = g''_{f,p}(\tau^2 x + \tau x_1 + x_1) \\ &= \dots = g''_{f,p}(\tau^m x + (\tau^{m-1} + \tau^{m-2} + \dots + 1)x_1) \end{aligned} \quad (42)$$

for all $m \in \mathbb{N}$. By taking the limit $m \rightarrow \infty$ in (42) and using the assumptions that $\tau \in (0, 1)$, that $g''_{f,p}$ is continuous at x_0 and that $x_1/(1 - \tau) = x_0$, we obtain $g''_{f,p}(x) = g''_{f,p}(x_1/(1 - \tau)) = g''_{f,p}(x_0)$ for all $x \in X$. Hence $g''_{f,p}$ is constant. Thus $g_{f,p}$ is at most quadratic.

Consider now the case $\tau > 1$. This case follows from (41) again by first denoting $y := \tau x + x_1$ and then observing that this notation and (41) lead to

$$g''_{f,p}(y) = g''_{f,p}(\tau^{-1}y - \tau^{-1}x_1) = g_{f,p}(\alpha y + y_1) \quad \forall y \in X, \quad (43)$$

where $\alpha := \tau^{-1}$ and $y_1 := -\tau^{-1}x_1$. The equality $x_1 = (1 - \tau)x_0$ and the definition of y_1 imply that $y_0 := y_1/(1 - \alpha) = x_0$. Our assumption on $g''_{f,p}$ thus implies that $g''_{f,p}$ is continuous at y_0 . This observation, (43), and the inequality $0 < \alpha < 1$ imply, as in (42) and the derivation after it, that $g_{f,p}$ must be at most quadratic. Therefore $g_{f,p}$ is at most quadratic in both cases $\tau \in (0, 1)$ and $\tau \in (1, \infty)$.

Since p is quadratic and $f = p + g_{f,p}$, it follows that f is at most quadratic. Since f solves (1), Lemma 5.1 implies that $f \in \mathcal{C}(X)$. Hence f is convex and since it solves (1) we can conclude from Lemma 7.5 that f is strictly convex. Hence its leading coefficient cannot be the zero operator, and thus f is quadratic. $\square$

Proposition 10.2. *Assume that the Hilbert space X is finite dimensional. Given a positive definite and invertible linear operator $E: X \rightarrow X$, a positive number $\tau \neq 1$, and two vectors $c, w \in X$, consider the class of functions $f: X \rightarrow \mathbb{R}$ which are twice differentiable and their second derivative is continuous at the point $x_0 := (1/(1 - \tau))(E^{-1}w - E^{-1}c)$. Then there exists a unique solution f to (1) in this class. In fact, this unique solution is the quadratic and strictly convex solution p from Proposition 8.1.*

Proof. The quadratic function p from Proposition 8.1 solves (1) according to this proposition and it belongs to the considered class of functions. This shows the existence of a solution to (1) in this class of functions. For uniqueness, suppose that f belongs to the considered class of functions and that it satisfies (1). Lemma 10.1 implies that f is strictly convex and quadratic. Hence f satisfies (6). Corollary 7.6 implies that its leading coefficient is invertible. It follows from Proposition 8.1 that $f = p$ (and hence for $g_{f,p}$ from the proof of Lemma 10.1 we have $g_{f,p} \equiv 0$), as claimed. $\square$

11. E is not positive semidefinite: nonexistence

When E is not positive semidefinite, then even simple special cases of (1) may have no solutions.

Lemma 11.1. *If $0 \neq w \in X$, then there exists no solution $f \in \mathcal{C}(X)$ to the functional equation*

$$f(x) = f(x + w) + \langle w, x \rangle, \quad x \in X. \quad (44)$$

Proof. Suppose to the contrary that some $f \in \mathcal{C}(X)$ satisfies (44). Since f is proper, there is a point $x_0 \in X$ such that $f(x_0) \in \mathbb{R}$. Consider the function

$\phi: \mathbb{R} \rightarrow [-\infty, \infty]$ defined by $\phi(t) := f(x_0 + tw)$ for each $t \in \mathbb{R}$. By putting $x := x_0 + tw$, $t \in \mathbb{R}$ in (44) we see that ϕ satisfies the functional equation

$$\phi(t) = \phi(t+1) + \delta t + \rho, \quad t \in \mathbb{R} \quad (45)$$

where $\delta := \|w\|^2 > 0$ and $\rho := \langle w, x_0 \rangle$.

We claim that ϕ must be finite everywhere. Indeed, first $\phi(t) > -\infty$ for all $t \in \mathbb{R}$ because f is proper. It remains to show that $\phi(t) < \infty$ for all $t \in \mathbb{R}$. By the choice of x_0 we have $\phi(0) = f(x_0) \in \mathbb{R}$. By setting $t := 0$ in (45) we see that $\phi(1) = \phi(0) - \rho$ and therefore $\phi(1) \in \mathbb{R}$. By putting $t := -1$ in (45) we obtain that $\phi(-1) = \phi(0) - \delta + \rho$ and hence also $\phi(-1) \in \mathbb{R}$. Induction and (45) yield $\phi(m) \in \mathbb{R}$ for all integers m . From the convexity of f it follows that ϕ is convex, and thus, since any $t \in \mathbb{R}$ satisfies $t \in [m, m+1]$ for some integer m , we have $\phi(t) \leq \max\{\phi(m), \phi(m+1)\} < \infty$, and hence ϕ is indeed finite everywhere.

Since ϕ is convex, it has a left derivative ϕ_- which is an increasing function on $\mathbb{R}$. By taking the left derivative on both sides of (45) one sees that $\phi_-(t) = \phi_-(t+1) + \delta$ for each $t \in \mathbb{R}$. In particular, $\phi_-(0) = \phi_-(1) + \delta > \phi_-(1)$, a contradiction with the above-mentioned fact that ϕ_- is increasing. Hence (44) cannot have any solution $f \in \mathcal{C}(X)$. $\square$

Proposition 11.2. *If $w \neq 0$, then no $f: X \rightarrow [-\infty, \infty]$ solves the equation*

$$f(x) = f^*(-x) + \langle w, x \rangle, \quad x \in X. \quad (46)$$

In addition, if $c \neq 0$, then no $f: X \rightarrow [-\infty, \infty]$ satisfies the equation

$$f(x) = f^*(-x + c), \quad x \in X. \quad (47)$$

Proof. Consider first (46) and suppose to the contrary that it has a solution $f: X \rightarrow [-\infty, \infty]$. Lemma 5.1 implies that $f \in \mathcal{C}(X)$ and Lemma 5.2 (equation (14)) implies that f is a solution to (44). This contradicts Lemma 11.1 and proves that no $f: X \rightarrow [-\infty, \infty]$ can satisfy (46). Now consider (47) and assume to the contrary that some $f: X \rightarrow [-\infty, \infty]$ solves it. Let $F: X \rightarrow [-\infty, \infty]$ be defined by $F(x) := f(x + c)$ for all $x \in X$. A direct calculation based on Lemma 4.3 implies that F solves (46) with $w := -c \neq 0$, a contradiction to what we established above. Hence no $f: X \rightarrow [-\infty, \infty]$ solves (47). $\square$

12. E is not positive semidefinite: existence

Below we describe a case in which E is not positive definite but (1) does have solution. For the sake of a simpler exposition, we present this proposition only for n -dimensional spaces, $n \in \mathbb{N}$, but we mention that the result can be extended to separable Hilbert spaces X and invertible continuous linear operators $E: X \rightarrow X$ which can be diagonalized using a unitary operator.

Given a self-adjoint $E: \mathbb{R}^n \rightarrow \mathbb{R}^n$ (not necessarily positive semidefinite), we identify E with its associated symmetric matrix. Since E can be diagonalized, we can write $E = UDU^{-1}$, where U is unitary and D is a diagonal matrix with diagonal

elements $d_1, \dots, d_n$. We denote by $\text{abs}(D)$ the diagonal matrix with diagonal elements $|d_1|, \dots, |d_n|$ and by $\text{sign}(D)$ the diagonal matrix with diagonal elements $\text{sign}(d_1), \dots, \text{sign}(d_n)$. Since E is invertible, so are D and $\text{abs}(D)$. In particular, $d_i \neq 0$ for all $i \in \{1, \dots, n\}$. Define

$$A := \sqrt{\tau} U \text{abs}(D) U^{-1}. \quad (48)$$

We have the following result.

Proposition 12.1. *Consider (1) and suppose that the invertible linear operator E is self-adjoint (but possibly not positive semidefinite) and take A as in (48). Suppose that the set of solutions x to the equation $(\tau EA^{-1} + I)x = w + \tau EA^{-1}c$ is nonempty. Let b any such a solution and define*

$$\gamma := \frac{\beta + \frac{1}{2}\tau \langle c - b, A^{-1}(c - b) \rangle}{\tau + 1}.$$

Let $f: X \rightarrow \mathbb{R}$ be defined by $f(x) := \frac{1}{2}\langle Ax, x \rangle + \langle b, x \rangle + \gamma$ for all $x \in X$. Then f solves (1). In particular, there exists a solution to (1) if $w = -\tau EA^{-1}c$ (more particularly, when $c = w = 0$), and this solution is

$$f(x) := \frac{1}{2}\langle Ax, x \rangle + \frac{\beta + \frac{1}{2}\tau \langle c, A^{-1}c \rangle}{\tau + 1}, \quad x \in X. \quad (49)$$

Proof. The assumptions on A and E imply that $E = UDU^{-1} = E^*$, that

$$\tau EA^{-1} = \tau UDU^{-1}(1/\sqrt{\tau})U(\text{abs}(D))^{-1}U^{-1} = \sqrt{\tau}U\text{sign}(D)U^{-1},$$

and that $\tau EA^{-1}E = \sqrt{\tau}U\text{sign}(D)U^{-1}UDU^{-1} = \sqrt{\tau}U\text{abs}(D)U^{-1} = A$. The assumption on b implies that $(\tau EA^{-1} + I)b = w + \tau EA^{-1}c$. These equalities and the definition of γ imply that (28) is satisfied. Since A is invertible and positive definite and E is self-adjoint, Lemma 7.1 implies that f solves (1). Finally, if $w = -\tau EA^{-1}c$, then $w + \tau EA^{-1}c = 0$, and hence $b := 0$ satisfies $0 = (\tau EA^{-1} + I)b$. For this b we have $\gamma = (\beta + \frac{1}{2}\tau \langle c, A^{-1}c \rangle)/(\tau + 1)$. We conclude from the previous argument that the function $f: X \rightarrow \mathbb{R}$ defined by (49) solves (1). $\square$

Remark 12.2. Define $E: \mathbb{R}^n \rightarrow \mathbb{R}^n$ as $E := -I$. Let $\tau := 1$, $c := 0$ and $\beta := 0$. Given $w \in X$, if $w = 0$, then Proposition 12.1 ensures that (1) has a solution. However, if $w \neq 0$, then Proposition 11.2 ensures that (1) no $f: X \rightarrow [-\infty, \infty]$ solves (1). A similar conclusion holds when $E := -I$, $w := 0$, $\tau := 1$, $\beta := 0$ and $c = 0$ or $c \neq 0$. This phenomenon is another manifestation to the sensitivity of (1) with respect to the various parameters which appear in it.

13. E is not positive semidefinite: non-uniqueness

This section shows that there can be several (actually infinitely many) solutions to (1) when E is not positive semidefinite. We first consider the case of quadratic solutions (Example 13.1) and then of non-quadratic ones (Example 13.2).

Example 13.1. Suppose that $X = \mathbb{R}^2$. Assume that $E(x_1, x_2) = (x_2, -x_1)$ for all $x = (x_1, x_2) \in X$, that $\tau = 1$, that $c = w = 0$, and that $\beta = 0$. In other words, (1) becomes

$$f(x_1, x_2) = f^*(x_2, -x_1), \quad (x_1, x_2) \in X. \quad (50)$$

Let $B = (b_{ij})_{i,j=1,2}$ be an arbitrary symmetric positive semidefinite matrix having real entries and a determinant which is equal to 1. We look at B as an operator in $\mathbb{R}^2$, namely $B(x) = Bx$. It is straightforward to verify that the function $f: X \rightarrow \mathbb{R}$ defined by $f(x) := \frac{1}{2}\langle Bx, x \rangle$ for all $x \in X$ solves (50).

We note that using the above result we can find solutions to the equation

$$f(x_1, x_2, \dots, x_{2n-1}, x_{2n}) = f^*(x_2, -x_1, \dots, x_{2n-1}, -x_{2n}), \quad (x_i)_{i=1}^{2n} \in \mathbb{R}^{2n}, \quad (51)$$

where $1 < n \in \mathbb{N}$ is fixed. Indeed, for each $i \in \{1, \dots, n\}$ let B_i be any 2×2 symmetric positive semidefinite matrix with $\det(B_i) = 1$ let $g_i: \mathbb{R}^2 \rightarrow \mathbb{R}$ be defined by $g_i(x_1, x_2) := \frac{1}{2}\langle B_i(x_1, x_2), (x_1, x_2) \rangle$ for each $(x_1, x_2) \in \mathbb{R}^2$ and let $g: \mathbb{R}^{2n} \rightarrow \mathbb{R}$ be defined by $g(x) := \sum_{i=1}^n g_i(x_{2i-1}, x_{2i})$ for every $x = (x_i)_{i=1}^{2n} \in \mathbb{R}^{2n}$. Then, the well-known formula for the conjugate of a direct sum gives $g^*(x) = \sum_{i=1}^n g_i^*(x_{2i-1}, x_{2i})$. From the previous paragraph g_i solves (50) for each $i \in \{1, \dots, n\}$. Thus

$$g^*(x_2, -x_1, \dots, x_{2n}, -x_{2n-1}) = \sum_{i=1}^n g_i^*(x_{2i}, -x_{2i-1}) = \sum_{i=1}^n g_i(x_{2i-1}, x_{2i}) = g(x_i)_{i=1}^{2n}$$

for every $(x_i)_{i=1}^{2n} \in \mathbb{R}^{2n}$. Hence g solves (51).

Example 13.2. Consider the equation

$$f(x) = f^*(-x), \quad x \in X, \quad (52)$$

namely (1) with $E = -I$, $c = w = 0$ and $\beta = 0$. Assume first that $X = \mathbb{R}$. According to Proposition 12.1, the function $f_1: \mathbb{R} \rightarrow \mathbb{R}$ defined by $f_1(x) := \frac{1}{2}x^2$ for all $x \in X$ solves (52). However, as already mentioned in [22, p.106], the function $f_2: \mathbb{R} \rightarrow [-\infty, \infty]$ defined by

$$f_2(x) := \begin{cases} \infty, & x \in (-\infty, 0], \\ -\frac{1}{2} - \log(x), & x \in (0, \infty), \end{cases}$$

also solves (52). A simple verification shows that two other types of solutions to (52) are, respectively,

$$f_3(x) := \begin{cases} \infty, & x \in (-\infty, 0), \\ 0, & x \in [0, \infty), \end{cases} \quad \text{and} \quad f_{4,\lambda}(x) := \begin{cases} \frac{\lambda}{2}x^2, & x \in (-\infty, 0], \\ \frac{1}{2\lambda}x^2, & x \in [0, \infty), \end{cases} \quad (53)$$

where $\lambda > 0$ is arbitrary. In addition, if some $f: X \rightarrow [-\infty, \infty]$ solves (52), then so does the function $f_-(x) := f(-x)$, $x \in X$; indeed, using Lemma 4.3

and the fact that f satisfies (52), we have $f^*(x) = f^*(-x) = f(x) = f_-(-x)$ for every $x \in X$. Now suppose that $X = \mathbb{R}^n$ for some integer $n > 1$. Let $g: X \rightarrow [-\infty, \infty]$ be defined by $g(x) := \sum_{i=1}^n g_i(x_i)$ for all $x = (x_i)_{i=1}^n \in X$, where $g_i \in \{f_1, f_2, f_{2-}, f_3, f_{3-}\} \cup \{f_{4,\lambda} : \lambda \in (0, \infty)\}$ for every $i \in \{1, \dots, n\}$. Using the well-known formula for the conjugate of a direct sum we conclude that any one of the functions g mentioned above solves (52).

14. Proof of Theorem 3.1

Proof of Theorem 3.1. Part (a) follows from Lemma 5.1. Part (b) follows from Proposition 8.1. Part (c)(i) follows from Proposition 9.1. Part (c)(ii) follows from Proposition 10.2. Part (d) follows from Propositions 11.2 (non-existence), Proposition 12.1 (existence of certain quadratic solutions), Example 13.1 (infinitely many quadratic solutions), and Example 13.2 (non-quadratic solutions). $\square$

15. Concluding remarks and open problems

We conclude the paper with the following remarks.

Remark 15.1. At the moment it is not clear whether (1) always has a unique solution (namely, the quadratic function with coefficients defined in (8)) whenever E is positive definite. Actually, even in the simple cases, where $0 < \tau \neq 1$ and $f(x) = \tau f^*(x)$ for all $x \in X$ or $f(x) = f^*(x + c)$ for each $x \in X$, where $c \neq 0$ is fixed, it is not clear whether the above-mentioned quadratic function is the unique solution to one of these equations, and we suspect that non-uniqueness can hold. As we saw in Sections 11–13, even more substantial complications arise when E is not positive definite, and the task of giving a complete description of the structure of the solutions to (1) in this case, as a function of the various parameters which appear in (1), seems to be out of reach now even in the finite-dimensional case (an interesting open issue in this direction is whether there can be cases where the number of solutions to (1) is finite, but greater than one).

Remark 15.2. Suppose that we look for solutions f of (1) in the class of differentiable (Fréchet or Gâteaux) functions. The change of variables $y := Ex + c$ and (1) implies that f^* is also differentiable, and Lemma 5.1 ensures that f is convex, proper and lower semicontinuous. Functions $f: X \rightarrow \mathbb{R}$ having the property that they are convex, proper, lower semicontinuous, and they and their conjugates are Gâteaux differentiable were investigated recently in [21]. They are called *fully Legendre*. If f is fully Legendre, then its gradient f' is invertible and satisfies $(f')^{-1} = (f^*)'$ (see [21, Lemma 3.6]). Hence, given $x^* \in X$, the function $F(x) := f(x) - \langle x^*, x \rangle$, $x \in X$, is proper, lower semicontinuous, convex, and Gâteaux differentiable on X . These conditions ensure that $F'(x) = (\partial F)(x)$. Moreover, F' vanishes at the (unique) point $x(x^*) := (f')^{-1}(x^*)$ and hence $x(x^*)$ is a global minimizer of F . The previous discussion and (2) imply that $f^*(x^*) = \sup_{x \in X} [-F(x)] = -F(x(x^*)) = \langle x^*, (f')^{-1}(x^*) \rangle - f((f')^{-1}(x^*))$. We conclude that f satisfies the following functional-differential equation:

$$f(x) = \langle Ex + c, (f')^{-1}(Ex + c) \rangle - f((f')^{-1}(Ex + c)) + \langle w, x \rangle + \beta, \quad x \in X. \quad (54)$$

In particular, the functions mentioned in Propositions 9.1, 12.1, Example 13.1, and in (53) solve (54). Conversely, if we look for solutions f of (54) which are fully Legendre, then the previous discussion implies that f solves (1) too.

Remark 15.3. An interesting corollary of Lemma 7.5 is the following assertion: An invertible positive semi-definite linear operator A acting from a real Hilbert space X into itself must be positive definite. Indeed, let $f(x) := \frac{1}{2}\langle Ax, x \rangle$, $x \in X$. Then $f^*(x) = \frac{1}{2}\langle A^{-1}x, x \rangle$ for each $x \in X$ by Lemma 4.2. Hence $f^*(Ax) = f(x)$ for every $x \in X$, that is, f solves (1) with $E = A$, $c = w = 0$, $\beta = 0$. Lemma 7.5 ensures that f is strictly convex, and from Lemma 4.1 we conclude that A is positive definite.

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