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Fractoconvex Structures

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We define a new structure on a space endowed with convexities, and call it a fractoconvex structure (or, a space with fractoconvexity). We introduce two operations on a set of fractoconvexities and in a special case we show that they satisfy the laws for a distributive lattice. We establish a connection between fractoconvex sets and convex sets using the concept of independent convexities, based on the possibility of representing a fractoconvex set as the intersection of its convex hulls. Finally, we consider some examples of fractoconvexities on the 2-sphere and on $\mathbb{Z}$.

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1. Introduction

The concept of a convexity plays an important role in many topics of mathematics. In each of them the properties associated with convexity appear on an appropriate abstract level. The theory that deals with convexity and its applications from a general point of view was formed in the 1960–1980s and was called the theory of convex structures. A fairly complete exposition of the theory of convex structures can be found in the monographs [1, 2, 3]. Following [1], we recall the most general viewpoint of the notion of convexity.

Let X be a set. A *convexity* $\mathbf{G} \subset 2^X$ on X is defined, as a rule, in two equivalent ways. In the first way, a collection $\mathbf{G} \subset 2^X$ is called a convexity on X if $X \in \mathbf{G}$ and $\mathbf{G}$ is closed under arbitrary intersections of its elements. In the second way, a convexity is generated by a *convex hull operator* $\mathbf{g}\colon 2^X \to 2^X$ such that, for any sets A, B satisfying the inclusions $A \subset B \subset X$, the following conditions hold: $A \subset \mathbf{g}\, A$, $\mathbf{g}\, A \subset \mathbf{g}\, B$, and $\mathbf{g}\,\mathbf{g}\, A = \mathbf{g}\, A$; this convexity is in turn equal to $\mathbf{G} = \{A \subset X : \mathbf{g}\, A = A\}$. A convexity $\mathbf{G}$ is called *finitely defined* if for every $A \subset X$ it follows that $\mathbf{g}\, A = \bigcup\{\mathbf{g}\, B : B \subset A, |B| < \infty\}$. A convexity $\mathbf{G}$ is called *n-ary* if $A \in \mathbf{G} \Leftrightarrow \forall B \subset A\big(|B| \leqslant n \Rightarrow \mathbf{g}\, B \subset A\big)$ whenever $A \subset X$.

Now let the set X be endowed with a family of convexities on it. Clearly, in X we can construct new structures based on the convexities, and these structures are not necessarily convexities on X, but may have similarities with them. For example,

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in [4] the author proposed and investigated the notions of an n-semiconvex set and an n-biconvex set. We briefly recall these notions.

Let $\mathbf{G}_1$ and $\mathbf{G}_2$ be convexities on a set X, $\mathbf{g}_1$ and $\mathbf{g}_2$ be the convex hulls associated with them, respectively, and let n be a natural number. A set $A \subset X$ is called *n-semiconvex* with respect to $\mathbf{G}_1$ and $\mathbf{G}_2$ if

$$\forall B \subset A\big(|B| \leqslant n \Rightarrow \exists i \in \{1,2\}\ \ \mathbf{g}_i\, B \subset A\big). \tag{1}$$

A set $A \subset X$ is called *n-biconvex* with respect to $\mathbf{G}_1$ and $\mathbf{G}_2$ if

$$\forall B \subset A\big(|B| \leqslant n \Rightarrow \forall i \in \{1,2\}\ \ \mathbf{g}_i\, B \subset A\big). \tag{2}$$

From (1) and (2) (see the quantifiers in the round brackets) it follows that, for fixed n, $\mathbf{G}_1$, and $\mathbf{G}_2$, the family of all n-semiconvex sets is not stable for intersections, but the family of all n-biconvex sets is in turn stable for intersections. Therefore, the family of all n-biconvex sets is a convexity on X. At the same time, it has been shown in [4] that

- Some special n-semiconvex sets can be represented as the intersection of their convex hulls.
- In some cases the theorems which are similar to the hyperplane separation theorems in $\mathbb{R}^n$ can be applied to n-semi- and n-biconvex sets.

In this paper the notions of n-semi- and n-biconvex set are generalized for an arbitrary, not necessarily finite, family of convexities. We define the notions of "fractoconvexity" (fractional convexity) and "multiconvexity" and investigate their properties. We shall briefly consider the question of how fractoconvex sets can be represented by convex sets belonging to given convexities. Finally, we provide four examples to illustrate the notions and statements proposed in our paper.

2. Fractoconvexities

Let Λ be an index set, $\mathbf{G}_\lambda$, $\lambda \in \Lambda$ be convexities on a set X, $\{M_i\}$ be a partition of Λ such that

$$\forall i\big(\lambda_{1,2} \in M_i, \lambda_1 \neq \lambda_2 \Rightarrow \mathbf{G}_{\lambda_1} \neq \mathbf{G}_{\lambda_2}\big).$$

and, for every i, let m_i be a cardinal number such that $m_i \leqslant |M_i|$. (Here and in what follows, by $|\cdot|$ we denote the cardinality of a set).

We say that a set A is an *n-ary fractoconvex set of the type* $\{(M_i, m_i)\}$ *in* X *with respect to the convexities* $\mathbf{G}_\lambda$ (briefly, (n) - $\bigvee_i \dfrac{m_i}{\{\mathbf{G}_\lambda, \lambda \in M_i\}}$*-fractoconvex set in* X),

$$\text{if} \qquad \forall B \subset A\Big(|B| \leqslant n \Rightarrow \big(\exists\, i\ \exists\, \Omega \subset M_i,\ |\Omega| = m_i : \bigcup_{\lambda \in \Omega} \mathbf{g}_\lambda\, B \subset A\big)\Big).$$

Similarly, we say that a set A is an *n-ary multiconvex set in X with respect to the convexities* $\mathbf{G}_\lambda$ (briefly, (n)-$\{\mathbf{G}_\lambda, \lambda \in \Lambda\}$*-multiconvex set in X*), if

$$\forall B \subset A\Big(|B| \leqslant n \Rightarrow \bigcup_{\lambda\in\Lambda} \mathbf{g}_\lambda\, B \subset A\Big).$$

The collection of all (n)-$\bigvee_i \dfrac{m_i}{\{\mathbf{G}_\lambda, \lambda \in M_i\}}$-fractoconvex (resp., (n)-$\{\mathbf{G}_\lambda, \lambda \in \Lambda\}$-multiconvex) sets in X will be called a *fractoconvexity* (resp., *multiconvexity*) and will be denoted by (n)-$\bigvee_i \dfrac{m_i}{\{\mathbf{G}_\lambda, \lambda \in M_i\}}$ (resp., (n)-$\{\mathbf{G}_\lambda, \lambda \in \Lambda\}$). The pair $\left(X, (n)\text{-}\bigvee_i \dfrac{m_i}{\{\mathbf{G}_\lambda, \lambda \in M_i\}}\right)$ will be called a *fractoconvex structure* (or, a *space with fractoconvexity*).

From these definitions we obtain that a multiconvexity is a particular case of a fractoconvexity, i.e. any (n)-$\{\mathbf{G}_\lambda, \lambda \in \Lambda\}$-multiconvex set is an (n)-$\dfrac{|\Lambda|}{\{\mathbf{G}_\lambda, \lambda \in \Lambda\}}$-fractoconvex set. Also, a set that is n-semiconvex with respect to $\mathbf{G}_1$ and $\mathbf{G}_2$ is an (n)-$\dfrac{1}{\{\mathbf{G}_1, \mathbf{G}_2\}}$-fractoconvex set, a set that is n-biconvex with respect to $\mathbf{G}_1$ and $\mathbf{G}_2$ is an (n)-$\dfrac{2}{\{\mathbf{G}_1, \mathbf{G}_2\}}$-fractoconvex set and is an (n)-$\{\mathbf{G}_1, \mathbf{G}_2\}$-multiconvex set, a set that is convex with respect to an n-ary convexity $\mathbf{G}_1$ is an (n)-$\dfrac{1}{\{\mathbf{G}_1\}}$-fractoconvex set and is an (n)-$\{\mathbf{G}_1\}$-multiconvex set.

Let $\mathfrak{G}^{(n)}(X)$ be the collection of all n-ary convexities on X. By $\mathfrak{F}^{(n)}(X)$ (resp., $\mathfrak{F}^{(n)}_{\text{fin}}(X)$) denote the collection of all (n)-fractoconvexities on X with respect to all (resp., all finite) subsets of $\mathfrak{G}^{(n)}(X)$. It is obvious that $\mathfrak{F}^{(n)}_{\text{fin}}(X) \subset \mathfrak{F}^{(n)}(X)$ and $\mathfrak{F}^{(n-1)}(X) \subset \mathfrak{F}^{(n)}(X)$. If not otherwise stated, *we assume that all convexities $\mathbf{G}_\lambda$, $\lambda \in \Lambda$ considered below are n-ary; additionally, the prefix (n)- for the fracto- and multiconvexities will be omitted.*

Take arbitrary fractoconvexities $\mathbf{F}_j = \bigvee_i \dfrac{m_i^j}{\{\mathbf{G}_\lambda^j, \lambda \in M_i^j\}} \in \mathfrak{F}^{(n)}(X)$, $j \in \{1, 2\}$. According to the definition of a fractoconvexity, we can also define a new fractoconvexity $\mathbf{F}_1 \vee \mathbf{F}_2 \in \mathfrak{F}^{(n)}(X)$ and a corresponding *operation* $\vee\colon \mathfrak{F}^{(n)}(X) \times \mathfrak{F}^{(n)}(X) \to \mathfrak{F}^{(n)}(X)$ as follows:

$$\begin{gathered} A \in \mathbf{F}_1 \vee \mathbf{F}_2 \\ \Leftrightarrow \forall B \subset A \ \Big(|B| \leqslant n \Rightarrow \big(\exists j \in \{1,2\}\ \exists i\ \exists \Omega \subset M_i^j, |\Omega| = m_i^j : \bigcup_{\lambda\in\Omega} \mathbf{g}_\lambda\, B \subset A\big)\Big). \end{gathered}$$

It is readily seen that the operation $\vee$ on the set $\mathfrak{F}^{(n)}_{\text{fin}}(X)$ possesses the following properties:

(i) $\vee$ is commutative and associative;

(ii) if $k < l$, then $\dfrac{k}{\{\mathbf{G}_1, \ldots, \mathbf{G}_k\}} \vee \dfrac{l}{\{\mathbf{G}_1, \ldots, \mathbf{G}_l\}} = \dfrac{k}{\{\mathbf{G}_1, \ldots, \mathbf{G}_k\}}$;

(iii) $\dfrac{k}{\{\mathbf{G}_1, \ldots, \mathbf{G}_m\}} = \bigvee_{1 \leqslant i_1 < \ldots < i_k \leqslant m} \dfrac{k}{\{\mathbf{G}_{i_1}, \ldots, \mathbf{G}_{i_k}\}}$.

Property (iii), in particularly, implies the equality $\dfrac{1}{\{\mathbf{G}_1, \ldots, \mathbf{G}_k\}} = \bigvee_{i=1}^{k} \dfrac{1}{\{\mathbf{G}_i\}}$.

We now consider the set-theoretic intersection operation on fractoconvexities.

Proposition 2.1. *For any convexities* $\mathbf{G}_1, \ldots, \mathbf{G}_k$, $k < \infty$, *we have*

$$\frac{k}{\{\mathbf{G}_1, \ldots, \mathbf{G}_k\}} = \frac{1}{\{\mathbf{G}_1 \cap \ldots \cap \mathbf{G}_k\}} = \frac{1}{\{\mathbf{G}_1\}} \cap \ldots \cap \frac{1}{\{\mathbf{G}_k\}}.$$

Proof. Without loss of generality, we put $k = 2$. We shall show that the convexity $\mathbf{G}_1 \cap \mathbf{G}_2$ is n-ary. Indeed, let $\tilde{\mathbf{g}}$ be the convex hull associated with $\mathbf{G}_1 \cap \mathbf{G}_2$. From the properties of the convex hull operator it follows that

$$A \in \mathbf{G}_1 \cap \mathbf{G}_2 \Rightarrow \forall B \subset A\big(|B| \leqslant n \Rightarrow \tilde{\mathbf{g}}B \subset A\big).$$

On the other hand, if $\forall B \subset A\big(|B| \leqslant n \Rightarrow \tilde{\mathbf{g}}B \subset A\big)$, then, in view of the inclusions $\mathbf{g}_1\, B, \mathbf{g}_2\, B \subset \tilde{\mathbf{g}}B$, we have

$$\forall B \subset A\big(|B| \leqslant n \Rightarrow \mathbf{g}_1\, B \subset A, \mathbf{g}_2\, B \subset A\big).$$

Since the convexities $\mathbf{G}_1$ and $\mathbf{G}_2$ is n-ary, we obtain $A \in \mathbf{G}_1 \cap \mathbf{G}_2$, whence $\mathbf{G}_1 \cap \mathbf{G}_2$ is n-ary.

Now, in our notation, we can write $\mathbf{G}_1 \cap \mathbf{G}_2 = \dfrac{1}{\{\mathbf{G}_1 \cap \mathbf{G}_2\}}$. Since the convexities $\mathbf{G}_1$ and $\mathbf{G}_2$ are n-ary, the convexity $\mathbf{G}_1 \cap \mathbf{G}_2$ coincides with the set $\dfrac{1}{\{\mathbf{G}_1\}} \cap \dfrac{1}{\{\mathbf{G}_2\}}$. Hence, the second required equality has been proved. In the same way, we obtain $A \in \dfrac{2}{\{\mathbf{G}_1, \mathbf{G}_2\}} \Leftrightarrow A \in \mathbf{G}_1 \cap \mathbf{G}_2$, whence the first equality is true. $\square$

Thus, if the cardinality in the numerator equals the cardinality of the denominator and $\mathbf{G}_i$ are n-ary for all $i = 1, \ldots, k$, $k < \infty$, then the fractoconvexity $\dfrac{k}{\{\mathbf{G}_1, \ldots, \mathbf{G}_k\}}$ is the n-ary convexity $\mathbf{G}_1 \cap \ldots \cap \mathbf{G}_k$. In other words, as in the case of biconvex sets, the collection of all n-ary multiconvex sets is an n-ary convexity.

Corollary 2.2. *For any fractoconvexity* $\mathbf{F} = \bigvee_i \dfrac{m_i}{\{\mathbf{G}_\lambda, \lambda \in M_i\}} \in \mathfrak{F}^{(n)}_{\text{fin}}(X)$ *we have*

$$\mathbf{F} = \bigvee_{\substack{i;\\ \Omega \subset M_i, |\Omega| = m_i}} \bigcap_{\lambda_i \in \Omega} \frac{1}{\{\mathbf{G}_{\lambda_i}\}}.$$

Proof. Evidently, this equality follows from (iii) and Proposition 2.1. $\square$

Proposition 2.3. *The operations* $\vee$ *and* $\cap$ *satisfy the distributive laws on* $\mathfrak{F}^{(n)}_{\text{fin}}(X)$*:* $\forall\, \mathbf{F}_1, \mathbf{F}_2, \mathbf{F}_3 \in \mathfrak{F}^{(n)}_{\text{fin}}(X)$ *we have*

$$(\mathbf{F}_1 \vee \mathbf{F}_2) \cap \mathbf{F}_3 = (\mathbf{F}_1 \cap \mathbf{F}_3) \vee (\mathbf{F}_2 \cap \mathbf{F}_3),$$
$$(\mathbf{F}_1 \cap \mathbf{F}_2) \vee \mathbf{F}_3 = (\mathbf{F}_1 \vee \mathbf{F}_3) \cap (\mathbf{F}_2 \vee \mathbf{F}_3).$$

Proof. These laws follow simply from (i)–(iii), Proposition 2.1, and the distributivity of the operations $\vee$ and $\cap$ for the fractoconvexities $\dfrac{1}{\{\mathbf{G}_\lambda, \lambda \in \Lambda\}} \in \mathfrak{F}^{(n)}_{\text{fin}}(X)$. The last property follows from the definition of $\vee$. For example, we show the distributivity of $\cap$ over $\vee$.

$$\begin{aligned}
&A \in \left(\frac{1}{\{\mathbf{G}_\lambda, \lambda \in \Lambda_1\}} \vee \frac{1}{\{\mathbf{G}_\lambda, \lambda \in \Lambda_2\}}\right) \cap \frac{1}{\{\mathbf{G}_\lambda, \lambda \in \Lambda_3\}} \Leftrightarrow \\
&\Leftrightarrow \forall B \subset A\Big(|B| \leqslant n \Rightarrow \big(\big((\exists \lambda_1 \in \Lambda_1 : \mathbf{g}_{\lambda_1} B \subset A) \vee (\exists \lambda_2 \in \Lambda_2 : \mathbf{g}_{\lambda_2} B \subset A)\big) \\
&\qquad\qquad \wedge (\exists \lambda_3 \in \Lambda_3 : \mathbf{g}_{\lambda_3} B \subset A)\big)\Big) \Leftrightarrow \\
&\Leftrightarrow \forall B \subset A\Big(|B| \leqslant n \Rightarrow \big((\exists \lambda_1 \in \Lambda_1 \exists \lambda_3 \in \Lambda_3 : \mathbf{g}_{\lambda_1} B \cup \mathbf{g}_{\lambda_3} B \subset A) \\
&\qquad\qquad \vee (\exists \lambda_2 \in \Lambda_2 \exists \lambda_3 \in \Lambda_3 : \mathbf{g}_{\lambda_2} B \cup \mathbf{g}_{\lambda_3} B \subset A)\big)\Big) \Leftrightarrow \\
&\Leftrightarrow A \in \left(\frac{1}{\{\mathbf{G}_\lambda, \lambda \in \Lambda_1\}} \cap \frac{1}{\{\mathbf{G}_\lambda, \lambda \in \Lambda_3\}}\right) \vee \left(\frac{1}{\{\mathbf{G}_\lambda, \lambda \in \Lambda_2\}} \cap \frac{1}{\{\mathbf{G}_\lambda, \lambda \in \Lambda_3\}}\right).
\end{aligned}$$

$\square$

In the same way, one can prove the absorption law:

$$(\mathbf{F}_1 \vee \mathbf{F}_2) \cap \mathbf{F}_1 = \mathbf{F}_1, \quad (\mathbf{F}_1 \cap \mathbf{F}_2) \vee \mathbf{F}_1 = \mathbf{F}_1.$$

Let $\hat{\mathfrak{F}}^{(n)}_{\text{fin}}(X)$ be some subfamily of $\mathfrak{F}^{(n)}_{\text{fin}}(X)$ which is closed under a finite number of applications of the operations $\vee$ and $\cap$. Obviously, from the properties obtained above for these operations it follows that $\hat{\mathfrak{F}}^{(n)}_{\text{fin}}(X)$ is a distributive lattice.

3. Independent convexities.

Convexities $\mathbf{G}_\lambda \in \mathfrak{G}^{(n)}(X)$, $\lambda \in \Lambda$ will be called *mutually* (n)*-independent* or, briefly, *independent*, if the following condition holds

$$\forall A \in (n)\text{-}\frac{1}{\{\mathbf{G}_\lambda, \lambda \in \Lambda\}} : A = \bigcap_{\lambda \in \Lambda} \mathbf{g}_\lambda A.$$

If the equality does not necessarily hold for all $A \in (n)\text{-}\dfrac{1}{\{\mathbf{G}_\lambda, \lambda \in \Lambda\}}$, then the maximal subfamily of $(n)\text{-}\dfrac{1}{\{\mathbf{G}_\lambda, \lambda \in \Lambda\}}$ for which the equality holds will be called the (n)*-independence domain* of the convexities $\mathbf{G}_\lambda, \lambda \in \Lambda$, and will be denoted by $(n)\text{-}\mathrm{idc}(\mathbf{G}_\lambda, \lambda \in \Lambda)$. The sets belonging to $(n)\text{-}\mathrm{idc}(\mathbf{G}_\lambda, \lambda \in \Lambda)$ will be called the *elements of* (n)*-independence* of the convexities $\mathbf{G}_\lambda, \lambda \in \Lambda$. In what follows, if not otherwise stated, the prefix (n)- will be omitted.

Given a set $M \subset \dfrac{1}{\{\mathbf{G}_\lambda, \lambda \in \Lambda\}}$ we consider the question whether the inclusion $M \subset \mathrm{idc}(\mathbf{G}_\lambda, \lambda \in \Lambda)$ is true. This question can be answered in two ways. First, the condition of convexities independence is directly verified for every $A \in M$. However, as was shown by examples in [4], this way can be very laborious. Second, the required inclusion may be checked by using some uncomplicated sufficient condition of convexities independence. One of such conditions will be given by us in Proposition 3.2.

The importance of the notion of independent convexities is that, in special cases, we can obtain statements about the separation property for two certain elements of independence of convexities, and the statements are analogous to the separation theorems for two convex sets. Moreover, these elements are separated by a set represented by intersection of some $\mathbf{g}_\lambda$-halfspaces. A recent investigation on this topic for semiconvex sets in S^2 has been carried out in [4] by the author.

Now we shall give the following definition, which will play an important role in Lemma 3.1 and in Proposition 3.2.

Two finitely defined (not necessarily n-ary) convexities $\mathbf{G}_1$ and $\mathbf{G}_2$ are said to be *conically independent* provided the following condition is true

$$\forall n \geqslant 4\ \forall x_1, \dots, x_n \in X\ \forall x \in \mathbf{g}^\cap\{x_1, \dots, x_n\}\ \exists y_1, y_2 \in \mathbf{g}^\cap\{x_1, \dots, x_{n-1}\} : \\ x \in \mathbf{g}_1\{y_1, x_n\} \cap \mathbf{g}_2\{y_2, x_n\}.$$

(Here and in what follows, we use the notation: $\mathbf{g}^\cap A = \mathbf{g}_1\, A \cap \mathbf{g}_2\, A$.)

Lemma 3.1. *Let the convexities* $\mathbf{G}_1$ *and* $\mathbf{G}_2$ *be conically independent, and let the set* A *satisfy the condition*

$$\forall x_1, x_2, x_3 \in A : \mathbf{g}^\cap\{x_1, x_2, x_3\} \subset A. \tag{3}$$

Then we have $A = \mathbf{g}^\cap A$.

Proof. Since the convexities $\mathbf{G}_1$ and $\mathbf{G}_2$ are finitely defined, we see that

$$\forall x \in \mathbf{g}^\cap A\ \exists x_1, \dots, x_n \in A : x \in \mathbf{g}^\cap\{x_1, \dots, x_n\}. \tag{4}$$

From (3) it follows that the proof is trivial if $n < 4$; therefore, we may put $n \geqslant 4$. Iterating the conical independence of $\mathbf{G}_1$ and $\mathbf{G}_2$ in (4), we get the following chain of implications:

$$\begin{aligned}
&\exists\, y_1, y_2 \in \mathbf{g}^{\cap}\{x_1, \dots, x_{n-1}\} : x \in \mathbf{g}_1\{y_1, x_n\} \cap \mathbf{g}_2\{y_2, x_n\} \Rightarrow \\
&\;\Rightarrow \exists\, y_{11}, y_{12}, y_{21}, y_{22} \in \mathbf{g}^{\cap}\{x_1, \dots, x_{n-2}\} : y_1 \in \mathbf{g}_1\{y_{11}, x_{n-1}\} \cap \mathbf{g}_2\{y_{12}, x_{n-1}\}, \\
&\qquad y_2 \in \mathbf{g}_1\{y_{21}, x_{n-1}\} \cap \mathbf{g}_2\{y_{22}, x_{n-1}\} \Rightarrow \dots \Rightarrow \exists y_{\underbrace{1\dots11}_{n-4}}, y_{\underbrace{1\dots12}_{n-4}}, \dots, y_{\underbrace{2\dots21}_{n-4}}, \\
&\qquad y_{\underbrace{2\dots22}_{n-4}} \in \mathbf{g}^{\cap}\{x_1, \dots, x_4\} : y_{\underbrace{1\dots11}_{n-5}} \in \mathbf{g}_1\{y_{\underbrace{1\dots11}_{n-4}}, x_5\} \cap \mathbf{g}_2\{y_{\underbrace{1\dots12}_{n-4}}, x_5\}, \dots \Rightarrow \\
&\;\Rightarrow \exists\, y_{\underbrace{1\dots11}_{n-3}}, y_{\underbrace{1\dots12}_{n-3}}, \dots \in \mathbf{g}^{\cap}\{x_1, x_2, x_3\} : y_{\underbrace{1\dots11}_{n-4}} \in \mathbf{g}_1\{y_{\underbrace{1\dots11}_{n-3}}, x_4\} \cap \mathbf{g}_2\{y_{\underbrace{1\dots12}_{n-3}}, x_4\} \dots
\end{aligned} \tag{5}$$

Without loss of generality, consider the points $u = y_{\underbrace{1\dots11}_{n-4}}$, $v_1 = y_{\underbrace{1\dots11}_{n-3}}$, and $v_2 = y_{\underbrace{1\dots12}_{n-3}}$. From (3) it follows that $x_4, v_1, v_2 \in A$; hence, again from (3), $\mathbf{g}^{\cap}\{x_4, v_1, v_2\} \subset A$. Obviously, if $u \in \{x_4, v_1, v_2\}$, then $u \in A$. Let $u \notin \{x_4, v_1, v_2\}$. Taking into account the properties of a convex hull operator, for each $i \in \{1, 2\}$, we obtain

$$\mathbf{g}_1\{v_1, x_4\} \cap \mathbf{g}_2\{v_2, x_4\} \subset \mathbf{g}_i\{v_i, x_4\} \subset \mathbf{g}_i\{x_4, v_1, v_2\}.$$

whence $\mathbf{g}_1\{v_1, x_4\} \cap \mathbf{g}_2\{v_2, x_4\} \subset \mathbf{g}^{\cap}\{x_4, v_1, v_2\}$. Since $u \in \mathbf{g}_1\{v_1, x_4\} \cap \mathbf{g}_2\{v_2, x_4\}$, we see again that $u \in A$.

Moving back along the chain of implications (5) in which the previous procedure applies to all points $y_{\dots}$ and $x_{\cdot}$, we obtain $\mathbf{g}_1\{y_1, x_n\} \cap \mathbf{g}_2\{y_2, x_n\} \subset \mathbf{g}^{\cap}\{x_n, y_1, y_2\} \subset A$, whence $x \in A$. From the arbitrariness of x, we conclude that $\mathbf{g}_1 A \cap \mathbf{g}_2 A \subset A$. The reverse inclusion is evident. □

Proposition 3.2. *Let the convexities $\mathbf{G}_1, \mathbf{G}_2 \in \mathfrak{G}^{(3)}(X)$ be conically independent; then $\mathbf{G}_1$ and $\mathbf{G}_2$ are independent.*

Proof. Consider an arbitrary set $A \in (3)\text{-}\dfrac{1}{\{\mathbf{G}_1, \mathbf{G}_2\}}$. The 3-arity of $\mathbf{G}_1$ and $\mathbf{G}_2$ imply that for any $x_1, x_2, x_3 \in A$ there exists an $i \in \{1, 2\}$ for which $\mathbf{g}_i\{x_1, x_2, x_3\} \subset A$. Since $\mathbf{g}^{\cap}\{x_1, x_2, x_3\} \subset \mathbf{g}_i\{x_1, x_2, x_3\}$, it follows that A satisfy (3). Thus, by Lemma 3.1 we have $A = \mathbf{g}^{\cap} A$. □

4. Examples of spaces with fractoconvexity

To describe the fractoconvexities of the Examples 4.1–4.3, we need the following construction [4].

Let S be the 2-sphere in $\mathbb{R}^3$ with center at the origin, B be the closed ball bounded by S, C be an arbitrary fixed set in the interior of B, and the symbol $[,]$ denotes the line segment operator in $\mathbb{R}^3$. The convexities $\mathbf{G}(c)$, $c \in C$, on S are defined analogously to the convexity in the sense of Robinson [5] but with respect to the points $c \in C$, respectively. This means that a set $A \subset S$ is $\mathbf{G}(c)$-convex iff, for two distinct points x_1, x_2 such that the straight line determined by them does not pass through the point c, the set cut out by the 2-dimensional cone with vertex c

and base $[x_1, x_2]$ is entirely contained in A. It is readily seen that all convexities $\mathbf{G}(c), c \in C$ are binary, and the segment $\mathbf{g}_c\{x_1, x_2\}$ joining two points $x_1, x_2 \in S$ coincides with the subset of S mentioned above if x_1, x_2, and c are non-collinear and equals $\{x_1, x_2\}$ otherwise.

Example 4.1. Let $C = \{c_0, c_1\}$. The fractoconvexity $\mathbf{F}_1 = (2)\text{-}\dfrac{1}{\{\mathbf{G}(c_0), \mathbf{G}(c_1)\}}$ is the family of all 2-semiconvex sets with respect to $\mathbf{G}(c_0)$ and $\mathbf{G}(c_1)$. Among these 2-semiconvex sets, the so-called $\mathbf{g}_{01}$-regular 2-semiconvex sets are very important. (A set A is called $\mathbf{g}_{01}$-*regular* if there exists an open halfspace H in $\mathbb{R}^3$ such that $c_0, c_1 \in H$ and $A \subset H^c$.) It has been shown in [4] that every $\mathbf{g}_{01}$-regular 2-semiconvex set belongs to the independence domain of $\mathbf{G}(c_0)$ and $\mathbf{G}(c_1)$.

Example 4.2. Let $C = \{c_\lambda := \lambda c_0 + (1-\lambda)c_1 \mid \lambda \in [0,1]\}$ and let us introduce the convexities $\mathbf{G}'(c_\lambda)$, $c_\lambda \in C$, which are the restriction of $\mathbf{G}(c_\lambda)$, $c_\lambda \in C$ to the subfamilies of the sets that are $\mathbf{g}_{01}$-regular with respect to some fixed open halfspace H parallel to the segment $[c_0, c_1]$. Considering the fractoconvexity $\mathbf{F}_2 = (2)\text{-}\dfrac{1}{\{\mathbf{G}'(c_\lambda), c_\lambda \in C\}}$, by analogy with the previous example, one can prove that the convexities $\mathbf{G}'(c_\lambda)$, $c_\lambda \in C$ are independent.

Example 4.3. Let C be as in Example 4.2. Consider the multiconvexities

$$\begin{aligned} \mathbf{G}_1 &= (2)\text{-}\ \{\mathbf{G}'(c_\lambda) \mid c_\lambda \in [c_0, (c_0+c_1)/2]\}, \\ \mathbf{G}_2 &= (2)\text{-}\ \{\mathbf{G}'(c_\lambda) \mid c_\lambda \in [(c_0+c_1)/2, c_1]\}, \end{aligned}$$

and put $\mathbf{F}_3 = (2)\text{-}\dfrac{1}{\{\mathbf{G}_1, \mathbf{G}_2\}}$. Obviously, these multiconvexities are binary. By definition of an n-ary convexity, they are 3-ary as well.

Consider the convexity $\mathbf{G}'((c_0+c_1)/2)$. By $\tilde{\mathbf{g}}$ denote the corresponding convex hull operator. It is easily seen that for all $n \geqslant 2$

$$\forall x_1, \ldots, x_n \in X\ \forall x \in \tilde{\mathbf{g}}\{x_1, \ldots, x_n\}\ \exists y \in \tilde{\mathbf{g}}\{x_1, \ldots, x_{n-1}\} : x \in \tilde{\mathbf{g}}\{y, x_n\}. \tag{6}$$

Moreover, the operators $\mathbf{g}_1$, $\mathbf{g}_2$, and $\tilde{\mathbf{g}}$ are related to each other by the condition

$$\forall n \in \mathbb{N}\ \forall x_1, \ldots, x_n \in X\ \ \mathbf{g}^{\cap}\{x_1, \ldots, x_n\} = \tilde{\mathbf{g}}\{x_1, \ldots, x_n\}. \tag{7}$$

Combining (6) and (7), we see that the 3-ary convexities $\mathbf{G}_1$ and $\mathbf{G}_2$ are conically independent; therefore, considering the fractoconvexity $\mathbf{F}_3 = (2)\text{-}\dfrac{1}{\{\mathbf{G}_1, \mathbf{G}_2\}}$ and using Proposition 3.2, we conclude that these convexities are independent.

Example 4.4. Suppose that the binary convexities $\mathbf{G}_1$ and $\mathbf{G}_2$ and the fractoconvexity $\mathbf{F}_4$ are defined on $\mathbb{Z}$ as follows. Let $\mathbf{G}_1$ be the collection of all sets $A \cap \mathbb{Z}$, where $A \subset \mathbb{R}$ is a standard convex set. Suppose f is a bijective function from $\mathbb{Z}$ to itself. Using f, for any $x_1, x_2 \in \mathbb{Z}$ we define the segment $\mathbf{g}_2\{x_1, x_2\}$ by the formula $\mathbf{g}_2\{x_1, x_2\} := f\big(\mathbf{g}_1\{f^{-1}(x_1), f^{-1}(x_2)\}\big)$.

We put $\mathbf{G}_2 = \{A \subset \mathbb{Z} \mid \forall x_1, x_2 \in A \ \mathbf{g}_2\{x_1, x_2\} \subset A\}$.

If a set $A \subset \mathbb{Z}$ is bounded, then the operators $\mathbf{g}_1$ and $\mathbf{g}_2$ satisfy the following equalities:

(1) $\mathbf{g}_1 A = \mathbb{Z} \cap \{a_1\lambda + b_1(1-\lambda) \mid \lambda \in [0,1], \quad a_1 = \min A, b_1 = \max A\}$;

(2) $\mathbf{g}_2 A := f\big(\mathbf{g}_1\{a_2, b_2\}\big) = \mathbf{g}_2\big\{f(a_2), f(b_2)\big\}, \ a_2 = \min\limits_{x\in A} f^{-1}(x), b_2 = \max\limits_{x\in A} f^{-1}(x)$.

Consider the fractoconvexity $\mathbf{F}_4 = (2)\text{-}\dfrac{1}{\{\mathbf{G}_1, \mathbf{G}_2\}}$. Then we have:

Proposition 4.5. *If a set $A \in \mathbf{F}_4$ is bounded, then $A \in \operatorname{idc}(\mathbf{G}_1, \mathbf{G}_2)$.*

Proof. Since $A \subset \mathbf{g}_1 A \cap \mathbf{g}_2 A$, it suffices to verify the reverse inclusion. If $\mathbf{g}_1 A \subset A$ or $\mathbf{g}_2 A \subset A$, then proof is trivial. Therefore, we assume that $\mathbf{g}_1 A \not\subset A$ and $\mathbf{g}_2 A \not\subset A$.

From the invertibility of the function f and from the definition of the points a_2 and b_2 it follows that the points $f(a_2)$ and $f(b_2)$ belong to A. The set A is semiconvex; hence, from the relations $\mathbf{g}_1\{a_1, b_1\} = \mathbf{g}_1 A \not\subset A$ and $\mathbf{g}_2\big\{f(a_2), f(b_2)\big\} = \mathbf{g}_2 A \not\subset A$, we obtain

$$\mathbf{g}_2\{a_1, b_1\} = f\big(\mathbf{g}_1\{f^{-1}(a_1), f^{-1}(b_1)\}\big) \subset A \quad \text{and} \quad \mathbf{g}_1\big\{f(a_2), f(b_2)\big\} \subset A. \tag{8}$$

Suppose that the set $(\mathbf{g}_1 A \cap \mathbf{g}_2 A) \setminus A$ is nonempty, that is,

$$\exists x \in \big(\mathbf{g}_1\{a_1, b_1\} \cap \mathbf{g}_2\big\{f(a_2), f(b_2)\big\}\big) \setminus A. \tag{9}$$

Under this assumption and by (8), we see that $x \notin \mathbf{g}_1\big\{f(a_2), f(b_2)\big\} \subset A$. But from (9) it follows that $x \in \mathbf{g}_1\{a_1, b_1\}$; hence, we have the following possible locations of the point x: either $x \in \mathbf{g}_1\{a_1, f(\cdot)\}$ or $x \in \mathbf{g}_1\{f(\cdot), b_1\}$, where $f(\cdot)$ belongs to $\{f(a_2), f(b_2)\}$ and is determined depending on the mutual position of $f(a_2)$ and $f(b_2)$ (see below). Both these cases are investigated equally. For example, consider only the first of them. We have two subcases:

either $x \in \mathbf{g}_1\{a_1, f(a_2)\}$ if $f(a_2) \leqslant f(b_2)$; or $x \in \mathbf{g}_1\{a_1, f(b_2)\}$ if $f(a_2) > f(b_2)$.

Here we have taken into account that $f(a_2) = f\big(\min\limits_{x\in A} f^{-1}(x)\big) \geqslant \min A = a_1$.

Without loss of generality it suffices to consider one subcase, e.g., $f(a_2) \leqslant f(b_2)$.

Since $x \notin A$ and $x \in \mathbf{g}_1\{a_1, f(a_2)\}$, we obtain

$$\mathbf{g}_1\{a_1, f(a_2)\} \not\subset A. \tag{10}$$

Therefore, we have $\mathbf{g}_2\{a_1, f(a_2)\} \subset A$, since A is semiconvex. Hence, by the definition of $\mathbf{g}_2$, we get

$$f\big(\mathbf{g}_1\{a_2, f^{-1}(a_1)\}\big) \subset A. \tag{11}$$

From (9) it follows that $x \in \mathbf{g}_2\big\{f(a_2), f(b_2)\big\}$, in other words, that $x \in f(\mathbf{g}_1\{a_2, b_2\})$. The definitions of the points a_2 and b_2 imply that $f^{-1}(a_1) \in \mathbf{g}_1\{a_2, b_2\}$. Combining these remarks with (11) and $x \notin A$, we obtain $x \in f\big(\mathbf{g}_1\{f^{-1}(a_1), b_2\}\big) \not\subset A$. At the same time, $f\big(\mathbf{g}_1\{f^{-1}(a_1), b_2\}\big) = \mathbf{g}_2\big\{a_1, f(b_2)\big\}$; hence, $\mathbf{g}_1\{a_1, f(b_2)\} \subset A$, since A is semiconvex.

By virtue of the assumption $f(a_2) \leqslant f(b_2)$, we have $\mathbf{g}_1\{a_1, f(a_2)\} \subset \mathbf{g}_1\{a_1, f(b_2)\}$, whence $\mathbf{g}_1\{a_1, f(a_2)\} \subset A$. But the last contradicts the inclusion (10). Thus, $(\mathbf{g}_1 A \cap \mathbf{g}_2 A) \setminus A = \varnothing$, and since $A \subset \mathbf{g}_1 A \cap \mathbf{g}_2 A$, the last equality is true iff $A = \mathbf{g}_1 A \cap \mathbf{g}_2 A$. □

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