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Characterization of Strong Ball Proximinality in L_1-Predual Spaces

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We characterize finite co-dimensional strongly ball proximinal subspaces of L_1-predual spaces and prove that they are precisely the finite co-dimensional strongly proximinal subspaces.

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1. Introduction

The concept of best approximation is one of the important concepts in approximation theory. The present paper deals with problems related to best approximation in L_1-predual spaces by elements of the closed unit ball of its closed subspaces. A Banach space X is said to be an *L_1-predual space or a Lindenstrauss space* if X^* is isometric to $L_1(\mu)$ for some positive measure μ. It is well-known that $C(K)$, the space of all continuous functions on a compact Hausdorff space K equipped with the supremum norm, is an L_1-predual space (see [8] for details).

In this article, we consider only Banach spaces over the real field $\mathbb{R}$ and all subspaces we consider are assumed to be closed. For a Banach space X, B_X denotes the closed unit ball of X.

A non-empty closed subset K of a Banach space X is said to be *proximinal* in X, if for every $x \in X$, the set $P_K(x) = \{k \in K : d(x, K) = \|x - k\|\}$ is non-empty, where $d(x, K) = \inf\{\|x-k\| : k \in K\}$ is the the distance of x from K. A subspace Y of X is said to be *ball proximinal* in X if B_Y is proximinal in X (see [2] for details).

The compactness assumption on a subset K of a Banach space X is sufficient to ensure the existence of best approximation from K for all points of X and for the metric projection $P_K\colon X \to 2^K$ to be upper Hausdorff semicontinuous. But, as the compactness condition is too restrictive in infinite dimensional Banach spaces, being not satisfied by various important sets, one can ask whether there is any substitute for compactness. Towards this, we recall a stronger form of proximinality.

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Definition 1.1. ([5]) A proximinal subset K of a Banach space X is said to be *strongly proximinal* at $x \in X$ if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that $P_K(x,\delta) \subseteq P_K(x) + \varepsilon B_X$, where $P_K(x,\delta) = \{y \in K : \|x-y\| < d(x,K)+\delta\}$. We say that K is strongly proximinal in X if it is strongly proximinal at all points of X. A subspace Y of X is said to be *strongly ball proximinal* in X if B_Y is strongly proximinal in X.

It is known that if K is a strongly proximinal subset of a Banach space X, then the metric projection P_K is upper Hausdorff semicontinuous (see [5]). So any result on strong proximinality will automatically imply some continuity properties of metric projection. In [6], it is proved that the metric projection onto a finite co-dimensional strongly proximinal subspace of an L_1-predual space is Hausdorff metric continuous. We now recall the following differentiability notion, introduced in [4], which characterizes strongly proximinal hyperplanes.

Definition 1.2. The norm of a Banach space X is said to be *strongly subdifferentiable* (in short SSD) at $x \in X$ if the one sided limit

$$d^+(x)(y) := \lim_{t\to 0^+} \frac{\|x+ty\| - \|x\|}{t}$$

exists uniformly for $y \in B_X$. In this case, x is said to be an SSD point of X.

In [5], it is proved that for a Banach space X and for $f \in X^*$, f is an SSD-point of X^* if and only if $\ker(f)$ is strongly proximinal in X, where $\ker(f)$ denotes the kernel of f in X.

The following polyhedral concept, introduced in [1], is stronger than the notion of an SSD-point.

Definition 1.3. A vector x in a Banach space X is said to be a *quasi-polyhedral* (in short QP) point of X if there exists a $\delta > 0$ such that $J_{X^*}(z) \subseteq J_{X^*}(x)$ for $\|z-x\| < \delta$ and $\|z\| = \|x\|$, where $J_{X^*}(x) = \{f \in B_{X^*} : f(x) = \|x\|\}$.

In [5], it is proved that if Y is a finite co-dimensional strongly proximinal subspace of a Banach space X, then the annihilator of Y, denoted by $Y^\perp$, is contained in the set of all SSD-points of X^*. Moreover, if Y is a finite co-dimensional subspace of a Banach space X such that $Y^\perp$ is contained in the set of all QP-points of X^*, then Y is strongly proximinal in X. In [7, Proposition 3.20], it is proved that the converse of these two results hold in L_1-predual spaces.

In connection with the proximinality properties of subspaces of Banach spaces, one can investigate the relation, if any, that exists between (strong) proximinality and (strong) ball proximinality. It is easy to see that (strongly) ball proximinal subspaces are (strongly) proximinal. In [9], Saidi gave an example of a proximinal subspace of co-dimension one which is not ball proximinal and later in [2, Example 3.3], it is proved that this subspace is in fact strongly proximinal. An interesting problem in approximation theory is to find a class of Banach spaces in which the convexity concept strong ball proximinality coincide with the linearity concept strong proximinality. In [6, Theorem 4.3], it is proved that finite co-dimensional strongly proximinal subspaces of $C(K)$ are strongly ball proximinal. In this article, we prove that strong proximinality and strong ball proximinality coincide in the case of finite co-dimensional subspaces of L_1-predual spaces. Using this result we characterize finite co-dimensional strongly ball proximinal subspaces of L_1-predual spaces in terms of SSD-points and QP-points. We also prove that a finite co-dimensional subspace Y of an L_1-predual space X is strongly ball proximinal in X if and only if $Y^{\perp\perp}$ is strongly ball proximinal in X^{**}.

2. Strong ball proximinality in L_1-predual spaces

We first prove two distance formulas which will be needed in the proofs of the subsequent results.

Lemma 2.1. *Let C be a closed and convex subset of a Banach space X. Then $d(x,C)=d(x,\overline{C}^{w^*})$ for all $x\in X$, where $\overline{C}^{w^*}$ is the weak*-closure of C in X^{**} (under the canonical embedding of X in X^{**}). In particular, $d(x,B_Y)=d(x,B_{Y^{\perp\perp}})$.*

Proof. Let $x\notin\overline{C}^{w^*}$ and $0<\varepsilon<d(x,C)$. Then, by the Hahn-Banach separation theorem, there exists a linear functional $f\in X^*$ such that $\|f\|=1$ and $f(x)>\sup_C(f)+d(x,C)-\varepsilon$. Since $\sup_C(f)=\sup_{\overline{C}^{w^*}}(f)$ and $f(x)\leq d(x,\overline{C}^{w^*})+\sup_{\overline{C}^{w^*}}(f)$, we have $d(x,\overline{C}^{w^*})\geq d(x,C)-\varepsilon$. Thus $d(x,\overline{C}^{w^*})\geq d(x,C)$ and hence $d(x,C)=d(x,\overline{C}^{w^*})$. The final conclusion follows from the fact that $B_{Y^{\perp\perp}}=\overline{B_Y}^{w^*}$ □

Lemma 2.2. *Let Y be a subspace of a Banach space X such that $Y^{\perp\perp}$ is strongly ball proximinal at all points of X. Then $d(y,P_{B_Y}(x))=d(y,P_{B_{Y^{\perp\perp}}}(x))$ for all $y\in Y$ and $x\in X$.*

Proof. Let $x\in X$ and $y\in Y$. Also, let $d=d(y,P_{B_{Y^{\perp\perp}}}(x))$ and $d'=d(x,B_{Y^{\perp\perp}})$. Since $P_{B_{Y^{\perp\perp}}}(x)$ is weak*-compact, it is proximinal in X^{**}. Choose $\phi\in P_{B_{Y^{\perp\perp}}}(x)$ such that $d(y,P_{B_{Y^{\perp\perp}}}(x))=\|y-\phi\|$.

As $d(x,B_Y)=d(x,B_{Y^{\perp\perp}})$, we observe that $P_{B_Y}(x,\delta)\subseteq P_{B_{Y^{\perp\perp}}}(x,\delta)$ for all $\delta>0$. Since $B_{Y^{\perp\perp}}$ is strongly proximinal at x, for every $\varepsilon>0$, there exists a $\delta_\varepsilon>0$ such that $d(v,P_{B_{Y^{\perp\perp}}}(x))<\varepsilon$ whenever $v\in P_{B_Y}(x,\delta_\varepsilon)$.

Now let $\varepsilon>0$ be arbitrary and let $\varepsilon_1>0$ be such that $0<\varepsilon_1<\min\left\{\frac{\varepsilon}{2(d+1)},\frac{\delta_{\varepsilon/2^2}}{d'+1}\right\}$.

Let $E_1=\operatorname{span}\{x,y,\phi\}\subseteq X^{**}$. Then, by an extended version of principle of local reflexivity (see [3, Theorem 3.2]), there exists a bounded linear map $T_1\colon E_1\to X$ such that $T_1(x)=x$, $T_1(y)=y$, $T_1(\phi)\in Y$ and $\|T_1\|\leq 1+\varepsilon_1$.

Now let $y_1 = \dfrac{T_1(\phi)}{1+\varepsilon_1}$. Then

$$\begin{aligned}\|y - y_1\| &\le \|T_1(y) - T_1(\phi)\| + \|T_1(\phi) - \frac{T_1(\phi)}{1+\varepsilon_1}\| \\ &\le (1+\varepsilon_1)d + \varepsilon_1 = d + \varepsilon_1(1+d) < d + \frac{\varepsilon}{2},\end{aligned}$$

and

$$\begin{aligned}\|x - y_1\| &\le \|T_1(x) - T_1(\phi)\| + \|T_1(\phi) - \frac{T_1(\phi)}{1+\varepsilon_1}\| \\ &\le (1+\varepsilon_1)d' + \varepsilon_1 = d' + \varepsilon_1(1+d') < d' + \delta_{\varepsilon/2^2}\,.\end{aligned}$$

Thus $y_1 \in P_{B_Y}(x, \delta_{\varepsilon/2^2})$ and hence $d(y_1, P_{B_{Y^{\perp\perp}}}(x)) < \frac{\varepsilon}{2^2}$. Now let $\phi_1 \in P_{B_{Y^{\perp\perp}}}(x)$ such that $\|y_1 - \phi_1\| < \frac{\varepsilon}{2^2}$. Then, again by principle of local reflexivity, there exists an element $y_2 \in B_Y$ such that $\|y_1 - y_2\| < \frac{\varepsilon}{2^2}$ and $\|x - y_2\| \le d + \delta_{\varepsilon/2^3}$.

Proceeding inductively, we obtain a sequence (y_n) in B_Y such that $\|y_{n-1} - y_n\| < \varepsilon/2^n$ and $\|x - y_n\| < d(x, B_Y) + \delta_{\varepsilon/2^{n+1}}$. Without loss of generality, we assume that $\delta_{\varepsilon/2^n} \to 0$.

Clearly, (y_n) is a Cauchy sequence in B_Y and hence there exists a $y_0 \in B_Y$ such that $y_0 = \lim_{n\to\infty} y_n$. Then $d(x, B_Y) = \lim_{n\to\infty} \|x - y_n\| = \|x - y_0\|$. Thus $y_0 \in P_{B_Y}(x)$. Since $\|y - y_n\| \le d + \varepsilon/2 + \ldots + \varepsilon/2^n$ for all $n \in \mathbb{N}$, by letting $n \to \infty$, it follows that $\|y - y_0\| \le d + \varepsilon$. Thus $d(y, P_{B_Y}(x)) \le \|y - y_0\| \le d(y, P_{B_{Y^{\perp\perp}}}(x)) + \varepsilon$. Since $\varepsilon > 0$ is arbitrary, $d(y, P_{B_Y}(x)) \le d(y, P_{B_{Y^{\perp\perp}}}(x))$ and hence the result follows. □

The following result connects strong ball proximinality of the subspace with its bidual.

Proposition 2.3. *Let Y be a subspace of a Banach space X such that $Y^{\perp\perp}$ is strongly ball proximinal at all points of X. Then Y is strongly ball proximinal in X.*

Proof. It follows from the proof of Lemma 2.2 that Y is ball proximinal in X. Now let $x \in X$ and (y_n) be a sequence in B_Y such that $\|x - y_n\| \to d(x, B_Y)$. Since $d(x, B_Y) = d(x, B_{Y^{\perp\perp}})$ and $Y^{\perp\perp}$ is strongly ball proximinal at x, we have $d(y_n, P_{B_{Y^{\perp\perp}}}(x)) \to 0$. Then, by Lemma 2.2, $d(y_n, P_{B_Y}(x)) \to 0$ and hence Y is strongly ball proximinal in X. □

Our next result shows that in L_1-predual spaces, strong ball proximinality of a finite co-dimensional subspace can be characterized in terms of the strong ball proximinality of its bidual.

Proposition 2.4. *Let Y be a finite co-dimensional subspace of an L_1-predual space X. Then Y is strongly ball proximinal in X if and only if $Y^{\perp\perp}$ is strongly ball proximinal in X^{**}.*

Proof. Suppose Y is strongly ball proximinal in X. Then, by [7, Theorem 3.10], $Y^{\perp\perp}$ is strongly proximinal in X^{**}. Since X is an L_1-predual space, by [8, Theorem 6.1], X^{**} is isometric to $C(K)$ for some compact Hausdorff space K. Then, by [6, Theorem 4.3], $Y^{\perp\perp}$ is strongly ball proximinal in X^{**}.

The converse follows from Proposition 2.3. □

In general, existence of best approximation for all points of a dense subset of a Banach space does not imply proximinality. But our next result exhibits strong ball proximinality through the weak*-dense subset X of X^{**}.

Corollary 2.5. *Let X be an L_1-predual space and W be a finite co-dimensional weak*-closed subspace of X^{**}. If W is strongly ball proximinal at all $x \in X$, then W is strongly ball proximinal in X^{**}.*

Proof. Since W is a finite co-dimensional weak*-closed subspace of X^{**}, there exists a basis $\{g_1, \dots, g_n\} \subset X^*$ of $W^\perp$. Now let $Y = \bigcap_i \ker(g_i)$. Then $Y^{\perp\perp} = W$. Hence, by Proposition 2.3, Y is strongly ball proximinal in X and the result follows from Proposition 2.4. □

We now state and prove the main result of the paper.

Theorem 2.6. *Let Y be a finite co-dimensional subspace of an L_1-predual space X. Then the following are equivalent.*

(i) *Y is strongly proximinal in X.*

(ii) *Y is strongly ball proximinal in X.*

(iii) *$Y^\perp \subseteq \{x^* \in X^* : x^*$ is an SSD-point of $X^*\}$.*

(iv) *$Y^\perp \subseteq \{x^* \in X^* : x^*$ is a QP-point of $X^*\}$.*

Proof. The implications (i) ⇒ (iii) and (iv) ⇒ (i) follow from [5]. The equivalence of (iii) and (iv) follows from [7, Proposition 3.18] and [5, Lemma 3.3].

Now to prove (i) ⇒ (ii) assume Y is strongly proximinal in X. Then, by [7, Theorem 3.10], $Y^{\perp\perp}$ is strongly proximinal in X^{**}. Since, by [8, Theorem 6.1], X^{**} is isometric to $C(K)$ for some compact Hausdorff space K, it follows from [6, Theorem 4.3] that $Y^{\perp\perp}$ is strongly ball proximinal in X^{**}. Then, by Proposition 2.4, Y is strongly ball proximinal in X.

Finally, (ii) ⇒ (i) follows from [2, Proposition 3.2(c)]. □

Our next result follows from Theorem 2.6 and [7, Corollary 3.21].

Corollary 2.7. *Let Y be a finite co-dimensional subspace of an L_1-predual space X. Then Y is strongly ball proximinal in X if and only if Y is the intersection of finitely many strongly ball proximinal hyperplanes in X.*

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