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On Approximate Birkhoff-James Orthogonality and Normal Cones in a Normed Space*

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We study two notions of approximate Birkhoff-James orthogonality in a normed space, from a geometric point of view, and characterize them in terms of normal cones. We further explore the interconnection between normal cones and approximate Birkhoff-James orthogonality to obtain a complete characterization of normal cones in a two-dimensional smooth Banach space. We also obtain a uniqueness theorem for approximate Birkhoff-James orthogonality set in a normed space.

Keywords: Approximate Birkhoff-James orthogonality, normal cones.

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1. Introduction.

Birkhoff-James orthogonality is arguably the most natural and well-studied notion of orthogonality in a normed space. Indeed, in the normed space setting, the close connection shared by Birkhoff-James orthogonality with various geometric properties like strict convexity, smoothness etc. can hardly be over emphasized. As a consequence, Birkhoff-James orthogonality plays a crucial role in exploring the geometry of normed spaces [5]. In view of this, it is perhaps not surprising that various generalizations (and approximations) of Birkhoff-James orthogonality have been introduced and studied by several mathematicians, including Dragomir [4] and Chmieliński [2]. In this paper, our aim is to study two different approximations of Birkhoff-James orthogonality, in order to have a better understanding of the geometry of normed spaces. Among other things, we exhibit that both types

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of approximate Birkhoff-James orthogonality have a close connection with normal cones in a normed space. Without further ado, let us establish our notations and terminologies to be used throughout this paper.

Let $\mathbb{X}$ be a normed space defined over $\mathbb{R}$, the field of real numbers. Let $B_{\mathbb{X}}$ and $S_{\mathbb{X}}$ denote the *unit ball* and the *unit sphere* of $\mathbb{X}$ respectively, i.e., $B_{\mathbb{X}} = \{x \in \mathbb{X} : \|x\| \leq 1\}$ and $S_{\mathbb{X}} = \{x \in \mathbb{X} : \|x\| = 1\}$. For any two elements x and y in $\mathbb{X}$, x is said to be *orthogonal* to y in the sense of Birkhoff-James [1] if $\|x + \lambda y\| \geq \|x\|$ for all real scalars λ. Recently, Sain [6] introduced the notions of $x^+,\ x^-$ for a given element x of $\mathbb{X}$, in order to completely characterize Birkhoff-James orthogonality of linear operators on a finite dimensional Banach space. An element $y \in \mathbb{X}$ is said to be in x^+ if $\|x + \lambda y\| \geq \|x\|$ for all $\lambda \geq 0$ and $y \in \mathbb{X}$ is said to be in x^- if $\|x + \lambda y\| \geq \|x\|$ for all $\lambda \leq 0$. The notion of *Birkhoff-James orthogonality* has been generalized by Dragomir [4] in the following way, in order to obtain a suitable definition of approximate Birkhoff-James orthogonality in normed spaces.

Let $\epsilon \in [0, 1)$. Then for $x, y \in \mathbb{X}$, x is said to be *approximate ϵ-Birkhoff-James orthogonal* to y if

$$\|x + \lambda y\| \geq (1 - \epsilon)\|x\| \ \forall \lambda \in \mathbb{R}.$$

Later on, Chmieliński [2] slightly modified the definition in the following way: Let $\epsilon \in [0, 1)$. Then for $x, y \in \mathbb{X}$, x is said to be *approximate ϵ-Birkhoff-James orthogonal* to y if

$$\|x + \lambda y\| \geq \sqrt{1 - \epsilon^2}\|x\| \ \forall \lambda \in \mathbb{R}.$$

In this case, we write $x \perp_D^{\epsilon} y$.

Chmieliński [2] also introduced another notion of approximate Birkhoff-James orthogonality, defined in the following way: Let $\epsilon \in [0, 1)$. Then for $x, y \in \mathbb{X}$, x is said to be *approximate ϵ-Birkhoff-James orthogonal* to y if

$$\|x + \lambda y\|^2 \geq \|x\|^2 - 2\epsilon\|x\|\|\lambda y\| \ \forall \lambda \in \mathbb{R}.$$

In this case, we write $x \perp_B^{\epsilon} y$.

It should be noted that in an inner product space, both types of approximate Birkhoff-James orthogonality coincide. However, this is not necessarily true in a normed space. We would also like to remark that in a normed space, both types of approximate Birkhoff-James orthogonality are homogeneous.

Recently, Chmieliński et al. [3] characterized "$x \perp_B^{\epsilon} y$" for real normed spaces by means of the following theorem:

Theorem 1.1 (Theorem 2.3,[3]). *Let $\mathbb{X}$ be a real normed space. For $x, y \in \mathbb{X}$ and $\epsilon \in [0, 1)$:*

$$x \perp_B^{\epsilon} y \Leftrightarrow \exists\ z \in \ Lin\{x, y\} : x \perp_B z, \|z - y\| \leq \epsilon\|y\|.$$

The above characterization of "$x \perp_B^{\epsilon} y$" is essentially analytic in nature. In this paper, our aim is to explore the structure and properties of both types of approximate orthogonal vectors from a geometric point of view. In order to serve this purpose, we introduce the following two notations:

Given $x \in \mathbb{X}$ and $\epsilon \in [0, 1)$, we define

$$F(x, \epsilon) = \{y \in \mathbb{X} : x \perp_D^{\epsilon} y\}, \quad \text{and} \quad G(x, \epsilon) = \{y \in \mathbb{X} : x \perp_B^{\epsilon} y\}.$$

In this context, the concept of normal cones in a normed space play a very important role. Therefore, at this point of our discussion, the following definition is in order:

A subset K of $\mathbb{X}$ is said to be a *normal cone* in $\mathbb{X}$ if

$$(i)\ K + K \subset K, \quad (ii)\ \alpha K \subset K \text{ for all } \alpha \geq 0 \quad \text{and} \quad (iii)\ K \cap (-K) = \{\theta\}.$$

Normal cones are important in the study of geometry of normed spaces because there is a natural partial ordering $\geq$ associated with K, namely, for any two elements $x, y \in \mathbb{X}$, $x \geq y$ if $x - y \in K$. It is easy to observe that in a two-dimensional Banach space $\mathbb{X}$, any normal cone K is completely determined by the intersection of K with the unit sphere $S_{\mathbb{X}}$. Keeping this in mind, when we say that K is a normal cone in $\mathbb{X}$, determined by v_1, v_2, what we really mean is that

$$K \cap S_{\mathbb{X}} = \left\{ \frac{(1-t)v_1 + tv_2}{\|(1-t)v_1 + tv_2\|} : t \in [0, 1] \right\}.$$

Of course, in this case $K = \{\alpha v_1 + \beta v_2 : \alpha, \beta \geq 0\}$. We prove that in a two-dimensional Banach space $\mathbb{X}$, given any $x \in \mathbb{X}$ and any $\epsilon \in [0, 1)$, $F(x, \epsilon) = K_1 \cup (-K_1)$ and $G(x, \epsilon) = K_2 \cup (-K_2)$, where K_1, K_2 are normal cones in $\mathbb{X}$. In fact, in case of $F(x, \epsilon)$, we prove something more. We show that in a two-dimensional smooth Banach space $\mathbb{X}$, given any normal cone K, there exists $x \in S_{\mathbb{X}}$ and $\epsilon \in [0, 1)$ such that $F(x, \epsilon) = K \cup (-K)$. Indeed, this interconnection between normal cones and approximate ϵ-Birkhoff-James orthogonality sets is a major highlight of the present paper. Equipped with the above mentioned interconnection, we proceed to obtain a complete geometric description of $F(x, \epsilon)$ and $G(x, \epsilon)$, when $\mathbb{X}$ is a normed space of any dimension. In order to accomplish this goal, let us introduce the following two notations:

Let $\mathbb{X}$ be a normed space. For $x, y \in \mathbb{X}$ and $\epsilon \in [0, 1)$, let $P_{x,y}(\epsilon)$, $Q_{x,y}(\epsilon)$ denote the restriction of $F(x, \epsilon)$ and $G(x, \epsilon)$, respectively, to the subspace spanned by x and y. Using these notations, we obtain a complete geometric description of $F(x, \epsilon)$ and $G(x, \epsilon)$, when $\mathbb{X}$ is any normed space. We show that both $F(x, \epsilon)$ and $G(x, \epsilon)$ are union of two-dimensional normal cones. We also prove a uniqueness theorem for $F(x, \epsilon)$, first in the case of a two-dimensional Banach space and then for any normed space.

2. Main results

Theorem 2.1. *Let $\mathbb{X}$ be a two-dimensional Banach space. Then for any $x \in S_{\mathbb{X}}$ and $\epsilon \in [0, 1)$, there exists a normal cone K in $\mathbb{X}$ such that $F(x, \epsilon) = K \cup (-K)$.*

Proof. For $x \in S_{\mathbb{X}}$ there exists $y \in S_{\mathbb{X}}$ such that $x \perp_B y$. Consider

$$A = \{t \in [0, 1] : x \perp_D^{\epsilon} \{(1-t)x + ty\}\} \text{ and } B = \{t \in [0, 1] : x \perp_D^{\epsilon} \{-(1-t)x + ty\}\}.$$

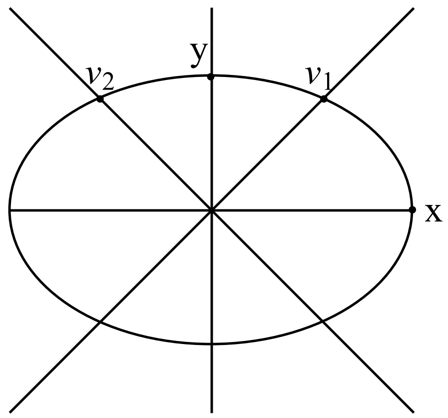

Figure 2.1

Clearly, both A and B are non empty proper subsets of $[0,1]$, as $0 \notin A \cup B$ and $1 \in A \cap B$. We next show that both A and B are closed. Suppose $\{t_n\} \subseteq A$ be such that $t_n \longrightarrow t$. Then

$$x \perp_D^{\epsilon} \{(1-t_n)x + t_n y\} \Longrightarrow \|x + \lambda\{(1-t_n)x + t_n y\}\| \geq \sqrt{1-\epsilon^2}$$

for all $\lambda \in \mathbb{R}$. Letting $n \longrightarrow \infty$ we have, $\|x + \lambda\{(1-t)x + ty\}\| \geq \sqrt{1-\epsilon^2}$ for all $\lambda \in \mathbb{R}$. Then $x \perp_D^{\epsilon} \{(1-t)x + ty\}$ and so $t \in A$. This proves that A is closed. Similarly, one can show that B is closed.

Let $t_1 = \inf A,\ t_2 = \inf B$. Clearly $t_1 \neq 0,\ t_2 \neq 0$.

Let $u_1 = (1-t_1)x + t_1 y$, $u_2 = -(1-t_2)x + t_2 y$ and $v_1 = u_1/\|u_1\|$, $v_2 = u_2/\|u_2\|$. Let K be the normal cone determined by v_1, v_2. We next show that $F(x,\epsilon) = K \cup (-K)$.

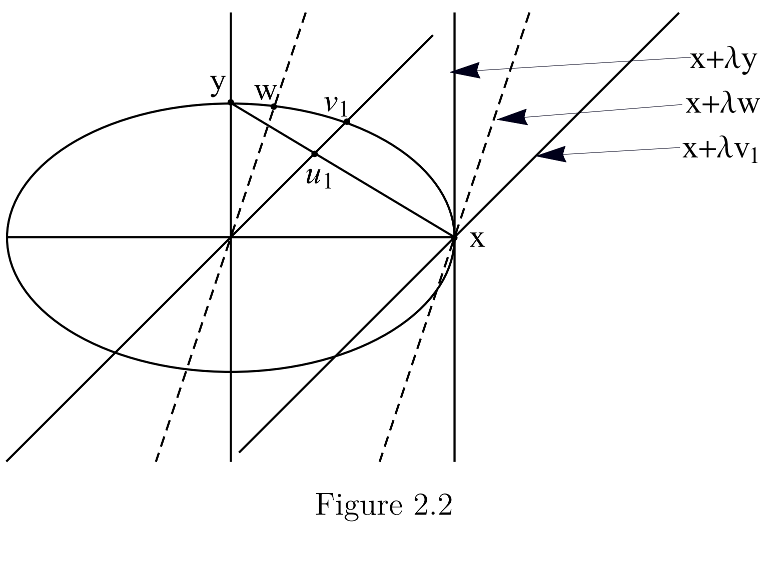

Figure 2.2

If $z \in \mathbb{X}$ is such that $z = c\{(1-t)x + ty\}$, where $c > 0$ and $t < t_1$ then from the definition of infimum, $x \not\perp_D^{\epsilon} z$. Let $w \in \mathbb{X}$ be such that $w = c\{(1-t)x + ty\}$, where $c \geq 0$ and $1 \geq t \geq t_1$. We show that $x \perp_D^{\epsilon} w$. Let $\lambda > 0$. Choose

$\eta_1 = 1 + \lambda c(1-t) \geq 1$ and choose $\lambda_1 = \frac{\lambda ct}{\eta_1}$. Then $x + \lambda w = \eta_1(x + \lambda_1 y)$. Hence

$$\|x + \lambda w\| = |\eta_1|\|x + \lambda_1 y\| \geq \|x + \lambda_1 y\| \geq 1 \geq \sqrt{1-\epsilon^2}$$

Let $\lambda < 0$. Write $w = b\{(1-s)x + sv_1\}$, where $b \geq 0$, $s \geq 1$. Choose $\eta_2 = 1 + \lambda b(1-s) \geq 1$ and $\lambda_2 = \frac{\lambda bs}{\eta_2}$. Then $x + \lambda w = \eta_2(x + \lambda_2 v_1)$. Hence

$$\|x + \lambda w\| = |\eta_2|\|x + \lambda_2 v_1\| \geq \|x + \lambda_2 v_1\| \geq \sqrt{1-\epsilon^2}$$

Thus, for all $\lambda \in \mathbb{R}$, $\|x + \lambda w\| \geq \sqrt{1-\epsilon^2} \Rightarrow x \perp_D^{\epsilon} w$. Similarly, if $z = c\{-(1-t)x + ty\}$, where $c > 0$ and $t < t_2$ then $x \not\perp_D^{\epsilon} z$ and if $w = d\{-(1-t)x + ty\}$, where $d \geq 0$ and $1 \geq t \geq t_2$ then $x \perp_D^{\epsilon} w$. Therefore, $F(x, \epsilon) = K \cup (-K)$. $\square$

Remark 2.2. We further observe that if $\epsilon \in (0,1)$ then $t_1 < 1$, for if $z \in S_{\mathbb{X}}$ and $\|z - y\| \leq \frac{1-\sqrt{1-\epsilon^2}}{1+\sqrt{1-\epsilon^2}}$ then $\|x + \lambda z\| \geq |\|\lambda z\| - \|x\|| \geq |\lambda| - 1 > \sqrt{1-\epsilon^2}$, whenever $|\lambda| > 1 + \sqrt{1-\epsilon^2}$. Also, if $|\lambda| \leq 1 + \sqrt{1-\epsilon^2}$ then

$$\begin{aligned}\|x + \lambda z\| &= \|x + \lambda y + \lambda(z-y)\| \geq \|x + \lambda y\| - |\lambda|\|z - y\| \\ &\geq 1 - (1 + \sqrt{1-\epsilon^2})\|z - y\| \geq \sqrt{1-\epsilon^2},\end{aligned}$$

so that $x \perp_D^{\epsilon} z$. This shows that there exists some t, $0 < t < 1$, such that $x \perp_D^{\epsilon} \{(1-t)x + ty\}$. Similarly, if $\epsilon \in (0,1)$ then $t_2 < 1$.

Remark 2.3. From the proof of the Theorem 2.1, it is clear that if $\epsilon > 0$ then both v_1, v_2 can not be simultaneously in x^+ (or x^-). In fact, for $\epsilon > 0$, exactly one of v_1, v_2 will be in x^+ and the other one will be in x^-. On the other hand, it is easy to observe that in two-dimensional smooth Banach spaces, $v_1 = v_2$, if $\epsilon = 0$.

Next, let us consider the set $S(x, \epsilon) = \{z \in S_{\mathbb{X}} : \ \inf_{\lambda \in \mathbb{R}} \|x + \lambda z\| = \sqrt{1-\epsilon^2}\}$. In context of the previous theorem, it is possible to obtain a nice characterization of $S(x, \epsilon)$. We accomplish the goal in the following theorem:

Theorem 2.4. *Let $\mathbb{X}$ be a two-dimensional Banach space and v_1, v_2 be as in Theorem 2.1. Then $S(x, \epsilon) = \{\pm v_1, \pm v_2\}$, if $\epsilon \in (0,1)$ and $S(x, \epsilon) = \{y \in S_{\mathbb{X}} : x \perp_B y\}$ if $\epsilon = 0$.*

Proof. First, let $\epsilon \in (0,1)$. Note that every element in $S_{\mathbb{X}}$ can be written as some multiple of a convex combination of v_2, y or y, v_1 or v_1, x or $x, -v_2$. From Theorem 2.1, it is clear that $\inf_{\lambda} \|x + \lambda z\| < \sqrt{1-\epsilon^2}$ if z is some multiple of a convex combination of v_1, x or $x, -v_2$. Now, suppose $z = \frac{(1-t)y + tv_1}{\|(1-t)y + tv_1\|}$ for some $0 \leq t < 1$. Let $\lambda < 0$. Write $z = a\{(1-s)x + sv_1\}$, where $s > 1$ and $a = \frac{1}{\|(1-s)x + sv_1\|} > 0$. Let $\eta = 1 + a\lambda(1-s) > 1$ and $\lambda' = \frac{a\lambda s}{\eta}$. Then $x + \lambda z = \eta\{x + \lambda' v_1\} \Rightarrow \|x + \lambda z\| > \|x + \lambda' v_1\| \geq \sqrt{1-\epsilon^2}$. Now, let $\lambda \geq 0$. Write $z = b\{(1-s')x + s'y\}$, where $0 < s' \leq 1$ and $b = \frac{1}{\|(1-s')x + s'y\|}$. Let $\eta_1 = 1 + \lambda b(1-s') \geq 1$ and $\lambda_1 = \frac{\lambda bs'}{\eta_1}$. Then $x + \lambda z = \eta_1\{x + \lambda_1 y\} \Rightarrow \|x + \lambda z\| \geq \|x + \lambda_1 y\| \geq 1 > \sqrt{1-\epsilon^2}$. Therefore,

$\inf_{\lambda\in\mathbb{R}} \|x + \lambda z\| > \sqrt{1-\epsilon^2}$. Similarly, if $z = \frac{(1-t)y+tv_2}{\|(1-t)y+tv_2\|}$ for some $0 \leq t < 1$ then we can show that $\inf_\lambda \|x + \lambda z\| > \sqrt{1-\epsilon^2}$. Thus, $z \in S(x,\epsilon)$ if and only if $z \in \{\pm v_1, \pm v_2\}$.

The second part of the theorem is obvious. □

Let $P_{x,y}(\epsilon)$ denote the restriction of $F(x,\epsilon)$ to the subspace spanned by x and y. Equipped with the characterization of $F(x,\epsilon)$ in two-dimensional Banach spaces, it is now possible to obtain a complete description of $F(x,\epsilon)$ in any normed space. The following theorem, the proof of which is immediate, illustrates our claim.

Theorem 2.5. *Let $\mathbb{X}$ be a normed space. Let $x \in \mathbb{X}$ and $\epsilon \in [0,1)$. Then $F(x,\epsilon) = \bigcup_{y\in\mathbb{X}} P_{x,y}(\epsilon)$. In particular, $F(x,\epsilon)$ is a union of two-dimensional normal cones.*

Let us now turn our attention to the converse of Theorem 2.1. As promised in the introduction, we prove that for any normal cone K in a two-dimensional smooth Banach space $\mathbb{X}$, there exists some $x \in S_{\mathbb{X}}$ and some $\epsilon \in [0,1)$ such that $K \cup (-K) = F(x,\epsilon)$.

Theorem 2.6. *Let $\mathbb{X}$ be a two-dimensional smooth Banach space. Let K be a normal cone in $\mathbb{X}$. Then there exists $x \in S_{\mathbb{X}}$ and $\epsilon \in [0,1)$ such that $F(x,\epsilon) = K \cup (-K)$.*

Proof. Let the cone K be determined by $v_1,\ v_2 \in S_{\mathbb{X}}$. Note that since K is a normal cone, $v_1 \neq -v_2$. First, let us assume that $v_1 \neq v_2$. Consider the two sets

$$W_1 = \{x \in S_{\mathbb{X}} :\ \inf_\lambda \|x + \lambda v_1\| > \inf_\lambda \|x + \lambda v_2\|\},$$
$$W_2 = \{x \in S_{\mathbb{X}} :\ \inf_\lambda \|x + \lambda v_2\| > \inf_\lambda \|x + \lambda v_1\|\}.$$

Clearly, $W_1 \neq \emptyset$ and $W_2 \neq \emptyset$, since $v_2 \in W_1$ and $v_1 \in W_2$. We show that W_1 is open. Let $z \in W_1$. Let $l_1 = \inf_\lambda \|z + \lambda v_1\| > \inf_\lambda \|z + \lambda v_2\| = l_2$. Choose $\epsilon' > 0$ such that $\epsilon' < \frac{l_1 - l_2}{2}$. We claim that $B(z,\epsilon') \cap S_{\mathbb{X}} \subseteq W_1$. By standard compactness argument, there exists λ_0 such that $\|z + \lambda_0 v_2\| = l_2$. Now, suppose that $w \in B(z,\epsilon') \cap S_{\mathbb{X}}$. Then for any $\lambda \in \mathbb{R}$, we have,

$$\|w + \lambda v_1\| = \|(z + \lambda v_1) - (z - w)\| \geq \|z + \lambda v_1\| - \|z - w\| > l_1 - \epsilon'.$$

Also,

$$\|w + \lambda_0 v_2\| = \|(z + \lambda_0 v_2) + (w - z)\| \leq \|z + \lambda_0 v_2\| + \|w - z\| < l_2 + \epsilon'.$$

Therefore, $\inf_\lambda \|w + \lambda v_1\| > l_1 - \epsilon' > l_2 + \epsilon' > \inf_\lambda \|w + \lambda v_2\|$. This implies that $w \in W_1$. Hence, $B(z,\epsilon') \cap S_{\mathbb{X}} \subseteq W_1$. Therefore, W_1 is open in $S_{\mathbb{X}}$. Similarly it can be shown that W_2 is open. Again, $W_1 \cap W_2 = \emptyset$. Since $S_{\mathbb{X}}$ is connected, $W_1 \cup W_2 \neq S_{\mathbb{X}}$. Choose $x \in S_{\mathbb{X}} \setminus (W_1 \cup W_2)$. Then clearly $\inf_\lambda \|x + \lambda v_1\| = \inf_\lambda \|x + \lambda v_2\|$. We would like to remark that since $\mathbb{X}$ is smooth, and $v_1 \neq \pm v_2$, x can not be Birkhoff-James orthogonal to both v_1 and v_2. Since $\inf_\lambda \|x + \lambda v_1\| = \inf_\lambda \|x + \lambda v_2\|$, it is now easy to see that x is not Birkhoff-James orthogonal to either of v_1, v_2. Therefore, there exists $\epsilon \in (0,1)$ such that $\inf_\lambda \|x+\lambda v_1\| = \inf_\lambda \|x+\lambda v_2\| = \sqrt{1-\epsilon^2}$. Now, by Theorem 2.1 and Theorem 2.4, it is clear that $F(x,\epsilon) = K \cup (-K)$.

Let us now consider the case $v_1 = v_2$. Clearly, K is simply a half-line in this case, given by $K = \{\lambda v_1 : \lambda \geq 0\}$. By Theorem 2.3 of James [5], there exists $x \in S_{\mathbb{X}}$ such that $x \perp_B v_1 (= v_2)$. Furthermore, since $\mathbb{X}$ is smooth, if $y \in S_{\mathbb{X}}$ is such that $x \perp_B y$, then $y = \pm v_1$. Therefore, by choosing $\epsilon = 0$, it is now immediate that $F(x,\epsilon) = K \cup (-K)$. This establishes the theorem. $\square$

In the following example, we illustrate the fact that the smoothness assumption in Theorem 2.6 can not be dropped.

Example 2.7. Let $\mathbb{X} = l_\infty^2$. Let K be the normal cone determined by $(-\frac{1}{2}, 1)$ and $(-1,1)$. If possible, suppose that there exists $x \in S_{\mathbb{X}}$ and $\epsilon \in [0,1)$ such that $F(x,\epsilon) = K \cup (-K)$. Let $x \perp_B z$ and $z \in S_{\mathbb{X}}$. Then $z \in K \cup (-K)$. Without loss of generality assume that $z \in K$. Then $z = (1-t)(-\frac{1}{2}, 1) + t(-1,1)$ for some $t \in [0,1]$. Then clearly $x = \pm(1,1)$. Since $x \perp_B (-1,0)$, we must have, $(-1,0) \in F(x,\epsilon)$. But $(-1,0) \notin K \cup (-K)$. This contradiction proves that there does not exist any $x \in S_{\mathbb{X}}$ and $\epsilon \in [0,1)$ such that $F(x,\epsilon) = K \cup (-K)$.

In the next theorem, we prove a uniqueness result for $F(x,\epsilon)$ in a two-dimensional Banach space for $\epsilon \in (0,1)$.

Theorem 2.8. *Let $\mathbb{X}$ be a two-dimensional Banach space. Let $x_1, x_2 \in S_{\mathbb{X}}$ and $\epsilon_1, \epsilon_2 \in (0,1)$. If $F(x_1,\epsilon_1) = F(x_2,\epsilon_2)$ then $x_1 = \pm x_2$ and $\epsilon_1 = \epsilon_2$.*

Proof. We prove the theorem in the following two steps.

Step 1: If $F(x,\epsilon) = F(y,\epsilon)$ for some $x, y \in S_{\mathbb{X}}$, then $x = \pm y$.

Let $F(x,\epsilon) = F(y,\epsilon) = K \cup (-K)$, where K is the normal cone determined by $v_1, v_2 \in S_{\mathbb{X}}$. It is clear from the proof of the Theorem 2.1 that x is either in the cone determined by $v_1, -v_2$ or in the cone determined by $-v_1, v_2$. Moreover, the same is true for y also. Assume that x is in the cone determined by $v_1, -v_2$ so that $x = a\{(1-t_1)v_1 - t_1 v_2\}$ for some $t_1 \in (0,1)$ and $a > 0$. Suppose y is in the cone determined by $v_1, -v_2$ and $y = b\{(1-t)x - tv_2\}$ where $b > 0$ and $0 \leq t \leq 1$. Clearly $t \neq 1$. If possible, suppose that $t \neq 0$. Let $(x + \beta v_2) \perp_B v_2$ and $(y + \mu v_1) \perp_B v_1$. Then from the Theorem 2.4, it is clear that $\|x + \beta v_2\| = \|y + \mu v_1\| = \sqrt{1-\epsilon^2}$. Clearly, $x + \beta v_2 = k(y + \lambda v_2)$ for some scalars k, λ. Thus, $x + \beta v_2 = kb(1-t)x - kbtv_2 + k\lambda v_2 \Rightarrow k = \frac{1}{b(1-t)}$. If $k > 1$ then $\sqrt{1-\epsilon^2} = \|x + \beta v_2\| = |k|\|y + \lambda v_2\| > \|y + \lambda v_2\| \geq \sqrt{1-\epsilon^2}$, a contradiction. Therefore, $k = \frac{1}{b(1-t)} \leq 1$. Again, $y + \mu v_1 = k_1(x + \lambda_1 v_1)$ for some scalars k_1, λ_1. Therefore, $b(1-t)x - bt[\frac{1-t_1}{t_1} v_1 - \frac{1}{at_1} x] + \mu v_1 = k_1 x + k_1 \lambda_1 v_1 \Rightarrow k_1 = b(1-t) + \frac{bt}{at_1} \Rightarrow k_1 > 1$. Therefore, $\sqrt{1-\epsilon^2} = \|y + \mu v_1\| = |k_1|\|x + \lambda_1 v_1\| > \|x + \lambda_1 v_1\| \geq \sqrt{1-\epsilon^2}$, a contradiction. Hence we must have $t = 0$. However, since $x, y \in S_{\mathbb{X}}$, this implies that $x = y$. Similarly, $y = b\{(1-t)x + tv_1\}$ for some $b > 0$ implies that $t = 0$ and once again we have, $y = x$. On the other hand, if y is in the cone determined by $-v_1, v_2$ then it can be shown using similar arguments that $x = -y$.

Step 2: Claim $F(x_1, \epsilon_1) = F(x_2, \epsilon_2) \Rightarrow \epsilon_1 = \epsilon_2$.

Let $F(x_1, \epsilon_1) = F(x_2, \epsilon_2) = K \cup (-K)$, where K is the normal cone determined by $v_1, v_2 \in S_{\mathbb{X}}$. Then by Theorem 2.4, it is clear that $\inf_{\lambda \in \mathbb{R}} \|x_2 + \lambda v_1\| = \inf_{\lambda \in \mathbb{R}} \|x_2 + \lambda v_2\| = \sqrt{1 - {\epsilon_2}^2}$ and $\inf_{\lambda \in \mathbb{R}} \|x_1 + \lambda v_1\| = \inf_{\lambda \in \mathbb{R}} \|x_1 + \lambda v_2\| = \sqrt{1 - {\epsilon_1}^2}$. Therefore, if $x_1 = \pm x_2$ then $\epsilon_1 = \epsilon_2$ and we are done.

Hence assume that $x_1 \neq \pm x_2$. Clearly $x_1,\ x_2 \notin K \cup (-K)$. From remark 2.3, we have, either $v_1 \in {x_1}^+,\ v_2 \in {x_1}^-$ or $v_1 \in {x_1}^-,\ v_2 \in {x_1}^+$. Without loss of generality, assume that $v_1 \in {x_1}^+,\ v_2 \in {x_1}^-$. First suppose that $x_2 = \frac{(1-t)x_1 + tv_1}{\|(1-t)x_1 + tv_1\|}$ for some $0 < t < 1$. Then it can be easily verified that $v_1 \in {x_2}^+,\ v_2 \in {x_2}^-$. Now,

$$\begin{aligned} 1 = \Big\|\frac{(1-t)x_1 + tv_1}{\|(1-t)x_1 + tv_1\|}\Big\| &= \frac{1-t}{\|(1-t)x_1 + tv_1\|}\Big\|x_1 + \frac{t}{1-t}v_1\Big\| \\ &\geq \frac{1-t}{\|(1-t)x_1 + tv_1\|}, \end{aligned}$$

since $v_1 \in {x_1}^+$. Now for any $\lambda \in \mathbb{R}$,

$$\begin{aligned} \|x_2 + \lambda v_1\| &= \Big\|\frac{(1-t)x_1 + tv_1}{\|(1-t)x_1 + tv_1\|} + \lambda v_1\Big\| \\ &= \frac{1-t}{\|(1-t)x_1 + tv_1\|}\Big\|x_1 + \Big(\frac{t}{1-t} + \frac{\lambda\|(1-t)x_1 + tv_1\|}{1-t}\Big)v_1\Big\| \\ \Rightarrow \inf_{\lambda \in \mathbb{R}} \|x_2 + \lambda v_1\| &= \frac{1-t}{\|(1-t)x_1 + tv_1\|} \inf_{\lambda \in \mathbb{R}} \|x_1 + \lambda v_1\| \leq \inf_{\lambda \in \mathbb{R}} \|x_1 + \lambda v_1\| \\ \Rightarrow \sqrt{1 - {\epsilon_2}^2} &\leq \sqrt{1 - {\epsilon_1}^2}. \end{aligned}$$

We can write $x_2 = \frac{(1-t)x_1 + tv_2}{\|(1-t)x_1 + tv_2\|}$ for some $0 < t < 1$. This implies that

$$\begin{aligned} x_1 &= \frac{\|(1-t)x_1 + tv_2\|}{1-t}x_2 - \frac{t}{1-t}v_2 \\ \Rightarrow 1 &= \frac{\|(1-t)x_1 + tv_2\|}{1-t}\Big\|x_2 - \frac{t}{\|(1-t)x_1 + tv_2\|}v_2\Big\| \geq \frac{\|(1-t)x_1 + tv_2\|}{1-t}, \end{aligned}$$

since $v_2 \in {x_2}^-$. Therefore, for all $\lambda \in \mathbb{R}$,

$$\begin{aligned} x_1 + \lambda v_2 &= \frac{\|(1-t)x_1 + tv_2\|}{1-t}x_2 - \frac{t}{1-t}v_2 + \lambda v_2 \\ \Rightarrow \inf_{\lambda \in \mathbb{R}} \|x_1 + \lambda v_2\| &= \frac{\|(1-t)x_1 + tv_2\|}{1-t} \inf_{\lambda \in \mathbb{R}} \|x_2 + \lambda v_2\| \leq \inf_{\lambda \in \mathbb{R}} \|x_2 + \lambda v_2\| \\ \Rightarrow \sqrt{1 - {\epsilon_1}^2} &\leq \sqrt{1 - {\epsilon_2}^2}. \end{aligned}$$

Therefore, $\epsilon_1 = \epsilon_2$. Similarly, if we assume that $x_2 = \frac{(1-t)x_1 - tv_2}{\|(1-t)x_1 - tv_2\|}$, for some $0 < t < 1$, then we can apply similar arguments to prove that $\epsilon_1 = \epsilon_2$. This completes the proof of the theorem. $\square$

Remark 2.9. For $\epsilon = 0$, $F(x_1, 0) = F(x_2, 0) \Rightarrow x_1 = \pm x_2$ holds if the space is strictly convex . Suppose $F(x_1, 0) = F(x_2, 0)$. Then there exists z such that $x_1 \perp_B z$ and $x_2 \perp_B z$. Being strictly convex, Birkhoff-James orthogonality is left unique and so $x_1 = \pm x_2$.

Remark 2.10. It follows from Theorem 2.6 and Theorem 2.8 that given any normal cone K in a two-dimensional smooth Banach space $\mathbb{X}$, there exists a unique (upto multiplication by ± 1) $x \in S_{\mathbb{X}}$ and a unique $\epsilon \in [0, 1)$ such that $F(x, \epsilon) = K \cup (-K)$. This observation strengthens Theorem 2.6 to a considerable extent.

Now we are in a position to prove the promised uniqueness theorem for $F(x, \epsilon)$ in any normed space.

Theorem 2.11. *Let $\mathbb{X}$ be a normed space. Let $x_1, x_2 \in S_{\mathbb{X}}$ and $\epsilon_1, \epsilon_2 \in (0, 1)$. If $F(x_1, \epsilon_1) = F(x_2, \epsilon_2)$ then $x_1 = \pm x_2$ and $\epsilon_1 = \epsilon_2$.*

Proof. The proof of the theorem essentially follows from Theorem 2.5 and Theorem 2.8. □

Let us now shift our attention to the structure of $G(x, \epsilon)$ in normed spaces. The following two lemmas are required to obtain the desired characterization.

Lemma 2.12. *Let $\mathbb{X}$ be a two-dimensional Banach space. Let $(\theta \neq) z \in \mathbb{X}$, $\epsilon \in [0, 1)$. Then $\overline{B}(z, \epsilon) \cap S_{\mathbb{X}}$ is either empty or path connected.*

Proof. Note that $\overline{B}(z, \epsilon) \cap S_{\mathbb{X}} = \emptyset$ if and only if $\|z - \frac{z}{\|z\|}\| > \epsilon$. Assume $\overline{B}(z, \epsilon) \cap S_{\mathbb{X}} \neq \emptyset$. Let $z_0 = \frac{z}{\|z\|}$. Then $z_0 \in \overline{B}(z, \epsilon) \cap S_{\mathbb{X}}$.

Suppose that $y \in \overline{B}(z, \epsilon) \cap S_{\mathbb{X}}$. We show that there exists a path in $\overline{B}(z, \epsilon) \cap S_{\mathbb{X}}$ joining y and z_0. Let $u = \frac{z+y}{2}$. Then $\|z - u\| = \|y - u\| \leq \frac{\epsilon}{2}$. Let $v_t = \frac{(1-t)y+tz}{\|(1-t)y+tz\|}$, where $t \in [0, 1]$. Claim that $v_t \in \overline{B}(z, \epsilon) \cap S_{\mathbb{X}}$. Let $v'_t = (1-t)y + tz$. Then either $v'_t = (1-r)z + ru$ for some $r \in [0, 1]$ or $v'_t = (1-s)u + sy$ for some $s \in [0, 1]$. Assume $v'_t = (1-r)z + ru$. Then

$$\begin{aligned}\|z - v'_t\| + \|v'_t - u\| &= \|z - (1-r)z - ru\| + \|(1-r)z + ru - u\| \\ &= r\|z - u\| + (1-r)\|z - u\| = \|z - u\| \leq \frac{\epsilon}{2}.\end{aligned}$$

Now,
$$\begin{aligned}\|v_t - v'_t\| &= \|v_t - \|v'_t\| v_t\| = |1 - \|v'_t\|| = |\|y\| - \|v'_t\|| \\ &\leq \|y - v'_t\| \leq \|y - u\| + \|u - v'_t\| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} - \|z - v'_t\| \\ \implies \|z - v_t\| &\leq \|z - v'_t\| + \|v_t - v'_t\| \leq \epsilon.\end{aligned}$$

Hence $v_t \in \overline{B}(z, \epsilon) \cap S_{\mathbb{X}}$. Next assume that $v'_t = (1-s)u + sy$ for some $s \in [0, 1]$.

Then
$$\begin{aligned}\|y - v'_t\| + \|v'_t - u\| &= \|y - (1-s)u - sy\| + \|(1-s)u + sy - u\| \\ &= (1-s)\|y - u\| + s\|y - u\| = \|y - u\| \leq \frac{\epsilon}{2}.\end{aligned}$$

Again $\quad \|v_t - v_t'\| = \|v_t - \|v_t'\|v_t\| = |1 - \|v_t'\|| = |\|y\| - \|v_t'\|| \leq \|y - v_t'\|.$

Hence
$$\|z - v_t\| \leq \|z - u\| + \|u - v_t\| \leq \frac{\epsilon}{2} + \|u - v_t'\| + \|v_t' - v_t\|$$
$$\leq \frac{\epsilon}{2} + \|u - v_t'\| + \|v_t' - y\| \leq \epsilon.$$

Thus, $v_t \in \overline{B}(z,\epsilon) \cap S_{\mathbb{X}}$. Consider the map $f : [0,1] \longrightarrow \overline{B}(z,\epsilon) \cap S_{\mathbb{X}}$ by $f(t) = v_t$. Then f is a path in $\overline{B}(z,\epsilon) \cap S_{\mathbb{X}}$ joining $f(0) = y$ and $f(1) = z_0$. Hence $\overline{B}(z,\epsilon) \cap S_{\mathbb{X}}$ is path connected. This proves the lemma. □

Lemma 2.13. *Let $\mathbb{X}$ be a two-dimensional Banach space. Let $z \in S_{\mathbb{X}}$ and $\epsilon \in [0,1)$. Then $A = [\bigcup_{\alpha \in \mathbb{R}} \overline{B}(\alpha z, \epsilon)] \cap S_{\mathbb{X}}$ has at most two components.*

Proof. If $0 \leq \alpha < 1 - \epsilon$, then for any $w \in S_{\mathbb{X}}$, we have, $\|\alpha z - w\| \geq \|w\| - \|\alpha z\| = 1 - \alpha > \epsilon$. If $\alpha > 1 + \epsilon$, then for any $w \in S_{\mathbb{X}}$, $\|\alpha z - w\| \geq \|\alpha z\| - \|w\| = \alpha - 1 > \epsilon$. Similarly, if $-1 + \epsilon < \alpha \leq 0$ or $\alpha < -1 - \epsilon$, then for any $w \in S_{\mathbb{X}}$, we have, $\|\alpha z - w\| > \epsilon$. Therefore,

$$A = [\{\bigcup_{\alpha \in [1-\epsilon, 1+\epsilon]} \overline{B}(\alpha z, \epsilon)\} \cap S_{\mathbb{X}}] \bigcup [\{\bigcup_{\alpha \in [-1-\epsilon, -1+\epsilon]} \overline{B}(\alpha z, \epsilon)\} \cap S_{\mathbb{X}}].$$

Let $A_1 = [\{\bigcup_{\alpha \in [1-\epsilon, 1+\epsilon]} \overline{B}(\alpha z, \epsilon)\} \cap S_{\mathbb{X}}]$, and $A_2 = [\{\bigcup_{\alpha \in [-1-\epsilon, -1+\epsilon]} \overline{B}(\alpha z, \epsilon)\} \cap S_{\mathbb{X}}]$.

Clearly, to prove the lemma, it is sufficient to show that both A_1, A_2 are connected. Now, $A_1 = [\{\bigcup_{\alpha \in [1-\epsilon, 1+\epsilon]} \overline{B}(\alpha z, \epsilon)\} \cap S_{\mathbb{X}}]$. It is easy to see that for each $\alpha \in [1-\epsilon, 1+\epsilon]$, $z \in \overline{B}(\alpha z, \epsilon) \cap S_{\mathbb{X}}$. Also, by Lemma 2.12, $\overline{B}(\alpha z, \epsilon) \cap S_{\mathbb{X}}$ is path connected for each $\alpha \in [1-\epsilon, 1+\epsilon]$. Since union of connected sets having nonempty intersection is connected, A_1 is connected. Similarly, we can show that A_2 is connected. This completes the proof of the lemma. □

Let us now exhibit the interconnection between $G(x,\epsilon)$ and normal cones in a two-dimensional Banach space.

Theorem 2.14. *Let $\mathbb{X}$ be a two-dimensional Banach space. Let $x \in S_{\mathbb{X}}$ be a smooth point and $\epsilon \in [0,1)$. Then there exists a normal cone K in $\mathbb{X}$ such that $G(x,\epsilon) = K \cup (-K)$.*

Proof. It follows from the characterization of approximate ϵ-Birkhoff-James orthogonality given by Chmieliński et al. in [3] that

$$G(x,\epsilon) \cap S_{\mathbb{X}} = \{y \in S_{\mathbb{X}} : \|\alpha z - y\| \leq \epsilon \text{ for some } \alpha \in \mathbb{R}\},$$

where $z \in S_{\mathbb{X}}$ is the unique (up to multiplication by ± 1) vector such that $x \perp_B z$, since $x \in S_{\mathbb{X}}$ is smooth. Therefore, $G(x,\epsilon) \cap S_{\mathbb{X}} = [\bigcup_{\alpha \in \mathbb{R}} \overline{B}(\alpha z, \epsilon)] \cap S_{\mathbb{X}}$. Applying Lemma 2.13, we see that $G(x,\epsilon) \cap S_{\mathbb{X}}$ has at most two connected components. However, it is easy to observe that $\pm z \in G(x,\epsilon) \cap S_{\mathbb{X}}$ and $\pm x \notin G(x,\epsilon) \cap S_{\mathbb{X}}$.

Therefore, $G(x,\epsilon) \cap S_{\mathbb{X}}$ has exactly two connected components, one of them containing z and the other containing $-z$. Since $G(x,\epsilon) \cap S_{\mathbb{X}}$ is closed, each of the components must be closed in $S_{\mathbb{X}}$. It also follows from the property of $G(x,\epsilon)$ that the two components of $G(x,\epsilon)$ are symmetric with respect to origin. Since $\mathbb{X}$ is a two-dimensional Banach space, any connected component of $S_{\mathbb{X}}$, not containing two antipodal points, must be an arc of $S_{\mathbb{X}}$. Therefore, there exists $v_1, v_2 \in S_{\mathbb{X}}$ such that

$$G(x,\epsilon) \cap S_{\mathbb{X}} = \{\frac{(1-t)v_1 + tv_2}{\|(1-t)v_1 + tv_2\|} : t \in [0,1]\} \cup \{\frac{-(1-t)v_1 - tv_2}{\| -(1-t)v_1 - tv_2\|} : t \in [0,1]\}.$$

Let K be the normal cone determined by v_1, v_2. Then from the properties of $G(x,\epsilon)$, it is clear that $G(x,\epsilon) = K \cup (-K)$. This completes the proof of the theorem. $\square$

Let $Q_{x,y}(\epsilon)$ denote the restriction of $G(x,\epsilon)$ to the subspace spanned by x and y. The following theorem, that describes the structure of $G(x,\epsilon)$ in any normed space, is immediate.

Theorem 2.15. *Let $\mathbb{X}$ be a normed space. Let $x \in \mathbb{X}$ be a smooth point and $\epsilon \in [0,1)$. Then $G(x,\epsilon) = \bigcup_{y \in \mathbb{X}} Q_{x,y}(\epsilon)$. In particular, $G(x,\epsilon)$ is a union of two-dimensional normal cones.*

Remark 2.16. Theorem 2.5 and Theorem 2.15 together imply that in a normed space, both $F(x,\epsilon)$ and $G(x,\epsilon)$ are unions of two-dimensional normal cones. However, since these two different types of approximate Birkhoff-James orthogonality do not coincide in a general normed space, it follows that the constituent two-dimensional normal cones for $F(x,\epsilon)$ and $G(x,\epsilon)$ may not be identical.

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References

[1] G. Birkhoff: Orthogonality in linear metric spaces, Duke Math. J. 1 (1935) 169–172.

[2] J. Chmieliński: On an ϵ-Birkhoff orthogonality, J. Inequal. Pure Appl. Math. 6(3) (2005), article no. 79.

[3] J. Chmieliński, T. Stypuła, P. Wójcik: Approximate orthogonality in normed spaces and its applications, Linear Algebra Appl. 531 (2017) 305–317.

[4] S. S. Dragomir: On approximation of continuous linear functionals in normed linear spaces, An. Univ. Timisoara Ser. Stiint. Mat. 29 (1991) 51–58.

[5] R. C. James: Orthogonality and linear functionals in normed linear spaces, Trans. Amer. Math. Soc. 61 (1947) 265–292.

[6] D. Sain: Birkhoff-James orthogonality of linear operators on finite dimensional Banach spaces, J. Math. Anal. Appl. 447 (2017) 860–866.