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Asymptotic Hyers-Ulam Stability or Superstability for Generalized Linear Equations by Unilateral Perturbations

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In relation to the famous problem of Ulam "Give conditions in order for a linear mapping near an approximated linear mapping to exist", we consider the stability or superstability of generalized linear equation

$$f(x+y) - f(x) - f(y) = B[\phi(x) + \phi(y)]$$

by left or right perturbations with some hypotheses of convexity or concavity, and – in a forthcoming paper – apply our conclusions to the generalized exponential equation

$$\frac{f(x+y)}{f(x)f(y)} = [\phi(x)\phi(y)]^B.$$

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1. Introduction

Compared to some results of Aoki, Hyers, Gajda, Gavruta, Rassias and others ([1], [2], [3], [4], [6], [7]), we propose three changes:

(1) Firstly, when the range of f is ordered and satisfies properties similar to those of the set of real numbers, R, in place of $||f(x+y) - f(x) - f(y)|| \leq H(x,y)$, we propose to explore unilateral perturbations, namely the right ones, given by

$$f(x+y) - f(x) - f(y) \leq H(x,y),$$

and the left ones, given by

$$f(x+y) - f(x) - f(y) \geq H(x,y)$$

where $H(x,y) = B[\phi(x)+\phi(y)]$ (with their analogues for the exponential equation in a forthcoming paper).

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For reasons that will become evident after Theorem 3.15 we call the right perturbations, by ”convexity side” and after a similar theorem in a forthcoming paper, the left ones, by ”concavity side”.

(2) We introduce the following particular meaning of ”closeness”: two functions f and g are said to be *near at infinity* if they are asymptotically close, that is if

$$\lim_{x\to\infty} [f(x) - g(x)] = 0\,.$$

Using the above concept, we obtain some stability or superstability results for the generalized linear equation

$$f(x+y) - f(x) - f(y) = B[\phi(x) + \phi(y)],$$

and then, for the generalized exponential equation

$$\frac{f(x+y)}{f(x)f(y)} = [\phi(x)\phi(y)]^B$$

in the presence of some convexity or concavity hypothesis.

(3) Finally we remark that the functions in Hyers' theorem [6] and those used by Rassias, Gajda, Gavruta et al., [2], [3], [7] implicitly satisfy the condition $f(0) = 0$, whereas our theorems are applicable to a broader class of functions.

Our paper is organized as follows: we begin with Theorem 2.1 and Corollary 2.2 that give necessary and sufficient conditions for a convex function to have an asymptote, that is to be asymptotically close to a linear or affine function.

In the following, we will deal more generally with functions satisfying

$$f(x+y) - f(x) - f(y) \le B[\phi(x) + \phi(y)], \tag{1}$$

but given that we first establish our results for convex functions, for which the difference $f(2x) - 2f(x)$ is particularly important, (see Lemma 2.4), in Lemmas 3.1 and 3.2 we will consider this relation only for $x = y$, that is

$$f(2x) - 2f(x) \le 2B\phi(x).$$

Then, we generalize the above theorems for functions $f\colon [0; +\infty[\to R$ satisfying (1), where $\phi\colon [0; +\infty[\to R$ satisfies

$$\phi(2x) \le \beta\phi(x) \text{ or } \phi(2x) \ge \beta\phi(x)$$

for $\beta \ne 2$, in order to examine if f can be asymptotically close, or even equal, to any $ax - b - k\phi(x)$, which can be interpreted as Hyers-Ulam stability or superstability. See for this Theorem 3.15, one of our main results. Some special cases of this theorem, where $\phi(2x) = \beta\phi(x)$, or $\phi(x) = x^p$ are considered in Theorem 3.17 and Theorem 3.18.

The condition $\lim_{x\to\infty}\phi(x)=+\infty$ that appears in our lemmas and in some parts of our theorems, generalizes Aoki's results [1], that first allowed growth of the form $K(||x^p||+||y||^p)$ for the norm of the Cauchy difference $f(x+y)-f(x)-f(y)$ where $0\le p<1$, generalizing so Hyers' idea in [6], where he considered the case of $p=0$. These results will be carried over with a concavity hypotheses in a forthcoming paper, where they will also be applied to obtain some asymptotic Hyers-Ulam stability or superstability results for the exponential equation.

$$\frac{f(x+y)}{f(x)f(y)}=1, \quad \text{or its generalized version,} \quad \frac{f(x+y)}{f(x)f(y)}=[\phi(x)\phi(y)]^B$$

by means of unilateral perturbations.

2. Stability or superstability by unilateral perturbations on the convexity side for generalized linear mappings

Theorem 2.1. *A convex function $f\colon [0;+\infty[\to R$ has an asymptote in $+\infty$ if and only if, $b=sup\{f(x+y)-f(x)-f(y)\}$ is finite. In this case, the equation of the asymptote is $y=ax-b$, where $a=\lim_{x\to\infty}(f(x)/x)$ (asymptotic stability). If moreover $f(0)=-b$, then $f(x)=ax-b$ (superstability result).*

Corollary 2.2. *A convex function $f\colon [0;+\infty[\to R$ is additive if and only if, for every $x>0$ and every $y>0$*

$$f(x+y)-f(x)-f(y)\le 0 \quad \textit{and} \quad f(0)=0.$$

Corollary 2.3. *Let $f\colon [0;+\infty[\to R$ be a convex function satisfying*

$$f(x+y)-f(x)-f(y)\le B. \tag{2}$$

Then f is asymptotically close to an affine function $y=ax-b$, where

$$a=\lim_{x\to+\infty}\frac{f(x)}{x} \quad \textit{and} \quad b=sup\{f(x+y)-f(x)-f(y)\}.$$

If moreover, $f(0)=-B$, then $f(x)=ax-B$.

We begin the proof of the above theorem and consequently, of the above corollaries, by proving some auxiliary results stated in the following lemmas:

For any function f, let f_r' denotes its right derivative, if it exists.

Lemma 2.4. *For a convex function $f\colon [0;+\infty[\to R$, we have that*

$$\sup\{f(x+y)-f(x)-f(y): x\ge 0,\ y\ge 0\}=\lim_{x\to\infty}[f(2x)-2f(x)],$$

and

$$\inf\{f(x+y)-f(x)-f(y): x\ge 0,\ y\ge 0\}=-f(0).$$

Proof of Lemma 2.4. Denote $F(x,y) = f(x+y) - f(x) - f(y)$ and for a fixed $a > 0$ we search for its supremum and infimum in the set

$$D_a = \{(x,y) : x \geq 0,\ y \geq 0,\ x+y \leq a\}.$$

For fixed $y \in [0,a]$, $\phi_y(x) = f(x+y) - f(x) - f(y)$ is right differentiable on $[0;+\infty[$, with $\phi'_{yr}(x) = f'_r(x+y) - f'_r(x)$, and, since f'_r is increasing, $\phi'_{yr}(x) \geq 0$. Thus $\phi_y(x)$ is increasing in $[0;a-y]$, hence

$$\sup\{\phi_y(x) : x \in [0;a-y]\} = \phi_y(a-y) = f(a) - f(a-y) - f(y),$$

and
$$\inf\{\phi_y(x) : x \in [0;a-y]\} = \phi_y(0) = -f(0).$$

It follows that for every $a > 0$,

$$\begin{aligned}\sup\{F(x,y) : (x,y) \in D_a\} &= \sup\{\phi_y(a-y) : 0 \leq y \leq a\}\\ &= \sup\{f(a) - f(a-y) - f(y) : 0 \leq y \leq a\}\end{aligned}$$

and $\inf\{F(x,y) : (x,y) \in D_a\} = -f(0)$. Hence, $\inf\{F(x,y) : x \geq 0,\ y \geq 0\} = -f(0)$. Next denoting $\psi_a(y) = f(a) - f(a-y) - f(y)$ we have that ψ_a is right differentiable and $\psi'_{ar}(y) = f'_r(a-y) - f'_r(y)$. Since f'_r is increasing, $\psi'_{ar}(y) \geq 0$ for $0 \leq y \leq (a/2)$ and $\psi'_{ar}(y) < 0$ for $(a/2) < y \leq a$. Therefore it follows that

$$\sup\{\psi_a(y) : 0 \leq y \leq a\} = \psi_a(\frac{a}{2}) = f(a) - 2f(\frac{a}{2})$$

and
$$\sup\{F(x,y) : (x,y) \in D_a\} = f(a) - 2f(\frac{a}{2}).$$

Note that since $F(\frac{a}{2},\frac{a}{2}) = f(a) - 2f(\frac{a}{2})$, the "sup" is in fact a "max".

Now the function $h(x) = f(x) - 2f(\frac{x}{2})$ is right differentiable and we have that $h'_r(x) = f'_r(x) - f'_r(\frac{x}{2}) \geq 0$, so it is increasing and

$$B := \sup\{h(x) : x \geq 0\} = \lim_{x\to+\infty} h(x) = \lim_{x\to+\infty} [f(x) - 2f(\frac{x}{2})].$$

Letting $a \to +\infty$ in $\sup\{F(x,y) : (x,y) \in D_a\} = f(a) - 2f(\frac{a}{2})$ it follows that

$$\sup\{F(x,y) : x \geq 0, y \geq 0\} = \lim_{a\to\infty} [f(a) - 2f(\frac{a}{2})] = \lim_{x\to\infty} [f(2x) - 2f(x)]$$

This completes the proof. □

Lemma 2.5. *If $f\colon [0;+\infty[\to R$ and $g\colon [0;+\infty[\to R$ are two convex functions satisfying $\lim_{x\to\infty} [f(x) - g(x)] = 0$, then*

$$\sup\{f(x+y) - f(x) - f(y) : x \geq 0, y \geq 0\} = \sup\{g(x+y) - g(x) - g(y) : x \geq 0, y \geq 0\}$$

In particular, if $g(x) = ax - b$ is an asymptote for f, then

$$\sup\{f(x+y) - f(x) - f(y) : x \geq 0, y \geq 0\} = b = -g(0).$$

Proof of Lemma 2.5. From

$$g(2x) - 2g(x) = [g(2x) - f(2x)] + [2f(x) - 2g(x)] + [f(2x) - 2f(x)]$$

and $\lim_{x\to\infty}[f(x) - g(x)] = 0$, we deduce that

$$\lim_{x\to\infty}[g(2x) - 2g(x)] = \lim_{x\to\infty}[f(2x) - 2f(x)],$$

that is

$$\sup\{f(x+y)-f(x)-f(y)\colon x \geq 0, y \geq 0\} = \sup\{g(x+y)-g(x)-g(y)\colon x \geq 0, y \geq 0.$$

If $g(x) = ax - b$ is an asymptote for f, then

$$\sup\{f(x+y) - f(x) - f(y)\colon x \geq 0, y \geq 0\} = \sup\{a(x+y)-b-ax+b-ay+b\} = b.$$

In particular, if $\lim_{x\to\infty}[f(2x) - 2f(x)] = \lim_{x\to\infty}[g(2x) - 2g(x)] \leq 0$, f and g are subadditive in the sense of Hille-Phillips [5]. This completes the proof. □

We have now all the ingredients to prove the statement of Theorem 2.1.

Proof of Theorem 2.1.

Necessity: Assume that $f\colon [0, +\infty[\to R$ is convex and has $y = ax - b$ as an asymptote. From Lemma 2.5, we have that

$$\sup\{f(x + y) - f(x) - f(y) : x \geq 0, y \geq 0\} = b$$

is finite.

Sufficiency: Suppose that $\sup\{f(x + y) - f(x) - f(y) : x \geq 0, y \geq 0\} = b$ is finite. The function $g\colon [0; +\infty[\to R$ defined by $g(x) = f(x) + b$ is convex and

$$\sup\{g(x+y)-g(x)-g(y)\colon x \geq 0, y \geq 0\}=\sup\{f(x+y)-f(x)-f(y)-b\} = b-b = 0.$$

From Theorem 7.6.1 of Hille and Philips [5], for the subadditive function g, we have that $a := \lim_{x\to+\infty}(g(x)/x)$ and consequently $a := \lim_{x\to+\infty}(f(x)/x)$, exists and is finite. Now from Theorem 7.2.4 of Hille and Philips, [5] we obtain that $g(x)/x = (f(x) + b)/x$ is decreasing for $x > 0$, hence $(g_r'(x)x - g(x))/x^2 \leq 0$, and $g_r'(x) \leq g(x)/x$. Moreover, for $x > 0$, $g(x)/x = (f(x) + b)/x \geq a$, whence $f(x) \geq ax - b$. Since, by convexity, $g_r'(x)$ is increasing, it has a limit that satisfies

$$\lim_{x\to+\infty} g_r'(x) \leq \lim_{x\to+\infty} \frac{g(x)}{x} = a$$

(in fact, $\lim_{x\to+\infty} g_r'(x) = a$, see Remark 2.6 below). Thus, it follows that $g_r'(x) \leq a$, and consequently, $\phi(x) = g(x) - ax$ is decreasing and positive, therefore it has a finite limit. Denoting $\lim_{x\to+\infty}[g(x) - ax] = b_1$, that is

$$\lim_{x\to+\infty}[f(x) - (ax + b_1 - b] = 0,$$

it follows that $y = ax + b_1 - b$ is an asymptote for f. From Lemma 2.5, $b = -(b_1 - b)$, which is equivalent to $b_1 = 0$. Therefore the equation of the asymptote is $y = ax - b$.

Suppose now that $f(0) = -b$.

From the aforementioned argument we have that for $x \geq 0$, $f(x) \geq ax - b$. If for any $x_0 > 0$, $f(x_0) > ax_0 - b$, from $\lim_{x\to+\infty}[f(x) - (ax - b)] = 0$, we can find an $x_1 > x_0$ satisfying $f(x_1)-ax_1 < f(x_0)-ax_0$, that implies $f(x_1)-f(x_0) < a(x_1-x_0)$ and $(f(x_1) - f(x_0))/(x_1 - x_0) < a$. But, by the properties of convex functions,

$$\frac{f(x_1) - f(x_0)}{x_1 - x_0} > \frac{f(x_0) - f(0)}{x_0 - 0} = \frac{f(x_0) + b}{x_0} > a.$$

This contradiction proves that $f(x) = ax - b$ for every $x \geq 0$, which completes the proof. $\square$

Remark 2.6. Indeed, $\lim_{x\to+\infty} g'_r(x) = a$, because for any $x_0 > 0$, and $x > x_0$, by the properties of convex functions, we have $(g(x) - g(x_0))/(x - x_0) \leq g'_r(x)$. Hence letting $x \to +\infty$ in

$$\frac{\dfrac{g(x)}{x} - \dfrac{g(x_0)}{x}}{1 - \dfrac{x_0}{x}} \leq g'_r(x),$$

we get $\displaystyle\lim_{x\to+\infty} \frac{g(x)}{x} \leq \lim_{x\to+\infty} g'_r(x)$, and finally, $\displaystyle\lim_{x\to+\infty} g'_r(x) = \lim_{x\to+\infty} \frac{g(x)}{x} = a$.

Remark 2.7. Without the convexity of f, the result of Theorem 2.1 does not hold: Indeed, the function $f(x) = \sqrt{x}$ clearly satisfies $f(x+y) - f(x) - f(y) \leq 0$ for every $x \geq 0$ and $y \geq 0$, so $\sup\{f(x+y) - f(x) - f(y) : x \geq 0,\ y \geq 0\}$ is finite, but it has no asymptote.

Remark 2.8. If $y = ax - b$ is an asymptote for a function $f\colon [0; +\infty[\to R$, and f is convex, by Lemma 2.5 $\sup\{f(x+y) - f(x) - f(y) : x \geq 0,\ y \geq 0\} = b$, but if f is not convex; $\sup\{f(x+y) - f(x) - f(y) : x \geq 0,\ y \geq 0\}$ may be different from b, even if f satisfies $f(x+y) - f(x) - f(y) \leq b$, for every $x \geq 0$ and $y \geq 0$, as is the case for the next function defined by:

$$f(x) = \begin{cases} cx + d & \text{if } 0 \leq x \leq \gamma \\ ax - b & \text{if } x > \gamma \end{cases}$$

with $0 < a < c$, $\gamma > 0$ and $a\gamma - b = c\gamma + d$. It has asymptote $y = ax - b$ and is concave, hence, making use of $\sup(-A) = -\inf(A)$ and applying Lemma 2.4 to the convex function $F = -f$, we have

$$\begin{gathered} \sup\{f(x+y) - f(x) - f(y) : x \geq 0,\ y \geq 0\} \\ = -\inf\{F(x+y) - F(x) - F(y)\} = -(-F(0)) = F(0) = -f(0) = -d, \end{gathered}$$

but the previous conditions imply $-d \neq b$!

Proof of Corollary 2.2. The necessary part of the claim is clear.

Conversely, the conditions $f(x+y)-f(x)-f(y)\le 0$ and $f(0)=0$ imply

$$b=\sup\{f(x+y)-f(x)-f(y):x\ge 0,y\ge 0\}=0.$$

Hence, Theorem 2.1 implies, $f(x)=ax$. □

Proof of Corollary 2.3. We let $b=\sup\{f(x+y)-f(x)-f(y)\}$ which is finite by (2). From Theorem 2.1, f is asymptotically close to the affine function $y=ax-b$, where $a=\lim_{x\to+\infty}(f(x)/x)$. Suppose now that $f(0)=-B$. This implies that $\sup\{f(x+y)-f(x)-f(y)\}=B$, whence from Theorem 2.1, $f(x)=ax-B$. □

3. Main results

In the following we will consider the generalized, right perturbed, linear equation

$$f(x+y)-f(x)-f(y)\le B[\phi(x)+\phi(y)],$$

with $\phi\colon [0;+\infty[\to R$ satisfying relations as $\phi(2x)\le\beta\phi(x)$ or $\phi(2x)\ge\beta\phi(x)$, and with a convexity hypothesis, in order to obtain stability or superstability results. Since the difference $f(2x)-2f(x)$ is a very important feature for convex functions (see Lemma 2.4), in Lemmas 3.1, 3.5 and 3.6 we consider this relation only for $x=y$, that is $f(2x)-2f(x)\le 2B\phi(x)$.

The simplest functions satisfying these relations are the functions $\phi(x)=x^p$ and $f(x)=ax+k\phi(x)$ with $\beta=2^p$ and $2B=k(2^p-2)$ (in fact $\phi(2x)=\beta\phi(x)$). Given the expected results, it is natural to begin with the study of functions $g(x)=f(x)+k\phi(x)$ that have an asymptote.

This leads us to the following Lemmas 3.1, 3.2, 3.5, 3.6 that justify, among others, the special role of $k=(2B)/(2-\beta)$ in Theorem 3.15.

Lemma 3.1. *Let f and ϕ be two functions defined on $[0;+\infty[$ and let $(x_n)\to+\infty$ be a sequence, satisfying $f(2x_n)-2f(x_n)\le 2B\phi(x_n)$.*

Suppose that for any real k the function $g(x)=f(x)+k\phi(x)$ has an asymptote in $+\infty$ and that there exists a real β, satisfying for all $n\ge n_0\in N$, one of the following relations

(i) $\phi(2x_n)\le\beta\phi(x_n)$ *and* $k>0$*; or*

(ii) $\phi(2x_n)\ge\beta\phi(x_n)$ *and* $k<0$*; or*

(iii) $\phi(2x_n)=\beta\phi(x_n)$.

Then the following statements hold true:

(1) *if* $\lim\limits_{n\to+\infty}\phi(x_n)=+\infty$, *then* $2B+k(\beta-2)\ge 0$, *that is*

$$\begin{cases} k\le\dfrac{2B}{2-\beta} & \text{if } \beta<2\\[2mm] k\ge\dfrac{2B}{2-\beta} & \text{if } \beta>2\\[2mm] B\ge 0 & \text{if } \beta=2\end{cases}$$

(2) *if* $\lim_{n\to+\infty}\phi(x_n) = -\infty$, *then* $2B + k(\beta-2) \le 0$, *that is*

$$\begin{cases} k \ge \dfrac{2B}{2-\beta} & \textit{if } \beta < 2 \\ k \le \dfrac{2B}{2-\beta} & \textit{if } \beta > 2 \\ B \le 0 & \textit{if } \beta = 2 \end{cases}$$

Proof of Lemma 3.1. By hypothesis, let $y = ax - b$ be the asymptote of g. It follows that $\gamma(x) = g(x) - (ax - b) \to 0$ for $x \to +\infty$, $g(x) = ax - b + \gamma(x)$, $g(2x) - 2g(x) = b + \gamma(2x) - 2\gamma(x)$, and

$$\lim_{n\to+\infty} \frac{g(2x_n) - 2g(x_n)}{\phi(x_n)} = \lim_{n\to+\infty} \frac{b + \gamma(2x_n) - 2\gamma(x_n)}{\phi(x_n)} = 0.$$

On the other hand, from $g(x) = f(x) + k\phi(x)$ we get

$$\frac{g(2x_n) - 2g(x_n)}{\phi(x_n)} = \frac{f(2x_n) - 2f(x_n)}{\phi(x_n)} + k\left[\frac{\phi(2x_n)}{\phi(x_n)} - 2\right].$$

(1) Suppose now that $\lim_{n\to+\infty}\phi(x_n) = +\infty$, so that $\phi(x_n)$ are positive for n large enough, and assume that one of the conditions (i), (ii) or (iii) is fulfilled. It follows that

$$\frac{f(2x_n) - 2f(x_n)}{\phi(x_n)} \le 2B, \quad k\left[\frac{\phi(2x_n)}{\phi(x_n)} - 2\right] \le k(\beta - 2)$$

$$\left(\text{in the third case, } k\left[\frac{\phi(2x_n)}{\phi(x_n)} - 2\right] = k(\beta-2)\right), \quad \text{and}$$

$$\frac{g(2x_n) - 2g(x_n)}{\phi(x_n)} = \frac{f(2x_n) - 2f(x_n)}{\phi(x_n)} + k[\frac{\phi(2x_n)}{\phi(x_n)} - 2] \le 2B + k(\beta-2).$$

Passing to the limit the desired statement follows.

(2) Suppose now that $\lim_{n\to+\infty}\phi(x_n) = -\infty$, so that $\phi(x_n)$ are negative for n large enough, and assume that one of the conditions (i), (ii) or (iii) is fulfilled. It follows that

$$\frac{f(2x_n) - 2f(x_n)}{\phi(x_n)} \ge 2B, \quad k\left[\frac{\phi(2x_n)}{\phi(x_n)} - 2\right] \ge k(\beta - 2)$$

$$\left(\text{in the third case, } k\left[\frac{\phi(2x_n)}{\phi(x_n)} - 2\right] = k(\beta-2)\right), \quad \text{and}$$

$$\frac{g(2x_n) - 2g(x_n)}{\phi(x_n)} = \frac{f(2x_n) - 2f(x_n)}{\phi(x_n)} + k\left[\frac{\phi(2x_n)}{\phi(x_n)} - 2\right] \ge 2B + k(\beta-2).$$

Passing to the limit the desired statement follows. $\square$

Lemma 3.2. *Let ϕ and f be two functions defined on $[0;+\infty[$, and k a real number for which the function $g(x) = f(x) + k\phi(x)$ has an asymptote in $+\infty$. Suppose moreover that there exists a sequence $(x_n) \to +\infty$ for which $\lim_{n\to+\infty} |\phi(x_n)| = +\infty$. Then the following statements hold true:*

(a) *If* $\lim_{n\to+\infty} \dfrac{\phi(2x_n)}{\phi(x_n)} = \beta$ *and* $\lim_{n\to+\infty} \dfrac{f(2x_n) - 2f(x_n)}{\phi(x_n)} = 2B$,

then $\begin{cases} k = \frac{2B}{2-\beta} & \text{if } \beta \neq 2 \\ B = 0 & \text{if } \beta = 2 \end{cases}$

(b) $\lim_{n\to+\infty} \dfrac{\phi(2x_n)}{\phi(x_n)} = \beta$ *is equivalent to* $\lim_{n\to+\infty} \dfrac{f(2x_n)-2f(x_n)}{\phi(x_n)} = k(2-\beta)$ *for* $k \neq 0$.

Proof of Lemma 3.2. By hypothesis, let $y = ax - b$ be the asymptote of g. This implies that $\gamma(x) = g(x) - (ax - b) \to 0$ as $x \to +\infty$, $g(x) = ax - b + \gamma(x)$, $g(2x) - 2g(x) = b + \gamma(2x) - 2\gamma(x)$, and

$$\lim_{n\to+\infty} \frac{g(2x_n) - 2g(x_n)}{\phi(x_n)} = \lim_{n\to+\infty} \frac{b + \gamma(2x_n) - 2\gamma(x_n)}{\phi(x_n)} = 0.$$

On the other hand, the fact that $g(x) = f(x) + k\phi(x)$ gives

$$\frac{g(2x_n) - 2g(x_n)}{\phi(x_n)} = \frac{f(2x_n) - 2f(x_n)}{\phi(x_n)} + k\left[\frac{\phi(2x_n)}{\phi(x_n)} - 2\right] \tag{3}$$

(a) Passing to the limit in (3), we obtain $0 = 2B + k(\beta - 2)$, that is

$$\begin{cases} k = \dfrac{2B}{2-\beta} & \text{if } \beta \neq 2 \\ B = 0 & \text{if } \beta = 2 \end{cases}$$

which proves the first statement.

(b) If the limit of one of the two ratios on the right side of (3) exists, the other exists too, hence passing to the limit in (3), we obtain

$$\lim_{n\to+\infty} \frac{f(2x_n) - 2f(x_n)}{\phi(x_n)} + k\left[\lim_{n\to+\infty} \frac{\phi(2x_n)}{\phi(x_n)} - 2\right] = 0,$$

that is, if $\lim_{n\to+\infty} \dfrac{\phi(2x_n)}{\phi(x_n)} = \beta$ then $\lim_{n\to+\infty} \dfrac{f(2x_n) - 2f(x_n)}{\phi(x_n)} = k(2-\beta)$ and,

if $\lim_{n\to+\infty} \dfrac{f(2x_n) - 2f(x_n)}{\phi(x_n)} = k(2-\beta)$ then $\lim_{n\to+\infty} \dfrac{\phi(2x_n)}{\phi(x_n)} = \dfrac{k(\beta-2)}{k} + 2 = \beta$.

The condition $k \neq 0$ is needed only in the last part, but, for $k = 0$, Lemma 3.2 is of no particular interest. $\square$

Remark 3.3. If the asymptote of g has for equation $y = ax$, that is if $b = 0$, we can replace the condition "there exists a sequence $(x_n) \to +\infty$ for which $\lim_{n\to+\infty} |\phi(x_n)| = +\infty$", by "there exists a sequence $(x_n) \to +\infty$ for which $|\phi(x_n)| \geq c > 0$" and keep the same proof.

Remark 3.4. The simpliest examples satisfying the requirements of Lemma 3.2 are the functions $\phi\colon [0;+\infty[\to R$, defined by $\phi(x)=x^p$ for $p>0$ and $f:[0;+\infty[\to R$ defined by $f(x) = cx + d\phi(x)$. In these cases, $\beta = 2^p$, $2B = d(2^p - 2)$ and $k = \frac{2B}{2-\beta} = -d$.

The following lemma is in some sense the inverse of Part (a) of Lemma 3.2:

Lemma 3.5. *Let f and ϕ be two functions defined on $[0;+\infty[$ and $(x_n) \to +\infty$ a sequence satisfying*

(i) $\lim\limits_{n\to+\infty} \phi(x_n) = +\infty$ *or* $-\infty$, *and*

(ii) $f(2x_n) - 2f(x_n) \leq 2B\phi(x_n)$.

If there exists $\beta \neq 2$, for which either

(iii) *for all $n \geq n_0$, $\phi(2x_n) \leq \beta\phi(x_n)$ and $\frac{2B}{2-\beta} > 0$, or*

(iv) *for all $n \geq n_0$, $\phi(2x_n) \geq \beta\phi(x_n)$ and $\frac{2B}{2-\beta} < 0$, or*

(v) *for all $n \geq n_0$, $\phi(2x_n) = \beta\phi(x_n)$,*

and, furthermore, if the function $g(x) = f(x) + k\phi(x)$ has an asymptote at $+\infty$ for $k = \frac{2B}{2-\beta}$, then

$$\lim_{n\to+\infty} \frac{\phi(2x_n)}{\phi(x_n)} = \beta \quad \textit{and} \quad \lim_{n\to+\infty} \frac{f(2x_n) - 2f(x_n)}{\phi(x_n)} = 2B.$$

Proof of Lemma 3.5. Denoting $\gamma(x) = g(x) - (ax - b)$, where $y = ax - b$ is the asymptote of g, we have $\gamma(x) \to 0$ as $x \to +\infty$, $f(x) = ax - b - k\phi(x) + \gamma(x)$ and

$$f(2x_n) - 2f(x_n) = 2k\phi(x_n) - k\phi(2x_n) + \gamma(2x_n)) - 2\gamma(x_n) + b \leq 2B\phi(x_n),$$

from which one finds

$$k\phi(2x_n) \geq (2k - 2B)\phi(x_n) + \gamma(2x_n) - 2\gamma(x_n) + b$$

that is
$$k\phi(2x_n) \geq k\beta\phi(x_n) + \gamma(2x_n) - 2\gamma(x_n) + b.$$

Assume first that $k > 0$ and $\phi(2x_n) \leq \beta\phi(x_n)$; dividing by k yields:

$$\phi(2x_n) \geq \beta\phi(x_n) + \frac{\gamma(2x_n) - 2\gamma(x_n) + b}{k}.$$

In the case when $\lim_{n\to+\infty} \phi(x_n) = +\infty$, dividing by $\phi(x_n)$ (in this case $\phi(x_n) > 0$ for n large enough), we have

$$\frac{\phi(2x_n)}{\phi(x_n)} \geq \beta + \frac{\gamma(2x_n) - 2\gamma(x_n) + b}{k\phi(x_n)}.$$

From $\beta + \dfrac{\gamma(2x_n) - 2\gamma(x_n) + b}{k\phi(x_n)} \le \dfrac{\phi(2x_n)}{\phi(x_n)} \le \beta$, it results that $\lim\limits_{n\to+\infty} \dfrac{\phi(2x_n)}{\phi(x_n)} = \beta$.

In the case when $\lim_{n\to+\infty}\phi(x_n) = -\infty$, dividing by $\phi(x_n)$ (in this case $\phi(x_n) < 0$ for n large enough), we have

$$\frac{\phi(2x_n)}{\phi(x_n)} \le \beta + \frac{\gamma(2x_n) - 2\gamma(x_n) + b}{k\phi(x_n)}$$

From the hypothesis $\phi(2x_n) \le \beta\phi(x_n)$ we get $\dfrac{\phi(2x_n)}{\phi(x_n)} \ge \beta$.

The fact $\beta \le \dfrac{\phi(2x_n)}{\phi(x_n)} \le \beta + \dfrac{\gamma(2x_n)-2\gamma(x_n)+b}{k\phi(x_n)}$, now implies $\lim\limits_{n\to+\infty} \dfrac{\phi(2x_n)}{\phi(x_n)} = \beta$.

Next we consider the case $k < 0$ and $\phi(2x_n) \ge \beta\phi(x_n)$; dividing by k the inequality $k\phi(2x_n) \ge k\beta\phi(x_n) + \gamma(2x_n) - 2\gamma(x_n) + b$, leads to

$$\phi(2x_n) \le \beta\phi(x_n) + \frac{\gamma(2x_n) - 2\gamma(x_n) + b}{k}$$

Again, if $\lim_{n\to+\infty}\phi(x_n) = +\infty$, dividing by $\phi(x_n)$ (in this case $\phi(x_n) > 0$ for n large enough), we have

$$\frac{\phi(2x_n)}{\phi(x_n)} \le \beta + \frac{\gamma(2x_n) - 2\gamma(x_n) + b}{k\phi(x_n)}.$$

From the hypothesis $\phi(2x_n) \ge \beta\phi(x_n)$ we get $\dfrac{\phi(2x_n)}{\phi(x_n)} \ge \beta$.

From $\beta \le \dfrac{\phi(2x_n)}{\phi(x_n)} \le \beta + \dfrac{\gamma(2x_n) - 2\gamma(x_n) + b}{k\phi(x_n)}$, results $\lim\limits_{n\to+\infty} \dfrac{\phi(2x_n)}{\phi(x_n)} = \beta$.

If $\lim_{n\to+\infty}\phi(x_n) = -\infty$, dividing by $\phi(x_n)$ (in this case $\phi(x_n) < 0$ for n large enough), we have

$$\frac{\phi(2x_n)}{\phi(x_n)} \ge \beta + \frac{\gamma(2x_n) - 2\gamma(x_n) + b}{k\phi(x_n)}.$$

From the hypothesis we have $\phi(2x_n) \ge \beta\phi(x_n)$, in other words $\dfrac{\phi(2x_n)}{\phi(x_n)} \le \beta$.

From $\beta + \dfrac{\gamma(2x_n) - 2\gamma(x_n) + b}{k\phi(x_n)} \le \dfrac{\phi(2x_n)}{\phi(x_n)} \le \beta$, results $\lim\limits_{n\to+\infty} \dfrac{\phi(2x_n)}{\phi(x_n)} = \beta$.

In both cases, by Lemma 3.2, part (b) implies

$$\lim_{n\to+\infty} \frac{f(2x_n) - 2f(x_n)}{\phi(x_n)} = k(2 - \beta) = 2B,$$

which completes the proof. □

Very slight modifications of Lemma 3.5 yield the following statement:

Lemma 3.6. *Let f and ϕ be two functions defined on $[0;+\infty[$, satisfying*

$$\lim_{x\to+\infty}\phi(x)=+\infty, \quad \textit{or} \quad \lim_{x\to+\infty}\phi(x)=-\infty \tag{4}$$

and $f(2x)-2f(x)\leq 2B\phi(x)$. If there exists $\beta\neq 2$, such that for all $x\geq x_0$ either one of the following conditions holds

(i) $\phi(2x)\leq\beta\phi(x)$ *and* $\dfrac{2B}{2-\beta}>0$, *or*

(ii) $\phi(2x)\geq\beta\phi(x)$ *and* $\dfrac{2B}{2-\beta}<0$, *or*

(iii) $\phi(2x)=\beta\phi(x)$

and the function $g(x)=f(x)+k\phi(x)$ has an asymptote at $+\infty$ for $k=\frac{2B}{2-\beta}$, then

$$\lim_{x\to+\infty}\frac{\phi(2x)}{\phi(x)}=\beta \quad \textit{and} \quad \lim_{x\to+\infty}\frac{f(2x)-2f(x)}{\phi(x)}=2B.$$

Remark 3.7. If the asymptote of g has the equation $y=ax$, i.e. $b=0$ in Lemma 3.5, we can replace condition (i) by: ($\forall n\geq n_0$, $\phi(x_n)\geq c>0$) or ($\forall n\geq n_0$, $\phi(x_n)\leq c<0$). Similarly, in Lemma 3.6 the condition (4) can be replaced by: ($\forall x\geq x_0$, $\phi(x)\geq c>0$) or, $\forall x\geq x_0$, $\phi(x)\leq c<0$). Exactly the same proof works.

Remark 3.8. In relation with Lemma 3.6 (similarly with Lemma 3.5), we ask if its statements remain true if, without changing the other conditions of Lemma 3.6, we change only the sign of $\frac{2B}{2-\beta}$ like below:

(i') for all $x\geq x_0$, $\phi(2x)\leq\beta\phi(x)$ and $\dfrac{2B}{2-\beta}<0$, or

(ii') for all $x\geq x_0$, $\phi(2x)\geq\beta\phi(x)$ and $\dfrac{2B}{2-\beta}>0$.

We will prove that in these cases there are infinitely many couples (B,β), for which these conditions are satisfied. Of course, the conditions $\frac{2B}{2-\beta}<0$, or $\frac{2B}{2-\beta}>0$ exclude that $B=0$, so in the following we will always suppose $B\neq 0$.

Suppose first that $\phi(x)\geq 0$. We fix some $B'>B$ such that the conditions $f(2x)-2f(x)\leq 2B'\phi(x)$ and $\beta'=2-(2-\beta)\gamma$ are fulfilled, where $\gamma=B'/B$, which yields

$$\frac{2B}{2-\beta}=\frac{2B'}{2-\beta'}.$$

But, in the first case, when for all $x\geq x_0$, $\phi(2x)\leq\beta\phi(x)$ and $\frac{2B}{2-\beta}<0$, we do not yet know if $\phi(2x)\leq\beta'\phi(x)$; it would be so, if $\beta'\geq\beta$.

Two cases are possible for $\frac{2B}{2-\beta} < 0$: ($B < 0$ and $\beta < 2$) or ($B > 0$ and $\beta > 2$).

Distinguishing the cases $B < 0$ or $B > 0$, from $\beta' - \beta = (2-\beta)(1-\gamma)$, it follows, in both cases, that $\beta' - \beta > 0$, that is, $\beta' > \beta$ and, consequently, $\phi(2x) < \beta'\phi(x)$.

While, in the second case, when

$$\forall x \geq x_0, \quad \phi(2x) \geq \beta\phi(x) \ \text{ and } \ \frac{2B}{2-\beta} > 0,$$

we do not yet know if $\phi(2x) \geq \beta'\phi(x)$; it would be so, if $\beta' \leq \beta$.

Two cases are possible: ($B > 0$ and $\beta < 2$) or ($B < 0$ and $\beta > 2$).

Distinguishing the cases $B < 0$ or $B > 0$, from $\beta' - \beta = (2-\beta)(1-\gamma)$, it follows in both cases that $\beta' - \beta < 0$, that is, $\beta' < \beta$ and, consequently, $\phi(2x) > \beta'\phi(x)$.

Suppose now that $\phi(x) \leq 0$.

By a similar reasoning, we fix some $B' < B$ for getting $f(2x) - 2f(x) \leq 2B'\phi(x)$ and $\beta' = 2 - (2-\beta)\gamma$, where $\gamma = B'/B$, for having

$$\frac{2B}{2-\beta} = \frac{2B'}{2-\beta'}.$$

In the first case, when for all $x \geq x_0$, $\phi(2x) \leq \beta\phi(x)$ and $\frac{2B}{2-\beta} < 0$, we do not yet know if $\phi(2x) \leq \beta'\phi(x)$; it would be so, if $\beta' \leq \beta$.

Two cases are possible for $\frac{2B}{2-\beta} < 0$: ($B < 0$ and $\beta < 2$) or ($B > 0$ and $\beta > 2$).

Distinguishing the cases $B < 0$ or $B > 0$, from $\beta' - \beta = (2-\beta)(1-\gamma)$, it follows in both cases that $\beta' - \beta < 0$, that is, $\beta' < \beta$ and, consequently, $\phi(2x) < \beta'\phi(x)$.

In the second case, when for all $x \geq x_0$, $\phi(2x) \geq \beta\phi(x)$ and $\frac{2B}{2-\beta} > 0$, we do not yet know if $\phi(2x) \geq \beta'\phi(x)$; it would be so, if $\beta' \geq \beta$.

Two cases are possible: ($B > 0$ and $\beta < 2$) or ($B < 0$ and $\beta > 2$).

Distinguishing the cases $B < 0$ or $B > 0$, from $\beta' - \beta = (2-\beta)(1-\gamma)$, it follows in both cases that $\beta' - \beta > 0$, that is, $\beta' > \beta$ and, consequently, $\phi(2x) > \beta'\phi(x)$.

So, if in Lemma 3.6 (and similarly in Lemma 3.5) the condition (i') or (ii') is fulfilled, all the conditions of these lemmas will be satisfied even if we replace $k = 2B/(2-\beta)$ by $k' = 2B'/(2-\beta')$, where $B' > B$ if $\phi(x) > 0$ and $B' < B$ if $\phi(x) < 0$, and, in both cases, $\beta' = \beta + (2-\beta)(1-\gamma)$, $\gamma = B'/B$.

This means that, if

$$\lim_{x\to+\infty} \frac{\phi(2x)}{\phi(x)} = \beta \ \text{ and / or } \ \lim_{x\to+\infty} \frac{f(2x) - 2f(x)}{\phi(x)} = 2B,$$

it is impossible to have

$$\lim_{x\to+\infty} \frac{\phi(2x)}{\phi(x)} = \beta' \ \text{ and / or } \ \lim_{x\to+\infty} \frac{f(2x) - 2f(x)}{\phi(x)} = 2B',$$

if $\beta \neq \beta'$ and $B \neq B'$.

This is an indirect argument proving that, without supplementary conditions, if

$$\left(\phi(2x) \leq \beta\phi(x) \;\text{ and }\; \frac{2B}{2-\beta} < 0\right) \;\text{ or }\; \left(\phi(2x) \geq \beta\phi(x) \;\text{ and }\; \frac{2B}{2-\beta} > 0\right),$$

the conclusions of these lemmas are not true in general (see also Example 3.9 below). However, with supplementary conditions as for example $\phi(2x) = \beta\phi(x)$, which implies that the sign of k is not important, they may be true (see Example 3.10 below).

Example 3.9. Let $\phi(x) = \sqrt{x}+1$ and $f(x) = x+\sqrt{x}+2$. A simple study shows

$$\phi(2x) = \sqrt{2x}+1 = \sqrt{2}\sqrt{x}+1 \leq \beta(\sqrt{x}+1) = \beta\phi(x)$$

if and only if $\beta \geq \sqrt{2}$. Let, for example, $\beta = \sqrt{2}+1$. On the other hand

$$\begin{aligned} f(2x) - 2f(x) \leq 2B\phi(x) \;&\Leftrightarrow\; (\sqrt{2}-2)\sqrt{x} - 2 \leq 2B(\sqrt{x}+1) \\ &\Leftrightarrow\; (\sqrt{2}-2-2B)\sqrt{x} \leq 2B+2 \end{aligned}$$

is satisfied if and only if $\sqrt{2}-2-2B \leq 0$ and $2B+2 \geq 0$, that is, $2B \geq \sqrt{2}-2$. Choose, for example, $2B = \sqrt{2}-1$. For $\beta = \sqrt{2}+1$, $2B = \sqrt{2}-1$, and $k = 2B/(2-\beta) = -1 < 0$, the function $g(x) = f(x)+k\phi(x) = f(x)-\phi(x) = x+1$ has clearly an asymptote, but

$$\lim_{x\to+\infty} \frac{\phi(2x)}{\phi(x)} = \lim_{x\to+\infty} \frac{\sqrt{2x}+1}{\sqrt{x}+1} = \sqrt{2} \neq \beta, \quad \text{and}$$

$$\lim_{x\to+\infty} \frac{f(2x)-2f(x)}{\phi(x)} = \lim_{x\to+\infty} \frac{(\sqrt{2}-2)\sqrt{x}-2}{\sqrt{x}+1} = \sqrt{2}-2 \neq 2B.$$

Example 3.10. Let $\phi(x) = \sqrt{x}$ and $f(x) = x+\sqrt{x}$. Obviously

$$\phi(2x) = \sqrt{2x} = \beta\phi(x) \;\text{ for }\; \beta = \sqrt{2}, \;\text{ and}$$

$$f(2x) - 2f(x) = (\sqrt{2}-2)\sqrt{x} = 2B\phi(x) \;\text{ for }\; 2B = \sqrt{2}-2.$$

Therefore, for $k = 2B/(2-\beta) = -1$, the function

$$g(x) = f(x) + k\phi(x) = f(x) - \phi(x) = x$$

has clearly an asymptote, and,

$$\lim_{x\to+\infty} \frac{\phi(2x)}{\phi(x)} = \lim_{x\to+\infty} \frac{\sqrt{2x}}{\sqrt{x}} = \sqrt{2} = \beta, \quad \text{and}$$

$$\lim_{x\to+\infty} \frac{f(2x)-2f(x)}{\phi(x)} = \lim_{x\to+\infty} \frac{(\sqrt{2}-2)\sqrt{x}}{\sqrt{x}} = \sqrt{2}-2 = 2B.$$

This is the case for $\phi(2x) = \beta\phi(x)$, for which the sign of k is not important.

The following example is used to answer several questions posed in the Remarks 3.12, 3.13 and 3.14.

Example 3.11. Let ϕ: $[0;+\infty[\to R$ be the function defined by

$$\phi(x) = \begin{cases} \sqrt{x} & \text{if } x \neq 2^n, n \in Z \\ \sqrt[3]{x} & \text{if } x = 2^n, n \in Z \end{cases}$$

and $f(x) = x - \phi(x)$. Since

$$\phi(2x) = \begin{cases} \sqrt{2}\sqrt{x} & \text{if } x \neq 2^n, n \in Z \\ \sqrt[3]{2}\ \sqrt[3]{x} & \text{if } x = 2^n, n \in Z \end{cases}$$

and $\sqrt[3]{2} \leq \sqrt{2}$, we obtain that for every $x \geq 0$, $\sqrt[3]{2}\phi(x) \leq \phi(2x) \leq \sqrt{2}\phi(x)$.

So, if we need $\phi(2x) \leq \beta\phi(x)$ we must have $\beta = \sqrt{2}$, and, if we need $\phi(2x) \geq \beta\phi(x)$ we must have $\beta = \sqrt[3]{2}$. On the other hand,

$$f(2x) - 2f(x) = 2\phi(x) - \phi(2x) = \begin{cases} (2-\sqrt{2})\phi(x) & \text{if } x \neq 2^n,\ n \in Z, \\ (2-\sqrt[3]{2})\phi(x) & \text{if } x = 2^n,\ n \in Z. \end{cases}$$

So,
$$(2-\sqrt{2})\phi(x) \leq f(2x) - 2f(x) \leq (2-\sqrt[3]{2})\phi(x)$$

and if we need $2B\phi(x) \leq f(2x) - 2f(x) \leq 2B$, as in Lemma 3.5 or 3.6, then $2B = 2 - \sqrt[3]{2}$.

Remark 3.12. If there exists at least one k for which $f(x)+k\phi(x)$ has an asymptote, this does not imply that $\lim\limits_{x\to+\infty} \frac{\phi(2x)}{\phi(x)}$ and/or $\lim\limits_{x\to+\infty} \frac{f(2x)-2f(x)}{\phi(x)}$ exist:

Indeed, if we take $k = 1$ in Example 3.11 above, $f(x) + k\phi(x) = x$ has clearly an asymptote, but none of the limits $\lim\limits_{x\to+\infty} \frac{\phi(2x)}{\phi(x)}$ and/or $\lim\limits_{x\to+\infty} \frac{f(2x)-2f(x)}{\phi(x)}$ exist, because for a sequence $(x_n) \to +\infty$, such that $x_n \neq 2^n$ for every $n \in N$, we have $\frac{\phi(2x_n)}{\phi(x_n)} = \sqrt{2} \to \sqrt{2}$ and $\frac{f(2x_n)-2f(x_n)}{\phi(x_n)} = 2-\sqrt{2} \to 2-\sqrt{2}$, but for the sequence $(x'_n) \to +\infty$, such that $x'_n = 2^n$ for every $n \in N$, $\frac{\phi(2x'_n)}{\phi(x'_n)} = \sqrt[3]{2} \to \sqrt[3]{2} \neq \sqrt{2}$ and $\frac{f(2x'_n)-2f(x'_n)}{\phi(x'_n)} = 2-\sqrt[3]{2} \to 2-\sqrt[3]{2} \neq 2-\sqrt{2}$.

Remark 3.13. Note that if there exists a sequence $(x_n) \to +\infty$ with the property $\lim\limits_{n\to+\infty} \frac{\phi(2x_n)}{\phi(x_n)} = \beta$, this does not imply $\lim\limits_{n\to+\infty} \frac{f(2x_n)-2f(x_n)}{\phi(x_n)} = 2B$ in general.

Indeed, in Example 3.11 above, for any sequence $(x_n) \to +\infty$, such that $x_n \neq 2^n$ for every $n \in N$, we have $\frac{\phi(2x_n)}{\phi(x_n)} = \sqrt{2}$ and also $\lim\limits_{n\to+\infty} \frac{\phi(2x_n)}{\phi(x_n)} = \sqrt{2} = \beta$, but $\frac{f(2x_n)-2f(x_n)}{\phi(x_n)} = 2-\sqrt{2}$ and $\lim\limits_{n\to+\infty} \frac{f(2x_n)-2f(x_n)}{\phi(x_n)} = 2-\sqrt{2} \neq 2-\sqrt[3]{2} = 2B$.

However, if for $k = \frac{2B}{2-\beta}$ the function $g(x) = f(x) + k\phi(x)$ has an asymptote, $\lim\limits_{x\to+\infty} \phi(x) = +\infty$ or $\lim\limits_{x\to+\infty} \phi(x) = -\infty$, and $\lim\limits_{x\to+\infty} \frac{\phi(2x)}{\phi(x)} = \beta$, then for every sequence $(x_n) \to +\infty$, satisfying $\phi(x_n) \to +\infty$, or $\phi(x_n) \to -\infty$, it follows from Part (b) of Lemma 3.2 that $\lim\limits_{n\to+\infty} \frac{f(2x_n)-2f(x_n)}{\phi(x_n)} = k(2-\beta) = 2B$, regardless of the sign of $k = \frac{2B}{2-\beta}$.

Remark 3.14. If for $k = \frac{2B}{2-\beta}$ the function $f(x) + k\phi(x)$ has no asymptote, this does not mean that it has no asymptote for another value of k. In Example 3.11, for example, the conditions of Lemma 3.6 are satisfied for $\beta = \sqrt[3]{2}$ and $2B = 2 - \sqrt{2}$, the function $f(x) + k\phi(x)$ has clearly no asymptote for the value $k = \frac{2B}{2-\beta} = \frac{2-\sqrt[3]{2}}{2-\sqrt{2}} > 1$, but for $k_1 = 1$, $f(x) + k_1\phi(x) = x$ has the asymptote $y = x$.

Lemma 3.2(a) shows that, if there exists a sequence $(x_n) \to +\infty$, satisfying $\lim\limits_{n\to+\infty} |\phi(x_n)| = +\infty$, $\lim\limits_{n\to+\infty} \frac{\phi(2x_n)}{\phi(x_n)} = \beta$ and $\lim\limits_{n\to+\infty} \frac{f(2x_n)-2f(x_n)}{\phi(x_n)} = 2B$, then the only candidate, among the reals k, for which $g(x) = f(x) + k\phi(x)$ can have an asymptote, is $k = \frac{2B}{2-\beta}$.

This conclusion leads us to formulate the next Theorem 3.15, which is one of our main results, and is based on the previous lemmas and discussion, only for the case of $k = \frac{2B}{2-\beta}$. However different versions of this theorem for other values of k, notably when $\phi(2x) = \beta\phi(x)$ may be useful (see, for example, Theorem 3.17 below).

Note that in all the following theorems, the first part of the statement may be regarded as stability result, while the second part of the statement, as superstability result of the affine equation, and in particular of the linear equation, in the sense of Hyers-Ulam.

Theorem 3.15. *Let f and ϕ be two functions defined on $[0; +\infty[$, satisfying*

$$f(x+y) - f(x) - f(y) \le B\,[\phi(x) + \phi(y)].$$

Assume that there exists a real number $\beta \ne 2$, for which one of the following conditions is satisfied for all $x \ge 0$:

(i) $\phi(2x) \le \beta\phi(x)$ *and* $\dfrac{2B}{2-\beta} \ge 0$, *or*

(ii) $\phi(2x) \ge \beta\phi(x)$ *and* $\dfrac{2B}{2-\beta} \le 0$, *or*

(iii) $\phi(2x) = \beta\phi(x)$.

Let $k = \frac{2B}{2-\beta}$. If the function $g(x) = f(x) + k\phi(x)$ is convex, then f is asymptotically close to $ax - b - k\phi(x)$, where

$$a = \lim_{x\to+\infty} \frac{g(x)}{x} \quad \textit{and} \quad b = \lim_{x\to+\infty} [g(2x) - 2g(x)].$$

If moreover $f(0) + k\phi(0) = -b$, then for all $x \ge 0$, $f(x) = ax - b - k\phi(x)$.

In the particular case $b = 0$, we have additionally for all $x \geq 0$

$$\phi(2x) = \beta\phi(x) \quad \textit{and} \quad f(2x) - 2f(x) = 2B\phi(x).$$

Proof of Theorem 3.15. The function $g(x) = f(x) + k\phi(x)$ is convex by hypothesis, hence from Theorem 2.1, it has an asymptote if and only if

$$b = \sup\{g(x+y) - g(x) - g(y) : x \geq 0,\ y \geq 0\} = \lim_{x\to\infty} [g(2x) - 2g(x)]$$

is finite. Now setting $x = y$ in (1) gives $f(2x) - 2f(x) \leq 2B\phi(x)$ and in all cases of this theorem, $k\phi(2x) \leq k\beta\phi(x)$, hence, since $k = \frac{2B}{2-\beta}$ implies $2B - 2k + k\beta = 0$, we obtain

$$\begin{aligned} g(2x) - 2g(x) &= f(2x) - 2f(x) + k[\phi(2x) - 2\phi(x)] \\ &\leq (2B - 2k)\phi(x) + k\phi(2x) \leq (2B - 2k + k\beta)\phi(x) = 0. \end{aligned}$$

It follows that $g(2x) - 2g(x) \leq 0$ and $b \leq 0$ is finite.

From Theorem 2.1, g has an asymptote $y = ax - b$, with $a = \lim_{x\to+\infty} (g(x)/x)$.

This means that f is asymptotically close to $ax-b-k\phi(x)$, which is the asymptotic Hyers-Ulam stability.

Suppose now that in addition, $g(0) = f(0) + k\phi(0) = -b$. From Theorem 2.1, $g(x) = ax - b$, that is $f(x) = ax - b - k\phi(x)$ (Hyers-Ulam superstability).

In the special case when $b = 0$ and $g(0) = 0$, replacing $f(x) = ax - k\phi(x)$ in $f(2x) - 2f(x) \leq 2B\phi(x)$, yields $2k\phi(x) - k\phi(2x) \leq 2B\phi(x)$ and moreover $k\phi(2x) \geq (2k - 2B)\phi(x)$, that is $k\phi(2x) \geq k\beta\phi(x)$.

Now, if $k > 0$, and, as required in Theorem 3.15, $\phi(2x) \leq \beta\phi(x)$, the above relation yields $\phi(2x) \geq \beta\phi(x)$, and then $\phi(2x) = \beta\phi(x)$.

Similarly, if $k < 0$, and, as required in Theorem 3.15, $\phi(2x) \geq \beta\phi(x)$, the relation $k\phi(2x) \geq k\beta\phi(x)$ yields $\phi(2x) \leq \beta\phi(x)$, and then, in both cases, $\phi(2x) = \beta\phi(x)$.

Furthermore, in this special case,

$$f(2x) - 2f(x) = 2k\phi(x) - k\phi(2x) = 2k\phi(x) - k\beta\phi(x) = k(2-\beta)\phi(x) = 2B\phi(x).$$

The equalities $f(x) = ax - b - k\phi(x)$, or $f(x) = ax - k\phi(x)$ for all $x \geq 0$, in Theorem 3.15 may be regarded as a superstability result for the linear equation.

This completes the proof of the theorem. $\square$

Remark 3.16. If the function $g(x) = f(x)+k\phi(x)$ in the formulation of Theorem 3.15, is convex and if

$$(k > 0 \text{ and } \phi \text{ is concave}) \quad \text{or} \quad (k < 0 \text{ and } \phi \text{ is convex})$$

then the function $f(x) = g(x) - k\phi(x)$ of this Theorem is convex.

In the special case when $\phi(2x) = \beta\phi(x)$ (for example if $\phi(x) = x^p$), we can prove the following result:

Theorem 3.17. *Let f and ϕ be two functions defined on $[0;+\infty[$, satisfying*

$$f(x+y)-f(x)-f(y)\leq B\,[\phi(x)+\phi(y)].$$

Assume that there exists a real number β, such that $\phi(2x)=\beta\phi(x)$ for all $x\geq 0$. If for any real number k, either

$$\big(\phi(x)\geq 0 \;\; and \;\; k(2-\beta)\geq 2B\big) \quad or \quad \big(\phi(x)\leq 0 \;\; and \;\; k(2-\beta)\leq 2B\big),$$

and the function $g(x)=f(x)+k\phi(x)$ is convex, then f is asymptotically close to $ax-b-k\phi(x)$, where

$$a=\lim_{x\to+\infty}\frac{g(x)}{x} \quad and \quad b=\lim_{x\to+\infty}[g(2x)-2g(x)].$$

If moreover $f(0)+k\phi(0)=-b$, then for all $x\geq 0$, $f(x)=ax-b-k\phi(x)$.

In the case $b=0$ we have moreover $f(2x)-2f(x)=2B\phi(x)$ for all $x\geq 0$.

Proof of Theorem 3.17. The function $g(x)=f(x)+k\phi(x)$ is convex by hypothesis, hence from Theorem 2.1, it has an asymptote if and only if

$$b=\sup\{g(x+y)-g(x)-g(y):x\geq 0,\ y\geq 0\}=\lim_{x\to\infty}[g(2x)-2g(x)]$$

is finite. Letting $x=y$ in $f(x+y)-f(x)-f(y)\leq B[\phi(x)+\phi(y)]$, we have $f(2x)-2f(x)\leq 2B\phi(x)$ and $k\phi(2x)=k\beta\phi(x)$, hence

$$g(2x)-2g(x)=f(2x)-2f(x)+k[\phi(2x)-2\phi(x)]\leq(2B-2k+k\beta)\phi(x)\leq 0,$$

because $\big(\phi(x)\geq 0$ and $k(2-\beta)\geq 2B\big)$, or $\big(\phi(x)\leq 0$ and $k(2-\beta)\leq 2B\big)$. It follows that $g(2x)-2g(x)\leq 0$ and $b\leq 0$ is finite. From Theorem 2.1, g has an asymptote $y=ax-b$ with $a=\lim_{x\to+\infty}\frac{g(x)}{x}$. This means that f is asymptotically close to $ax-b-k\phi(x)$ (asymptotic Hyers-Ulam stability).

Suppose now that, moreover, $g(0)=f(0)+k\phi(0)=-b$. From Theorem 2.1, $g(x)=ax-b$, that is $f(x)=ax-b-k\phi(x)$ (Hyers-Ulam superstability).

In the special case when, moreover, $b=0$, we obviously get

$$f(2x)-2f(x)=2k\phi(x)-k\phi(2x)=2k\phi(x)-k\beta\phi(x)=k(2-\beta)\phi(x).$$

Given that $\big(\phi(x)\geq 0$ and $k(2-\beta)\geq 2B\big)$, or $\big(\phi(x)\leq 0$ and $k(2-\beta)\leq 2B\big)$, in both cases we have

$$k(2-\beta)\phi(x)\geq 2B\phi(x), \quad \text{and consequently,} \quad f(2x)-2f(x)\geq 2B\phi(x).$$

From $f(2x)-2f(x)\leq 2B\phi(x)$ and $f(2x)-2f(x)\geq 2B\phi(x)$, it follows that $f(2x)-2f(x)=2B\phi(x)$.

Note that the equalities $f(x) = ax - b - k\phi(x)$, or $f(x) = ax - k\phi(x)$ for all $x \geq 0$, in the statement of Theorem 3.17 are indeed superstability results for the linear equation. □

The next theorem, which is a special case of Theorem 3.15, provides a *stability* or *superstability* real unilateral version of the theorems by Aoki [1], or by Rassias's [7] (for $0 \leq p < 1$), and Gajda's theorems (for $p > 1$), [2].

Theorem 3.18. *Let $f\colon [0;+\infty[\to R$ satisfy $f(x+y) - f(x) - f(y) \leq \theta(x^p + y^p)$, for $0<p\neq 1$ and $f(0) = 0$. If for $k = 2\theta/(2-2^p)$ the function $g(x)=f(x)+kx^p$ is convex, then f is asymptotically close to $ax - b - kx^p$, where*

$$a = \lim_{x\to+\infty} \frac{f(x) + kx^p}{x} \quad \textit{and} \quad b = \lim_{x\to+\infty} [g(2x) - 2g(x)].$$

If moreover, $b = 0$, then $f(x) = ax - kx^p$, for $x \geq 0$.

Proof of Theorem 3.18. First notice that the function $\phi(x) = x^p$ for $p \neq 1$, satisfies $\phi(2x) = 2^p x^p = 2^p\phi(x)$, whence $\phi(2x) = \beta\phi(x)$ for $\beta = 2^p \neq 2$. Then Theorem 3.15 implies that f is asymptotically close to $ax - b - kx^p$, where

$$a = \lim_{x\to+\infty} \frac{f(x)+kx^p}{x} \text{ and } b = \lim_{x\to+\infty} [g(2x)-2g(x)] = \lim_{x\to+\infty} [f(2x)-2f(x)+2\theta x^p].$$

If moreover, $b = 0$, since $g(0) = f(0) + k\phi(0) = 0$ for $p > 0$, it suffices to apply Theorem 3.15 in order to have $f(x) = ax - kx^p$, and the proof is complete. □

Remark 3.19. The additional condition $f(0) = 0$ that we added in Theorem 3.18, is also present in both Rassias' and Gajda's theorems, because for $x = y = 0$, the condition $||f(x+y)-f(x)-f(y)|| \leq \theta(||x||^p+||y||^p)$ of these Theorems, implies $f(0) = 0$.

Remark 3.20. It is important to note that in the above real unilateral version of Rassias' or Gajda's theorems, the additional assumption of g-convexity is guaranteed by using only the right, also called *convexity side* of the real Rassias's or Gajda's condition:

$$-\theta(||x||^p + ||y||^p) \leq f(x+y) - f(x) - f(y) \leq \theta(||x||^p + ||y||^p).$$

We get in exchange an asymptotic Hyers-Ulam stability result, and in certain cases, the equality $f(x) = ax-kx^p$, that is a superstability result of the generalized linear equation.

Remark 3.21. We also note, as the function $\phi(x) = x^p$ is concave for $0 \leq p < 1$ and convex for $p > 1$, that the functions $f(x)$ satisfying the conditions of this theorem, are convex if ($0 \leq p < 1$ and $k > 0$) or if ($p > 1$ and $k > 0$).

Remark 3.22. Gajda [2] showed that Theorem 3.18 is false for $p = 1$, by constructing a counterexample.

The theorem below is a version of Theorem 3.17 adapted to the case $\phi(x) = x^p$:

Theorem 3.23. *Let* $f\colon [0;+\infty[\to R$ *satisfy* $f(x+y) - f(x) - f(y) \leq \theta(x^p + y^p)$, $0 \leq p \neq 1$ *and* $f(0) = 0$. *If for any* k *satisfying* $k(2 - 2^p) \geq 2\theta$, *that is*

$$\begin{cases} k \geq \dfrac{2\theta}{2 - 2^p} & \text{if } p < 1, \\ k \leq \dfrac{2\theta}{2 - 2^p} & \text{if } p > 1, \end{cases}$$

the function $g(x) = f(x) + kx^p$ *is convex, then the function* f *is asymptotically close to* $ax - b - kx^p$, *where*

$$a = \lim_{x\to+\infty} \frac{f(x)+kx^p}{x} \quad \text{and}$$

$$b = \lim_{x\to+\infty} g(2x) - 2g(x) = \lim_{x\to+\infty} f(2x) - 2f(x) + k(2^p - 2)x^p.$$

If moreover $b = 0$, *then* $f(x) = ax - kx^p$ *for all* $x \geq 0$.

Remark 3.24. It is important to recognize that, in the assumptions of Theorem 3.23, if there exists a k satisfying $k(2-2^p) \geq 2\theta$ for which the function $f(x)+kx^p$ is convex, and $b = 0$, it must be unique, because $f(x) = a_1x - k_1x^p = a_2x - k_2x^p$ implies $(k_1 - k_2)x^p = (a_1 - a_2)x$ and the last equality is possible only if $k_1 = k_2$ and $a_1 = a_2$.

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