

Valadier-like Formulas for the Supremum Function II: The Compactly Indexed Case*

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Continuing with the work on the subdifferential of the pointwise supremum of convex functions, started in part I of this paper [R. Correa, A. Hantoute, M. A. López, *Valadier-like formulas for the supremum function I*, J. Convex Analysis 25 (2018) 1253–1278], we focus now on the compactly indexed case. We assume that the index set is compact and that the data functions are upper semicontinuous with respect to the index variable (actually, this assumption will only affect the set of ε -active indices at the reference point). As in the previous work, we do not require any continuity assumption with respect to the decision variable. The current compact setting gives rise to more explicit formulas, which only involve subdifferentials at the reference point of active data functions. Other formulas are derived under weak continuity assumptions. These formulas reduce to the characterization given by M. Valadier [*Sous-différentiels d'une borne supérieure et d'une somme continue de fonctions convexes*, C. R. Acad. Sci. Paris Sér. A-B Math. 268 (1969) 39–42, Theorem 2], when the supremum function is continuous.

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1. Introduction

Let us consider a family of convex functions $f_t: X \rightarrow \overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$, $t \in T$, defined in a locally convex topological vector space X . The aim of this paper, which continues [3], is to give characterizations of the subdifferential of the supremum function

$$f := \sup_{t \in T} f_t, \quad (1)$$

which only involve the exact subdifferentials of data functions at the reference point rather than at nearby ones. Our results are based on some compactness assumptions of certain subsets of the index set T , and some upper semicontinuity assumptions of the mappings $t \mapsto f_t(z)$, $z \in \text{dom } f$. In Theorem 3.6 we establish that, under a natural closure condition (10), for all $x \in X$

$$\partial f(x) = \bigcap_{L \in \mathcal{F}(x)} \overline{\text{co}} \left\{ \bigcup_{t \in T(x)} \partial(f_t + \mathbf{I}_{L \cap \text{dom } f})(x) \right\},$$

where $T(x) := \{t \in T \mid f_t(x) = f(x)\}$, and

$$\mathcal{F}(x) := \{\text{finite-dimensional linear subspaces of } X \text{ such that } x \in L\}. \quad (2)$$

The following characterization of $\partial f(x)$, given in Theorem 3.8, uses the ε -subdifferentials of the functions f_t :

$$\partial f(x) = \bigcap_{\substack{\varepsilon > 0 \\ L \in \mathcal{F}(x)}} \overline{\text{co}} \left\{ \bigcup_{t \in T(x)} \partial_\varepsilon f_t(x) + \mathbf{N}_{L \cap \text{dom } f}(x) \right\}.$$

Condition (10) (see Theorem 3.1) covers the case when the functions f_t are lower semicontinuous (lsc). In particular, if the restriction of f to the affine hull of $\text{dom } f$ is continuous on the relative interior of $\text{dom } f$ (assumed to be nonempty), the intersection over $L \in \mathcal{F}(x)$ can be removed, giving rise to the following formula, established in Theorem 3.1:

$$\partial f(x) = \overline{\text{co}} \left\{ \bigcup_{t \in T(x)} \partial(f_t + \mathbf{I}_{\text{dom } f})(x) \right\}.$$

Our results generalize and improve the well-known formula due to Valadier [18, Theorem 2], which establishes that, under the continuity of f at x ,

$$\partial f(x) = \overline{\text{co}} \left\{ \bigcup_{t \in T(x)} \partial f_t(x) \right\}. \quad (3)$$

Actually, we show that if f is continuous at some point (not necessarily the reference point x), then we get

$$\partial f(x) = \mathbf{N}_{\text{dom } f}(x) + \overline{\text{co}} \left\{ \bigcup_{t \in T(x)} \partial f_t(x) \right\},$$

which reduces to (3) whenever $N_{\text{dom } f}(x) = \{\theta\}$. Observe that the continuity of f at the reference point x is equivalent to the continuity of f at some point together with $N_{\text{dom } f}(x) = \{\theta\}$. Indeed, the last condition implies that x is in the *quasi-interior* of $\text{dom } f$ which coincides with its interior (see, i.e., [1]).

There is a wide literature dealing with subdifferential calculus rules for the supremum of convex functions; we refer for instance to [2, 6, 7, 9, 10, 11, 12, 17, 18, 19, 20, 21], among many others. We also refer to [14], and references therein, for the nonconvex case. The supremum function plays a crucial role in many fields, including semi-infinite optimization ([8], [22]).

The paper is organized as follows. After Section 2, devoted to preliminaries, the main Section 3 provides the desired characterization of $\partial f(x)$ in different settings: Theorem 3.1 deals with a finite-dimensional-like setting, where function f satisfies a weak continuity condition, which is always held in finite dimensions; Theorem 3.5 concerns the supremum of lsc convex functions, while the most general result is given in Theorem 3.6 under the closure-type condition (10). All these results use the exact subdifferential of the data functions at the nominal point. Another formula using approximate subdifferentials of data functions is given in Theorem 3.8. Finally, Theorem 3.12 provides a simpler formula similar to (3) when additional continuity assumptions are imposed.

2. Preliminaries

In this paper X stands for a (real) separated locally convex (lcs, shortly) space, whose topological dual space is denoted by X^* and endowed with the weak*-topology. Hence, X and X^* form a dual pair by means of the canonical bilinear form $\langle x, x^* \rangle := \langle x^*, x \rangle := x^*(x)$, $(x, x^*) \in X \times X^*$. The zero vectors are denoted by θ , and the convex, closed and balanced neighborhoods of θ are called θ -neighborhoods. The family of such θ -neighborhoods in X and in X^* are denoted by $\mathcal{N}_X$ and $\mathcal{N}_{X^*}$, respectively.

Given a nonempty set A in X (or in X^*), by $\text{co } A$ and $\text{aff } A$ we denote the *convex hull* and the *affine hull* of A , respectively. Moreover, $\text{cl } A$ and $\overline{A}$ are indistinctly used for denoting the *closure* of A (*weak*-closure* if $A \subset X^*$). Thus, $\overline{\text{co } A} := \text{cl}(\text{co } A)$, $\overline{\text{aff } A} := \text{cl}(\text{aff } A)$, etc. We use $\text{ri } A$ to denote the (topological) *relative interior* of A (i.e., the interior of A in the topology relative to $\text{aff } A$ when this set is closed, and the empty set otherwise). We define the *orthogonal* of A by

$$A^\perp := \{x^* \in X^* \mid \langle x^*, x \rangle = 0 \text{ for all } x \in A\}.$$

We say that a convex function $\varphi: X \rightarrow \overline{\mathbb{R}}$ is proper if its (*effective*) *domain*, $\text{dom } \varphi := \{x \in X \mid \varphi(x) < +\infty\}$, is nonempty and it does not take the value $-\infty$. The *lsc envelope* of φ is denoted by $\text{cl } \varphi$. We adopt the convention $(+\infty) + (-\infty) = (-\infty) + (+\infty) = +\infty$.

If $\psi: X \rightarrow \mathbb{R} \cup \{+\infty\}$ is another proper convex function, which is finite and continuous at some point in $\text{dom } \varphi$, then we have [7, Corollary 9(iii)]

$$\text{cl}(\max \{\varphi, \psi\}) = \max \{\text{cl } \varphi, \text{cl } \psi\}. \quad (4)$$

For $\varepsilon \geq 0$, the ε -subdifferential of φ at a point x where $\varphi(x)$ is finite is the weak*-closed convex set

$$\partial_\varepsilon \varphi(x) := \{x^* \in X^* \mid \varphi(y) - \varphi(x) \geq \langle x^*, y - x \rangle - \varepsilon \text{ for all } y \in X\}.$$

If $\varphi(x) \notin \mathbb{R}$, then we set $\partial_\varepsilon \varphi(x) = \emptyset$. In particular, for $\varepsilon = 0$ we get the *Fenchel subdifferential* of φ at x , $\partial \varphi(x) := \partial_0 \varphi(x)$. When $x \in \text{dom}(\partial \varphi) := \{y \in X \mid \partial \varphi(y) \neq \emptyset\}$, we know that

$$\varphi(x) = (\text{cl } \varphi)(x) \text{ and } \partial_\varepsilon \varphi(x) = \partial_\varepsilon (\text{cl } \varphi)(x), \text{ for all } \varepsilon \geq 0. \quad (5)$$

The *indicator* and the *support* functions of $A \subset X$ are, respectively, defined as

$$\begin{aligned} I_A(x) &:= 0, \text{ if } x \in A; \quad +\infty, \text{ if } x \in X \setminus A, \\ \sigma_A(x^*) &:= \sup\{\langle x^*, a \rangle \mid a \in A\}, \quad x^* \in X^*, \end{aligned} \quad (6)$$

with the convention $\sigma_\emptyset \equiv -\infty$.

If A is convex and $x \in X$, we define the *normal cone* to A at x as

$$N_A(x) := \{x^* \in X^* \mid \langle x^*, y - x \rangle \leq 0 \text{ for all } y \in A\}, \text{ if } x \in A,$$

and $N_A(x) = \emptyset$, if $x \in X \setminus A$.

Now, we review the results given in [3], which constitute the main foundations of the present work. We have proved there that the subdifferential of the supremum function $f = \sup_{t \in T} f_t$ is expressed in terms of appropriate enlargements of the Fenchel subdifferential, $\check{\partial}_p^\varepsilon(f_t + I_{\overline{L \cap \text{dom } f}})$, $t \in T_\varepsilon(x)$, $p \in \mathcal{P}$, and $L \in \mathcal{F}(x)$, where

$$T_\varepsilon(x) := \{t \in T \mid f_t(x) \geq f(x) - \varepsilon\}, \quad \varepsilon \geq 0,$$

$$\mathcal{P} = \{\text{continuous seminorms on } X\},$$

and $\mathcal{F}(x)$ is the family of finite-dimensional subspaces introduced in (2). Such enlargements involve the exact subdifferentials of functions $f_t + I_{\overline{L \cap \text{dom } f}}$ at nearby points of x . Precisely, for a convex function φ , $\check{\partial}_p^\varepsilon \varphi(x)$ is defined by

$$\check{\partial}_p^\varepsilon \varphi(x) := \{y^* \in \partial \varphi(y) \mid p(y - x) \leq \varepsilon, \quad |\varphi(y) - \varphi(x)| \leq \varepsilon, \text{ and } |\langle y^*, y - x \rangle| \leq \varepsilon\}.$$

Observe that $\check{\partial}_p^\varepsilon \varphi$ provides an outer approximation of $\partial \varphi$ as

$$\partial \varphi(x) \subset \check{\partial}_p^\varepsilon \varphi(x) \subset \partial_{2\varepsilon} \varphi(x). \quad (7)$$

When the functions f_t are proper and lsc, we proved in [3, Theorem 8] that

$$\partial f(x) = \bigcap_{\substack{\varepsilon > 0, \, p \in \mathcal{P} \\ L \in \mathcal{F}(x)}} \overline{\text{co}} \left\{ \bigcup_{t \in T_\varepsilon(x)} \check{\partial}_p^\varepsilon (f_t + I_{\overline{L \cap \text{dom } f}})(x) \right\}. \quad (8)$$

In particular, if the restriction of f to $\text{aff}(\text{dom } f)$ is continuous on $\text{ri}(\text{dom } f)$ (assumed to be nonempty), the intersection over $L \in \mathcal{F}(x)$ can be removed to obtain [3, Theorem 9]:

$$\partial f(x) = \bigcap_{\varepsilon > 0, p \in \mathcal{P}} \overline{\text{co}} \left\{ \bigcup_{t \in T_\varepsilon(x)} \check{\partial}_p^\varepsilon(f_t + I_{\overline{\text{dom } f}})(x) \right\}. \quad (9)$$

Moreover, if the f_t 's are proper but not necessarily lsc, and f is finite and continuous at some point, we obtained from (9) that

$$\partial f(x) = N_{\text{dom } f}(x) + \bigcap_{\varepsilon > 0, p \in \mathcal{P}} \overline{\text{co}} \left\{ \bigcup_{t \in \tilde{T}_\varepsilon(x)} \check{\partial}_p^\varepsilon(\text{cl } f_t)(x) \right\},$$

where $\tilde{T}_\varepsilon(x) := \{t \in T \mid (\text{cl } f_t)(x) \geq f(x) - \varepsilon\}$, see [3, Theorem 10].

In the particular case when f is continuous at the point x , we recover in [3, Corollary 12] the Valadier formula [18, Theorem 1]:

$$\partial f(x) = \bigcap_{\varepsilon > 0, p \in \mathcal{P}} \overline{\text{co}} \left\{ \bigcup_{t \in T_\varepsilon(x), p(y-x) \leq \varepsilon} \partial f_t(y) \right\}.$$

3. Compactly indexed case

In this section we characterize the subdifferential of the supremum function

$$f = \sup_{t \in T} f_t$$

of a compactly indexed family of convex functions $f_t: X \rightarrow \mathbb{R} \cup \{\pm\infty\}$, $t \in T$, where X is a lcs space whose family of continuous seminorms is denoted by $\mathcal{P}$, and the dual space X^* is endowed with the weak*-topology.

First, we state our result in a finite dimensional-like setting. Recall that

$$T_\varepsilon(x) = \{t \in T \mid f_t(x) \geq f(x) - \varepsilon\}, \quad \varepsilon \geq 0,$$

and

$$T(x) = \{t \in T \mid f_t(x) = f(x)\}.$$

Theorem 3.1. *Assume that the family of convex functions $\{f_t, t \in T\}$ is such that the function $f|_{\text{aff}(\text{dom } f)}$ is finite and continuous on $\text{ri}(\text{dom } f)$, assumed to be nonempty. Suppose that*

$$\text{cl } f = \sup_{t \in T} (\text{cl } f_t). \quad (10)$$

Let $x \in X$ be such that for some $\varepsilon_0 > 0$:

- (i) the set $T_{\varepsilon_0}(x)$ is compact,

(ii) the functions $t \mapsto f_t(z)$, $z \in \text{dom } f$, are upper semicontinuous (usc) on $T_{\varepsilon_0}(x)$.

$$\text{Then :} \quad \partial f(x) = \overline{\text{co}} \left\{ \bigcup_{t \in T(x)} \partial(f_t + \mathbf{I}_{\text{dom } f})(x) \right\}. \quad (11)$$

The proof of this theorem uses the following technical lemma:

Lemma 3.2. *Given the family of convex functions $\{f_t, t \in T\}$ and $x_0 \in \text{dom}(\partial f)$, consider the functions*

$$\ell_t := \max\{f_t, f(x_0) - c\}, \quad t \in T,$$

where $c > 0$, and $\ell := \sup_{t \in T} \ell_t$.

Then, under condition (10), there exists an open neighborhood U of x_0 such that the proper functions ℓ_t satisfy the following:

- (i) $\text{cl } f = \text{cl } \ell$ and $f = \max\{f, f(x_0) - c\} = \ell$, on U .
- (ii) $\text{cl } \ell = \sup_{t \in T} (\text{cl } \ell_t)$.
- (iii) $\{t \in T \mid \ell_t(x_0) \geq \ell(x_0) - \varepsilon\} = T_\varepsilon(x_0)$ for all $\varepsilon \in [0, c[$.
- (iv) $\partial f(x_0) = \partial \ell(x_0)$.

Proof. We may suppose that $x_0 = \theta$ and $f(\theta) = 0$. Observe that $\ell = \max\{f, -c\}$ and, so, $\text{dom } \ell = \text{dom } f (\neq \emptyset)$ and the ℓ_t 's are proper. Since f is lsc at θ (because it is subdifferentiable at θ) there exists an open neighborhood $U \in \mathcal{N}_X$ such that

$$f(x) \geq -c \quad \text{for all } x \in U.$$

Hence, $f = \ell$ and $\text{cl } \ell = \text{cl } f$ on U ; consequently, $\partial f(\theta) = \partial \ell(\theta)$ and we have proved (i) and (iv). Now we proceed by proving (ii):

$$\begin{aligned} \text{cl } \ell &= \text{cl}(\sup_{t \in T} (\max\{f_t, -c\})) = \text{cl}(\max\{\sup_{t \in T} f_t, -c\}) \\ &= \max\{\text{cl}(\sup_{t \in T} f_t), -c\} \quad (\text{by (4)}) \\ &= \max\{\sup_{t \in T} (\text{cl } f_t), -c\} \quad (\text{by (10)}) \\ &= \sup_{t \in T} \max\{\text{cl } f_t, -c\} = \sup_{t \in T} \text{cl}(\max\{f_t, -c\}) \quad (\text{by (4)}) \\ &= \sup_{t \in T} (\text{cl } \ell_t), \end{aligned}$$

leading to (ii).

Finally, to prove (iii), observe that for every $t \in T$ such that $\ell_t(\theta) > -c$ we have

$$\ell_t(\theta) = \max\{f_t(\theta), -c\} = f_t(\theta),$$

and so for all $\varepsilon \in [0, c[$

$$\{t \in T \mid \ell_t(\theta) \geq \ell(\theta) - \varepsilon\} = \{t \in T \mid \ell_t(\theta) \geq -\varepsilon\} = \{t \in T \mid f_t(\theta) \geq -\varepsilon\} = T_\varepsilon(\theta),$$

yielding (iii). $\square$

Proof of Theorem 3.1. First, we show the inclusion " $\supset$ ". We start by verifying that for every $t \in T$

$$\partial(f_t + I_{\text{dom } f})(x) \subset \partial(f_t + I_{\overline{\text{dom } f}})(x). \quad (12)$$

We fix $x_0 \in \text{ri}(\text{dom } f)$ and pick $z^* \in \partial(f_t + I_{\text{dom } f})(x)$. Given $y \in \overline{\text{dom } f}$, we define

$$y_\lambda := \lambda x_0 + (1 - \lambda)y, \quad \lambda \in]0, 1[,$$

so that $y_\lambda \in \text{dom } f$ by the accessibility lemma, and

$$\langle z^*, y_\lambda - x \rangle \leq f_t(y_\lambda) - f_t(x) \leq \lambda f(x_0) + (1 - \lambda)f_t(y) - f_t(x).$$

As $\lambda \downarrow 0$ we get

$$\langle z^*, y - x \rangle \leq f_t(y) - f_t(x), \quad (13)$$

and, so, $z^* \in \partial(f_t + I_{\overline{\text{dom } f}})(x)$.

Next, for every $p \in \mathcal{P}$ and $\varepsilon > 0$ we have, by (12) and (7),

$$\begin{aligned} \overline{\text{co}} \left\{ \bigcup_{t \in T(x)} \partial(f_t + I_{\text{dom } f})(x) \right\} &\subset \overline{\text{co}} \left\{ \bigcup_{t \in T(x)} \partial(f_t + I_{\overline{\text{dom } f}})(x) \right\} \\ &\subset \overline{\text{co}} \left\{ \bigcup_{t \in T_\varepsilon(x)} \check{\partial}_p^\varepsilon(f_t + I_{\overline{\text{dom } f}})(x) \right\}. \end{aligned} \quad (14)$$

So, due to (9) we obtain

$$\overline{\text{co}} \left\{ \bigcup_{t \in T(x)} \partial(f_t + I_{\text{dom } f})(x) \right\} \subset \bigcap_{\varepsilon > 0, p \in \mathcal{P}} \overline{\text{co}} \left\{ \bigcup_{t \in T_\varepsilon(x)} \check{\partial}_p^\varepsilon(f_t + I_{\overline{\text{dom } f}})(x) \right\} = \partial f(x).$$

To prove the inclusion " $\subset$ ", it suffices to consider the nontrivial case $\partial f(x) \neq \emptyset$, entailing that $x \in \text{dom } f$ and (see (5))

$$f(x) = (\text{cl } f)(x) \text{ and } \partial_\varepsilon f(x) = \partial_\varepsilon(\text{cl } f)(x) \text{ for all } \varepsilon \geq 0; \quad (15)$$

and because $f(x) \in \mathbb{R}$ we can suppose that $x = \theta$ and $f(\theta) = (\text{cl } f)(\theta) = 0$. Moreover, as $\partial f(\theta) \neq \emptyset$, both functions f and $\text{cl } f$ are proper. In addition, since we have that $\text{ri}(\text{dom } f) = \text{ri}(\text{dom}(\text{cl } f))$, $\text{aff}(\text{dom } f) = \text{aff}(\text{dom}(\text{cl } f))$, and $\text{cl } f \leq f$, we deduce that

$$(\text{cl } f)|_{\text{aff}(\text{dom}(\text{cl } f))} \text{ is finite and continuous on } \text{ri}(\text{dom}(\text{cl } f)). \quad (16)$$

In a first step, we suppose that $\text{cl } f_t$ is proper for all $t \in T$. We define

$$g_t := (\text{cl } f_t) + I_{\overline{\text{dom } f}}, \quad t \in T, \quad (17)$$

so that, by (10) and the relation $\text{dom}(\text{cl } f) \subset \overline{\text{dom } f}$,

$$\sup_{t \in T} g_t = \sup_{t \in T} (\text{cl } f_t) + I_{\overline{\text{dom } f}} = \text{cl } f + I_{\overline{\text{dom } f}} = \text{cl } f. \quad (18)$$

Thus, taking into account (16), we are in position to apply (9) to the lsc convex proper functions g_t , and we get

$$\begin{aligned} \partial f(\theta) &= \partial(\text{cl } f)(\theta) \quad (\text{by (15)}) = \partial(\sup_{t \in T} g_t)(\theta) \quad (\text{by (18)}) \\ &= \bigcap_{\varepsilon > 0, p \in \mathcal{P}} \overline{\text{co}} \left\{ \bigcup_{t \in \tilde{T}_\varepsilon(\theta)} \check{\partial}_p^\varepsilon(g_t + I_{\overline{\text{dom}(\text{cl } f)}})(\theta) \right\} \quad (\text{by (9)}) \\ &= \bigcap_{\varepsilon > 0, p \in \mathcal{P}} \overline{\text{co}} \left\{ \bigcup_{t \in \tilde{T}_\varepsilon(\theta)} \check{\partial}_p^\varepsilon g_t(\theta) \right\} \quad (\text{since } \text{dom } g_t \subset \overline{\text{dom } f} = \overline{\text{dom}(\text{cl } f)}) \\ &= \bigcap_{\varepsilon > 0, p \in \mathcal{P}} \overline{\text{co}} \left\{ \bigcup_{t \in \tilde{T}_\varepsilon(\theta)} \check{\partial}_p^\varepsilon g_t(\theta) \cap \partial_{2\varepsilon} g_t(\theta) \right\} \quad (\text{by (7)}), \end{aligned} \quad (19)$$

where

$$\begin{aligned} \tilde{T}_\varepsilon(\theta) &:= \{t \in T \mid g_t(\theta) \geq (\sup_{t \in T} g_t)(\theta) - \varepsilon\} \\ &= \{t \in T \mid (\text{cl } f_t)(\theta) \geq -\varepsilon\} \subset T_\varepsilon(\theta). \end{aligned} \quad (20)$$

Moreover, for every $t \in \tilde{T}_\varepsilon(\theta)$ we have that $\partial_{2\varepsilon} g_t(\theta) \subset \partial_{3\varepsilon} f(\theta)$, which comes from the following inequalities: for $z \in \text{dom } f$ and $z^* \in \partial_{2\varepsilon} g_t(\theta)$,

$$\begin{aligned} f(z) &\geq f_t(z) \geq (\text{cl } f_t)(z) = g_t(z) \geq g_t(\theta) + \langle z^*, z \rangle - 2\varepsilon \\ &\geq (\sup_{t \in T} g_t)(\theta) + \langle z^*, z \rangle - 3\varepsilon = (\text{cl } f)(\theta) + \langle z^*, z \rangle - 3\varepsilon \quad (\text{by (18)}) \\ &= \langle z^*, z \rangle - 3\varepsilon. \end{aligned}$$

So, (19) yields

$$\partial f(\theta) = \bigcap_{\varepsilon > 0, p \in \mathcal{P}} \overline{\text{co}} \left\{ \bigcup_{t \in \tilde{T}_\varepsilon(\theta)} \check{\partial}_p^\varepsilon g_t(\theta) \cap \partial_{3\varepsilon} f(\theta) \right\}. \quad (21)$$

Take $x^* \in \partial f(\theta)$ and fix $u \in X$. Given $k \in \mathbb{N}$ and $p \in \mathcal{P}$, (21) ensures the following relation in $\mathbb{R}^2$

$$\begin{aligned} \left(\begin{array}{c} \langle x^*, u \rangle \\ \langle x^*, x_0 \rangle \end{array} \right) &\in \overline{\text{co}} \left\{ \bigcup_{t \in \tilde{T}_{\frac{1}{k}}(\theta)} \left\{ \left(\begin{array}{c} \langle y^*, u \rangle \\ \langle y^*, x_0 \rangle \end{array} \right) \mid y^* \in \check{\partial}_p^{\frac{1}{k}} g_t(\theta) \cap \partial_{\frac{3}{k}} f(\theta) \right\} \right\} \\ &\subset \text{co} \left\{ \bigcup_{t \in \tilde{T}_{\frac{1}{k}}(\theta)} \left\{ \left(\begin{array}{c} \langle y^*, u \rangle \\ \langle y^*, x_0 \rangle \end{array} \right) \mid y^* \in \check{\partial}_p^{\frac{1}{k}} g_t(\theta) \cap \partial_{\frac{3}{k}} f(\theta) \right\} \right\} + \left[-\frac{1}{k}, \frac{1}{k} \right]^2. \end{aligned}$$

Therefore, taking into account Carathéodory's Theorem, there are

$$(\lambda_{1,k}, \lambda_{2,k}, \lambda_{3,k}) \in \Delta_3 := \{(\lambda_1, \lambda_2, \lambda_3) \mid \lambda_i \geq 0, \lambda_1 + \lambda_2 + \lambda_3 = 1\}$$

and

$$t_{i,k} \in \tilde{T}_{1/k}(\theta), \text{ and } y_{i,k}^* \in \check{\partial}_p^{1/k} g_{t_{i,k}}(\theta) \cap \partial_{3/k} f(\theta), \quad i = 1, 2, 3, \quad (22)$$

such that

$$\langle x^*, u \rangle \leq \langle \lambda_{1,k} y_{1,k}^* + \lambda_{2,k} y_{2,k}^* + \lambda_{3,k} y_{3,k}^*, u \rangle + 1/k, \quad (23)$$

$$|\langle x^*, x_0 \rangle - \langle \lambda_{1,k} y_{1,k}^* + \lambda_{2,k} y_{2,k}^* + \lambda_{3,k} y_{3,k}^*, x_0 \rangle| \leq 1/k. \quad (24)$$

Then, by the definition of $\check{\partial}_p^{1/k}$, there exist $y_{i,k} \in \text{dom } g_{t_{i,k}} \subset \overline{\text{dom } f}$ ($i = 1, 2, 3$) such that

$$p(y_{i,k}) \leq 1/k, \quad |(\text{cl } f_{t_{i,k}})(y_{i,k}) - (\text{cl } f_{t_{i,k}})(\theta)| \leq 1/k, \quad |\langle y_{i,k}^*, y_{i,k} \rangle| \leq 1/k, \quad (25)$$

and (recall (22))

$$y_{i,k}^* \in \partial g_{t_{i,k}}(y_{i,k}) \cap \partial_{3/k} f(\theta). \quad (26)$$

From the continuity assumption of $f|_{\text{aff}(\text{dom } f)}$ at $x_0 \in \text{ri}(\text{dom } f)$ we choose $m \geq 0$ and $W \in \mathcal{N}_X$ such that

$$x_0 + W \cap \text{aff}(\text{dom } f) \subset \text{dom } f \text{ and } \sup_{w \in W \cap \text{aff}(\text{dom } f)} f(x_0 + w) \leq m.$$

Then, using (26), for each $i = 1, 2, 3$, and for all $z \in W \cap \text{aff}(\text{dom } f)$

$$\langle y_{i,k}^*, x_0 + z \rangle \leq f(x_0 + z) - f(\theta) + 3/k = f(x_0 + z) + 3/k \leq m + 3/k, \quad (27)$$

and so, since $\theta \in W \cap \text{aff}(\text{dom } f)$,

$$\langle y_{i,k}^*, x_0 \rangle \leq m + 3/k. \quad (28)$$

Thus, from (24) we deduce that the sequences $(\langle \lambda_{i,k} y_{i,k}^*, x_0 \rangle)_k$, $i = 1, 2, 3$, are bounded. Consequently, by (27) and (28) we see that there exists a positive number r such that

$$\langle \lambda_{i,k} y_{i,k}^*, z \rangle \leq \lambda_{i,k}(m + 3/k - \langle y_{i,k}^*, x_0 \rangle) \leq r \text{ for all } z \in W \cap \text{aff}(\text{dom } f); \quad (29)$$

that is,

$$r^{-1}(\lambda_{i,k} y_{i,k}^*)_k \subset (W \cap \text{aff}(\text{dom } f))^\circ := \{u^* \in X^* \mid \langle u^*, z \rangle \leq 1 \ \forall z \in W \cap \text{aff}(\text{dom } f)\}.$$

We endow the lcs space $Y := \text{aff}(\text{dom } f) (= \overline{\text{aff}(\text{dom } f)})$ with the induced topology from X , and denote by Y^* its topological dual space. By the Alaoglu-Bourbaki Theorem applied to the dual pair (Y, Y^*) , there exists a subnet of the sequence

of the restrictions of $(\lambda_{i,k}y_{i,k}^*)_k$ to Y , denoted by $(\lambda_{i,k_\alpha}y_{i,k_\alpha}^*)_{\alpha \in \Upsilon}$, which weak*-converges to some $\hat{y}_i^* \in Y^*$ ($i = 1, 2, 3$). Thus, if $y_i^* \in X^*$ is an extension of $\hat{y}_i^*$ to X , it satisfies for every $z \in \text{dom } f$

$$\begin{aligned} \langle y_i^*, z \rangle &= \langle \hat{y}_i^*, z \rangle = \lim_{\alpha \in \Upsilon} \langle \lambda_{i,k_\alpha}y_{i,k_\alpha}^*, z \rangle \\ &= \lim_{\alpha \in \Upsilon} (\langle \lambda_{i,k_\alpha}y_{i,k_\alpha}^*, z - y_{i,k_\alpha} \rangle + \langle \lambda_{i,k_\alpha}y_{i,k_\alpha}^*, y_{i,k_\alpha} \rangle) \\ &\leq \liminf_{\alpha \in \Upsilon} (\lambda_{i,k_\alpha} (g_{t_{i,k_\alpha}}(z) - g_{t_{i,k_\alpha}}(y_{i,k_\alpha})) + 1/k_\alpha) \quad (\text{by (26) and (25)}) \\ &= \liminf_{\alpha \in \Upsilon} (\lambda_{i,k_\alpha} ((\text{cl } f_{t_{i,k_\alpha}})(z) - (\text{cl } f_{t_{i,k_\alpha}})(y_{i,k_\alpha})) + 1/k_\alpha) \\ &\leq \liminf_{\alpha \in \Upsilon} (\lambda_{i,k_\alpha} ((\text{cl } f_{t_{i,k_\alpha}})(z) - (\text{cl } f_{t_{i,k_\alpha}})(\theta)) + 2/k_\alpha) \quad (\text{by (25)}) \\ &\leq \liminf_{\alpha \in \Upsilon} (\lambda_{i,k_\alpha} (\text{cl } f_{t_{i,k_\alpha}})(z) + 3/k_\alpha) \quad (\text{as } t_{i,k_\alpha} \in \tilde{T}_{1/k_\alpha}(\theta), \text{ by (22)}) \\ &= \liminf_{\alpha \in \Upsilon} \lambda_{i,k_\alpha} (\text{cl } f_{t_{i,k_\alpha}})(z) \end{aligned} \tag{30}$$

$$\leq \liminf_{\alpha \in \Upsilon} \lambda_{i,k_\alpha} f_{t_{i,k_\alpha}}(z), \tag{31}$$

where $(k_\alpha)_{\alpha \in \Upsilon} \subset \mathbb{N}$ is such that $\lim_{\alpha \in \Upsilon} k_\alpha = +\infty$ and $1/k_\alpha \leq \varepsilon_0$, eventually.

We may suppose that $\lambda_{i,k_\alpha} \rightarrow \lambda_i$ for some $\lambda_i \in [0, 1]$, so that $\lambda := (\lambda_1, \lambda_2, \lambda_3) \in \Delta_3$. We denote $I_0 := \{i = 1, 2, 3 \mid \lambda_i = 0\}$ (this set can be empty). If $i \in I_0$, from (31) we get

$$\langle y_i^*, z \rangle \leq \liminf_{\alpha \in \Upsilon} \lambda_{i,k_\alpha} f_{t_{i,k_\alpha}}(z) \leq \liminf_{\alpha \in \Upsilon} \lambda_{i,k_\alpha} f(z) = 0 \quad \text{for all } z \in \text{dom } f,$$

showing that

$$y_i^* \in N_{\text{dom } f}(\theta) = N_{\overline{\text{dom } f}}(\theta). \tag{32}$$

Otherwise, if $i \notin I_0$ (observe that $\{1, 2, 3\} \setminus I_0$ is nonempty because $\sum_i \lambda_i = 1$), (30) gives rise to

$$\langle \lambda_i^{-1} y_i^*, z \rangle \leq \liminf_{\alpha \in \Upsilon} (\text{cl } f_{t_{i,k_\alpha}})(z) \leq \liminf_{\alpha \in \Upsilon} f_{t_{i,k_\alpha}}(z) \quad \text{for all } z \in \overline{\text{dom } f}. \tag{33}$$

But, since $1/k_\alpha \leq \varepsilon_0$ eventually, by (20)

$$t_{i,k_\alpha} \in \tilde{T}_{1/k_\alpha}(\theta) \subset T_{1/k_\alpha}(\theta) \subset T_{\varepsilon_0}(\theta), \tag{34}$$

the last set being compact by assumption (i), we may assume that $(t_{i,k_\alpha})_{\alpha \in \Upsilon}$ converges to some $t_i \in T_{\varepsilon_0}(\theta)$. Thus,

$$\begin{aligned} f(\theta) &\geq f_{t_i}(\theta) \geq \limsup_{\alpha \in \Upsilon} f_{t_{i,k_\alpha}}(\theta) \quad (\text{by (ii)}) \geq \limsup_{\alpha \in \Upsilon} (\text{cl } f_{t_{i,k_\alpha}})(\theta) \\ &\geq \limsup_{\alpha \in \Upsilon} (f(\theta) - 1/k_\alpha) = f(\theta) \quad (\text{by (34)}); \end{aligned}$$

that is, $t_i \in T(\theta)$. So, (33) together with hypothesis (ii) yield, for all $z \in \text{dom } f$,

$$\langle \lambda_i^{-1} y_i^*, z \rangle \leq \liminf_{\alpha \in \Upsilon} f_{t_{i,k_\alpha}}(z) \leq \limsup_{\alpha \in \Upsilon} f_{t_{i,k_\alpha}}(z) \leq f_{t_i}(z) = f_{t_i}(z) - f_{t_i}(\theta);$$

so that

$$\lambda_i^{-1} y_i^* \in \partial(f_{t_i} + I_{\text{dom } f})(\theta). \tag{35}$$

By taking limits in (23) when $u \in Y$ we get (recall that $k_\alpha \rightarrow +\infty$)

$$\begin{aligned} \langle x^*, u \rangle &\leq \limsup_{\alpha \in \Upsilon} (\langle \lambda_{1,k_\alpha} y_{1,k_\alpha}^* + \lambda_{2,k_\alpha} y_{2,k_\alpha}^* + \lambda_{3,k_\alpha} y_{3,k_\alpha}^*, u \rangle + 1/k_\alpha) \\ &\leq \sum_{i \in I_0} \limsup_{\alpha \in \Upsilon} \langle \lambda_{i,k_\alpha} y_{i,k_\alpha}^*, u \rangle + \sum_{i \notin I_0} \limsup_{\alpha \in \Upsilon} \langle \lambda_{i,k_\alpha} y_{i,k_\alpha}^*, u \rangle \\ &= \sum_{i \in I_0} \langle y_i^*, u \rangle + \left\langle \sum_{i \notin I_0} \lambda_i (\lambda_i^{-1} y_i^*), u \right\rangle \quad (\text{as } (\lambda_{i,k_\alpha} y_{i,k_\alpha}^*)|_Y \xrightarrow{w^*} \hat{y}_i^* = (y_i^*)|_Y), \end{aligned}$$

which gives us, due to (32) and (35),

$$\langle x^*, u \rangle \leq \sigma_{N_{\text{dom } f}(\theta)}(u) + \sigma_{\sum_{i \notin I_0} \lambda_i \partial(f_{t_i} + I_{\text{dom } f})(\theta)}(u). \quad (36)$$

Observe that when $u \notin Y$ ($= \text{aff}(\text{dom } f)$), we have that

$$u \notin \overline{\mathbb{R}_+(\text{dom } f)} = (N_{\text{dom } f}(\theta))^\perp, \quad \text{and so } \sigma_{N_{\text{dom } f}(\theta)}(u) = +\infty;$$

hence, thanks to (36), for every $u \in X$ it holds

$$\begin{aligned} \langle x^*, u \rangle &\leq \sigma_{N_{\text{dom } f}(\theta)}(u) + \sigma_{\sum_{i \notin I_0} \lambda_i \partial(f_{t_i} + I_{\text{dom } f})(\theta)}(u) \\ &= \sigma_{N_{\text{dom } f}(\theta) + \sum_{i \notin I_0} \lambda_i \partial(f_{t_i} + I_{\text{dom } f})(\theta)}(u). \end{aligned} \quad (37)$$

Additionally, for each $i \notin I_0$

$$N_{\text{dom } f}(\theta) + \lambda_i \partial(f_{t_i} + I_{\text{dom } f})(\theta) \subset \partial(\lambda_i f_{t_i} + I_{\text{dom } f})(\theta) = \lambda_i \partial(f_{t_i} + I_{\text{dom } f})(\theta),$$

and so (37) yields (recall that $t_i \in T(\theta)$)

$$\langle x^*, u \rangle \leq \sigma_{\sum_{i \in \{1,2,3\} \setminus I_0} \lambda_i \partial(f_{t_i} + I_{\text{dom } f})(\theta)}(u) \leq \sigma_{\overline{\text{co}}\{\bigcup_{t \in T(\theta)} \partial(f_t + I_{\text{dom } f})(\theta)\}}(u).$$

As u is arbitrary in X we conclude that

$$x^* \in \overline{\text{co}} \left\{ \bigcup_{t \in T(\theta)} \partial(f_t + I_{\text{dom } f})(\theta) \right\},$$

proving the desired inclusion when the $\text{cl } f_t$'s are proper.

Finally, to deal with the case when the $(\text{cl } f_t)$'s are not all necessarily proper, we consider the functions

$$\ell_t := \max\{f_t, -2\varepsilon_0\}, \quad t \in T, \quad \text{and} \quad \ell := \sup_{t \in T} \ell_t.$$

According to Lemma 3.2, the $\text{cl } \ell_t$'s are proper, $\{t \in T \mid \ell_t(\theta) \geq \ell(\theta) - \varepsilon\} = T_\varepsilon(\theta)$ for all $\varepsilon \in [0, \varepsilon_0[$ ($\subset [0, 2\varepsilon_0]$), $\text{dom } \ell = \text{dom } f$, $\text{cl } \ell := \sup_{t \in T} (\text{cl } \ell_t)$, and $\partial f(\theta) = \partial \ell(\theta)$. It follows that the family $\{\ell_t, t \in T\}$ satisfies the current conditions (i) and

(ii). Consequently, from the previous part of the proof, applied to the functions ℓ_t , $t \in T$, we get (observe that $\ell(\theta) = \max\{f_t(\theta), -2\varepsilon_0\} = f(\theta) = 0$)

$$\partial f(\theta) = \partial \ell(\theta) = \overline{\text{co}} \left\{ \bigcup_{t \in \{t \in T \mid \ell_t(\theta) = 0\}} \partial(\ell_t + \text{I}_{\text{dom } \ell})(\theta) \right\} = \overline{\text{co}} \left\{ \bigcup_{t \in T(\theta)} \partial(\ell_t + \text{I}_{\text{dom } f})(\theta) \right\}.$$

Take $t \in T(\theta)$ such that $\partial(\ell_t + \text{I}_{\text{dom } \ell})(\theta) \neq \emptyset$ (such a t always exists because $\partial f(\theta) \neq \emptyset$). We show that

$$\partial(\ell_t + \text{I}_{\text{dom } f})(\theta) = \partial(f_t + \text{I}_{\text{dom } f})(\theta). \quad (38)$$

Since $\ell_t + \text{I}_{\text{dom } \ell} = \max\{f_t + \text{I}_{\text{dom } \ell}, -2\varepsilon_0\}$ and $(f_t + \text{I}_{\text{dom } \ell})(\theta) = 0 > -2\varepsilon_0$, it suffices to verify that $f_t + \text{I}_{\text{dom } \ell}$ is lsc at θ , because then the two functions $\ell_t + \text{I}_{\text{dom } \ell}$ and $f_t + \text{I}_{\text{dom } \ell}$ coincide in a neighborhood of θ . Indeed, using (4) and the lower semicontinuity of $\ell_t + \text{I}_{\text{dom } \ell}$ at θ (a consequence of the nonemptiness of $\partial(\ell_t + \text{I}_{\text{dom } \ell})(\theta)$), we have

$$\begin{aligned} \max\{\text{cl}(f_t + \text{I}_{\text{dom } \ell})(\theta), -2\varepsilon_0\} &= \text{cl}(\max\{f_t + \text{I}_{\text{dom } \ell}, -2\varepsilon_0\})(\theta) \\ &= \text{cl}(\ell_t + \text{I}_{\text{dom } \ell})(\theta) = (\ell_t + \text{I}_{\text{dom } \ell})(\theta) = 0; \end{aligned}$$

that is, $\text{cl}(f_t + \text{I}_{\text{dom } \ell})(\theta) = 0$. But $t \in T(\theta)$, and so $(f_t + \text{I}_{\text{dom } \ell})(\theta) = 0 = \text{cl}(f_t + \text{I}_{\text{dom } \ell})(\theta)$, yielding the lower semicontinuity of $\ell_t + \text{I}_{\text{dom } \ell}$ at θ , and (38) follows.

The proof of the inclusion " $\subset$ " is complete. $\square$

Remark 3.3. In Theorem 3.1 one can replace (i)–(ii) by the following weaker pair of conditions (i')–(ii'), which emphasizes the role played by the ε -active sets at x :

(i') the sets $T_\varepsilon(x)$ are compact for all $\varepsilon \in [0, \varepsilon_0]$,

(ii') the functions $t \mapsto f_t(z)$, $z \in \text{dom } f$, are usc on $T(x)$.

Indeed, from (34) we have that $t_{i, k_\alpha} \in \tilde{T}_{1/k_\alpha}(x) \subset T_{1/k_\alpha}(x)$ for all $\alpha \in \Upsilon$, and so (i') gives rise to $t_i \in \cap_\alpha \text{cl}(T_{1/k_\alpha}(x)) = \cap_\alpha T_{1/k_\alpha}(x) = T(x)$.

Thus, the proof also follows by using the weaker assumption (ii'), instead of (ii).

Remark 3.4. [7, Corollary 9] The closure condition (10) holds in each one of the following situations:

(1) the functions f_t , $t \in T$, are lsc.

(2) the f_t 's have a common continuity point; this follows if, for instance, the supremum function f is finite and continuous at some point.

(3) T is finite and all but one of the functions f_t 's have a common continuity point in $\text{dom } f$ (this includes (4)).

(4) X is finite-dimensional, and the relative interiors $\text{ri}(\text{dom } f_t)$, $t \in T$, have a common point in $\text{dom } f$.

Theorem 3.5. Assume that the convex functions f_t , $t \in T$, are proper and lsc. Let $x \in X$ be such that for some $\varepsilon_0 > 0$:

- (i) the set $T_{\varepsilon_0}(x)$ is compact,
- (ii) the functions $t \mapsto f_t(z)$, $z \in \text{dom } f$, are usc on $T_{\varepsilon_0}(x)$.

Then
$$\partial f(x) = \bigcap_{L \in \mathcal{F}(x)} \overline{\text{co}} \left\{ \bigcup_{t \in T(x)} \partial(f_t + I_{L \cap \text{dom } f})(x) \right\},$$

where $\mathcal{F}(x) := \{\text{finite-dimensional linear subspaces of } X \text{ such that } x \in L\}$.

Proof. We start by verifying the inclusion " $\supset$ ". Fix $L \in \mathcal{F}(x)$, $p \in \mathcal{P}$ and $\varepsilon > 0$. By arguing as in the proof of the inclusion " $\supset$ " in Theorem 3.1 (see (14)) we can show that

$$\begin{aligned} \overline{\text{co}} \left\{ \bigcup_{t \in T(x)} \partial(f_t + I_{L \cap \text{dom } f})(x) \right\} &\subset \overline{\text{co}} \left\{ \bigcup_{t \in T(x)} \partial(f_t + I_{\overline{L \cap \text{dom } f}})(x) \right\} \\ &\subset \overline{\text{co}} \left\{ \bigcup_{t \in T_\varepsilon(x)} \check{\partial}_p^\varepsilon(f_t + I_{\overline{L \cap \text{dom } f}})(x) \right\}. \end{aligned}$$

So, due to (8) we obtain

$$\begin{aligned} \bigcap_{L \in \mathcal{F}(x)} \overline{\text{co}} \left\{ \bigcup_{t \in T(x)} \partial(f_t + I_{L \cap \text{dom } f})(x) \right\} &\subset \bigcap_{\substack{\varepsilon > 0, p \in \mathcal{P} \\ L \in \mathcal{F}(x)}} \overline{\text{co}} \left\{ \bigcup_{t \in T_\varepsilon(x)} \check{\partial}_p^\varepsilon(f_t + I_{\overline{L \cap \text{dom } f}})(x) \right\} \\ &= \partial f(x). \end{aligned}$$

Observe that this argument does not use the lower semicontinuity of the f_t 's.

We first prove the inclusion " $\subset$ " by considering the nontrivial case $\partial f(x) \neq \emptyset$, entailing that $x \in \text{dom } f$. Let us suppose that $x = \theta$, $f(\theta) = (\text{cl } f)(\theta) = 0$.

For a fixed $L \in \mathcal{F}(\theta)$ we apply Theorem 3.1 to the lsc proper convex functions

$$h_t := f_t + I_{\overline{L \cap \text{dom } f}}, \quad t \in T, \quad \text{and} \quad h := \sup_{t \in T} h_t = f + I_{\overline{L \cap \text{dom } f}}.$$

Obviously, $h \geq f$, $\text{dom } h \subset \text{dom } f$, $h(\theta) = f(\theta) = 0$, $\text{ri}(\text{dom } h) \neq \emptyset$ (because $\text{dom } h \subset L$), and, therefore $h|_{\text{aff}(\text{dom } h)}$ is continuous on $\text{ri}(\text{dom } h)$, together with

$$\partial f(\theta) \subset \partial h(\theta) \tag{39}$$

and

$$\{t \in T \mid h_t(\theta) \geq h(\theta) - \varepsilon\} = \{t \in T \mid f_t(\theta) \geq -\varepsilon\} = T_\varepsilon(\theta) \quad \text{for all } \varepsilon \geq 0. \tag{40}$$

Since, for $z \in \text{dom } h$ ($\subset \text{dom } f$), the function $t \mapsto h_t(z) = f_t(z) + \mathbf{I}_{\overline{L \cap \text{dom } f}}(z)$ is usc on $T_{\varepsilon_0}(\theta)$, Theorem 3.1 applies and yields, taking into account (40),

$$\begin{aligned} \partial h(\theta) &= \overline{\text{co}} \left\{ \bigcup_{\{t \in T \mid h_t(\theta) = h(\theta)\}} \partial(h_t + \mathbf{I}_{\text{dom } h})(\theta) \right\} \\ &= \overline{\text{co}} \left\{ \bigcup_{t \in T(\theta)} \partial(f_t + \mathbf{I}_{\overline{L \cap \text{dom } f}} + \mathbf{I}_{\text{dom } f \cap \overline{L \cap \text{dom } f}})(\theta) \right\} \\ &= \overline{\text{co}} \left\{ \bigcup_{t \in T(\theta)} \partial(f_t + \mathbf{I}_{\overline{L \cap \text{dom } f}} + \mathbf{I}_{L \cap \text{dom } f \cap \overline{L \cap \text{dom } f}})(\theta) \right\}, \end{aligned}$$

and hence, by (39),

$$\partial f(\theta) \subset \partial h(\theta) \subset \overline{\text{co}} \left\{ \bigcup_{t \in T(\theta)} \partial(f_t + \mathbf{I}_{L \cap \text{dom } f})(\theta) \right\}. \quad (41)$$

Thus, due to the arbitrariness of the L 's, the desired inclusion follows. $\square$

Theorem 3.6. *Assume that the convex functions f_t , $t \in T$, satisfy*

$$\text{cl } f = \sup_{t \in T} (\text{cl } f_t).$$

Let $x \in X$ be such that for some $\varepsilon_0 > 0$:

- (i) *the set $T_{\varepsilon_0}(x)$ is compact,*
- (ii) *the functions $t \mapsto (\text{cl } f_t)(z)$, $z \in \text{dom}(\text{cl } f)$, are usc on $T_{\varepsilon_0}(x)$.*

$$\text{Then} \quad \partial f(x) = \bigcap_{L \in \mathcal{F}(x)} \overline{\text{co}} \left\{ \bigcup_{t \in T(x)} \partial(f_t + \mathbf{I}_{L \cap \text{dom } f})(x) \right\}. \quad (42)$$

Proof. The inclusion " $\supset$ " is proved as in Theorem 3.5. To prove the inclusion " $\subset$ " we suppose that $\partial f(x) \neq \emptyset$, allowing us to take $x = \theta$ and $f(\theta) = 0$. Let us consider the proper convex functions

$$\ell_t := \max\{f_t, -2\varepsilon_0\}, \quad t \in T, \quad \text{and} \quad \ell := \sup_{t \in T} \ell_t = \max\{f, -2\varepsilon_0\}. \quad (43)$$

According to Lemma 3.2, in some open neighborhood U of θ we have that $f \geq -2\varepsilon_0$ and f coincides with ℓ , entailing $(\emptyset \neq) \partial f(\theta) = \partial \ell(\theta)$ and $(\text{cl } \ell)(\theta) = \ell(\theta) = f(\theta) = 0$. In addition, we have $\text{dom } \ell = \text{dom } f$ and

$$\text{cl } \ell = \text{cl } f \quad \text{on } U. \quad (44)$$

From (43) we also have that $\text{cl } \ell = \max\{\text{cl } f, -2\varepsilon_0\}$, and so $\text{dom}(\text{cl } \ell) = \text{dom}(\text{cl } f)$.

Now, taking into account that $\text{cl } \ell = \sup_{t \in T} (\text{cl } \ell_t)$ by Lemma 3.2(ii), we apply Theorem 3.5 to the family $\{\text{cl } \ell_t, t \in T\}$. To this aim we need to verify the conditions of that theorem. Indeed, it is clear that each $\text{cl } \ell_t$ is a proper convex lsc function. Moreover, we have that for all $\varepsilon \in [0, \varepsilon_0]$

$$\begin{aligned} T'_\varepsilon(\theta) &:= \{t \in T \mid (\text{cl } \ell_t)(\theta) \geq (\text{cl } \ell)(\theta) - \varepsilon\} \\ &= \{t \in T \mid (\text{cl } \ell_t)(\theta) = \text{cl}(\max\{f_t(\theta), -2\varepsilon_0\}) = \max\{(\text{cl } f_t)(\theta), -2\varepsilon_0\} \geq -\varepsilon\} \\ &= \{t \in T \mid (\text{cl } f_t)(\theta) \geq -\varepsilon\} \subset T_\varepsilon(\theta). \end{aligned} \quad (45)$$

Then, since that for every $z \in \text{dom}(\text{cl } \ell) = \text{dom}(\text{cl } f)$ the function $t \mapsto (\text{cl } \ell_t)(z)$ is usc on $T'_{\varepsilon_0}(\theta) \subset T_{\varepsilon_0}(\theta)$, the set $T'_{\varepsilon_0}(\theta)$ is closed and so compact, by (45) and the current hypothesis (i). We apply Theorem 3.5 to get

$$\partial f(\theta) = \partial \ell(\theta) = \partial(\text{cl } \ell)(\theta) = \bigcap_{L \in \mathcal{F}(\theta)} \overline{\text{co}} \left\{ \bigcup_{t \in T'(\theta)} \partial((\text{cl } \ell_t) + \text{I}_{\overline{L \cap \text{dom}(\text{cl } \ell)}})(\theta) \right\}, \quad (46)$$

where (recall (45))

$$T'(\theta) = \{t \in T \mid (\text{cl } f_t)(\theta) = 0\} \subset \{t \in T \mid f_t(\theta) = 0\}. \quad (47)$$

Since, for every $L \in \mathcal{F}(\theta)$,

$$(\text{cl } \ell_t) + \text{I}_{\overline{L \cap \text{dom}(\text{cl } \ell)}} = \max \left\{ (\text{cl } f_t) + \text{I}_{\overline{L \cap \text{dom}(\text{cl } \ell)}}, -2\varepsilon_0 \right\},$$

for all $t \in T'(\theta)$ we have

$$\partial((\text{cl } \ell_t) + \text{I}_{\overline{L \cap \text{dom}(\text{cl } \ell)}})(\theta) = \partial((\text{cl } f_t) + \text{I}_{\overline{L \cap \text{dom}(\text{cl } \ell)}})(\theta) \subset \partial((\text{cl } f_t) + \text{I}_{\overline{L \cap \text{dom } f}})(\theta),$$

and similarly, $\partial((\text{cl } f_t) + \text{I}_{\overline{L \cap \text{dom } f}})(\theta) \subset \partial(f_t + \text{I}_{\overline{L \cap \text{dom } f}})(\theta)$.

Using successively the last two relations together with (47), relation (46) implies

$$\partial f(\theta) \subset \bigcap_{L \in \mathcal{F}(\theta)} \overline{\text{co}} \left\{ \bigcup_{t \in T'(\theta)} \partial((\text{cl } f_t) + \text{I}_{\overline{L \cap \text{dom } f}})(\theta) \right\} \quad (48)$$

$$\subset \bigcap_{L \in \mathcal{F}(\theta)} \overline{\text{co}} \left\{ \bigcup_{t \in T'(\theta)} \partial((\text{cl } f_t) + \text{I}_{\overline{L \cap \text{dom } f}})(\theta) \right\} \quad (49)$$

$$\subset \bigcap_{L \in \mathcal{F}(\theta)} \overline{\text{co}} \left\{ \bigcup_{t \in T(\theta)} \partial(f_t + \text{I}_{\overline{L \cap \text{dom } f}})(\theta) \right\},$$

which gives the desired inclusion “ $\supset$ ”.

□

Remark 3.7. As (48) shows, we have proved the following equivalent formula (under the assumptions of Theorem 3.6)

$$\partial f(x) = \bigcap_{L \in \mathcal{F}(x)} \overline{\text{co}} \left\{ \bigcup_{t \in T'(x)} \partial((\text{cl } f_t) + \text{I}_{\overline{L \cap \text{dom } f}})(x) \right\} \quad (50)$$

$$= \bigcap_{L \in \mathcal{F}(x)} \overline{\text{co}} \left\{ \bigcup_{t \in T'(x)} \partial((\text{cl } f_t) + \text{I}_{L \cap \text{dom } f})(x) \right\}, \quad (51)$$

where $T'(x) = \{t \in T \mid (\text{cl } f_t)(x) = f(x)\}$.

In order to characterize the subdifferential of the supremum function f by using only the functions f_t , rather than the augmented ones $f_t + \text{I}_{L \cap \text{dom } f}$, we provide another formula in the following theorem.

Theorem 3.8. *Let $\{f_t, t \in T\}$ and $x \in X$ be as in Theorem 3.6. Then*

$$\partial f(x) = \bigcap_{\substack{\varepsilon > 0 \\ L \in \mathcal{F}(x)}} \overline{\text{co}} \left\{ \bigcup_{t \in T(x)} \partial_\varepsilon f_t(x) + \text{N}_{L \cap \text{dom } f}(x) \right\}.$$

Proof. The inclusion " $\supset$ " is easy to prove. Indeed, we have that, for all $t \in T(x)$ and $\varepsilon > 0$, $\partial_\varepsilon f_t(x) \subset \partial_\varepsilon f(x)$, and so, for all $L \in \mathcal{F}(x)$,

$$\partial_\varepsilon f_t(x) + \text{N}_{L \cap \text{dom } f}(x) \subset \partial_\varepsilon f(x) + \text{N}_{L \cap \text{dom } f}(x) \subset \partial_\varepsilon (f + \text{I}_{L \cap \text{dom } f})(x) = \partial_\varepsilon (f + \text{I}_L)(x).$$

Hence,

$$\begin{aligned} \bigcap_{\varepsilon > 0, L \in \mathcal{F}(x)} \overline{\text{co}} \left\{ \bigcup_{t \in T(x)} \partial_\varepsilon f_t(x) + \text{N}_{L \cap \text{dom } f}(x) \right\} &\subset \bigcap_{L \in \mathcal{F}(x)} \bigcap_{\varepsilon > 0} \partial_\varepsilon (f + \text{I}_L)(x) \\ &= \bigcap_{L \in \mathcal{F}(x)} \partial(f + \text{I}_L)(x) = \partial f(x). \end{aligned}$$

Let us prove the inclusion " $\subset$ ". As in the previous theorems we suppose that $x = \theta$, $\partial f(\theta) \neq \emptyset$, and $f(\theta) = (\text{cl } f)(\theta) = 0$. Then, by (50),

$$\partial f(\theta) = \bigcap_{L \in \mathcal{F}(\theta)} \overline{\text{co}} \left\{ \bigcup_{t \in T'(\theta)} \partial((\text{cl } f_t) + \text{I}_{\overline{L \cap \text{dom } f}})(\theta) \right\}, \quad (52)$$

where $T'(\theta) = \{t \in T \mid (\text{cl } f_t)(\theta) = 0\}$. We are going to apply [5, Theorem 12] and to this purpose we see, from the one hand, that for every $t \in T$

$$\emptyset \neq \text{ri}(\overline{L \cap \text{dom } f}) = \text{ri}(L \cap \text{dom } f) \subset \text{dom } f_t \subset \text{dom}(\text{cl } f_t),$$

and this entails $\text{dom}(\text{cl } f_t) \cap \text{ri}(\text{dom } \text{I}_{\overline{L \cap \text{dom } f}}) \neq \emptyset$.

On the other hand, the restriction of the function $I_{\overline{L \cap \text{dom } f}}$ to the affine hull of its domain is continuous on $\text{ri}(\text{dom } I_{\overline{L \cap \text{dom } f}})$, and so, [5, Theorem 12] applies and yields

$$\partial((\text{cl } f_t) + I_{\overline{L \cap \text{dom } f}})(\theta) = \bigcap_{\varepsilon > 0} \text{cl} \left(\partial_\varepsilon(\text{cl } f_t)(\theta) + N_{\overline{L \cap \text{dom } f}}(\theta) \right). \quad (53)$$

If $t \in T'(\theta)$ we have $(\text{cl } f_t)(\theta) = f_t(\theta) = 0$, and for all $\varepsilon > 0$

$$\partial_\varepsilon(\text{cl } f_t)(\theta) + N_{\overline{L \cap \text{dom } f}}(\theta) \subset \partial_\varepsilon f_t(\theta) + N_{\overline{L \cap \text{dom } f}}(\theta) = \partial_\varepsilon f_t(\theta) + N_{L \cap \text{dom } f}(\theta),$$

so that (53) yields $\partial((\text{cl } f_t) + I_{\overline{L \cap \text{dom } f}})(\theta) \subset \text{cl}(\partial_\varepsilon f_t(\theta) + N_{L \cap \text{dom } f}(\theta))$.

Therefore, by (52),

$$\begin{aligned} \partial f(\theta) &\subset \bigcap_{L \in \mathcal{F}(\theta)} \overline{\text{co}} \left\{ \bigcup_{t \in T'(\theta)} \text{cl}(\partial_\varepsilon f_t(\theta) + N_{L \cap \text{dom } f}(\theta)) \right\} \\ &= \bigcap_{L \in \mathcal{F}(\theta)} \overline{\text{co}} \left\{ \bigcup_{t \in T'(\theta)} \partial_\varepsilon f_t(\theta) + N_{L \cap \text{dom } f}(\theta) \right\}. \end{aligned}$$

Since $T'(\theta) \subset T(\theta)$, the aimed inclusion follows by intersecting over $\varepsilon > 0$. $\square$

To avoid in Theorem 3.6 the intersection over sets $L \in \mathcal{F}(x)$, one has to require extra conditions relying either on the space X or on the function f . We start with the following result, whose first part is similar to the finite-dimensional-like result established in Theorem 3.1.

Corollary 3.9. *Let $\{f_t, t \in T\}$ and $x \in X$ be as in Theorem 3.6. Suppose that $\text{ri}(\text{dom } f) \neq \emptyset$ and $f|_{\text{aff}(\text{dom } f)}$ is continuous on $\text{ri}(\text{dom } f)$. Then*

$$\partial f(x) = \overline{\text{co}} \left\{ \bigcup_{t \in T(x)} \partial(f_t + I_{\text{dom } f})(x) \right\}.$$

Moreover, if the following two conditions hold for all $t \in T(x)$:

- (a) $\text{ri}(\text{dom } f_t) \cap \text{dom } f \neq \emptyset$,
- (b) $(f_t)|_{\text{aff}(\text{dom } f_t)}$ is continuous on $\text{ri}(\text{dom } f_t)$, then

$$\partial f(x) = \overline{\text{co}} \left\{ \bigcup_{t \in T(x)} \partial f_t(x) + N_{\text{dom } f}(x) \right\}.$$

Proof. The inclusion " $\supset$ " comes from Theorem 3.6, due to the following relation which is true for every $L \in \mathcal{F}(x)$ and $t \in T(x)$,

$$\partial(f_t + I_{\text{dom } f})(x) \subset \partial(f_t + I_{L \cap \text{dom } f})(x),$$

so that

$$\overline{\text{co}} \left\{ \bigcup_{t \in T(x)} \partial(f_t + I_{\text{dom } f})(x) \right\} \subset \bigcap_{L \in \mathcal{F}(x)} \overline{\text{co}} \left\{ \bigcup_{t \in T(x)} \partial(f_t + I_{L \cap \text{dom } f})(x) \right\} = \partial f(x).$$

To prove the inclusion " $\subset$ " we may suppose that $x \in \text{dom}(\partial f)$.

First, we assume that the functions $f_t + I_{L \cap \text{dom } f}$, $t \in T$, are proper. Take $t \in T(x)$. By the current assumption, we choose a point $x_0 \in \text{ri}(\text{dom } f)$. Fix an $L \in \mathcal{F}(x)$ such that $x_0 \in L$. If

$$q_t := f_t + I_{\text{dom } f},$$

then $\text{dom } q_t = \text{dom } f$ and

$$\text{ri}(\text{dom } q_t) \cap \text{ri}(\text{dom } I_L) = \text{ri}(\text{dom } f) \cap L \neq \emptyset,$$

as $x_0 \in \text{ri}(\text{dom } f) \cap L$. Moreover, since $q_t|_{\text{aff}(\text{dom } q_t)} \equiv q_t|_{\text{aff}(\text{dom } f)} \leq f|_{\text{aff}(\text{dom } f)}$, the (proper convex) function $q_t|_{\text{aff}(\text{dom } q_t)}$ is continuous on $\text{ri}(\text{dom } q_t)$. It is clear that $I_L|_{\text{aff}(\text{dom } I_L)} (\equiv 0|_L)$ is continuous on L . Then from [4, Theorem 5] it follows that

$$\begin{aligned} \partial(f_t + I_{L \cap \text{dom } f})(x) &= \partial(q_t + I_L)(x) \\ &= \text{cl}(\partial q_t(x) + \partial I_L(x)) = \text{cl}(\partial(f_t + I_{\text{dom } f})(x) + L^\perp). \end{aligned} \quad (54)$$

Therefore, by Theorem 3.6 we get

$$\begin{aligned} \partial f(x) &= \bigcap_{L \in \mathcal{F}(x)} \overline{\text{co}} \left\{ \bigcup_{t \in T(x)} \partial(f_t + I_{L \cap \text{dom } f})(x) \right\} \\ &\subset \bigcap_{L \in \mathcal{F}(x), x_0 \in L} \overline{\text{co}} \left\{ \bigcup_{t \in T(x)} \partial(f_t + I_{L \cap \text{dom } f})(x) \right\} \\ &= \bigcap_{L \in \mathcal{F}(x), x_0 \in L} \overline{\text{co}} \left\{ \bigcup_{t \in T(x)} \text{cl}(\partial(f_t + I_{\text{dom } f})(x) + L^\perp) \right\} \quad (\text{by (54)}) \\ &= \bigcap_{L \in \mathcal{F}(x), x_0 \in L} \overline{\text{co}} \left\{ \bigcup_{t \in T(x)} \partial(f_t + I_{\text{dom } f})(x) + L^\perp \right\}. \end{aligned}$$

Next, given a $U \in \mathcal{N}_{X^*}$ we choose an $F \in \mathcal{F}(x)$ such that $x_0 \in F$ and $F^\perp \subset U$. Hence, from the last inclusion we obtain

$$\begin{aligned} \partial f(x) &\subset \overline{\text{co}} \left\{ \bigcup_{t \in T(x)} \partial(f_t + I_{\text{dom } f})(x) + F^\perp \right\} \\ &\subset \text{co} \left\{ \bigcup_{t \in T(x)} \partial(f_t + I_{\text{dom } f})(x) \right\} + F^\perp + U \\ &\subset \text{co} \left\{ \bigcup_{t \in T(x)} \partial(f_t + I_{\text{dom } f})(x) \right\} + 2U, \end{aligned} \quad (55)$$

and the desired inclusion follows by intersecting over $U \in \mathcal{N}_{X^*}$.

Now, to deal with the case when not necessarily all the f_t 's, $t \in T$, are proper, we consider the functions

$$\hat{f}_t := \max\{f_t, f(x) - 2\varepsilon_0\}, \quad t \in T, \quad \hat{f} := \max \hat{f}_t = \max\{f, f(x) - 2\varepsilon_0\}.$$

By Lemma 3.2, and taking into account Remark 3.4, it follows that the proper functions $\hat{f}_t$, $t \in T$, satisfy the assumptions of the current theorem. So, from the previous part of the proof applied to the $\hat{f}_t$'s we get

$$\begin{aligned} \partial f(x) = \partial \hat{f}(x) &= \overline{\text{co}} \left\{ \bigcup_{\{t \in T \mid \hat{f}_t(x) = f(x)\}} \partial(\hat{f}_t + \text{I}_{\text{dom } \hat{f}})(x) \right\} \\ &= \overline{\text{co}} \left\{ \bigcup_{t \in T(x)} \partial(\hat{f}_t + \text{I}_{\text{dom } f})(x) \right\}. \end{aligned} \quad (56)$$

Observe that for $t \in T(x)$ such that $\partial(\hat{f}_t + \text{I}_{\text{dom } f})(x) \neq \emptyset$ it holds

$$\begin{aligned} f(x) &= (\hat{f}_t + \text{I}_{\text{dom } f})(x) = \text{cl}(\hat{f}_t + \text{I}_{\text{dom } f})(x) \\ &= \max\{\text{cl}(f_t + \text{I}_{\text{dom } f})(x), f(x) - 2\varepsilon_0\} \\ &= \text{cl}(f_t + \text{I}_{\text{dom } f})(x) \leq (f_t + \text{I}_{\text{dom } f})(x) \leq f(x), \end{aligned}$$

which implies that $\text{cl}(f_t + \text{I}_{\text{dom } f})(x) = (f_t + \text{I}_{\text{dom } f})(x)$ and so, the function $f_t + \text{I}_{\text{dom } f}$ is lsc at x . Hence, $\partial(\hat{f}_t + \text{I}_{\text{dom } f})(x) = \partial(\max\{f_t + \text{I}_{\text{dom } f}, f(x) - 2\varepsilon_0\})(x) = \partial(f_t + \text{I}_{\text{dom } f})(x)$, and (56) implies

$$\partial f(x) = \overline{\text{co}} \left\{ \bigcup_{t \in T(x)} \partial(f_t + \text{I}_{\text{dom } f})(x) \right\},$$

and the proof the first assertion is finished.

The last assertion also comes from [4, Theorem 5]. By the accessibility lemma, condition (a) and the properties of f imply that

$$\text{ri}(\text{dom } f_t) \cap \text{ri}(\text{dom } f) \neq \emptyset \quad \text{for all } t \in T(x),$$

and then

$$\begin{aligned} \partial f(x) &= \overline{\text{co}} \left\{ \bigcup_{t \in T(x)} \partial(f_t + \text{I}_{\text{dom } f})(x) \right\} \\ &= \overline{\text{co}} \left\{ \bigcup_{t \in T(x)} \text{cl}(\partial f_t(x) + \text{N}_{\text{dom } f}(x)) \right\} = \overline{\text{co}} \left\{ \bigcup_{t \in T(x)} \partial f_t(x) + \text{N}_{\text{dom } f}(x) \right\}. \quad \square \end{aligned}$$

By dropping conditions (a) and (b) in Corollary 3.9 we derive the following result.

Corollary 3.10. *Let $\{f_t, t \in T\}$ and $x \in X$ be as in Theorem 3.6. Suppose that $\text{ri}(\text{dom } f) \neq \emptyset$. Then*

$$\partial f(x) = \bigcap_{\varepsilon > 0} \overline{\text{co}} \left\{ \bigcup_{t \in T(x)} \partial_\varepsilon f_t(x) + N_{\text{dom } f}(x) \right\}.$$

Proof. The inclusion " $\supset$ " follows from Theorem 3.8 in view of the following relation, for $t \in T(x)$, $L \in \mathcal{F}(x)$, and $\varepsilon > 0$,

$$\partial_\varepsilon f_t(x) + N_{\text{dom } f}(x) \subset \partial_\varepsilon f_t(x) + N_{L \cap \text{dom } f}(x).$$

To prove the inclusion " $\subset$ " we suppose that $x \in \text{dom}(\partial f)$. By Theorem 3.8 we have that

$$\partial f(x) = \bigcap_{\substack{\varepsilon > 0, \\ L \in \mathcal{F}(x)}} \overline{\text{co}} \left\{ \bigcup_{t \in T(x)} \partial_\varepsilon f_t(x) + N_{L \cap \text{dom } f}(x) \right\}. \quad (57)$$

The current assumption ensures, thanks to [4, Theorem 5], that for every $L \in \mathcal{F}(x)$ such that $L \cap \text{ri}(\text{dom } f) \neq \emptyset$

$$N_{L \cap \text{dom } f}(x) = \text{cl}(N_{\text{dom } f}(x) + L^\perp).$$

Hence, by arguing as in (55), relation (57) leads to

$$\begin{aligned} \partial f(x) &\subset \bigcap_{\substack{\varepsilon > 0, \\ L \in \mathcal{F}(x), \\ L \cap \text{ri}(\text{dom } f) \neq \emptyset}} \overline{\text{co}} \left\{ \bigcup_{t \in T(x)} \partial_\varepsilon f_t(x) + \text{cl}(N_{\text{dom } f}(x) + L^\perp) \right\} \\ &\subset \bigcap_{\substack{\varepsilon > 0, \\ U \in \mathcal{N}_{X^*}}} \overline{\text{co}} \left\{ \bigcup_{t \in T(x)} \partial_\varepsilon f_t(x) + N_{\text{dom } f}(x) + U \right\} \\ &= \bigcap_{\varepsilon > 0} \overline{\text{co}} \left\{ \bigcup_{t \in T(x)} \partial_\varepsilon f_t(x) + N_{\text{dom } f}(x) \right\}. \quad \square \end{aligned}$$

The following lemma is used in Theorem 3.12.

Lemma 3.11. *Assume that $f = \sup_{t \in T} f_t$ is finite and continuous at some point. If $x \in X$ satisfies conditions (i) and (ii) of Theorem 3.6, then $\text{cl } f = \sup_{t \in T} (\text{cl } f_t)$ and*

$$\partial f(x) = \overline{\text{co}} \left\{ \bigcup_{t \in T'(x)} \partial(\text{cl } f_t)(x) + N_{\text{dom } f}(x) \right\},$$

where $T'(x) = \{t \in T \mid (\text{cl } f_t)(x) = f(x)\}$.

Proof. By Moreau-Rockafellar's Theorem (see, i.e., [15, Proposition 10.3]), the continuity assumption ensures that, for every $t \in T'(x)$ such that $\partial((\text{cl } f_t) + \text{I}_{L \cap \text{dom } f})(x) \neq \emptyset$, and every $L \in \mathcal{F}(x)$ such that $L \cap \text{int}(\text{dom } f) \neq \emptyset$,

$$\partial((\text{cl } f_t) + \text{I}_{L \cap \text{dom } f})(x) = \partial(\text{cl } f_t)(x) + \text{N}_{\text{dom } f}(x) + L^\perp,$$

taking into account that the continuity of f is inherited by the function $\text{cl } f_t$, whose properness is a consequence of $\partial((\text{cl } f_t) + \text{I}_{L \cap \text{dom } f})(x) \neq \emptyset$. Observe that the equality $\text{cl } f = \sup_{t \in T}(\text{cl } f_t)$ follows according to Remark 3.4. Then, using (51),

$$\begin{aligned} \partial f(x) &= \bigcap_{L \in \mathcal{F}(x)} \overline{\text{co}} \left\{ \bigcup_{t \in T'(x)} \partial((\text{cl } f_t) + \text{I}_{L \cap \text{dom } f})(x) \right\} \\ &\subset \bigcap_{\substack{L \in \mathcal{F}(x) \\ L \cap \text{int}(\text{dom } f) \neq \emptyset}} \overline{\text{co}} \left\{ \bigcup_{t \in T'(x)} \partial(\text{cl } f_t)(x) + \text{N}_{\text{dom } f}(x) + L^\perp \right\} \\ &\subset \bigcap_{U \in \mathcal{N}_{X^*}} \overline{\text{co}} \left\{ \bigcup_{t \in T'(x)} \partial(\text{cl } f_t)(x) + \text{N}_{\text{dom } f}(x) + U \right\} \\ &= \overline{\text{co}} \left\{ \bigcup_{t \in T'(x)} \partial(\text{cl } f_t)(x) + \text{N}_{\text{dom } f}(x) \right\}. \end{aligned}$$

Thus, as we can easily check that

$$\overline{\text{co}} \left\{ \bigcup_{t \in T'(x)} \partial(\text{cl } f_t)(x) + \text{N}_{\text{dom } f}(x) \right\} \subset \partial f(x),$$

the conclusion follows. $\square$

The following result simplifies Theorem 3.6 when the supremum function is continuous on the interior of its domain.

Theorem 3.12. *Assume that the family of convex functions $\{f_t, t \in T\}$ is such that f is finite and continuous at some point. Let $x \in X$ satisfy conditions (i) and (ii) of Theorem 3.6. Then*

$$\begin{aligned} \partial f(x) &= \text{N}_{\text{dom } f}(x) + \overline{\text{co}} \left\{ \bigcup_{t \in T(x)} \partial f_t(x) \right\} \\ &= \text{N}_{\text{dom } f}(x) + \text{co} \left\{ \bigcup_{t \in T(x)} \partial f_t(x) \right\} \quad (\text{when } X = \mathbb{R}^n). \end{aligned}$$

Proof. The inclusion " $\supset$ " is straightforward, and thus we only need to check the converse inclusion in the nontrivial case $\partial f(x) \neq \emptyset$; therefore, we may assume

that $x = \theta$ and $f(\theta) = 0$. From Lemma 3.11 we have that

$$\partial f(\theta) = \overline{\text{co}} \left\{ \bigcup_{t \in T'(\theta)} \partial(\text{cl } f_t)(\theta) + N_{\text{dom } f}(\theta) \right\},$$

where $T'(x) = \{t \in T \mid (\text{cl } f_t)(x) = f(x)\}$. By the current continuity assumption we choose $x_0 \in X$, $W \in \mathcal{N}_X$ and $m \geq 0$ such that for all $w \in W$

$$f(x_0 + w) \leq m; \quad (58)$$

hence,

$$\begin{aligned} \sigma_{\cup_{t \in T'(\theta)} \partial(\text{cl } f_t)(\theta)}(x_0 + w) &\leq \sup_{t \in T'(\theta)} ((\text{cl } f_t)(x_0 + w) - (\text{cl } f_t)(\theta)) \\ &\leq \sup_{t \in T'(\theta)} f_t(x_0 + w) \leq f(x_0 + w) \leq m, \end{aligned}$$

showing that the proper lsc convex function $\sigma_{\cup_{t \in T'(\theta)} \partial(\text{cl } f_t)(\theta)}$ is also continuous at x_0 . Thus, by applying once again Moreau-Rockafellar's Theorem we obtain, since $\sigma_{N_{\text{dom } f}(\theta)}(x_0) \leq 0$,

$$\begin{aligned} \overline{\text{co}} \left\{ \bigcup_{t \in T'(\theta)} \partial(\text{cl } f_t)(\theta) + N_{\text{dom } f}(\theta) \right\} &= \partial \left(\sigma_{\cup_{t \in T'(\theta)} \partial(\text{cl } f_t)(\theta) + N_{\text{dom } f}(\theta)} \right) (\theta) \\ &= \partial \left(\sigma_{\cup_{t \in T'(\theta)} \partial(\text{cl } f_t)(\theta)} + \sigma_{N_{\text{dom } f}(\theta)} \right) (\theta) \\ &= \partial \sigma_{\cup_{t \in T'(\theta)} \partial(\text{cl } f_t)(\theta)}(\theta) + \partial \sigma_{N_{\text{dom } f}(\theta)}(\theta). \end{aligned}$$

Using again the well-known formula of the subdifferential of the support function, we obtain

$$\begin{aligned} \overline{\text{co}} \left\{ \bigcup_{t \in T'(\theta)} \partial(\text{cl } f_t)(\theta) + N_{\text{dom } f}(\theta) \right\} &= \overline{\text{co}} \left\{ \bigcup_{t \in T'(\theta)} \partial(\text{cl } f_t)(\theta) \right\} + N_{\text{dom } f}(\theta) \quad (59) \\ &\subset \overline{\text{co}} \left\{ \bigcup_{t \in T(\theta)} \partial f_t(\theta) \right\} + N_{\text{dom } f}(\theta), \end{aligned}$$

and the first formula follows.

To prove the second statment of the theorem, when $X = \mathbb{R}^n$, observe by (59) that

$$\partial f(\theta) = \overline{\text{co}} \left\{ \bigcup_{t \in T'(\theta)} \partial(\text{cl } f_t)(\theta) \right\} + N_{\text{dom } f}(\theta),$$

and thus, we shall show that

$$\overline{\text{co}} \left\{ \bigcup_{t \in T'(\theta)} \partial(\text{cl } f_t)(\theta) \right\} \subset \text{co} \left\{ \bigcup_{t \in T'(\theta)} \partial(\text{cl } f_t)(\theta) \right\} + N_{\text{dom } f}(\theta).$$

Fix u in the left-hand side. By taking into account Caratheodory's Theorem we choose

$$(\lambda_{1,i}, \dots, \lambda_{n+1,i}) \in \Delta_{n+1} := \left\{ (\lambda_1, \dots, \lambda_{n+1}) \mid \lambda_i \geq 0, \sum_{i=1}^{n+1} \lambda_i = 1 \right\},$$

$t_{1,i}, \dots, t_{n+1,i} \in T'(\theta)$, and $y_{1,i} \in \partial(\text{cl } f_{t_{1,i}})(\theta)$, $\dots$, $y_{n+1,i} \in \partial(\text{cl } f_{t_{n+1,i}})(\theta)$, $i = 1, 2, \dots$, such that

$$y_i := \lambda_{1,i} y_{1,i} + \dots + \lambda_{n+1,i} y_{n+1,i} \rightarrow u. \quad (60)$$

We may assume that for all $k = 1, \dots, n+1$, there are subnets $(\lambda_{k,\alpha})_{\alpha \in \Upsilon}$ and $(t_{k,\alpha})_{\alpha \in \Upsilon}$ of $(\lambda_{k,i})_i$ and $(t_{k,i})_i$, respectively, such that $\lambda_{k,\alpha} \rightarrow \lambda_k \in [0, 1]$ and $t_{k,\alpha} \rightarrow t_k \in T'(\theta)$ (remember that $T'(\theta)$ is a closed subset of the compact set $T_{\varepsilon_0}(\theta)$). Fix $\alpha \in \Upsilon$ and $k \in \{1, \dots, n+1\}$. Then, using (58), for every $w \in W$

$$\begin{aligned} \langle \lambda_{k,\alpha} y_{k,\alpha}, x_0 + w \rangle &\leq \lambda_{k,\alpha} ((\text{cl } f_{t_{k,\alpha}})(x_0 + w) - (\text{cl } f_{t_{k,\alpha}})(\theta)) \\ &\leq \lambda_{k,\alpha} f(x_0 + w) \leq \lambda_{k,\alpha} m \leq m, \end{aligned} \quad (61)$$

and, taking $w = \theta$, $\langle \lambda_{k,\alpha} y_{k,\alpha}, x_0 \rangle \leq m$. Due to (60), implying that

$$\langle \lambda_{1,\alpha} y_{1,\alpha} + \dots + \lambda_{n+1,\alpha} y_{n+1,\alpha}, x_0 \rangle \rightarrow \langle u, x_0 \rangle,$$

we deduce that the net $(\langle \lambda_{k,\alpha} y_{k,\alpha}, x_0 \rangle)_{\alpha \in \Upsilon}$ is bounded. Consequently, (61) ensures that for some $r > 0$

$$\langle \lambda_{k,\alpha} y_{k,\alpha}, w \rangle \leq r \quad \text{for all } w \in W,$$

showing that $(\lambda_{k,\alpha} y_{k,\alpha})_{\alpha \in \Upsilon} \subset rW^\circ$. Without loss of generality, we may suppose that $(\lambda_{k,\alpha} y_{k,\alpha})_{\alpha \in \Upsilon}$ converges to some $y_k \in \mathbb{R}^n$.

Let

$$K_+ := \{k = 1, \dots, n+1 \mid \lambda_k > 0\} \quad \text{and} \quad K_0 := \{k = 1, \dots, n+1 \mid \lambda_k = 0\}.$$

If $k \in K_+$, from $y_{k,\alpha} \in \partial(\text{cl } f_{t_{k,\alpha}})(\theta)$ we obtain, for all $z \in \text{dom } f$,

$$\langle y_{k,\alpha}, z \rangle \leq (\text{cl } f_{t_{k,\alpha}})(z) - (\text{cl } f_{t_{k,\alpha}})(\theta) = (\text{cl } f_{t_{k,\alpha}})(z),$$

which, by assumption (ii) and after passing to the limit on $\alpha \in \Upsilon$, yields

$$\langle \lambda_k^{-1} y_k, z \rangle \leq \limsup_{\alpha \in \Upsilon} (\text{cl } f_{t_{k,\alpha}})(z) \leq (\text{cl } f_{t_k})(z) \leq f_{t_k}(z)$$

for all $z \in \text{dom}(\text{cl } f)$ ($\supset \text{dom } f$).

By taking into account Moreau-Rockafellar's Theorem, this shows that

$$\lambda_k^{-1}y_k \in \partial(f_{t_k} + I_{\text{dom } f})(\theta) = \partial f_{t_k}(\theta) + N_{\text{dom } f}(\theta). \quad (62)$$

If $k \in K_0$, for all $z \in \text{dom } f$ we have

$$\langle \lambda_{k,\alpha} y_{k,\alpha}, z \rangle \leq \lambda_{k,\alpha} ((\text{cl } f_{t_k,\alpha})(z) - (\text{cl } f_{t_k,\alpha})(\theta)) \leq \lambda_{k,\alpha} f(z),$$

and by taking the limit on $\alpha \in \Upsilon$ we obtain that $y_k \in N_{\text{dom } f}(x)$. This, together with (62) and (60), leads us to

$$u = \sum_{k \in K_+} \lambda_k (\lambda_k^{-1} y_k) + \sum_{k \in K_0} y_k \in \text{co} \left\{ \bigcup_{t \in T(x)} \partial f_t(\theta) \right\} + N_{\text{dom } f}(\theta),$$

as the inclusion " $\subset$ " follows. □

Finally, we give Valadier's formula under slightly weaker conditions.

Corollary 3.13. *Assume that the family of convex functions $\{f_t, t \in T\}$ is such that f is finite and continuous at $x \in X$. Suppose that for some $\varepsilon_0 > 0$:*

- (i) *the set $T_{\varepsilon_0}(x)$ is compact,*
- (ii) *the functions $t \mapsto f_t(z)$, $z \in \text{dom } f$, are usc on $T_{\varepsilon_0}(x)$. Then*

$$\begin{aligned} \partial f(x) &= \overline{\text{co}} \left\{ \bigcup_{t \in T(x)} \partial f_t(x) \right\} \\ &= \text{co} \left\{ \bigcup_{t \in T(x)} \partial f_t(x) \right\} \quad (\text{when } X = \mathbb{R}^n). \end{aligned}$$

Proof. Since the inclusion " $\supset$ " is immediate, we only need to show the converse one when $\partial f(x) \neq \emptyset$; hence, we may suppose that $x = \theta$ and $f(\theta) = 0$. We choose an open θ -neighborhood $U \subset X$ and an $m \geq 0$ such that

$$-2\varepsilon_0 < f(u) \leq m \quad \text{for all } u \in U.$$

We denote,

$$\tilde{f}_t := \max\{f_t, -2\varepsilon_0\} + I_U, \quad t \in T, \quad \text{and} \quad \tilde{f} := \sup_{t \in T} \tilde{f}_t = \max\{f, -2\varepsilon_0\} + I_U.$$

It is clear that the proper convex function $\tilde{f}$ is finite and continuous at θ , with $\tilde{f}(\theta) = 0$, and that

$$\{t \in T \mid \tilde{f}_t(\theta) \geq \tilde{f}(\theta) - \varepsilon\} = T_\varepsilon(\theta) \quad \text{for all } \varepsilon \in [0, \varepsilon_0]. \quad (63)$$

Also, from the inequalities

$$\tilde{f}_t(u) \leq \tilde{f}(u) \leq \max\{m, -2\varepsilon_0\} = m \quad \text{for all } t \in T \text{ and } u \in U,$$

we deduce that all the proper convex functions $\tilde{f}_t$, $t \in T$, and $\tilde{f}$ are continuous on U . So,

$$\tilde{f}_t(u) = (\text{cl } \tilde{f}_t)(u), \quad t \in T, \quad \text{and} \quad \tilde{f}(u) = (\text{cl } \tilde{f})(u) \quad \text{for all } u \in U.$$

Therefore, by assumption (ii), for every $z \in \text{dom}(\text{cl } \tilde{f})$ ($= \text{dom}(\text{cl } f) \cap U \subset U \subset \text{dom } f$) the function

$$t \mapsto (\text{cl } \tilde{f}_t)(z) = \tilde{f}_t(z) = \max\{f_t(z), -2\varepsilon_0\}$$

is usc on $T_{\varepsilon_0}(\theta)$. Consequently, by Theorem 3.12 applied to the family $\{\tilde{f}_t, t \in T\}$ we obtain (recall (63))

$$\partial f(\theta) = \partial \tilde{f}(\theta) = \overline{\text{co}} \left\{ \bigcup_{t \in T(\theta)} \partial \tilde{f}_t(\theta) \right\}. \quad (64)$$

Moreover, if $t \in T(\theta)$ is such that $\partial \tilde{f}_t(\theta) \neq \emptyset$, then $\tilde{f}_t$ is lsc at θ , and so using Remark 3.4 (and (63)),

$$\begin{aligned} 0 &= \tilde{f}(\theta) = \tilde{f}_t(\theta) = (\text{cl } \tilde{f}_t)(\theta) = (\text{cl}(\max\{f_t, -2\varepsilon_0\}))(\theta) \\ &= \max\{(\text{cl } f_t)(\theta), -2\varepsilon_0\} \leq \max\{f_t(\theta), -2\varepsilon_0\} \leq \max\{f(\theta), -2\varepsilon_0\} = 0. \end{aligned}$$

Hence, $(\text{cl } f_t)(\theta) = f_t(\theta) = 0$ and so f_t is lsc at θ . This implies that $\partial \tilde{f}_t(\theta) = \partial f_t(\theta)$, and relation (64) gives

$$\partial f(\theta) = \overline{\text{co}} \left\{ \bigcup_{t \in T(\theta)} \partial f_t(\theta) \right\}.$$

The second assertion is obtained in the same way. □

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