Journal of Convex Analysis
Volume 26 (2019), No. 2, 515–526

Smoothness in some Banach Spaces of Operators and Vector Valued Functions

Enrique Jordá*
*Dep. de Matemática Aplicada, Universidad Politécnica de Valencia,
Plaza Ferrándiz y Carbonell 2, 03801 Alcoy, Spain
ejorda@mat.upv.es*

Ana María Zarco
*Dep. de Matemática Aplicada, Universidad Politécnica de Valencia,
Plaza Ferrándiz y Carbonell 2, 03801 Alcoy, Spain
anzargar@upv.es*

Received: October 25, 2017
Revised manuscript received: May 7, 2018
Accepted: May 8, 2018

A well known criterion of Šmulyan states that the norm $\|\cdot\|$ of a real Banach space X is Gâteaux differentiable at $x \in X$ if and only if there is $x^* \in S_{X^*}$ which is w^*-exposed by x in B_{X^*} and that the norm is Fréchet differentiable at x if and only if there is $x^* \in S_{X^*}$ which is w^*-strongly exposed in B_{X^*} by x. We show that in this criterion B_{X^*} can be replaced by a convenient smaller set, and we apply this extended criterion to characterize the points of Gâteaux and Fréchet differentiability of the norm in epsilon products of Banach spaces, extending previous work of Heinrich. As a consequence we get some results of smoothness of the norm in some Banach spaces of continuous and harmonic vector valued functions.

Keywords: Banach spaces, Frechet and Gateaux differentiability, epsilon products.

2010 Mathematics Subject Classification: 46B20, 46B50

1. Introduction and preliminaries

Throughout this paper X and Y stand for real Banach spaces, and X^* for the topological dual space of X. We make a common abuse of notation and $\|\cdot\|$ will denote the norm of different Banach spaces to which we refer in each situation. Let f be a real valued function defined on X. We say that f is *Gâteaux differentiable at x* if for each $h \in X$ the limit

$$f'(x)(h) := \lim_{t\to 0} \frac{f(x+th) - f(x)}{t}$$

exists and is a linear continuous function in h (i.e. $f'(x) \in X^*$). If, in addition, the limit above is uniform in $h \in S_X$, we say that f is *Fréchet differentiable at x*.

*The research of the first author was partially supported by the project MTM2016-76647-P.

ISSN 0944-6532 / \$ 2.50

Equivalently, f is Fréchet differentiable at x if there exists $f'(x) \in X^*$ such that

$$\lim_{y \to 0} \frac{f(x+y) - f(x) - f'(x)(y)}{\|y\|} = 0.$$

Let C be a bounded subset of X^*. A functional $x^* \in X^*$ is said to be *w^*-exposed* in $C \subseteq X^*$ by $x \in X$ when x^* is the unique functional in C such that $x^*(x) = \max\{y^*(x) : y^* \in C\}$. If $x^* \in X^*$ is w^*-exposed in C by x and $C \ni x_n^* \to x^*$ in norm whenever $x_n^*(x) \to x^*(x)$ then x^* is said to be *w^*-strongly exposed* in C. Šmulyan's theorem [4, Theorem I.1.4, Corollary I.1.5] asserts that the norm in X is Gâteaux differentiable at $x \in S_X$ if and only if there is a unique $x^* \in S_{X^*}$ such that $x^*(x) = 1$, i.e. x^* is w^*-exposed in B_{X^*} by x, and the norm in X is Fréchet differentiable at $x \in S_X$ if and only if there is a unique $x^* \in S_{X^*}$ satisfying: for every $\varepsilon > 0$ there exists $\delta > 0$ such that $y^* \in B_{X^*}$ and $y^*(x) > 1 - \delta$ imply $\|y^* - x^*\| < \varepsilon$, which is equivalent to say that x^* is w^*-strongly exposed by x.

In this paper we extend this criterion. More precisely we show that the ball B_{X^*} can be replaced by any proper subset $C \subsetneq B_{X^*}$ which is a *James boundary* (we give the definition in the next section) and w^*-closed. We apply this criterion to characterize the Fréchet and Gâteaux differentiability in epsilon products of Banach spaces, extending previous work of Heinrich [6] and also of Hennefeld and Holub [7, 9]. As a consequence we give results on closed subspaces $F \subseteq C(K, X)$, K being a compact space and X being a Banach space, extending previuos work of Sundaresan [12]. The same approach permits us to extend recent results of Boyd and Rueda [3] about smoothness in weighted Banach spaces of holomorphic functions (regarded as real Banach spaces) to the corresponding spaces of harmonic vector-valued functions.

Our notation for functional analysis and Banach space theory is standard. We refer to [5, 11]. Given a Banach space X we denote by w^* the weak star topology $\sigma(X^*, X)$ on the dual X^*. For a subset $S \subset X^*$ we denote by $\overline{\text{co}}^{w^*} S$ the w^*-closure of the convex hull of S. The subset formed by the extremal points of S is denoted by $\text{Ext}S$. Given another Banach space, $L(X, Y)$ stands for the space of bounded linear operators from X to Y and $K(X, Y)$ for the linear subspace (which is closed for the operator norm topology) of compact linear operators from X to Y. Given $T \in L(X, Y)$ we say that T attains its norm at a unique $x \in S_X$ *up to factor* if the equality $\|T(y)\| = \|T\|$ holds if and only if $y = \pm x$.

2. A Šmulyan type criterion for differentiability of the norm

Definition 2.1. A subset $J \subseteq B_{X^*}$ is called a *James boundary* of B_{X^*} if $\|x\| = \max\{x^*(x) : x^* \in J\}$ for all $x \in X$.

This is a particular case of the definition of James boundary of a weak* compact set, see, e.g.,[5, Definition 3.118].

The following result is well known.

Lemma 2.2. *Assume that* $\|\cdot\|$ *is Gâteaux differentiable at* $x \in S_X$, $x^*(x) = 1$ *and* $x^* \in S_{X^*}$. *Let* (x_n^*) *be a sequence in* B_{X^*} *such that* $x_n^*(x)$ *converges to* 1. *Then* (x_n^*) *is* w^**-convergent to* x^*.

Remark 2.3. Let $J \subseteq B_{X^*}$ be a James boundary. Hahn–Banach theorem easily implies that $\overline{\text{co}}^{w^*} J = B_{X^*}$. Milman's converse of the Krein–Milman theorem [5, Theorem 3.66] ensures that $\text{Ext}(B_{X^*}) \subseteq \overline{J}^{w^*}$. In particular if J is w^*-closed then $\text{Ext}(B_{X^*}) \subseteq J$. If $\|\cdot\|$ of X is Gâteaux differentiable at $x \in X \setminus \{0\}$ then x attains its supremum on B_{X^*} at a unique point x^*, which belongs necessarily to $\text{Ext}(B_{X^*})$.

Proposition 2.4. *Assume that there is a James boundary* $J \subseteq B_{X^*}$ *such that* $\overline{J}^{w^*} \cap S_{X^*} \subseteq J$ *and* $x^* \in X^*$ *is* w^**-exposed in* J *by* $x \in X$. *Then* $\|\cdot\|$ *is Gâteaux differentiable at* x.

Proof. The element $x \in S_X$ attains its supremum on S_{X^*} at an extreme point of B_{X^*}, say y^*. This point is in $\overline{J}^{w^*} \cap S_X^* \subseteq J$ by Remark 2.3 and the hypothesis, hence, $x^* = y^*$ by the assumption. The set $C := \{x^* \in B_{X^*} : x^*(x) = 1\}$ is w^*-compact, and the extremal points of C are easily checked to be extremal in B_{X^*}. Hence C has only one extremal point, and then Krein-Milman theorem implies that $C = \{x^*\}$, i.e. x^* is w^*-exposed in B_{X^*} by x. Therefore $\|\cdot\|$ is Gâteaux differentiable at x. □

Proposition 2.5. *Let* $J \subseteq B_{X^*}$ *be a James boundary. Assume that* $\|\cdot\|$ *is Gâteaux differentiable at* $x \in S_X$. *Let* x^* *be the differential of* $\|\cdot\|$ *at* x *(i.e.* $x^* \in S_{X^*}$ *and* $x^*(x) = 1$*). Assume that every sequence* (x_n^*) *in* $J \cap S_{X^*}$ *which* w^**-converges to* x^* *also converges to* x^* *in norm. Then the norm* $\|\cdot\|$ *of* X *is Fréchet differentiable at* x.

Proof. Assume that the norm of X is not Fréchet differentiable . Then we can find by [4, Lemma I.1.3] a sequence (h_n) of non-zero vectors in X which is norm convergent to 0 and $\varepsilon > 0$ such that

$$\|x + h_n\| + \|x - h_n\| \geq 2 + \varepsilon\|h_n\|.$$

Since J is a James boundary we can get x_n^*, $y_n^* \in J \cap S_{X^*}$ such that $x_n^*(x_0 + h_n) = \|x + h_n\|$ and $y_n^*(x - h_n) = \|x_0 - h_n\|$. Both x_n^* and y_n^* belong to S_{X^*} and also both $x_n^*(x)$ and $y_n^*(x)$ tend to 1, since $x_n^*(x) = \|x + h_n\| - x_n^*(h_n)$ and analogously for y_n^*. By Lemma 2.2 it follows that both (x_n^*) and (y_n^*) are w^*-convergent to x^*. Hence, by the assumption, $\|x_n^* - x^*\| \to 0$ and $\|y_n^* - x^*\| \to 0$. But

$$\begin{aligned}(x_n^* - y_n^*)(h_n) &= \|x + h_n\| + \|x - h_n\| - x_n^*(x) - y_n^*(x) \\ &\geq \|x + h_n\| + \|x - h_n\| - 2 \geq \varepsilon\|h_n\|.\end{aligned}$$

Hence $\|x_n^* - y_n^*\| \geq \varepsilon$ for all $n \in \mathbb{N}$, a contradiction. □

Definition 2.6. A subset $S \subseteq B_{X^*}$ is said to have the w^**-Kadets-Klee property* if a sequence (x_n^*) in S which is w*-convergent is also norm convergent.

Corollary 2.7. *If X is a Banach space and J is a James boundary such that $\overline{J}^{w^*} \cap S_{X^*} \subseteq J$ has the w^*-Kadets-Klee property then the norm of F is Fréchet differentiable at $x \in X$ if and only if it is Gâteaux differentiable. In particular this happens when J is w^*-compact and has the w^*-Kadets-Klee property.*

3. Tensor products and spaces of compact linear operators

We refer to [10, Chapter 44] for the following definitions and basic facts. Let X and Y be two Banach spaces. If $x \in X$ and $y \in Y$ the tensor $x \otimes y$ is defined as the linear operator from X^* to Y defined as $(x \otimes y)(x^*) = x^*(x)y$. This map is easily checked to be $w^* - \|\cdot\|$ continuous, and then compact. The subspace of $K(X^*, Y)$ formed by the linear operators which restricted to B_{X^*} are $w^* - \|\cdot\|$ continuous is called *ε-product of X and Y* and it is denoted by $X\varepsilon Y$. This space is complete when it is endowed by the norm operator topology. The ε- product is commutative, i.e. $X\varepsilon Y$ is isometric to $Y\varepsilon X$ by means of the transpose map. The tensor product of X and Y is the linear span of all the elements $x \otimes y$. It is denoted by $X \bigotimes Y$. The closure of the tensor product in $X\varepsilon Y$ is denoted by $X\widehat{\bigotimes}_\varepsilon Y$. X possesses the *approximation property* if and only if $X\widehat{\bigotimes}_\varepsilon Y = X\varepsilon Y$ for each Banach space Y. For a compact topological space K the space $C(K)$ of scalar valued continuous functions on K endowed with the supremum norm has the approximation property. The next lemma is proved in several steps in [10, 44 (6)]

Lemma 3.1. *$Q \in Y\varepsilon X^*$ if and only if there is $T \in K(X, Y)$ such that $Q = T^*$. Moreover, $K(X, Y)$ is isometric to $Y\varepsilon X^*$.*

As a consequence we get the following well known fact

Proposition 3.2. *The norm of $K(X, Y)$ is Gâteaux differentiable (Fréchet differentiable) at $T \in K(X, Y)$ if and only if the norm of $Y\varepsilon X^*$ is G-differentiable (F-differentiable) at T^*.*

Remark 3.3. $X^* \bigotimes Y^*$ is a subspace of $(X\varepsilon Y)^*$. An element $x^* \otimes y^*$ acts on $T \in X\varepsilon Y$ as follows: $(x^* \otimes y^*)(T) = y^*(T(x^*))$. The norm of $x^* \otimes y^*$ in $(X\varepsilon Y)^*$ is a cross norm, i.e. $\|x^* \otimes y^*\| = \|x^*\|\|y^*\|$.

Lemma 3.4. *The map $P : (B_{X^*}, w^*) \times (B_{Y^*}, w^*) \to (B_{(X\varepsilon Y)^*}, w^*)$, $(x^*, y^*) \mapsto x^* \otimes y^*$ is continuous.*

Proof. Let $(x_i^*) \subset B_{X^*}$ and $(y_i^*) \subset B_{Y^*}$ be two nets with the same index set such that (x_i^*) is w^*-convergent to $x^* \in B_{X^*}$ and (y_i^*) is w^*-convergent to $y^* \in B_{Y^*}$. We see that $(x_i^* \otimes y_i^*)$ is w^*-convergent to $x^* \otimes y^*$. Let $T \in X\varepsilon Y$. Since (x_i^*) is w^*-convergent to x^* it follows that $(T(x_i^*))$ converges in norm to $T(x^*)$. Given $\varepsilon > 0$ there exists i_0 such that $\|T(x_i^*) - T(x^*)\| \leq \varepsilon/2$ and $|y_i^*(T(x^*)) - y^*(T(x^*))| < \varepsilon/2$ for each $i \geq i_0$. Hence, for $i \geq i_0$

$$|(x_i^* \otimes y_i^*)(T) - (x^* \otimes y^*)(T)| = |y_i^*(T(x_i^*) - T(x^*)) + (y_i^* - y^*)(T(x^*))| \leq \varepsilon. \qquad \square$$

Proposition 3.5. *Let $J \subseteq B_{X^*}$ be such that $\widehat{J} := J \cup -J$ is a James boundary of B_{X^*} and $\overline{J}^{w^*} \cap S_{X^*} \subseteq J$. The set $J \bigotimes B_{Y^*} := \{x^* \otimes y^* : \ x^* \in J, \ y^* \in B_{Y^*}\}$ is a James boundary of $B_{(X\varepsilon Y)^*}$ such that $\overline{J \bigotimes B_{Y^*}}^{w^*} \cap S_{(X\varepsilon Y^*)} \subseteq J \bigotimes B_{Y^*}$.*

Proof. First we observe that $\widehat{J} \bigotimes B_{Y^*}$ is actually a James boundary of $B_{(X\varepsilon Y)^*}$. In fact, if $T \in X\varepsilon Y$, the transpose map $T^* \in Y\varepsilon X$ and satisfies that $T^*(B_{Y^*})$ is compact in X. Hence there is $y^* \in B_{Y^*}$ such that $\|T^*(y^*)\| = \|T^*\|$. Thus there is $x^* \in \widehat{J}$ such that $(x^* \otimes y^*)(T) = x^*(T^*(y^*)) = \|T^*\| = \|T\|$. From $x \otimes y = -x \otimes -y$ we get $J \bigotimes B_{X^*} = \widehat{J} \bigotimes B_{X^*}$.

Let $u \in \overline{J \bigotimes B_{Y^*}}^{w^*} \cap S_{(X\varepsilon Y)^*}$. Then there exists a net $(x_i^* \otimes y_i^*)$ in $J \bigotimes B_{Y^*}$ which is w^*-convergent to u. By passing to a convenient subnet if necessary, we can assume that there is x^* in $\overline{J}^{w^*}$ and $y^* \in B_{Y^*}$ such that (x_i^*) is w^*-convergent to x^* and (y_i^*) is w^*-convergent to y^*. By Lemma 3.4 it follows that $(x_i^* \otimes y_i^*)$ is w^*-convergent to $x^* \otimes y^*$, and hence $u = x^* \otimes y^*$. Since $u \in S_{(X\varepsilon Y)^*}$ we get $x^* \in S_{X^*}$. From the hypothesis on J we conclude $x^* \in J$. □

Theorem 3.6 and Theorem 3.8 below extend Heinrich work [6, Theorem 4.1, Corollary 4.2, Theorem 5.1, Theorem 5.2], where the same theorems are stated without proof for $X \widehat{\bigotimes}_\varepsilon Y$. The result is a proper extension when X and Y fail to have the approximation property.

Theorem 3.6. *Let X and Y be Banach spaces. Let $J \subseteq B_{X^*}$ be a James boundary such that $\overline{J}^{w^*} \cap S_{X^*} \subseteq J$. The following are equivalent*

(1) *The operator norm is Gâteaux differentiable at T in $X\varepsilon Y$ with $\|T\| = 1$.*
(2) *There exists a unique (up to a factor) $x^* \in J$ such that $\|T(x^*)\| = 1$ and the norm of Y is Gâteaux differentiable at $T(x^*)$.*

Proof. By Proposition 2.4 and Proposition 3.5 the operator norm of F is Gâteaux differentiable at T if and only if there is a unique $x^* \otimes y^* \in J \bigotimes B_{Y^*}$ such that $(x^* \otimes y^*)(T) = y^*(T(x^*)) = 1$. This implies $\|T(x^*)\| = 1$ and also that y^* is exposed in B_{Y^*} by $T(x^*)$. Both x^* and y^* are unique (up to a factor). □

As a consequence of Theorem 3.6 and Lemma 3.2 we get the following theorem also stated by Heinrich [6, Theorem 2.1,Corollary 2.2].

Theorem 3.7 (Heinrich). *Let X and Y be Banach spaces. The operator norm of $K(X, Y)$ is Gâteaux differentiable at $T \in K(X, Y)$ with $\|T\| = 1$ if and only if there exists a unique (up to factor) $y^* \in B_{Y^*}$ such that $\|T^*(y^*)\| = 1$ and the norm of X^* is Gâteaux differentiable at $T^*(y^*)$. Consequently, if X is reflexive, the norm is Gâteaux differentiable at $T \in K(X, Y)$ with $\|T\| = 1$ if and only if there exists a unique (up to factor) $x \in X$ such that $\|T(x)\| = 1$ and the norm of Y is is Gâteaux differentiable at $T(x)$.*

Theorem 3.7 is proved by Holub for $X = Y = l_2$ [9, Theorem 3.3] and by Hennefeld for $X = Y$ smooth, reflexive and with Schauder basis [7, Theorem 2.2].

Theorem 3.8. *Let X and Y be Banach spaces. Let $T \in X\varepsilon Y$ with $\|T\| = 1$, the following are equivalent*

(1) *The operator norm is Fréchet differentiable at T in $X\varepsilon Y$.*

(2) *There exists a unique (up to a factor) $x^* \in B_{X^*}$ such that $\|T(x^*)\| = 1$, the norm of Y is Fréchet differentiable at $T(x^*)$ and, the norm of X is Fréchet differentiable at $T^*(y^*)$ for y^* being the unique functional in S_{Y^*} such that $y^*(T(x^*)) = 1$ and y^* is the unique (up to a factor) vector in S_{Y^*} satisfying $\|T^*(y^*)\| = 1$.*

Proof. Suppose that the norm is Fréchet differentiable at T in X. Then the norm is Gâteaux differentiable at T and, by Theorem 3.6 there exists a unique (up to factor) $x^* \in B_{X^*}$ such that $\|T(x^*)\| = 1$ and a unique $y^* \in B_{Y^*}$ such that $y^*(T(x^*)) = 1$. Suppose that the norm of Y is not Fréchet differentiable at $T(x^*)$. Then, by the Šmulyan's theorem (see eg. [4, Corollary 1.1.5]) there exists $\varepsilon > 0$ and a sequence (y_n^*) in B_{Y^*} such that $y_n^*(T(x^*))$ tends to 1 (and this implies that (y_n^*) is w^*-convergent to y^* by Lemma 2.2), but $\|y_n^* - y^*\| > \varepsilon$. Now $y_n^* \otimes x^*$ is w^*-convergent to $y^* \otimes x^*$ but $\|y_n^* \otimes x^* - y^* \otimes x^*\| = \|y_n^* - y\|\|x^*\| > \varepsilon$, a contradiction. The same argument applied to T^* yields that the norm of X is Fréchet differentiable at $T^*(y^*)$. Hence (1) implies (2).

We assume now that (2) holds. Let x^* and y^* as in (2). By Theorem 3.6 the norm of $X\varepsilon Y$ is Gâteaux differentiable at T. From Lemma 3.4 and Proposition 3.5 it follows that $B_{X^*} \bigotimes B_{Y^*}$ is a w^*-compact James boundary of $B_{(X\varepsilon Y)^*}$. Let $(x_n^* \otimes y_n^*) \subset B_{X^*} \bigotimes B_{Y^*}$ be a sequence which is w^*-convergent to $x^* \otimes y^*$. Since $x_n^* \otimes y_n^* = -x_n^* \otimes -y_n^*$ for each $n \in \mathbb{N}$ we can assume that there exists a w^* neighborhood U of $-x$ in B_{X^*} such that (x_n^*) lies outside U. Let (x_{n_i}) a subnet of (x_n) which is w^*-convergent to some $\hat{x} \in B_{X^*}$. We assume without loss of generality that (y_{n_i}) is w^*-convergent to some $\hat{y}^* \in B_{Y^*}$. By Lemma 3.4 we have that $((x_{n_i}^* \otimes y_{n_i}^*)(T))$ converges to $\hat{x}^* \otimes \hat{y}^*(T) = \hat{y}^*(T(\hat{x}^*))$. Hence $\hat{x}^* = x^*$. Otherwise, since $-x^*$ is not a cluster point of (x_n^*) we would have $\|T(\hat{x}^*)\| < 1$, and $|(x_{n_i}^* \otimes y_{n_i}^*)(T)|$ converges to $|\hat{y}^* T(\hat{x}^*)| < 1$, contradicting the hypothesis $\lim(x_n^* \otimes y_n^*)(T) = y^*(T(x^*)) = 1$. Thus (x_n^*) is w^*-convergent to x^*, and therefore $(T(x_n^*))$ is norm convergent to $T(x^*)$. Assume now that (y_n^*) is not norm convergent to y^*. Then there exists $\varepsilon > 0$ a subsequence $(y_{n_k}^*)$ such that $y_{n_k}^* \notin B(y^*, \varepsilon)$ for each $k \in \mathbb{N}$. By the hypothesis on y^* and the Ŝmulyan's theorem there exists $\delta > 0$ such that $|y_{n_k}^*(T(x^*))| < 1 - \delta$ for each $k \in \mathbb{N}$. Hence we can use that $(T(x_n^*))$ converges in norm to $T(x^*)$ to get

$$1 = (x^* \otimes y^*)(T) = \lim_n |(x_n^* \otimes y_n^*)(T)| = \lim_k |(x_{n_k}^* \otimes y_{n_k}^*)(T)| \leq$$
$$\leq \limsup_k |y_{n_k}^*(T(x_{n_k}^*) - T(x^*))| + |y_{n_k}^*(T(x^*))| \leq 1 - \delta,$$

a contradiction. Hence the sequence (y_n^*) converges in norm to y^*. The same argument applied to T^* gives $\lim_n x_n^* = x$ in norm. Therefore

$$\|\cdot\| - \lim_n x_n^* \otimes y_n^* = x^* \otimes y^*,$$

and then the conclusion is obtained from Proposition 2.5 and Proposition 3.5. □

As an immediate consequence of Corollary 2.7 and Theorem 3.8 we get the following result.

Corollary 3.9. *Under the hypothesis of Theorem* 3.8, *if we assume that* $J \subset B_{X^*}$ *satisfies* $J \cup -J$, *is a James boundary of* B_{X^*}, $\overline{J}^{w^*} \cap S_{X^*} \subseteq J$ *and* $J \cap S_{X^*}$ *has the* w^**-Kadets-Klee property, then the norm is Fréchet differentiable at* T *if and only if there is a unique (up to factor)* $x^* \in J$ *such that* $\|T(x^*)\| = 1$ *and the norm of* Y *is Fréchet differentiable at* $T(x^*)$.

4. Banach spaces of vector-valued functions

4.1. Spaces of continuous functions

Our main theorem in this section is an extension of the vector-valued version of [4, Example I.1.6 (b)] given by Sundaresan [12, Theorem 1 and Theorem 2]. We use the representation $C(K,X) = C(K)\varepsilon X$ (cg [10, 44, Section 7]), where each $f \in C(K,X)$ is identified with $T_f : C(K)^* \to X$, $\delta_k \mapsto f(k)$. Notice that $\text{span}\{\delta_k : k \in K\}$ is w^*-dense and T_f is continuous on $\text{span}\{\delta_k : k \in K\}$ endowed with the w^*-topology since f is continuous on K. Here δ_k stands for the evaluation functional, i.e. $\delta_k : C(K) \to \mathbb{R}$, $f \mapsto f(k)$. Let $F \subset C(K)$ be a closed subspace. Let $J_K^F := \{(\delta_x)|_F : x \in K\}$ be endowed with its w^*-topology $\sigma(F^*, F)$. In case $F = C(K)$ we will write J_K.

Proposition 4.1. *Let* $F \subseteq C(K)$ *be a closed subspace. Then* J_K^F *is compact and therefore* $J := J_K^F \cup -J_K^F$ *is a* w^**-compact James boundary of* F.

Proof. J is trivially a James boundary. Moreover J_K^F is the restriction of the spectrum $((Sp(C(K)), w^*) = J_K)$ of $C(K)$, which is compact, to the elements of F. The restriction map $i : J_K \to J_K^F$, $\delta_x \mapsto \delta_x|_F$ is continuous. Hence J_K^F is compact. □

Definition 4.2. Let $F \subseteq C(K)$ be a closed subspace.

(a) The subset $M_F \subseteq K$ of *maximizing points* for F is the set $\{x \in K : \text{ exists } f \in F \text{ such that } |f(x)| = 1\}$, i.e., M_F is formed by the points $x \in K$ such that δ_x is norm attaining.

(b) The subset $P_F \subseteq K$ of *peaking points* for F is the set $\{x \in K : \text{there exists } f \in F \text{ such that } |f(x)| = 1 \text{ and } |f(y)| < 1 \text{ for } y \neq x\}$.

(c) F is said to *separate points* of K if for each $x, y \in K$, $x \neq y$ there is $f \in F$ such that $f(x) \neq f(y)$, i.e. if J_K^F is a Hausdorff space endowed with the $\sigma(F^*, F)$-topology.

Theorem 4.3 below is proved by Sundaresan in [12] for $F = C(K)$. In this case, condition (ii) is only satisfied if k_0 is an isolated point as a consequence of Urysohn's lemma.

Theorem 4.3. *Let* K *be a compact Hausdorff space and let* X *be a Banach space. Let* $F \subseteq C(K)$ *be a closed subspace separating points of* K *and let* $f \in F\widehat{\bigotimes}_\varepsilon X$ *be given.*

(i) *The norm of F is Gâteaux differentiable at f if and only if there is $k_0 \in K$, peaking f, such that the norm of X is Gâteaux differentiable at $f(k_0)$.*

(ii) *The norm is Fréchet differentiable at f if and only if there is a point $k_0 \in K$, peaking f, such that the norm of X is Fréchet differentiable at $f(k_0)$ and the map $M_F \to (F^*, \|\cdot\|)$, $k \mapsto \delta_k$ is sequentially continuous at k_0. In case that there exists $\varepsilon > 0$ such that $\|\delta_k - \delta_q\| > \varepsilon$ whenever $k, q \in K$, $k \neq q$, then this continuity only happens in isolated points.*

Proof. Let J_K^F and J as in Proposition 4.1. The Gâteaux part (i) is a consequence of Proposition 3.5, Theorem 3.6 and Proposition 4.1.

To prove (ii) we assume that the norm is Gâteaux differentiable at $f \in F$. Let k_0 as in (i) and let x^* be the unique functional satisfying $x^*(f(k_0)) = \|x^*\| = 1$. We see first the sufficency. The map $K \to (F^*, w^*)$, $k \mapsto \delta_k$ is obviously continuous. The hypothesis of K separating points of F implies that a sequence (δ_{k_n}) is convergent to δ_{k_0} in the w^*- topology if and only if (k_n) is convergent to k_0, and hence again the hypothesis imply that (δ_{k_n}) converges to δ_{k_0} in the norm topology. Since

$$\|\delta_{k_n} \otimes x^* - \delta_{k_0} \otimes x^*\| = \|\delta_{k_n} - \delta_{k_0}\|,$$

we conclude by Theorem 3.8. In order to prove the necessity we proceed by contradiction. Assume that there exists $\varepsilon > 0$ and a sequence $(k_n) \subset K$ convergent to k_0 such that $\|\delta_{k_n} - \delta_{k_0}\| > \varepsilon$. Thus $(\delta_{k_n} \otimes x^*)(f)$ is convergent to 1 and $\|\delta_{k_n} \otimes x^* - \delta_{k_0} \otimes x^*\| = \|\delta_{k_n} - \delta_{k_0}\| > \varepsilon$, a contradiction with the Šmulyan's theorem. Finally we suppose that $\|\delta_k - \delta_q\| > \varepsilon$ whenever $k, q \in K$, $k \neq q$ but k_0 is not isolated. Then there exists a net (k_i) convergent to k_0. This implies that there exists a sequence (k_n) such that $(\delta_{k_n} \otimes x^*(f))$ converges to 1. We are assuming Fréchet differentiability of f with differential $\delta_{k_0} \otimes x^*$, thus we have a contradiction with $\|\delta_{k_n} \otimes x^* - \delta_{k_0} \otimes x^*\| = \|\delta_{k_n} - \delta_{k_0}\| > \varepsilon$. □

If $H \subseteq C(K, X)$ is formed by functions with open range then there is no $F \subseteq C(K)$ such that $F \varepsilon X \subset H$. In case X is a unital Banach algebra we obtain below a result valid for more general subspaces of $C(K, X)$, which permits us to extract conclusions involving spaces of holomorphic functions.

Proposition 4.4. *Let X be a unital Banach algebra and let $F \subseteq C(K, X)$ be a subalgebra containing the constant functions and such that for each $k \in K$ there is $f_k \in F$ such that $f_k(k) = e$ and $\|f_k(q)\| < 1$ for each $q \in K$, $q \neq k$.*

(i) The norm of F is Gâteaux differentiable at $f \in F$ if and only if the norm of $C(K, X)$ is Gâteaux differentiable at f if and only if there is a unique $k \in K$ such that $\|f(k)\| = \|f\|$.

(ii) *If k is an accumulation point of K then the norm of F is not Fréchet differentiable at f.*

Proof. To prove (i), by Theorem 3.6, it is enough to show that the norm of f is Gâteaux differentiable at $f \in F$ if and only if there is $k \in K$, peaking

f, such that the norm of X is Gâteaux differentiable at $f(k)$. For $k \in K$, let $\delta_k^X : C(K,X), \to X,\ f \mapsto f(k)$ be the evaluation. Observe that

$$J := \{x^* \circ \delta_k^X :\ k \in K,\ x^* \in B_{X^*}\}$$

is a James boundary. Let $k, q \in K,\ k \neq q$. By the hypothesis there is $f \in S_F$ such that such that $f(k) = e$ and $\|f(q)\| < 1$. Let $x^*, y^* \in B_{X^*} \setminus \{0\}$. Let $\varepsilon > 0$ and $x \in S_X$ such that $x^*(x) > \varepsilon$. Let $n \in \mathbb{N}$ such that $\|f(q)\|^n < \varepsilon/4$. If we define $g := xf^n$ then we have $g \in S_F$,

$$\|x^* \circ \delta_k - y^* \delta_q\| \geq x^* \circ \delta_k(g) - y^* \circ \delta_q(g) > \varepsilon/2. \tag{1}$$

Hence $x^*\delta_k \neq y^*\delta_q$. If $x^*, y^* \in B_{X^*} \setminus \{0\}$, $x^* \neq y^*$ and $k \in K$ then there is $x \in S_X$ such that $x^*(x) \neq y^*(x)$. Hence, evaluating the constant function $f = x$ we obtain $x^* \circ \delta_k \neq y^* \circ \delta_k$. Thus two elements $x^* \circ \delta_k,\ y^* \circ \delta_q \in J$ satisfy $x^* \circ \delta_k = y^* \circ \delta_q \in J$ if and only if $k = q$ and $x^* = y^*$. Moreover J is $\sigma(F^*, F)$-compact, since it is the restriction to F of $J_K \otimes B_{X^*}$, which is $\sigma(C(K,X)^*, C(K,X))$-compact by Proposition 3.5. Hence we conclude by Proposition 2.4.

We prove now (ii). If k is an accumulation point of K, $x^* \in S_{X^*}$ satisfies $x^*(f(k)) = 1$ and (k_n) is a sequence which converges to k then $(x^* \circ \delta_{k_n})(f)$ converges to 1 but $\|x^* \circ \delta_{k_n} - x^* \circ \delta_k\| > 1/2$ by (1). Hence the norm of F is not Fréchet differentiable at f. □

Remark 4.5. Theorem 4.4 remains true if X is a Banach space and $F \subseteq C(K,X)$ satisfies that there exists $\varepsilon > 0$ such that $\|x^* \circ \delta_k - y^* \circ \delta_q\| > \varepsilon$ if $k \neq q$.

We apply below Proposition 4.4 to spaces of holomorphic functions regarded as real Banach spaces of functions with values in $\mathbb{C} = \mathbb{R}^2$, where we are also considering in $\mathbb{C}$ the structure of real Banach algebra. Since the unit ball of $\mathbb{C}$ is the euclidean ball, the norm is Fréchet differentiable at every $z \in \mathbb{C} \setminus \{0\}$.

Example 4.6. (a) Let $A(\overline{\mathbb{D}})$ be the disc algebra, i.e. the space of functions which are holomorphic in $\mathbb{D}$ and continuous in $\overline{\mathbb{D}}$ endowed with the supremum norm $\|\cdot\|$. For each $z_0 \in \partial\mathbb{D}$ the function $f(z) = (z + z_0)/2$ satisfies that $f(z_0) = 1$ and $|f(z)| < 1$ for each $z \in \mathbb{D} \setminus \{z_0\}$. We can apply Proposition 4.4 (i) to get that the norm of $A(\overline{\mathbb{D}})$ is Gâteaux differentiable at $f \in A(\overline{\mathbb{D}})$ if and only if there is $z \in \partial\mathbb{D}$ (by the Maximum Modulus Principle) such that $|f(z)| = \|f\|$ and $|f(y)| < \|f\|$ for each $y \in \overline{\mathbb{D}} \setminus \{z\}$. In this case the norm of $C(\overline{\mathbb{D}}, \mathbb{R}^2)$ and the norm $C(\partial\mathbb{D}, \mathbb{R}^2)$, if we identify in this last case f with its restriction to the circle, are both Gâteaux differentiable at f by Theorem 4.3.

(b) Let f be in the unit ball of $A(\overline{\mathbb{D}})$ such that the norm in $A(\overline{\mathbb{D}})$ is Gâteaux differentiable at f. The norm cannot be Fréchet differentiable. This is also a consequence of Proposition 4.4 (ii).

(c) For any function $f \in H^\infty$, the space regarded as a real Banach space formed by functions defined on $\mathbb{D}$ and with values in $\mathbb{R}^2$ which are continuous in the spectrum $\mathcal{M} \subseteq (H^\infty)^*$ (the dual of H^∞ endowed with its natural structure as a complex Banach space) and holomorphic on $\mathbb{D}$, with the canonical identification of $\mathbb{D}$ with the subset of the spectrum $\{\delta_z : z \in \mathbb{D}\}$. Any function f in the unit sphere of H^∞ satisfies that there exists $z \in \overline{\mathbb{D}}$ with $|z| = 1$ and a sequence (z_n) in $\overline{\mathbb{D}}$ convergent to z with $\lim_n |f(z_n)| = 1$. By a corollary of a theorem of Hayman and Newman [8, p. 204] we can consider the sequence to be interpolating in H^∞, i.e for each sequence (x_n) in l_∞ there is $g \in H^\infty$ such that $g(z_n) = x_n$. This means that $F := \{\delta_{z_n} : n \in \mathbb{N}\} \subset \mathcal{M}$ is discrete when we endow it with the w^*-topology. Interpolating property on (z_n) yields that for each bounded function g on F there is $\hat{g} \in H^\infty$ such that $g(\delta_{z_n}) = \hat{g}(z_n)$. From this it follows that the closure of F in the w^*-topology is homeomorphic to $\beta\mathbb{N}$. This yields that the cardinality of $N_f = \{x^* \in \mathcal{M} : x^*(f) = 1\}$ is uncountable. This implies that the set of real parts of the functionals which belong to N_f are also uncountable. Hence the norm of H^∞ is nowhere Gâteaux differentiable by Proposition 4.4, considering H^∞ as a subspace of $C(\mathcal{M}, \mathbb{R}^2)$.

4.2. Weighted Banach spaces of harmonic functions

Let $U \subseteq \mathbb{R}^d$ be an open and connected set. For a real Banach space X we denote by $h(U, X)$ the space of harmonic functions on U with values in X, $h(U)$ denotes the space of real valued harmonic functions. With this notation, the space of complex valued harmonic functions on U observed as a real topological vector space is written $h(U, \mathbb{R}^2)$. A function $f : U \to X$ is harmonic in a strong sense, i.e. it is a $C^\infty-$smooth function and is in the kernel of the vector-valued Laplacian, if and only if $x^* \circ f \in h(U)$ for all $x^* \in X^*$ (e.g. [2, Corollary 10]). A bounded continuous function $v : U \to]0, \infty[$ is called a weight on U. We define the weighted spaces of vector-valued functions as

$$h_v(U, X):\ = \{f \in h(U, X) :\ \|f\|_v := \sup_{z \in U} v(z)\|f(z)\| < \infty\},$$

$$h_v^0(U, X):\ = \{f \in h_v(U, X) :\ v(\cdot)\|f(\cdot)\| \text{ vanishes at infinity on } U\},$$

$$h_v^c(U, X):\ = \{f \in h_v(U, X) : (vf)(U) \text{ is relatively compact }\}.$$

A function $g : U \to \mathbb{R}$ is said to *vanish at infinity* on U if for each $\varepsilon > 0$ there exists a compact subset $K \subset U$ such that $|g(z)| \le \varepsilon$ for each $z \in U \setminus K$. Hence, $h_v^0(U, X)$ can be identified by the isometry $f \mapsto vf$ with a subspace of $C(\widehat{U}, X)$, $\widehat{U}$ being the Alexandroff compactification of U. We keep the notation $h_v(U)$ and $h_v^0(U)$ when $X = \mathbb{R}$. Obviously $h_v^c(U, X) = h_v(U, X)$ when X is finite dimensional. Let βU be the Stone Čech compactification of U. Also $h_v^c(U, X)$ is isomorphically isometric to a subspace of $C(\beta U, X)$ by means of $f \mapsto vf$. We have the inclusions

$$h_v^0(U, X) \subseteq h_v^c(U, X) \subseteq h_v(U, X).$$

We have the canonical identifications $h_v(U, X) = h_v(U)\varepsilon X$, $h_v^0(U, X) = h_v^0(U)\varepsilon X$ and $h_v^c(U, X) = h_v^c(U)\varepsilon X$ (see [1]).

Definition 4.7. Let U be a domain in $\mathbb{R}^d$ and let v a weight on U. A point $z \in U$ is said to be *peaking for* $h_v^0(U)$ if there exists $f \in h_v^0(U)$ such that $v(z)|f(z)| = 1$ and $v(x)|f(x)| < 1$ or each $x \neq z$.

Theorem 4.8. *Let U be domain in $\mathbb{R}^d$ and let v be a weight on U which vanishes at the boundary.*

(a) *The norm $\|\cdot\|_v$ is Gâteaux differentiable at $f \in h_v^0(U,X)$ if and only if there exists $z \in U$ peaking f with the norm of X being Gâteaux differentiable at $v(z)f(z)$. In this case $\|\cdot\|_v$ is also Gâteaux differentiable at f in $h_v^c(U,X)$.*

(b) *The norm $\|\cdot\|_v$ is Fréchet differentiable at $f \in h_v^0(U,X)$ if and only if there exists $z \in U$ peaking f with the norm of X being Fréchet differentiable at $v(z)f(z)$. In this case $\|\cdot\|_v$ is also Fréchet differentiable at f in $h_v^c(U,X)$.*

Proof. Let consider the isometry $i_v^0 : h_v^0(U) \to C(\widehat{U})$, $f \mapsto vf$. Let $J_{\widehat{U}} := \{\delta_x : x \in \widehat{U}\}$ be the spectrum of $C(\widehat{U})$. It is easy to check $(i_v^0)^*(J_{\widehat{U}}) = \{v(x)\delta_x : x \in \widehat{U}\} \subseteq h_v^0(U)^*$. Notice that the evaluation of vf at infinity is the null functional. Let consider also the isometry $i_v^c : h_v^c(U) \to C(\beta U)$, $f \mapsto vf$, when we identify vf with its unique extension to βU. Let $J_{\beta U} := \{\delta_x : x \in \beta U\}$ be the spectrum of $C(\beta U)$. Again we have $(i_v^c)^*(J_{\beta U}) = \{v(x)\delta_x : x \in \beta U\} \subseteq h_v^c(U)^*$. The hypothesis on v yields that h_v^0 contains the polynomials. From this fact it follows easily that $v(x)\delta_x \neq v(y)\delta_y$ for $x, y \in U$, $x \neq y$.

We observe the inclusion $(i_v^0)^*(J_{\widehat{U}}) \cap S_{h_v^0(U)^*} \subseteq J_v^0 := \{v(z)\delta_z : z \in U\}$. To prove (a) we apply the Theorem 4.3 to get that the norm of $h_v^0(U,X)$ is Gâteaux differentiable at $f \in h_v^0(U,X)$ if and only if there exists $z \in U$ peaking f such that the norm of X is Gâteaux differentiable at $v(z)f(z)$. The same argument gives that the norm is Gâteaux differentiable at $f \in h_v^c(U,X)$ if and only if there exists $z \in \beta U$ such that $v(z)\|f(z)\| = 1$, $v(x)\|f(x)\| < 1$ for each $x \in \beta U \setminus \{z\}$ and the norm of X is Gâteaux differentiable at $v(z)f(z)$. If $f \in h_v^0(U)$ then $z \in U$ since $v(x)f(x) = 0$ for each $x \in \beta U \setminus U$.

We consider (b). We prove that J_v^0 has the Kadets-Klee property. We consider the map $F : U \to h_v^0(U)^*$, $z \mapsto \delta_z$. Since $f \circ F = f \in h(U)$ for each $f \in h_v^0(U)$, we get that F is harmonic (and then continuous) for the norm operator topology by [2, Corollary 10, Remark 11]. Then $vF : U \to h_v^0(U)^*$ is continuous since v is also continuous. The hypothesis implies that $vF : U \to (J_v^0, \|\cdot\|)$ is bijective and continuous. One can easily check that $\widehat{vF} : \hat{U} \to (J_v^0 \cup \{0\}, w^*)$ is bijective and continuous if we define $\widehat{vF}(\infty) = 0$ and $\widehat{vF}|_U = vF$. Thus vF is an homeomorphism on the compact space $\hat{U}$. Altogether we find that in $vF(U)$ the norm and the w^*-topology agree. Hence the first equivalence follows from Corollary 3.9.

Assume that the norm $\|\cdot\|_v$ is Fréchet differentiable at $f \in h_v^0(U,X)$. By (a) there is $z_0 \in U$ such that $v(z_0)|f(z_0)| = 1$ and $v(z)|f(z)| < 1$ for each $z \in U \setminus \{z_0\}$. Let $J_v^c := \{v(z)\delta_z : z \in U\} \subseteq h_v^c(U)^*$. Again by [2, Corollary 10, Remark 11] we have that $vF : U \to (J_v^c, \|\cdot\|)$ is continuous. We conclude (ii) by application of Theorem 4.3 . □

We finish with the following corollary which extends [3, Theorem 6].

Corollary 4.9. *Let $U \subseteq \mathbb{R}^n$ be an open set and consider $\mathbb{R}^d$ endowed with the Euclidean norm. Let v be a weight on U which vanishes on the boundary. The norm $\|\cdot\|_v$ is Gâteaux differentiable at $f \in h_v^0(U, \mathbb{R}^d)$ if and only if $\|\cdot\|_v$ is Fréchet differentiable at f in $h_v(U, \mathbb{R}^d)$ if and only if there exists $z \in U$ such that $v(z)\|f(z)\| = 1$ and $v(y)\|f(y)\| < 1$ for any $y \neq z$.*

Corollary 4.9 and Example 4.6 show that the spaces $h_v^0(U, \mathbb{R}^2)$ and $h_v(U, \mathbb{R}^2)$, and the corresponding subspaces of holomorphic functions $H_v^0(U)$ and $H_v(U)$ behave differently from $A(\overline{\mathbb{D}})$ and H^∞ with respect to differentiability of the norm.

Acknowledgements. We thank our colleagues J. Bonet, M. Maestre and V. Montesinos for their careful reading of the preprint of this work, and for their suggestions and advice which really improved the original work.

References

[1] K.-D. Bierstedt: Gewichtete Räume stetiger vektorwertiger Funktionen und das injektive Tensorprodukt I, J. Reine Angew. Math. 259 (1973) 186–210.

[2] J. Bonet, L. Frerick, E. Jordá: Extension of vector valued holomorphic and harmonic functions, Studia Math. 183(3) (2007) 225–248.

[3] C. Boyd, P. Rueda: Complete weights and v-peak points of spaces of weighted holomorphic functions, Isr. J. Math. 155 (2006) 57–80.

[4] R. Deville, G. Godefroy, V. Zizler: Smoothness and Renormings in Banach Spaces, Oxford University Press, Oxford (1993).

[5] M. Fabian, P. Habala, P. Hájek, V. Montesinos, V. Zizler: Banach Space Theory, CMS Books in Mathematics, Springer, New York et al. (2011).

[6] S. Heinrich: The differentiability of the norm in spaces of operators, Funct. Anal. Appl. 9(4) (1975) 360–362.

[7] J. Hennefeld: Smooth, compact operators, Proc. Amer. Math. Soc. 77(1) (1979) 87–90.

[8] K. Hoffman: Banach Spaces of Analytic Functions, Prentice-Hall, Englewood Cliffs (1962).

[9] J. R. Holub: On the metric geometry of ideals of operators on Hilbert space, Math. Annalen 201 (1973) 157–163.

[10] G. Köthe: Topological Vector Spaces II, Grundlehren der Mathematischen Wissenschaften 237, Springer (1979).

[11] R. Meise, D. Vogt: Introduction to Functional Analysis, Oxford University Press, Oxford (1997).

[12] K. Sundaresan: Some geometric properties of the unit cell in spaces $C(X, B)$, Bull. Acad. Polon. Sci., Ser. Math. Astrom. Phys. 19 (1970) 1007–1012.