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On Linear Isometries on Strongly Regular Non-Archimedean Köthe Spaces

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We study when two strongly regular Köthe spaces $K(A)$ and $K(B)$ are isometrically isomorphic. Next we determine all linear isometries on a strongly regular Köthe space $K(A)$. Finally we prove that any linear isometry on a nuclear strongly regular Köthe space $K(A)$ is surjective. The most known and important examples of nuclear strongly regular Köthe spaces are the generalized power series spaces $D_f(a, r)$.

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1. Introduction

In this paper all linear spaces are over a non-Archimedean non-trivially valued field $\mathbb{K}$ which is complete under the metric induced by the valuation $|\cdot|: \mathbb{K} \to [0, \infty)$. Then for all $\alpha, \beta \in \mathbb{K}$ we have: $|\alpha| = 0$ if and only if $\alpha = 0$; $|\alpha\beta| = |\alpha||\beta|$; $|\alpha + \beta| \leq \max\{|\alpha|, |\beta|\}$ (the strong triangle inequality); $|\alpha + \beta| = \max\{|\alpha|, |\beta|\}$ if $|\alpha| \neq |\beta|$ (the isosceles triangle principle). By the strong triangle inequality, a series $\sum_{n=1}^{\infty} \alpha_n$ in $\mathbb{K}$ is convergent if and only if the sequence (α_n) is convergent to 0 in $\mathbb{K}$.

For fundamentals on locally convex Hausdorff spaces (lcs) and normed spaces we refer to [2, 3, 5, 6].

By a *Köthe space* we mean an infinite-dimensional Fréchet space with a Schauder basis and with a continuous norm.

An infinite matrix $A = (a_n^k)$ of positive real numbers is a *Köthe matrix* if $a_n^k \leq a_n^{k+1}$ for all $k, n \in \mathbb{N}$.

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The space $K(A) = \{(\alpha_n) \subset \mathbb{K} : \lim_n |\alpha_n| a_n^k = 0 \text{ for every } k \in \mathbb{N}\}$ with the base (p_k) of norms, where

$$p_k((\alpha_n)) = \max_n |\alpha_n| a_n^k, k \in \mathbb{N},$$

is a Köthe space. The sequence (e_j), where $e_j = (\delta_{j,n})$, is an unconditional Schauder basis in $K(A)$. Any Köthe space is isomorphic to the space $K(A)$ for some Köthe matrix A (see [1, Proposition 2.4]).

A Köthe space $K(A)$ is *nuclear* if and only if $A = (a_n^k)$ is nuclear i.e. for any $i \in \mathbb{N}$ there exists $j \in \mathbb{N}$ with $j > i$ such that $\lim_n a_n^i / a_n^j = 0$ ([1, Proposition 3.5]).

Let Γ be the family of all non-decreasing sequences $a = (a_n)$ of positive real numbers with $\lim_n a_n = \infty$. For $r \in (-\infty, \infty]$ we denote by Λ_r the family of all strictly increasing sequences $(r_k) \subset \mathbb{R}$ with $\lim_k r_k = r$ such that $r_k r_j > 0$ for all $k, j \in \mathbb{N}$. Let $a = (a_n) \in \Gamma, r \in (-\infty, \infty]$ and $(r_k) \in \Lambda_r$. Denote by $A(a, r)$ the Köthe space $K(A)$, where $A = (a_n^k) = (\exp(r_k a_n))$. It is said to be a *power series space* (*of finite type*, if $r < \infty$, and *of infinite type*, if $r = \infty$).

Denote by Φ_c the family of all strictly increasing odd functions $f : \mathbb{R} \to \mathbb{R}$ such that f is convex in $[0, \infty)$. Let $f \in \Phi_c, a = (a_n) \in \Gamma, r \in (-\infty, \infty]$ and $(r_k) \in \Lambda_r$. Denote by $D_f(a, r)$ the Köthe space $K(A)$, where $A = (a_n^k) = (\exp f(r_k a_n))$. It is said to be a *generalized power series space* (see [9, 10]). It is nuclear ([9, Proposition 2]).

Clearly, $D_f(a, r) = A(a, r)$, if $f : \mathbb{R} \to \mathbb{R}, f(x) = x$ and $r \in (-\infty, \infty], (r_k) \in \Lambda_r$.

The power series spaces $A(a, r)$ and the generalized power series spaces $D_f(a, r)$ are the most known and important examples of nuclear Köthe spaces (see [1, 7, 9, 10]).

In [8], we studied the linear isometries on the power series spaces. Let $a = (a_n), b = (b_n) \in \Gamma, r, s \in (-\infty, \infty]$ and $(r_k) \in \Lambda_r, (s_k) \in \Lambda_s$. We have shown that the power series spaces $A(a, r)$ and $A(b, s)$ are isometrically isomorphic if and only if (1) there exist $C, D \in \mathbb{R}$ such that $a_n = C b_n$ and $s_k = C r_k + D$ for all $n, k \in \mathbb{N}$; (2) for every $n \in \mathbb{N}$ there is $\psi_n \in \mathbb{K}$ with $|\psi_n| = \exp[-(D/C) a_n]$. In this case the linear map $P : A(a, r) \to A(b, s), (x_n) \to (\psi_n x_n)$ is an isometric isomorphism ([8, Theorem 3.1]).

Let $a = (a_n) \in \Gamma, r \in (-\infty, \infty]$ and $(r_k) \in \Lambda_r$. Let (N_s) be a partition of $\mathbb{N}$ into non-empty finite subsets such that $a_i = a_j$ for all $i, j \in N_s, s \in \mathbb{N}$ and $a_i < a_j$ for all $i \in N_s, j \in N_{s+1}, s \in \mathbb{N}$. We have proved that a continuous linear map $T : A(a, r) \to A(a, r)$ with $T e_j = \sum_{i=1}^{\infty} t_{i,j} e_i$, $j \in \mathbb{N}$, is an isometry if and only if

(1) $|t_{i,j}| \leq \exp[(a_j - a_i) r_1]$ when $a_i < a_j$, and $|t_{i,j}| \leq \exp[(a_j - a_i) r]$ when $a_i > a_j$;

(2) $\max_{i,j \in N_s} |t_{i,j}| = 1$ and $|\det[t_{i,j}]_{i,j \in N_s}| = 1$ for all $s \in \mathbb{N}$ ([8, Theorem 3.5].

Finally we have shown that for any power series space $A(a, r)$ every linear isometry $T : A(a, r) \to A(a, r)$ is surjective ([8, Corollary 3.10 and Theorem 3.12]).

A Köthe matrix $A = (a_n^k)$ is said to be *regular* if $a_n^{k+1} / a_n^k \leq a_{n+1}^{k+1} / a_{n+1}^k$ for all $k, n \in \mathbb{N}$; and *strictly regular* if $a_n^{k+1} / a_n^k < a_{n+1}^{k+1} / a_{n+1}^k$ for all $k, n \in \mathbb{N}$.

For non-empty subsets $X, Y \subset \mathbb{N}$ we write $X < Y$ if $x < y$ for all $x \in X, y \in Y$.

We will say that a Köthe matrix $A = (a_n^k)$ is *strongly regular* if there exists a partition (N_s) of $\mathbb{N}$ onto non-empty finite subsets such that (1) $a_i^k = a_j^k$ for all $i, j \in N_s$, $s, k \in \mathbb{N}$; (2) $a_i^{k+1}/a_i^k < a_j^{k+1}/a_j^k$ for all $i \in N_s$, $j \in N_{s+1}$, $s, k \in \mathbb{N}$; (3) $N_s < N_{s+1}$ for every $s \in \mathbb{N}$.

Clearly, any strictly regular Köthe matrix is strongly regular and any strongly regular Köthe matrix is regular.

We say that a Köthe space $K(A)$ is [*strongly*] *regular* if A is [strongly] regular. Any generalized power series space $D_f(a, r)$ is strongly regular (Proposition 3.2).

In this paper we study linear isometries on strongly regular Köthe spaces.

Let $A = (a_n^k)$ and $B = (b_n^k)$ be strongly regular Köthe matrices. The Köthe spaces $K(A)$ and $K(B)$ are isometrically isomorphic if and only if there exists a sequence $(\psi_n) \subset \mathbb{K}^*$ such that $a_n^k = |\psi_n| b_n^k$ for all $k, n \in \mathbb{N}$. In this case the linear map $P\colon K(A) \to K(B), x = (x_n) \to Px = (\psi_n x_n)$ is an isometric isomorphism (Theorem 3.4).

Let $A = (a_n^k)$ be a strongly regular Köthe matrix. Let (N_s) be a partition of $\mathbb{N}$ associated with A. We prove that a continuous linear map $T\colon K(A) \to K(A)$ with $Te_j = \sum_{i=1}^\infty t_{i,j} e_i, j \in \mathbb{N}$, is an isometry if and only if $|t_{i,j}| \le \inf_k a_j^k / a_i^k$ for all $i, j \in \mathbb{N}$ and $|\det[t_{i,j}]_{i,j \in N_s}| = 1$ for every $s \in \mathbb{N}$ (Theorem 3.8).

Finally we show that for any nuclear strongly regular Köthe matrix A, every linear isometry $T\colon K(A) \to K(A)$ is surjective (Theorem 3.14).

2. Preliminaries

Let $\mathbb{K}^* = \mathbb{K} \setminus \{0\}$. The *linear span* of a subset V of a linear space E is denoted by $\operatorname{lin} V$. By a *seminorm* on a linear space E we mean a function $p\colon E \to [0, \infty)$ such that $p(\alpha x) = |\alpha| p(x)$ for all $\alpha \in \mathbb{K}$, $x \in E$ and $p(x + y) \le \max\{p(x), p(y)\}$ for all $x, y \in E$ (the strong triangle inequality). Then $p(x + y) = \max\{p(x), p(y)\}$, if $x, y \in E$ with $p(x) \ne p(y)$ (the isosceles triangle principle). Indeed, if $p(x + y) < \max\{p(x), p(y)\}$ and $p(y) < p(x)$ (or $p(x) < p(y)$), then

$$\begin{aligned} p(x) &= p(x + y - y) \le \max\{p(x + y), p(-y)\} < p(x), \quad \text{or} \\ p(y) &= p(x + y - x) \le \max\{p(x + y), p(-x)\} < p(y), \end{aligned}$$

a contradiction.

A seminorm p on a linear space E is a *norm* if $\{x \in E : p(x) = 0\} = \{0\}$.

Remark 2.1. Let $A = (a_n^k)$ be a Köthe matrix and let $p \in [1, \infty)$. Let

$$\|(\alpha_n)\|_k = [\sum_{n=1}^\infty (|\alpha_n| a_n^k)^p]^{1/p} \quad \text{for all} \quad (\alpha_n) \in \mathbb{K}^\mathbb{N} \quad \text{and} \quad k \in \mathbb{N}.$$

Clearly, $\lambda^p(A) = \{(\alpha_n) \in \mathbb{K}^\mathbb{N} : \|(\alpha_n)\|_k < \infty \text{ for any } k \in \mathbb{N}\}$ is a linear subspace of $\mathbb{K}^\mathbb{N}$ but the functionals $\|\cdot\|_k : \lambda^p(A) \to [0, \infty)$, $(\alpha_n) \to \|(\alpha_n)\|_k, k \in \mathbb{N}$, are not seminorms on $\lambda^p(A)$. $\square$

A sequence (v_n) in a lcs E is a *Schauder basis* in E if each $x \in E$ can be written uniquely as $x = \sum_{n=1}^{\infty} \alpha_n v_n$ with $(\alpha_n) \subset \mathbb{K}$, and the coefficient functionals $f_n : E \to \mathbb{K}, x \to \alpha_n (n \in \mathbb{N})$ are continuous.

Let E and F be locally convex spaces. A linear map $T : E \to F$ is called an *isomorphism* if it is injective and surjective and the maps T, T^{-1} are continuous. If there exists an isomorphism $T : E \to F$, then we say that E is isomorphic to F. The family of all continuous linear maps from E to F will be denoted by $L(E, F)$.

The set of all continuous seminorms on a lcs E is denoted by $\mathcal{P}(E)$. A non-decreasing sequence (p_k) of continuous seminorms on a metrizable lcs E is a *base* in $\mathcal{P}(E)$ if for any $p \in \mathcal{P}(E)$ there are $C > 0$ and $k \in \mathbb{N}$ such that $p \leq C p_k$. A complete metrizable lcs is called a *Fréchet space*. By the strong triangle inequality of seminorms, a series $\sum_{n=1}^{\infty} x_n$ in a Fréchet space E is convergent if and only if the sequence (x_n) is convergent to 0 in E.

Let E and F be Fréchet spaces with fixed bases $(\|\cdot\|_k)$ and $(|||\cdot|||_k)$ in $\mathcal{P}(E)$ and $\mathcal{P}(F)$, respectively. A map $T : E \to F$ is an *isometry* if $|||Tx - Ty|||_k = \|x - y\|_k$ for all $x, y \in E$ and $k \in \mathbb{N}$; clearly, a linear map $T : E \to F$ is an isometry if and only if $|||Tx|||_k = \|x\|_k$ for all $x \in E$ and $k \in \mathbb{N}$. A linear map $T : E \to F$ is a *contraction* if $|||Tx|||_k \leq \|x\|_k$ for all $x \in E$ and $k \in \mathbb{N}$.

By [4, Corollary 1.7], we have the following

Proposition 2.2. *Let $m \in \mathbb{N}$. Equip the linear space $\mathbb{K}^m$ with the maximum norm. Let $T : \mathbb{K}^m \to \mathbb{K}^m$ be a linear map with $Te_j = \sum_{i=1}^{m} t_{i,j} e_i$ for $1 \leq j \leq m$. Then T is an isometry if and only if* $\max_{i,j} |t_{i,j}| = 1$ *and* $|\det[t_{i,j}]| = 1$.

3. Results

First we show that any generalized power series space $D_f(a, r)$ is strongly regular.

Lemma 3.1. *Let $f \in \Phi_c$ and $u_1, u_2, v_1, v_2 \in [0, \infty)$ with $u_1 < u_2$ and $v_1 < v_2$. Then*

$$f(u_1 v_2) + f(u_2 v_1) < f(u_1 v_1) + f(u_2 v_2). \tag{1}$$

Proof. If $u_1 = 0$ or $v_1 = 0$, then (1) is obvious. Assume that $u_1 > 0$ and $v_1 > 0$. Put $t = (u_1 v_2 - u_1 v_1)/(u_2 v_2 - u_1 v_1)$. Then $t \in (0, 1)$, $t u_2 v_2 + (1 - t) u_1 v_1 = u_1 v_2$ and $t u_1 v_1 + (1 - t) u_2 v_2 > u_2 v_1$. Indeed, $t u_1 v_1 + (1 - t) u_2 v_2 =$

$$\begin{aligned}
&= \frac{(u_1 v_2 - u_1 v_1) u_1 v_1}{u_2 v_2 - u_1 v_1} + \frac{(u_2 v_2 - u_1 v_2) u_2 v_2}{u_2 v_2 - u_1 v_1} = \frac{u_1^2 v_1 v_2 - u_1^2 v_1^2 + u_2^2 v_2^2 - u_1 u_2 v_2^2}{u_2 v_2 - u_1 v_1} \\
&= \frac{-u_1 v_2 (u_2 v_2 - u_1 v_1) + (u_2 v_2 - u_1 v_1)(u_2 v_2 + u_1 v_1)}{u_2 v_2 - u_1 v_1} \\
&= u_2 v_2 + u_1 v_1 - u_1 v_2 = (u_2 - u_1)(v_2 - v_1) + u_2 v_1 > u_2 v_1.
\end{aligned}$$

Thus $f(u_1 v_2) + f(u_2 v_1) < f(t u_2 v_2 + (1 - t) u_1 v_1) + f(t u_1 v_1 + (1 - t) u_2 v_2) \leq t f(u_2 v_2) + (1 - t) f(u_1 v_1) + t f(u_1 v_1) + (1 - t) f(u_2 v_2) = f(u_1 v_1) + f(u_2 v_2)$. $\square$

Proposition 3.2. *Any generalized power series space $D_f(a,r)$ is strongly regular.*

Proof. Let $f \in \Phi_c$, $a = (a_n) \in \Gamma$, $r \in (-\infty, \infty]$ and $(r_k) \in \Lambda_r$. Then we have $D_f(a,r) = K(A)$ where $A = (a_n^k) = (\exp f(r_k a_n))$.

Let (a_{n_s}) be a strictly increasing subsequence of (a_n) with $\{a_{n_s} : s \in \mathbb{N}\} = \{a_n : n \in \mathbb{N}\}$. Let $N_s = \{m \in \mathbb{N} : a_m = a_{n_s}\}$ for $s \in \mathbb{N}$. Then (N_s) is a partition of $\mathbb{N}$ onto non-empty finite subsets with $N_s < N_{s+1}$ for every $s \in \mathbb{N}$.

Clearly, $a_n^k = a_m^k$ if $k \in \mathbb{N}$ and $n, m \in N_s$, $s \in \mathbb{N}$.

If $k \in \mathbb{N}$ and $n \in N_s$, $m \in N_{s+1}$ for some $s \in \mathbb{N}$, then using Lemma 3.1 we get

$$f(r_k a_m) + f(r_{k+1} a_n) < f(r_k a_n) + f(r_{k+1} a_m), \text{ if } r > 0, \text{ and}$$

$$f(-r_{k+1} a_m) + f(-r_k a_n) < f(-r_{k+1} a_n) + f(-r_k a_m) \text{ if } r \leq 0.$$

Hence $f(r_{k+1} a_n) - f(r_k a_n) < f(r_{k+1} a_m) - f(r_k a_m)$, so $a_n^{k+1}/a_n^k < a_m^{k+1}/a_m^k$.

Thus $a_n^{k+1}/a_n^k < a_m^{k+1}/a_m^k$ if $k \in \mathbb{N}$ and $n \in N_s$, $m \in N_{s+1}$, $s \in \mathbb{N}$.

It follows that A is strongly regular, so $D_f(a,r)$ is strongly regular. □

Corollary 3.3. *Any generalized power series space $D_f(a,r)$ is regular. It is strictly regular if and only if the sequence $a = (a_n)$ is strictly increasing.*

Now we show when two strongly regular Köthe spaces $K(A)$ and $K(B)$ are isometrically isomorphic.

Theorem 3.4. *Let $A = (a_n^k)$ and $B = (b_n^k)$ be strongly regular Köthe matrices. The Köthe spaces $K(A)$ and $K(B)$ are isometrically isomorphic if and only if there exists a sequence $(\psi_n) \subset \mathbb{K}^*$ such that $a_n^k = |\psi_n| b_n^k$ for all $k, n \in \mathbb{N}$. In this case the linear map $P\colon K(A) \to K(B)$, $x = (x_n) \to Px = (\psi_n x_n)$ is an isometric isomorphism.*

Proof. ($\Rightarrow$) Let (N_r) and (M_r) be partitions of $\mathbb{N}$ associated with A and B, respectively. Let $v, w\colon \mathbb{N} \to \mathbb{N}$ be functions such that $j \in N_{v(j)}$ and $i \in M_{w(i)}$ for all $i, j \in \mathbb{N}$. Let $T\colon K(A) \to K(B)$ and $S\colon K(B) \to K(A)$ be linear isometries. Let $(t_{i,j}), (s_{l,i}) \subset \mathbb{K}$ with $Te_j = \sum_{i=1}^{\infty} t_{i,j} e_i$ and $Se_i = \sum_{l=1}^{\infty} s_{l,i} e_l$ for all $i, j \in \mathbb{N}$. For all $j, k \in \mathbb{N}$ we have $\|Te_j\|_k = \|e_j\|_k$, so $\max_i |t_{i,j}| b_i^k = a_j^k$. Let $k \in \mathbb{N}$ with $k > 1$. Let $j \in \mathbb{N}$. For some $i \in \mathbb{N}$ we have $|t_{i,j}| b_i^k = a_j^k$; clearly $|t_{i,j}| b_i^{k+1} \leq a_j^{k+1}$ and $|t_{i,j}| b_i^{k-1} \leq a_j^{k-1}$. Hence we get $a_j^k / b_i^k \leq a_j^{k+1}/b_i^{k+1}$ and $a_j^k / b_i^k \leq a_j^{k-1}/b_i^{k-1}$. Thus

$$\forall j \; \exists i : \frac{b_i^{k+1}}{b_i^k} \leq \frac{a_j^{k+1}}{a_j^k} \quad \text{and} \quad \frac{a_j^k}{a_j^{k-1}} \leq \frac{b_i^k}{b_i^{k-1}}. \tag{2}$$

Similarly, for all $i \in \mathbb{N}$ we have $\|Se_i\|_k = \|e_i\|_k$, so $\max_l |s_{l,i}| a_l^k = b_i^k$. Let $i \in \mathbb{N}$. For some $l \in \mathbb{N}$ we have $|s_{l,i}| a_l^k = b_i^k$; clearly, $|s_{l,i}| a_l^{k+1} \leq b_i^{k+1}$ and $|s_{l,i}| a_l^{k-1} \leq b_i^{k-1}$. Hence we get $b_i^k / a_l^k \leq b_i^{k+1}/a_l^{k+1}$ and $b_i^k / a_l^k \leq b_i^{k-1}/a_l^{k-1}$. Thus

$$\forall i \; \exists l : \frac{a_l^{k+1}}{a_l^k} \leq \frac{b_i^{k+1}}{b_i^k} \quad \text{and} \quad \frac{b_i^k}{b_i^{k-1}} \leq \frac{a_l^k}{a_l^{k-1}}. \tag{3}$$

Let $j \in \mathbb{N}$. By (2) and (3) there exists $l \in \mathbb{N}$ such that

$$\frac{a_l^{k+1}}{a_l^k} \leq \frac{a_j^{k+1}}{a_j^k} \quad \text{and} \quad \frac{a_j^k}{a_j^{k-1}} \leq \frac{a_l^k}{a_l^{k-1}}$$

Hence $v(l) \leq v(j)$ and $v(j) \leq v(l)$, so $v(j) = v(l)$ and $a_l^t = a_j^t$ for any $t \in \mathbb{N}$. Thus using (2) and (3) we get

$$\forall j\ \exists i : \frac{a_j^{k+1}}{a_j^k} = \frac{b_i^{k+1}}{b_i^k} \quad \text{and} \quad \frac{a_j^k}{a_j^{k-1}} = \frac{b_i^k}{b_i^{k-1}}. \tag{4}$$

Clearly, (2) is equivalent to

$$\forall l\ \exists j : \frac{b_j^{k+1}}{b_j^k} \leq \frac{a_l^{k+1}}{a_l^k} \quad \text{and} \quad \frac{a_l^k}{a_l^{k-1}} \leq \frac{b_j^k}{b_j^{k-1}}. \tag{5}$$

Let $i \in \mathbb{N}$. By (3) and (5) there exists $j \in \mathbb{N}$ such that

$$\frac{b_j^{k+1}}{b_j^k} \leq \frac{b_i^{k+1}}{b_i^k} \quad \text{and} \quad \frac{b_i^k}{b_i^{k-1}} \leq \frac{b_j^k}{b_j^{k-1}}\,.$$

Hence $w(j) \leq w(i)$ and $w(i) \leq w(j)$, so $w(j) = w(i)$ and $b_j^t = b_i^t$ for any $t \in \mathbb{N}$. Using (3) and (5) we get

$$\forall i\ \exists l : \frac{b_i^{k+1}}{b_i^k} = \frac{a_l^{k+1}}{a_l^k} \quad \text{and} \quad \frac{b_i^k}{b_i^{k-1}} = \frac{a_l^k}{a_l^{k-1}}. \tag{6}$$

By (4) and (6) we get $\{a_j^{k+1}/a_j^k : j \in \mathbb{N}\} = \{b_l^{k+1}/b_l^k : l \in \mathbb{N}\}$ for every $k \in \mathbb{N}$. If $(j_r) \in \prod_{r=1}^\infty N_r$ and $(l_r) \in \prod_{r=1}^\infty M_r$, then

$$\frac{a_{j_r}^{k+1}}{a_{j_r}^k} = \frac{b_{l_r}^{k+1}}{b_{l_r}^k} \quad \text{for all} \quad r, k \in \mathbb{N}, \tag{7}$$

since the sequences $(a_{j_r}^{k+1}/a_{j_r}^k)_{r=1}^\infty$ and $(b_{l_r}^{k+1}/b_{l_r}^k)_{r=1}^\infty$ are strictly increasing and

$$\left\{\frac{a_{j_r}^{k+1}}{a_{j_r}^k} : r \in \mathbb{N}\right\} = \left\{\frac{a_j^{k+1}}{a_j^k} : j \in \mathbb{N}\right\} = \left\{\frac{b_l^{k+1}}{b_l^k} : l \in \mathbb{N}\right\} = \left\{\frac{b_{l_r}^{k+1}}{b_{l_r}^k} : r \in \mathbb{N}\right\}.$$

Now we shall prove that $|N_r| = |M_r|$ for any $r \in \mathbb{N}$. Let $r \in \mathbb{N}$. Let $j_r \in N_r$. Let $(\varphi_j)_{j \in N_r} \subset \mathbb{K}$ with $\max_{j \in N_r} |\varphi_j| > 0$. For $k \in \mathbb{N}$ we have

$$\max_{j \in N_r} |\varphi_j| a_{j_r}^k = \max_{j \in N_r} |\varphi_j| a_j^k = \|\sum_{j \in N_r} \varphi_j e_j\|_k = \|T(\sum_{j \in N_r} \varphi_j e_j)\|_k = \|\sum_{j \in N_r} \varphi_j T e_j\|_k$$

$$= \|\sum_{j \in N_r} \varphi_j \sum_{i=1}^\infty t_{i,j} e_i\|_k = \|\sum_{i=1}^\infty (\sum_{j \in N_r} t_{i,j}\varphi_j) e_i\|_k = \max_i |\sum_{j \in N_r} t_{i,j}\varphi_j| b_i^k,$$

so $$\max_i |\sum_{j\in N_r} t_{i,j}\varphi_j| \frac{b_i^k}{a_{j_r}^k} = \max_{j\in N_r} |\varphi_j| \ \text{ for every } k\in\mathbb{N}. \tag{8}$$

Let $k\in\mathbb{N}$ with $k>1$. For some $i_r\in\mathbb{N}$ we have

$$|\sum_{j\in N_r} t_{i_r,j}\varphi_j| \frac{b_{i_r}^k}{a_{j_r}^k} = \max_{j\in N_r} |\varphi_j|; \tag{9}$$

clearly $$|\textstyle\sum_{j\in N_r} t_{i_r,j}\varphi_j| b_{i_r}^{k+1}/a_{j_r}^{k+1} \le \max_{j\in N_r}|\varphi_j|$$

and $$|\textstyle\sum_{j\in N_r} t_{i_r,j}\varphi_j| b_{i_r}^{k-1}/a_{j_r}^{k-1} \le \max_{j\in N_r}|\varphi_j|.$$

Hence $b_{i_r}^{k+1}/a_{j_r}^{k+1} \le b_{i_r}^k/a_{j_r}^k$ and $b_{i_r}^{k-1}/a_{j_r}^{k-1} \le b_{i_r}^k/a_{j_r}^k$, so $b_{i_r}^{k+1}/b_{i_r}^k \le a_{j_r}^{k+1}/a_{j_r}^k$ and $a_{j_r}^k/a_{j_r}^{k-1} \le b_{i_r}^k/b_{i_r}^{k-1}$. Therefore, using (7), we infer that $w(i_r)\le r$ and $r\le w(i_r)$, so $w(i_r)=r$. Thus $i_r\in M_r$ and

$$\frac{b_{i_r}^{k+1}}{b_{i_r}^k} = \frac{a_{j_r}^{k+1}}{a_{j_r}^k} \quad\text{and}\quad \frac{a_{j_r}^k}{a_{j_r}^{k-1}} = \frac{b_{i_r}^k}{b_{i_r}^{k-1}},$$

so, by (8),
$$\frac{a_{j_r}^{k+1}}{b_{i_r}^{k+1}} = \frac{a_{j_r}^k}{b_{i_r}^k} = \frac{a_{j_r}^{k-1}}{b_{i_r}^{k-1}} \in |\mathbb{K}^*|. \tag{10}$$

The linear map $P_r : \mathbb{K}^{N_r} \to \mathbb{K}^{M_r}$, $(\varphi_j)_{j\in N_r} \to (\sum_{j\in N_r} t_{i,j}\varphi_j)_{i\in M_r}$ is injective since

$$\begin{aligned}\|P_r\varphi\| &= \left(\max_{i\in M_r}|\sum_{j\in N_r} t_{i,j}\varphi_j|\frac{b_i^k}{a_{j_r}^k}\right)\frac{a_{j_r}^k}{b_{i_r}^k} = \left(\max_i|\sum_{j\in N_r} t_{i,j}\varphi_j|\frac{b_i^k}{a_{j_r}^k}\right)\frac{a_{j_r}^k}{b_{i_r}^k}\\ &= (\max_{j\in N_r}|\varphi_j|)\frac{a_{j_r}^k}{b_{i_r}^k} = \|\varphi\|\frac{a_{j_r}^k}{b_{i_r}^k}\end{aligned}$$

for any $\varphi\in\mathbb{K}^{N_r}$. Thus $|N_r|\le|M_r|$.

Similarly, using the linear isometry $S\colon K(B)\to K(A)$ we prove that the linear map
$$Q_r\colon \mathbb{K}^{M_r}\to\mathbb{K}^{N_r},\quad (\psi_i)_{i\in M_r} \to (\sum_{i\in M_r} s_{j,i}\psi_i)_{j\in N_r}$$

is injective. Thus $|M_r|\le|N_r|$. We have shown that $|N_r|=|M_r|$ for every $r\in\mathbb{N}$. Since $N_r<N_{r+1}$ and $M_r<M_{r+1}$ for any $r\in\mathbb{N}$, we infer that $N_r=M_r$ for any $r\in\mathbb{N}$. Let $r\in\mathbb{N}$ and $j\in N_r$. We have

$$\frac{a_j^{k+1}}{b_j^{k+1}} = \frac{a_{j_r}^{k+1}}{b_{i_r}^{k+1}} = \frac{a_{j_r}^k}{b_{i_r}^k} = \frac{a_j^k}{b_j^k} \ \text{ for any } k\in\mathbb{N},$$

so the sequence $(a_j^k/b_j^k)_{k=1}^\infty$ is constant. By (10) there exists $D_r \in \mathbb{K}^*$ such that $a_j^k/b_j^k = |D_r|$ for any $k \in \mathbb{N}$. Let $\psi_j = D_r$ for $j \in N_r, r \in \mathbb{N}$. Then $a_n^k = |\psi_n| b_n^k$ for all $k, n \in \mathbb{N}$.

($\Leftarrow$) Let $x = (x_n) \in K(A)$. Then $|x_n||\psi_n|b_n^k = |x_n|a_n^k$ for all $k, n \in \mathbb{N}$, so $\psi_n x_n e_n \to 0$ in $K(B)$. Thus the series $\sum_{n=1}^\infty \psi_n x_n e_n$ is convergent in $K(B)$ for any $x = (x_n) \in K(A)$. It follows that the operator

$$P\colon K(A) \to K(B), \quad x = (x_n) \to Px = \sum_{n=1}^\infty \psi_n x_n e_n$$

is well defined. Clearly, P is a linear isometry. Thus $P(K(A))$ is closed in $K(B)$. Since $\operatorname{lin}\{e_n : n \in \mathbb{N}\} \subset P(K(A))$ we infer that $P(K(A)) = K(B)$. Thus P is an isometric isomorphism, so $K(A)$ and $K(B)$ are isometrically isomorphic. □

Corollary 3.5. *Let $f, g \in \Phi_c$, $a = (a_n)$, $b = (b_n) \in \Gamma$, $r, s \in (-\infty, \infty]$ and $(r_k) \in \Lambda_r$, $(s_k) \in \Lambda_s$. The generalized power series spaces $D_f(a, r)$ and $D_g(b, s)$ are isometrically isomorphic if and only if there exists a sequence $(\psi_n) \subset \mathbb{K}^*$ such that $\ln|\psi_n| = f(r_k a_n) - g(s_k b_n)$ for all $k, n \in \mathbb{N}$.*

Hence for power series spaces we get the following (see [8, Theorem 3.1])

Corollary 3.6. *Let $a = (a_n)$, $b = (b_n) \in \Gamma$, $r, s \in (-\infty, \infty]$ and $(r_k) \in \Lambda_r$, $(s_k) \in \Lambda_s$. The power series spaces $A(a, r)$ and $A(b, s)$ are isometrically isomorphic if and only if there exist a sequence $(\psi_n) \subset \mathbb{K}^*$ and constants $C, D \in \mathbb{R}$ such that $a_n = Cb_n, s_k = Cr_k + D$ and $\ln|\psi_n| = -Db_n$ for all $k, n \in \mathbb{N}$.*

Proof. By Corollary 3.5, the power series spaces $A(a, r)$ and $A(b, s)$ are isometrically isomorphic if and only if

$$\exists\, (\psi_n) \subset \mathbb{K}^* \ \forall k, n \in \mathbb{N} : r_k a_n = \ln|\psi_n| + s_k b_n. \tag{11}$$

Assume that (11) holds. Put $C = (s_2 - s_1)/(r_2 - r_1)$ and $D = s_1 - Cr_1$. We have $(r_2 - r_1)a_n = (s_2 - s_1)b_n$ for $n \in \mathbb{N}$. Thus $a_n = Cb_n$ for $n \in \mathbb{N}$. Hence

$$\ln|\psi_n| = r_k a_n - s_k b_n = (Cr_k - s_k)b_n \quad \text{for all } k, n \in \mathbb{N},$$

so $\ln|\psi_n| = -Db_n$ and $Cr_k - s_k = -D$ for all $n, k \in \mathbb{N}$.

If there exist a sequence $(\psi_n) \subset \mathbb{K}^*$ and constants $C, D \in \mathbb{R}$ such that $a_n = Cb_n$, $s_k = Cr_k + D$ and $\ln|\psi_n| = -Db_n$ for all $k, n \in \mathbb{N}$, then (11) is obvious. □

As a consequence of Theorem 3.4 and its proof we get the following.

Proposition 3.7. *Let A and B be strongly regular Köthe matrices. If there exist linear isometries $T\colon K(A) \to K(B)$ and $S\colon K(B) \to K(A)$ then the Köthe spaces $K(A)$ and $K(B)$ are isometrically isomorphic.*

Now we determine all linear isometries on a strongly regular Köthe space $K(A)$.

Theorem 3.8. *Let $A = (a_n^k)$ be a strongly regular Köthe matrix. Let (N_s) be the partition of $\mathbb{N}$ associated with A. Let $T\colon K(A) \to K(A)$ be a continuous linear map and let $Te_j = \sum_{i=1}^{\infty} t_{i,j} e_i$ for $j \in \mathbb{N}$. Then T is an isometry if and only if*

$$|t_{i,j}| \leq \inf_k \frac{a_j^k}{a_i^k} \quad \textit{for all } i,j \in \mathbb{N} \textit{ and } \quad |\det[t_{i,j}]_{i,j\in N_s}| = 1 \textit{ for every } s \in \mathbb{N}. \tag{12}$$

Proof. $(\Rightarrow)$ For all $j,k \in \mathbb{N}$ we have $\|Te_j\|_k = \|e_j\|_k$, so

$$\max_i |t_{i,j}| a_i^k = a_j^k \text{ for all } j,k \in \mathbb{N}. \tag{13}$$

Hence $|t_{i,j}| \leq a_j^k / a_i^k$ for all $i,j,k \in \mathbb{N}$, so $|t_{i,j}| \leq \inf_k a_j^k / a_i^k$ for all $i,j \in \mathbb{N}$ and $|t_{i,j}| \leq 1$ for all $i,j \in N_s$, $s \in \mathbb{N}$.

Let $s \in \mathbb{N}$, $j_s \in N_s$ and $(\beta_j)_{j\in N_s} \subset \mathbb{K}$ with $\max_{j\in N_s} |\beta_j| > 0$. Then we have

$$\begin{aligned}(\max_{j\in N_s} |\beta_j|) a_{j_s}^k &= \max_{j\in N_s} |\beta_j| a_j^k = \| \sum_{j\in N_s} \beta_j e_j \|_k = \|T(\sum_{j\in N_s} \beta_j e_j)\|_k = \| \sum_{j\in N_s} \beta_j T e_j \|_k \\ &= \| \sum_{j\in N_s} \beta_j \sum_{i=1}^{\infty} t_{i,j} e_i \|_k = \| \sum_{i=1}^{\infty} (\sum_{j\in N_s} \beta_j t_{i,j}) e_i \|_k = \max_i | \sum_{j\in N_s} \beta_j t_{i,j} | a_i^k\end{aligned}$$

for any $k \in \mathbb{N}$. Thus

$$\max_i | \sum_{j\in N_s} \beta_j t_{i,j} | a_i^k / a_{j_s}^k = \max_{j\in N_s} |\beta_j| \quad \text{for } k \in \mathbb{N}. \tag{14}$$

Let $k > 1$. For some $i_s \in \mathbb{N}$ we have

$$| \sum_{j\in N_s} \beta_j t_{i_s,j} | \frac{a_{i_s}^k}{a_{j_s}^k} = \max_{j\in N_s} |\beta_j|; \tag{15}$$

clearly

$$| \sum_{j\in N_s} \beta_j t_{i_s,j} | a_{i_s}^{k+1} / a_{j_s}^{k+1} \leq \max_{j\in N_s} |\beta_j| \quad \text{and} \quad | \sum_{j\in N_s} \beta_j t_{i_s,j} | a_{i_s}^{k-1} / a_{j_s}^{k-1} \leq \max_{j\in N_s} |\beta_j|.$$

Hence
$$\frac{a_{i_s}^{k+1}}{a_{j_s}^{k+1}} \leq \frac{a_{i_s}^k}{a_{j_s}^k} \quad \text{and} \quad \frac{a_{i_s}^{k-1}}{a_{j_s}^{k-1}} \leq \frac{a_{i_s}^k}{a_{j_s}^k},$$

so
$$\frac{a_{i_s}^{k+1}}{a_{i_s}^k} \leq \frac{a_{j_s}^{k+1}}{a_{j_s}^k} \quad \text{and} \quad \frac{a_{j_s}^k}{a_{j_s}^{k-1}} \leq \frac{a_{i_s}^k}{a_{i_s}^{k-1}}.$$

Thus $v(i_s) \leq v(j_s)$ and $v(j_s) \leq v(i_s)$, so $v(i_s) = v(j_s)$. Therefore $i_s \in N_s$ and $a_{i_s}^t = a_{j_s}^t$ for any $t \in \mathbb{N}$. By (15) we have $| \sum_{j\in N_s} \beta_j t_{i_s,j} | = \max_{j\in N_s} |\beta_j|$. Hence $\max_{i\in N_s} | \sum_{j\in N_s} \beta_j t_{i,j} | = \max_{j\in N_s} |\beta_j|$. Thus the following map

$$S\colon \mathbb{K}^{N_s} \to \mathbb{K}^{N_s}, (\beta_j)_{j\in N_s} \to (\sum_{j\in N_s} \beta_j t_{i,j})_{i\in N_s}$$

is a linear isometry. By Proposition 2.2, $|\det[t_{i,j}]_{i,j\in N_s}| = 1$ and $\max_{i,j\in N_s} |t_{i,j}| = 1$.

($\Leftarrow$) Let $x = (x_j) \in K(A)$ and $k \in \mathbb{N}$. Let $V_m = \bigcup_{p=1}^m N_p$ for $m \in \mathbb{N}$. Clearly $x = \lim_m \sum_{j\in V_m} x_j e_j$, so $\|x\|_k = \lim_m \|\sum_{j\in V_m} x_j e_j\|_k$ and $\|Tx\|_k = \lim_m \|T(\sum_{j\in V_m} x_j e_j)\|_k$. Thus to prove that $\|Tx\|_k = \|x\|_k$ it is enough to show that $\|T(\sum_{j\in V_m} x_j e_j)\|_k = \|\sum_{j\in V_m} x_j e_j\|_k$ for any $m \in \mathbb{N}$. Let $m \in \mathbb{N}$. We have

$$T(\textstyle\sum_{j\in V_m} x_j e_j) = \sum_{j\in V_m} x_j T e_j = \sum_{j\in V_m} x_j \sum_{i=1}^\infty t_{i,j} e_i = \sum_{i=1}^\infty (\sum_{j\in V_m} x_j t_{i,j}) e_i.$$

Define $L := \|T(\sum_{j\in V_m} x_j e_j)\|_k = \max_i |\sum_{j\in V_m} x_j t_{i,j}| a_i^k$ and $P := \|\sum_{j\in V_m} x_j e_j\|_k = \max_{j\in V_m} |x_j| a_j^k$. We shall prove that $L = P$.

By (12) we have $|t_{i,j}| \le a_j^k / a_i^k$ for all $i, j \in \mathbb{N}$. Hence for $i \in \mathbb{N}$ we get

$$|\sum_{j\in V_m} x_j t_{i,j}| a_i^k \le \max_{j\in V_m} |x_j| a_j^k = P\,;$$

so $L \le P$. If $P = 0$, then $L = P$. Assume that $P > 0$. Put $s = \max\{1 \le p \le m \colon \max_{j\in N_p} |x_j| a_j^k = P\}$ and $j_0 \in N_s$ with $P = \max_{j\in N_s} |x_j| a_j^k = |x_{j_0}| a_{j_0}^k$. Then $\max_{j\in N_s} |x_j| = |x_{j_0}|$. Put $W_s = \bigcup\{N_p : 1 \le p < s\}$ and $M_s = \bigcup\{N_p : s < p \le m\}$. Then $|x_j| a_j^k \le |x_{j_0}| a_{j_0}^k$ for $j \in W_s$ and $|x_j| a_j^k < |x_{j_0}| a_{j_0}^k$ for $j \in M_s$. By (12) we have $|t_{i,j}| \le 1$ for $i, j \in N_s$ and $\max_{i,j\in N_s} |t_{i,j}| = 1$.

By Proposition 2.2, the linear map

$$S \colon \mathbb{K}^{N_s} \to \mathbb{K}^{N_s}, \quad (y_j)_{j\in N_s} \to (\sum_{j\in N_s} t_{i,j} y_j)_{i\in N_s}$$

is an isometry, so $\max_{i\in N_s} |\sum_{j\in N_s} t_{i,j} x_j| = \max_{j\in N_s} |x_j| = |x_{j_0}|$. Thus for some $i_0 \in N_s$ we have $|\sum_{j\in N_s} t_{i_0,j} x_j| = |x_{j_0}|$; clearly $a_{i_0}^k = a_{j_0}^k$. If $j \in W_s$, then

$$|x_j||t_{i_0,j}| \le |x_{j_0}| \frac{a_{j_0}^k}{a_j^k} \frac{a_j^{k+1}}{a_{i_0}^{k+1}} < |x_{j_0}| \frac{a_{j_0}^k}{a_j^k} \frac{a_j^k}{a_{j_0}^k} = |x_{j_0}|,$$

so $|\sum_{j\in W_s} x_j t_{i_0,j}| < |x_{j_0}|$. If $j \in M_s$, then

$$|x_j||t_{i_0,j}| < |x_{j_0}| \frac{a_{j_0}^k}{a_j^k} \frac{a_j^k}{a_{i_0}^k} = |x_{j_0}|,$$

so $|\sum_{j\in M_s} x_j t_{i_0,j}| < |x_{j_0}|$. Thus

$$|\sum_{j\in V_m} x_j t_{i_0,j}| = |\sum_{j\in W_s} x_j t_{i_0,j} + \sum_{j\in N_s} x_j t_{i_0,j} + \sum_{j\in M_s} x_j t_{i_0,j}| = |x_{j_0}|,$$

so $|\sum_{j\in V_m} x_j t_{i_0,j}| a_{i_0}^k = |x_{j_0}| a_{j_0}^k = P$. Hence $P \le L$. Thus $P = L$. We have shown that $\|Tx\|_k = \|x\|_k$ for all $x \in K(A), k \in \mathbb{N}$, so T is an isometry. $\square$

Remark 3.9. Let $A = (a_n^k)$ be a nuclear strongly regular Köthe matrix. Let (N_t) be a partition of $\mathbb{N}$ associated with A. For every $n \in \mathbb{N}$ there exists $v(n) \in \mathbb{N}$

such that $n \in N_{v(n)}$. Let $i, j \in \mathbb{N}$. The sequence $(a_j^k/a_i^k)_{k=1}^{\infty}$ is strictly increasing if $v(i) < v(j)$, constant if $v(i) = v(j)$ and strictly decreasing if $v(i) > v(j)$. Thus

$$\inf_k a_j^k/a_i^k = \begin{cases} a_j^1/a_i^1, & \text{if } v(i) < v(j); \\ 1, & \text{if } v(i) = v(j); \\ \lim_k a_j^k/a_i^k, & \text{if } v(i) > v(j). \end{cases}$$

Proposition 3.10. *Let $A=(a_n^k)$ be a nuclear Köthe matrix. Assume that $(t_{i,j}) \subset \mathbb{K}$ with $|t_{i,j}| \leq \inf_k a_j^k/a_i^k$ for all $i, j \in \mathbb{N}$. Then there exists a linear contraction $T\colon K(A) \to K(A)$ such that $Te_j = \sum_{i=1}^{\infty} t_{i,j}e_i$ for $j \in \mathbb{N}$. If additionally A is strongly regular and $|det[t_{i,j}]_{i,j\in N_s}| = 1$ for every $s \in \mathbb{N}$, then T is an isometry.*

Proof. First we prove that $\sum_{i=1}^{\infty} t_{i,j}e_i$ is convergent in $K(A)$ for any $j \in \mathbb{N}$. By the nuclearity of A, for every $k \in \mathbb{N}$ there exists $s(k)$ such that $a_i^k/a_i^{s(k)} \to_i 0$.

Let $j \in \mathbb{N}$. Then $|t_{i,j}|a_i^k \leq (a_j^{s(k)}/a_i^{s(k)})a_i^k = a_j^{s(k)}(a_i^k/a_i^{s(k)})$ for all $i, k \in \mathbb{N}$, so $|t_{i,j}|a_i^k \to_i 0$ for every $k \in \mathbb{N}$. Thus $t_{i,j}e_i \to_i 0$ in $K(A)$, so the series $\sum_{i=1}^{\infty} t_{i,j}e_i$ is convergent to some T_j in $K(A)$ and $\|T_j\|_k = \max_i |t_{i,j}|a_i^k \leq \max_i(a_j^k/a_i^k)a_i^k = a_j^k$ for every $k \in \mathbb{N}$.

Let $x = (x_j) \in K(A)$. For any $k \in \mathbb{N}$ we have $\|x_jT_j\|_k = |x_j|\|T_j\|_k \leq |x_j|a_j^k$, so $x_jT_j \to_j 0$ in $K(A)$, so the series $\sum_{j=1}^{\infty} x_jT_j$ is convergent in $K(A)$.

Thus the operator $T\colon K(A) \to K(A), x = (x_j) \to Tx = \sum_{j=1}^{\infty} x_jT_j$ is well defined; clearly it is linear. T is a contraction, since $\|Tx\|_k \leq \max_j \|x_jT_j\|_k \leq \max_j |x_j|a_j^k = \|x\|_k$ for all $x \in K(A)$, $k \in \mathbb{N}$. The use of Theorem 3.8 completes the proof. □

By the proof of Proposition 3.10 we get

Corollary 3.11. *Let $A = (a_n^k)$ be a Köthe matrix. Assume that $(d_{i,j}) \subset \mathbb{K}$ with $|d_{i,j}| \leq \inf_k a_j^k/a_i^k$ for all $i, j \in \mathbb{N}$. If $\forall j \exists s_j \forall i > s_j : d_{i,j} = 0$ then there exists a linear contraction $D\colon K(A) \to K(A)$ such that $De_j = \sum_{i=1}^{\infty} d_{i,j}e_i$ for $j \in \mathbb{N}$. If additionally A is strongly regular and $|det[d_{i,j}]_{i,j\in N_s}| = 1$ for every $s \in \mathbb{N}$, then D is an isometry.*

We shall need the next two propositions to prove that any linear isometry on a nuclear strongly regular Köthe space is surjective.

Proposition 3.12. *Let $A = (a_n^k)$ be a strongly regular Köthe matrix. Assume that $(d_{i,j}) \subset \mathbb{K}, |d_{i,j}| \leq \inf_k a_j^k/a_i^k$ if $v(i) \leq v(j)$, $d_{i,j} = 0$ if $v(i) > v(j)$ and $|det[d_{i,j}]_{i,j\in N_s}| = 1$ for every $s \in \mathbb{N}$. Then there exists an isometric isomorphism $D\colon K(A) \to K(A)$ such that $De_j = \sum_{i=1}^{\infty} d_{i,j}e_i, j \in \mathbb{N}$.*

Proof. By Corollary 3.11, there exists a linear isometry $D\colon K(A) \to K(A)$ such that $De_j = \sum_{i=1}^{\infty} d_{i,j}e_i, j \in \mathbb{N}$. We shall prove that D is surjective. Put $W_r = \bigcup_{k=1}^{r} N_k$ for $r \in \mathbb{N}$.

We have $\mathrm{lin}\{De_j : j \in W_k\} \subset \mathrm{lin}\{e_j : j \in W_k\}$ for every $k \in \mathbb{N}$, since $De_j = \sum_{i\in W_k} d_{i,j}e_i$ for $j \in N_k, k \in \mathbb{N}$.

D is a linear isometry, so $D(K(A))$ is closed in $K(A)$ and the sequence $(De_j)_{j\in W_k}$ is linearly independent for every $k \in \mathbb{N}$. Thus $\mathrm{lin}\{De_j : j \in W_k\} = \mathrm{lin}\{e_j : j \in W_k\}$ for every $k \in \mathbb{N}$, so $D(K(A)) \supset \mathrm{lin}\{e_j : j \in \mathbb{N}\}$. It follows that $D(K(A)) = K(A)$, so D is an isometric isomorphism. $\square$

Proposition 3.13. *Let $A = (a_n^k)$ be a nuclear strongly regular Köthe matrix. Assume that $(s_{i,j}) \subset \mathbb{K}$, $|s_{i,j}| \le \inf_k(a_j^k/a_i^k)$ if $v(i) \ge v(j)$, $s_{i,j} = 0$ if $v(i) < v(j)$ and $|det[s_{i,j}]_{i,j\in N_s}| = 1$ for every $s \in \mathbb{N}$. Then there exists an isometric isomorphism $S\colon K(A) \to K(A)$ such that $Se_j = \sum_{i=1}^{\infty} s_{i,j}e_i, j \in \mathbb{N}$.*

Proof. By Proposition 3.10, there exists a linear isometry $S\colon K(A) \to K(A)$ such that $Se_j = \sum_{i=1}^{\infty} s_{i,j}e_i, j \in \mathbb{N}$. We shall prove that S is surjective. Put $W_r = \bigcup_{k=1}^{r} N_k$ for $r \in \mathbb{N}$. Let $y = (y_i) \in K(A)$. We shall construct an $x \in K(A)$ such that $Sx = y$. By Proposition 2.2, the linear map

$$\mathbb{K}^{N_1} \to \mathbb{K}^{N_1}, (\beta_j)_{j\in N_1} \to (\sum_{j\in N_1} s_{i,j}\beta_j)_{i\in N_1}$$

is an isometry. Thus there exists $(x_j)_{j\in N_1} \subset \mathbb{K}$ with $\max_{j\in N_1}|x_j| = \max_{i\in N_1}|y_i|$ such that $\sum_{j\in N_1} s_{i,j}x_j = y_i$ for every $i \in N_1$. Assume that for some $l \in \mathbb{N}$ with $l > 1$ we have chosen $(x_j)_{j\in N_s} \subset \mathbb{K}$ for $1 \le s < l$. By Proposition 2.2, the linear map $\mathbb{K}^{N_l} \to \mathbb{K}^{N_l}$, $(\beta_j)_{j\in N_l} \to (\sum_{j\in N_l} s_{i,j}\beta_j)_{i\in N_l}$ is an isometry. Thus there exists $(x_j)_{j\in N_l} \subset \mathbb{K}$ with $\max_{j\in N_l}|x_j| = \max_{i\in N_l}|y_i - \sum_{j\in W_{l-1}} s_{i,j}x_j|$ such that $\sum_{j\in N_l} s_{i,j}x_j = y_i - \sum_{j\in W_{l-1}} s_{i,j}x_j$ for $i \in N_l$. Thus by induction we get $x = (x_j) \in \mathbb{K}^{\mathbb{N}}$ such that $\sum_{j\in W_l} s_{i,j}x_j = y_i$ for all $i \in N_l, l \in \mathbb{N}$ and $\max_{j\in N_1}|x_j| = \max_{i\in N_1}|y_i|$ and $\max_{j\in N_l}|x_j| = \max_{i\in N_l}|y_i - \sum_{j\in W_{l-1}} s_{i,j}x_j|$ for $l > 1$. We shall prove that $x \in K(A)$. Let $k \in \mathbb{N}$. Clearly, $\max_{j\in W_1}|x_j|a_j^k = \max_{i\in W_1}|y_1|a_i^k$. For $l > 1, i \in N_l$ and $j \in W_{l-1}$ we have $|s_{i,j}||x_j|a_i^k \le (a_j^k/a_i^k)|x_j|a_i^k = |x_j|a_j^k$. Hence

$$\max_{i\in N_l}|x_i|a_i^k = \max_{i\in N_l}|y_i - \sum_{j\in W_{l-1}} s_{i,j}x_j|a_i^k \le \max\{\max_{i\in N_l}|y_i|a_i^k, \max_{j\in W_{l-1}}|x_j|a_j^k\}.$$

Thus by induction we get $\max_{j\in W_l}|x_j|a_j^k \le \max_{i\in W_l}|y_i|a_i^k \le \|y\|_k$ for all $l \in \mathbb{N}$. It follows that $|x_j|a_j^k \le \|y\|_k$ for all $k, j \in \mathbb{N}$. Let $k \in \mathbb{N}$. Let $s(k) > k$ with $(a_j^k/a_j^{s(k)}) \to_j 0$. Then $|x_j|a_j^k = |x_j|a_j^{s(k)}(a_j^k/a_j^{s(k)}) \le \|y\|_{s(k)}(a_j^k/a_j^{s(k)})$ for every $j \in \mathbb{N}$. Thus $|x_j|a_j^k \to_j 0$ for every $k \in \mathbb{N}$, so $x \in K(A)$. We have

$$\begin{aligned} Sx &= \sum_{j=1}^{\infty} x_jSe_j = \sum_{j=1}^{\infty} x_j \sum_{i=1}^{\infty} s_{i,j}e_i = \sum_{i=1}^{\infty}(\sum_{j=1}^{\infty} s_{i,j}x_j)e_i \\ &= \sum_{l=1}^{\infty}\sum_{i\in N_l}(\sum_{j\in W_l} s_{i,j}x_j)e_i = \sum_{l=1}^{\infty}\sum_{i\in N_l} y_ie_i = \sum_{i=1}^{\infty} y_ie_i = y. \end{aligned}$$

Thus S is surjective, so S is an isometric isomorphism. $\square$

Now we can prove that every linear isometry on any nuclear strongly regular Köthe space is surjective.

Theorem 3.14. *Let $A = (a_n^k)$ be a nuclear strongly regular Köthe matrix. Then every linear isometry T on $K(A)$ is surjective.*

Proof. Let $l, m \in \mathbb{N}$ and $N_{l,m} = N_l \times N_m$. Denote by $\mathcal{M}_{l,m}$ the family of all matrices $B = [\beta_{i,j}]_{(i,j)\in N_{l,m}}$ with $\beta_{i,j} \in \mathbb{K}$ such that

(a) $|\beta_{i,j}| \le \inf_k a_j^k/a_i^k$ for all $(i,j) \in N_{l,m}$, if $l \ne m$;

(b) $|\beta_{i,j}| \le 1$ for all $(i,j) \in N_{l,m}$ and $|\det B| = 1$, if $l = m$.

By Proposition 2.2 , for every $m \in \mathbb{N}$ and $B \in \mathcal{M}_{m,m}$ we have $B^{-1} \in \mathcal{M}_{m,m}$. Let $Te_j = \sum_{i=1}^{\infty} t_{i,j} e_i$ for $j \in \mathbb{N}$. Put $T_{k,m} = [t_{i,j}]_{(i,j)\in N_{k,m}}$ and $I_{k,m} = [\delta_{i,j}]_{(i,j)\in N_{k,m}}$ for all $k, m \in \mathbb{N}$. We define matrices $D_{k,m}, S_{k,m} \in \mathcal{M}_{k,m}$ for $k \in \mathbb{N}$ and $m = 1, 2, 3, \ldots$.

Put $D_{k,1} = I_{k,1}$ and $S_{k,1} = T_{k,1}$ for $k \in \mathbb{N}$; clearly $D_{k,1}, S_{k,1} \in \mathcal{M}_{k,1}$ for $k \in \mathbb{N}$. Assume that for some $m \in \mathbb{N}$ with $m > 1$ we have $D_{k,j}, S_{k,j} \in \mathcal{M}_{k,j}$ for $k \in \mathbb{N}$ and $1 \le j < m$. Let $D_{1,m} = S_{1,1}^{-1} T_{1,m}$. It is easy to see that $D_{1,m} \in \mathcal{M}_{1,m}$, since $S_{1,1}^{-1} \in \mathcal{M}_{1,1}$ and $T_{1,m} \in \mathcal{M}_{1,m}$.

Let $D_{k,m} = S_{k,k}^{-1}[T_{k,m} - \sum_{v=1}^{k-1} S_{k,v} D_{v,m}]$ for $k = 2, 3, \ldots, m-1$. Let $1 < k < m$. Let $[s_{i,n}]_{(i,n)\in N_{k,v}} = S_{k,v}$ and $[d_{n,j}]_{(n,j)\in N_{v,m}} = D_{v,m}$ for $1 \le v < k$. Put $C_{k,m} = [c_{i,j}]_{(i,j)\in N_{k,m}} = \sum_{v=1}^{k-1} S_{k,v} D_{v,m}$. Then

$$|c_{i,j}| = |\sum_{v=1}^{k-1} \sum_{n\in N_v} s_{i,n} d_{n,j}| \le \max_{n\in W_{k-1}} |s_{i,n} d_{n,j}|$$

for $(i,j) \in N_{k,m}$. For $i \in N_k$, $j \in N_m$ and $n \in W_{k-1}$ we have

$$|s_{i,n} d_{n,j}| \le \inf_t a_n^t/a_i^t \inf_s a_j^s/a_n^s \le (a_n^l/a_i^l)(a_j^l/a_n^l) = a_j^l/a_i^l$$

for every $l \in \mathbb{N}$, so that $|s_{i,n} d_{n,j}| \le \inf_l a_j^l/a_i^l$. Hence $|c_{i,j}| \le \inf_l a_j^l/a_i^l$ for all $(i,j) \in N_{k,m}$, so $C_{k,m} \in \mathcal{M}_{k,m}$. Since $S_{k,k}^{-1} \in \mathcal{M}_{k,k}$ and $T_{k,m} \in \mathcal{M}_{k,m}$ we infer that $D_{k,m} \in \mathcal{M}_{k,m}$ for $k = 2, \ldots, m-1$.

Let $D_{k,m} = I_{k,m}$ for $k \ge m$; clearly $D_{k,m} \in \mathcal{M}_{k,m}$. Let $S_{k,m} = I_{k,m}$ for $1 \le k < m$ and let $S_{k,m} = T_{k,m} - \sum_{v=1}^{m-1} S_{k,v} D_{v,m}$ for $k \ge m$. Clearly, $S_{k,m} \in \mathcal{M}_{k,m}$ for $1 \le k < m$.

Let $k \ge m$. Let $[s_{i,n}]_{(i,n)\in N_{k,v}} = S_{k,v}$ and $[d_{n,j}]_{(n,j)\in N_{v,m}} = D_{v,m}$ for $1 \le v < m$. Put $C_{k,m} = [c_{i,j}]_{(i,j)\in N_{k,m}} = \sum_{v=1}^{m-1} S_{k,v} D_{v,m}$. Then

$$|c_{i,j}| = |\sum_{v=1}^{m-1} \sum_{n\in N_v} s_{i,n} d_{n,j}| \le \max_{n\in W_{m-1}} |s_{i,n} d_{n,j}|$$

for $(i,j) \in N_{k,m}$. For $i \in N_k$, $j \in N_m$ and $n \in W_{m-1}$ we have by Remark 3.9

$$|s_{i,n} d_{n,j}| \le \inf_t a_n^t/a_i^t \inf_s a_j^s/a_n^s < (a_n^l/a_i^l)(a_j^l/a_n^l) = a_j^l/a_i^l$$

for every $l \in \mathbb{N}$, so that $|s_{i,n}d_{n,j}| \leq \inf_l a_j^l/a_i^l$. Hence $|c_{i,j}| \leq \inf_l a_j^l/a_i^l$ for all $(i,j) \in N_{k,m}$, so $C_{k,m} \in \mathcal{M}_{k,m}$ for $k > m$. Since $|t_{i,j}| \leq \inf_l a_j^l/a_i^l$ for $(i,j) \in N_{k,m}$, we get $S_{k,m} \in \mathcal{M}_{k,m}$ for $k > m$ and $|c_{i,j}| < 1$ for all $(i,j) \in N_{m,m}$. Moreover for some $(A_\sigma)_{\sigma \in S(N_m)} \subset \{\alpha \in \mathbb{K} : |\alpha| < 1\}$ we have

$$|\det S_{m,m}| = |\sum_{\sigma \in S(N_m)} \operatorname{sgn} \sigma \prod_{i \in N_m} (t_{i,\sigma(i)} - c_{i,\sigma(i)})| = |\sum_{\sigma \in S(N_m)} \operatorname{sgn} \sigma [(\prod_{i \in N_m} t_{i,\sigma(i)}) - A_\sigma]|$$

$$= |\det(T_{m,m}) - \sum_{\sigma \in S(N_m)} \operatorname{sgn} \sigma A_\sigma| = |\det(T_{m,m})| = 1.$$

Thus $S_{m,m} \in \mathcal{M}_{m,m}$. By definition of $D_{k,m}$ and $S_{k,m}$ we get

(a) $T_{k,1} = S_{k,1} = \sum_{v=1}^k S_{k,v} D_{v,1}$ for $k \in \mathbb{N}$;

(b) $S_{1,1}D_{1,m} = T_{1,m}$ for $m \geq 2$ and $S_{k,k}D_{k,m} = T_{k,m} - \sum_{v=1}^{k-1} S_{k,v}D_{v,m}$ for $2 \leq k < m$, so that $T_{k,m} = \sum_{v=1}^k S_{k,v}D_{v,m}$ for $1 \leq k < m$;

(c) $S_{k,m}D_{m,m} = S_{k,m} = T_{k,m} - \sum_{v=1}^{m-1} S_{k,v}D_{v,m}$ for $k \geq m > 1$, so that $T_{k,m} = \sum_{v=1}^m S_{k,v}D_{v,m} = \sum_{v=1}^k S_{k,v}D_{v,m}$ for $k \geq m > 1$.

Thus
$$T_{k,m} = \sum_{v=1}^k S_{k,v} D_{v,m} \quad \text{for all} \quad k, m \in \mathbb{N}. \tag{16}$$

Let $[s_{i,j}]_{(i,j) \in \mathbb{N} \times \mathbb{N}}$ and $[d_{i,j}]_{(i,j) \in \mathbb{N} \times \mathbb{N}}$ be matrices such that $[s_{i,j}]_{(i,j) \in N_{k,m}} = S_{k,m}$ and $[d_{i,j}]_{(i,j) \in N_{k,m}} = D_{k,m}$ for all $k, m \in \mathbb{N}$.

By Theorem 3.8, Proposition 3.10 and Corollary 3.11, there exist linear isometries S and D on $K(A)$ such that $Se_j = \sum_{i=1}^\infty s_{ij} e_i$ and $De_j = \sum_{i=1}^\infty d_{i,j} e_i$ for all $j \in \mathbb{N}$; by Propositions 3.12 and 3.13, these isometries are surjective. Using (16) we get

$$t_{i,j} = \sum_{v=1}^k \sum_{n \in N_v} s_{i,n} d_{n,j} = \sum_{v=1}^\infty \sum_{n \in N_v} s_{i,n} d_{n,j} = \sum_{n=1}^\infty s_{i,n} d_{n,j}$$

for $(i,j) \in N_{k,m}$ and $k, m \in \mathbb{N}$. Hence for $j \in \mathbb{N}$ we get

$$SDe_j = S(\sum_{n=1}^\infty d_{n,j} e_n) = \sum_{n=1}^\infty d_{n,j} (\sum_{i=1}^\infty s_{i,n} e_i) = \sum_{i=1}^\infty (\sum_{n=1}^\infty s_{i,n} d_{n,j}) e_i = \sum_{i=1}^\infty t_{i,j} e_i = Te_j;$$

so $T = SD$. Thus T is surjective. □

Let A be a nuclear strongly regular Köthe matrix. By Theorem 3.14, the family $\mathcal{I}_A$ of all linear isometries $T\colon K(A) \to K(A)$ is a subgroup of the group of all automorphisms of $K(A)$. Put

$$\mathcal{D}_A = \{T \in \mathcal{I}_A : Te_j = \textstyle\sum_{i \in W_{v(j)}} t_{i,j} e_i \text{ for all } j \in \mathbb{N}\},$$

$$\mathcal{K}_A = \{T \in \mathcal{I}_A : Te_j = \textstyle\sum_{i \in N_{v(j)}} t_{i,j} e_i \text{ for all } j \in \mathbb{N}\}, \quad \text{and}$$

$$\mathcal{S}_A = \{T \in \mathcal{I}_A : Te_j = \textstyle\sum_{i \in M_{v(j)}} t_{i,j} e_i \text{ for all } j \in \mathbb{N}\} \quad \text{where}$$

$$W_k = \textstyle\bigcup_{p=1}^k N_p \quad \text{and} \quad M_k = \bigcup_{p=k}^\infty N_p \quad \text{for} \quad k \in \mathbb{N}.$$

Proposition 3.15. *Let A be a nuclear strongly regular Köthe matrix. Then $\mathcal{D}_A, \mathcal{K}_A, \mathcal{S}_A$ are subgroups of $\mathcal{I}_A$. Moreover for every $T \in \mathcal{I}_A$ there exist $D \in \mathcal{D}_A$ and $S \in \mathcal{S}_A$ such that $T = S \circ D$.*

Proof. Let $S, T \in \mathcal{S}_A$. Let $j \in \mathbb{N}$. We have

$$STe_j = S\Big(\sum_{i=1}^{\infty} t_{i,j}e_i\Big) = \sum_{i=1}^{\infty} t_{i,j}S(e_i) = \sum_{i=1}^{\infty} t_{i,j}\Big(\sum_{k=1}^{\infty} s_{k,i}e_k\Big) = \sum_{k=1}^{\infty}\Big(\sum_{i=1}^{\infty} s_{k,i}t_{i,j}\Big)e_k.$$

If $v(k) < v(j)$, then for every $i \in \mathbb{N}$ we have $v(k) < v(i)$ or $v(i) < v(j)$; so $s_{k,i} = 0$ or $t_{i,j} = 0$ for $i \in \mathbb{N}$. Thus $\sum_{i=1}^{\infty} s_{k,i}t_{i,j} = 0$ for $k \in \mathbb{N}$ with $v(k) < v(j)$; so $ST \in \mathcal{S}_A$.

Let $k \in \mathbb{N}$. Let $y = (y_i) = e_k$. Then $y_i = 0$ for $i \in W_l$ with $l < v(k)$. For some $x = (x_j) \in K(A)$ we have $Sx = y$. Using the proof of Proposition 3.13 we get $\sup\{|x_j|a_j^k : j \in W_l\} \leq \sup\{|y_i|a_i^k : i \in W_l\} = 0$, so $x_j = 0$ for $j \in W_l$ with $l < v(k)$. Hence $S^{-1}(e_k) = x = \sum_{j\in M_{v(k)}} x_je_j$ for $k \in \mathbb{N}$; so $S^{-1} \in \mathcal{S}_A$. Thus $\mathcal{S}_A$ is a subgroup of $\mathcal{I}_A$.

Let $D, T \in \mathcal{D}_A$. Let $j \in \mathbb{N}$. We have $DTe_j = \sum_{k=1}^{\infty}(\sum_{i=1}^{\infty} d_{k,i}t_{i,j})e_k$. If $v(k) > v(j)$ then for every $i \in \mathbb{N}$ we have $v(k) > v(i)$ or $v(i) > v(j)$; so $d_{k,i} = 0$ or $t_{i,j} = 0$ for $i \in \mathbb{N}$. Thus $\sum_{i=1}^{\infty} d_{k,i}t_{i,j} = 0$ for every $k \in \mathbb{N}$ with $v(k) > v(j)$, so $DT \in \mathcal{D}_A$.

Let $k \in \mathbb{N}$. Put $F_k = \operatorname{lin}\{e_i : v(i) \leq v(k)\}$. We have $D(F_k) = F_k$ (see the proof of Proposition 3.12). Thus there exists $x = (x_j) \in F_k$ such that $Dx = e_k$. Then $x_j = 0$ for $j \in \mathbb{N}$ with $v(j) > v(k)$ and $D^{-1}(e_k) = x = \sum_{j\in W_{v(k)}} x_je_j$, so $D^{-1} \in \mathcal{D}_A$. Thus $\mathcal{D}_A$ is a subgroup of $\mathcal{I}_A$. Clearly, $\mathcal{K}_A = \mathcal{S}_A \cap \mathcal{D}_A$, so $\mathcal{K}_A$ is subgroup of $\mathcal{I}_A$.

The last part of Proposition 3.15 follows by the proof of Theorem 3.14. □

Finally we state the following open problems for regular Köthe spaces.

Problem 3.16. *Let A and B be regular Köthe matrices. When the Köthe spaces $K(A)$ and $K(B)$ are isometrically isomorphic?*

Problem 3.17. *Let A be a regular Köthe matrix and let $T\colon K(A) \to K(A)$ be a continuous linear map with $Te_j = \sum_{i=1}^{\infty} t_{i,j}e_i$ for $j \in \mathbb{N}$. Find sufficient and necessary conditions on $(t_{i,j}) \subset \mathbb{K}$ under which T is an isometry.*

Problem 3.18. *Let A be a nuclear regular Köthe matrix. Does every linear isometry $T\colon K(A) \to K(A)$ is surjective?*

References

[1] N. De Grande-De Kimpe: Non-archimedean Fréchet spaces generalizing spaces of analytic functions, Indag. Math. 44 (1982) 423–439.

[2] C. Perez-Garcia: Locally convex spaces over non-archimedean valued fields, in: Ultrametric Functional Analysis, Contemporary Math. 319 (2003) 251–279.

[3] C. Perez-Garcia, W. Schikhof: Locally Convex Spaces over Non-Archimedean Valued Fields, Cambridge Studies in Advanced Mathematics, Cambridge University Press, Cambridge (2010).

[4] A. C. M. van Rooij: Notes on p-adic Banach spaces, Report 7633, Department of Mathematics, University of Nijmegen (1976).

[5] A. C. M. van Rooij: Non-Archimedean Functional Analysis, Marcel Dekker, New York (1978).

[6] W. H. Schikhof: Locally convex spaces over non-spherically complete valued fields, Bull. Soc. Math. Belg. 38 (1986) 187–224.

[7] W. Śliwa: On relations between non-archimedean power series spaces, Indag. Math., N.S. 17 (2006) 627–639.

[8] W. Śliwa, A. Ziemkowska: On linear isometries on non-Archimedean power series spaces, J. Convex Analysis 19 (2012) 453–466.

[9] W. Śliwa: On non-archimedean generalized power series spaces, Contemporary Math. 596 (2013) 281–291.

[10] W. Śliwa: On relations between non-Archimedean generalized power series spaces, J. Convex Analysis 22 (2015) 569–590.