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K-Midconvex and K-Midconcave Set-Valued Maps Bounded on "Large" Sets

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We prove that every K-midconvex set-valued map partially K-upper bounded on a "large" set (e.g. not null-finite, not Haar-null, not Haar-meager), as well as every K-midconcave set-valued map K-lower bounded on a "large" set, is K-continuous. These results generalize the solution of the problem posed by K. Baron and R. Ger in 1984 (see [3]), and also some results from [16].

Keywords: K-midconvex set-valued map, K-midconcave set-valued map, K-continuity, null-finite set, Haar-null set, Haar-meager set.

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1. Introduction

Let X and Y be real topological vector spaces. Assume that D is a convex subset of X and K is a convex cone in Y (i.e. $K + K \subset K$ and $tK \subset K$ for all $t \geq 0$). Denote by $\mathcal{S}(Y)$ the family of all nonempty subsets of Y.

A set-valued map $F\colon D \to \mathcal{S}(Y)$ is called:

- *K-convex*, if, for all $x_1, x_2 \in D$ and $t \in [0,1]$,
$$tF(x_1) + (1-t)F(x_2) \subset F(tx_1 + (1-t)x_2) + K; \tag{1}$$
- *K-concave*, if, for all $x_1, x_2 \in D$ and $t \in [0,1]$,
$$F(tx_1 + (1-t)x_2) \subset tF(x_1) + (1-t)F(x_2) + K; \tag{2}$$
- *K-midconvex* (or *K-Jensen convex*), if (1) is assumed only for $t = \frac{1}{2}$, i.e.
$$\frac{F(x_1) + F(x_2)}{2} \subset F\left(\frac{x_1 + x_2}{2}\right) + K \tag{3}$$
for all $x_1, x_2 \in D$;

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- *K-midconcave*, if

$$F\left(\frac{x_1+x_2}{2}\right) \subset \frac{F(x_1)+F(x_2)}{2} + K \tag{4}$$

for all $x_1, x_2 \in D$ (cf. e.g. [1], [4], [5], [16], [19]).

Note that if F is K-convex with $K = \{0\}$ then it is convex, that is its graph is a convex subset of $X \times Y$. If F is single-valued and Y is endowed with the relation $\leq_K$ of partial order defined by

$$x \leq_K y \Longleftrightarrow y - x \in K,$$

then conditions (1) and (2) reduce to the following conditions:

$$F(tx_1 + (1-t)x_2) \leq_K tF(x_1) + (1-t)F(x_2)$$

and
$$tF(x_1) + (1-t)F(x_2) \leq_K F(tx_1 + (1-t)x_2),$$

respectively. In particular, if $Y = \mathbb{R}$ and $K = [0, \infty)$, we obtain the standard definitions of convex and concave functions. We say that a set-valued map F is *midconvex* (*midconcave*) if it satisfies condition (3) (condition (4)) with $K = \{0\}$.

A set-valued map $F\colon D \to \mathcal{S}(Y)$ is said to be *K-continuous* at a point $x_0 \in D$ if for every neighbourhood W of zero in Y there exists a neighbourhood U of zero in X such that

$$F(x_0) \subset F(x) + W + K \quad \text{and} \quad F(x) \subset F(x_0) + W + K$$

for every $x \in (x_0 + U) \cap D$. In the case where $K = \{0\}$ the K-continuity means the continuity with respect to the Hausdorff topology on $\mathcal{S}(Y)$. If K is a normal cone (i.e. there exists a base $\mathcal{W}$ of neighbourhoods of zero in Y such that $W = (W - K) \cap (W + K)$ for every $W \in \mathcal{W}$) and F is a single-valued function, then the K-continuity means continuity.

It is known that K-convex set-valued maps defined on an infinite-dimensional space need not be K-continuous and K-midconvex set-valued maps may be discontinuous even if they are defined on an open interval in $\mathbb{R}$. However their K-continuity follows from other regularity assumptions, such as K-lower semicontinuity at a point, K-upper boundedness on a set with a non-empty interior (or on a set of positive Lebesgue measure, or on a second category set with the Baire property), measurability, closedness of the epigraph, etc. (cf. e.g. [1], [5], [12], [16], [17], [19], [20]).

In this paper we give some new results of this type which are tightly connected with the following problem formulated by K. Baron and R. Ger in 1983.

Problem 1.1. [3, P239] *Does the upper boundedness of an additive or midconvex function on some universally measurable set which is not Haar-null imply the continuity of the function?*

The notion of a Haar-null set was introduced by J. P. R. Christensen [6] in 1972. More precisely, he defined a universally measurable subset B in an abelian Polish group X to be *Haar-null* if there exists a σ-additive probability Borel measure μ on X such that $\mu(B+x)=0$ for all $x \in X$. Moreover, he proved that in each locally-compact abelian Polish group the notion of a Haar-null set is equivalent to the notion of a set of Haar measure zero.

In 2013 U. B. Darji [7] introduced a subideal of the σ-ideal of meager sets in an abelian Polish group, which is a topological analog to the σ-ideal of Haar-null sets. Darji defined a Borel subset B of an abelian Polish group X to be *Haar-meager* if there exists a continuous map $f\colon K \to X$ from a nonempty compact metric space K such that for every $x \in X$ the set $f^{-1}(B+x)$ is meager in K.

It turned out that Haar-null sets and Haar-meager sets have analogous properties (see e.g. [8]–[11], [13]). One of them is the fact that in any non-locally compact abelian Polish group there are two closed neither Haar-null nor Haar-meager sets A, B whose sum $A+B$ has empty interior (see [13], [14]). That is why Problem 1.1 is worth of extending to the case of a Borel set which is not Haar-meager.

The positive answer to Problem 1.1 has been recently given in [2] thanks to a new concept of null-finite sets.

Recall that in a real linear metric space X a set $A \subset X$ is called *null-finite* if there exists a null-sequence (i.e. a sequence tending to zero) $(x_n)_{n\in\mathbb{N}}$ in X such that the set $\{n \in \mathbb{N} : x + x_n \in A\}$ is finite for every $x \in X$.

The following crucial properties of null-finite sets have been proved in [2].

Theorem 1.2. [2, Theorems 5.1 and 6.1] *Each Borel null-finite set in an abelian Polish group is Haar-meager. Each universally measurable null-finite set in an abelian Polish group is Haar-null.*

In this paper we prove that every K-midconvex set-valued map which is partially K-upper bounded on a non-null-finite subset of D, as well as every K-midconcave set-valued map K-lower bounded on a non-null-finite subset of D, has to be K-continuous. Consequently, we characterize additive set-valued maps partially bounded on a non-null-finite set. Finally, using Theorem 1.2, we present a positive answer to Problem 1.1 in the case of set-valued maps.

2. Preliminaries

We recall here some known facts which will be needed in the proof of our main results.

Let X and Y be real (Hausdorff) topological vector spaces and $D \subset X$ be an open convex set. Denote by $\mathcal{B}(Y)$, $\mathcal{BC}(Y)$, $\mathcal{C}(Y)$ and $\mathcal{CC}(Y)$ the families of all nonempty bounded subsets, nonempty bounded convex subsets, nonempty compact subsets and nonempty compact convex subsets of Y, respectively.

It is well known that for any set $A \subset D$ and $s, t \in \mathbb{R}$ we have $(s+t)A \subset sA + tA$ and, if A is convex and $s, t \geq 0$, then $(s+t)A = sA + tA$.

Lemma 2.1. [16, Lemmas 3.1, 4.1, Theorems 3.1, 4.1] *Let $F\colon D \to \mathcal{S}(Y)$ be a set-valued map. Then the following conditions hold:*

(a) *if F is K-midconvex, then*

$$qF(x) + (1-q)F(y) \subset F(qx + (1-q)y) + K$$

for all $x, y \in D$ and all dyadic $q \in [0,1]$;

(b) *if F is K-midconcave and convex-valued, then*

$$F(qx + (1-q)y) \subset qF(x) + (1-q)F(y) + K$$

for all $x, y \in D$ and all dyadic $q \in [0,1]$;

(c) *if $F\colon D \to \mathcal{C}(Y)$ is K-midconvex and K-continuous, then it is* $\operatorname{cl} K$*-convex;*

(d) *if $F\colon D \to \mathcal{CC}(Y)$ is K-midconcave and K-continuous, then it is* $\operatorname{cl} K$*-concave.*

A set-valued map $F\colon D \to \mathcal{S}(Y)$ is called:

- *K-upper bounded* on a set $A \subset D$, if there exists a bounded set $B \subset Y$ such that
$$F(x) \subset B - K \quad \text{for all} \quad x \in A;$$
- *partially K-upper bounded* on a set $A \subset D$, if there exists a bounded set $B \subset Y$ such that
$$F(x) \cap (B - K) \neq \emptyset \quad \text{for all} \quad x \in A;$$
- *K-lower bounded* on a set $A \subset D$, if there exists a bounded set $B \subset Y$ such that
$$F(x) \subset B + K \quad \text{for all} \quad x \in A;$$
- *partially K-lower bounded* on a set $A \subset D$, if there exists a bounded set $B \subset Y$ such that
$$F(x) \cap (B + K) \neq \emptyset \quad \text{for all} \quad x \in A.$$

Clearly, K-upper (lower) bounded set-valued maps are partially K-upper (lower) bounded. Moreover, in the case when $K = \{0\}$, (partial) K-upper boundedness and (partial) K-lower boundedness are equivalent assumptions, called simply *(partial) boundedness.*

The following result of the Bernstein-Doetsch type will be useful in the sequel.

Lemma 2.2. [16, Corollary 3.3, Theorem 4.4] *Let $F\colon D \to \mathcal{B}(Y)$ be a set-valued map. If one of the following conditions holds:*

(a) *F is K-midconvex and partially K-upper bounded on some subset of D with nonempty interior;*

(b) *$F\colon D \to \mathcal{BC}(Y)$ is K-midconcave and K-lower bounded on some subset of D with nonempty interior,*

then F is K-continuous on D.

Remark 2.3. If a set-valued map $F\colon D \to \mathcal{B}(Y)$ is K-midconvex (K-midconcave, respectively), $x_0 \in D$ and $y_0 \in F(x_0)$, then the set-valued map $\widetilde{F}\colon D - x_0 \to \mathcal{B}(Y)$ defined by

$$\widetilde{F}(x) := F(x + x_0) - y_0, \quad x \in D - x_0,$$

is also K-midconvex (K-midconcave) and $0 \in \widetilde{F}(0)$. Moreover, $\widetilde{F}$ is K-lower bounded (partially K-upper bounded, respectively) on the set $A - x_0$ if and only if F is K-lower bounded (partially K-upper bounded) on A, and $\widetilde{F}$ is K-continuous on $D - x_0$ if and only if F is K-continuous on D. We will use this observation in the proofs of main results.

By the definition of K-midconvexity we get the following fact.

Lemma 2.4. *If $F\colon D \to \mathcal{S}(Y)$ is K-midconvex and $0 \in F(0)$, then, for all $x \in D$*

$$F(2x) \subset 2F(x) + K \quad \text{and} \quad \frac{1}{2}F(x) \subset F\Big(\frac{1}{2}x\Big) + K.$$

Now, let L be a convex cone in X. A set-valued map $A\colon L \to \mathcal{S}(Y)$ is called *additive* if it satisfies the Cauchy functional equation $A(x_1 + x_2) = A(x_1) + A(x_2)$ for all $x_1, x_2 \in L$.

Each additive set-valued map $A\colon L \to \mathcal{S}(Y)$ is midconcave. If, moreover, F is convex-valued, then it is midconvex (see [16, Remark 5.1.]).

A set-valued function $A\colon L \to \mathcal{S}(Y)$ is said to be *linear*, if it is additive and $A(tx) = tA(x)$ for all $x \in L$ and $t \in (0, \infty)$.

Lemma 2.5. [16, Theorem 5.3.] *If a set-valued function $A\colon L \to \mathcal{CC}(Y)$ is additive and continuous, then it is linear.*

3. Main results

Theorem 3.1. *Let X and Y be real linear metric spaces with invariant metrics. Assume that D is an open convex subset of X, $A \subset D$ is a set which is not null-finite and $\operatorname{cl} A \subset D$, and K is a convex cone in Y. If a set-valued map $F\colon D \to \mathcal{B}(Y)$ is K-midconvex and partially K-upper bounded on A, then F is K-continuous on D. If, moreover, all values of F belong to $\mathcal{C}(Y)$, then F is $\operatorname{cl} K$-convex.*

Proof. Let F be K-midconvex and partially K-upper bounded on A. Without loss of generality we may assume that $0 \in D$ and $0 \in F(0)$ (cf. Remark 2.3).

Since F is partially K-upper bounded on A, there exists a bounded set $B \subset Y$ such that $F(x) \cap (B - K) \neq \emptyset$ for every $x \in A$. Define

$$\widetilde{B} := \bigcup_{t \in [0,1]} tB. \tag{5}$$

Clearly, $\widetilde{B}$ is also bounded and $\widetilde{B} \subset p\widetilde{B}$ for every $p \geq 1$. Moreover,

$$F(x) \cap (\widetilde{B} - K) \neq \emptyset, \quad x \in A. \tag{6}$$

Suppose, contrary to our claim, that F is not K-continuous on D. Then, in view of Lemma 2.2, F is not partially K-upper bounded on any set with nonempty interior. In particular, F is not partially K-upper bounded on any neighbourhood

$$U_n := \frac{1}{4^n} B_X \cap \frac{1}{2^n} D \cap \left(-\frac{1}{2^n} D\right), \quad n \in \mathbb{N}, \tag{7}$$

where B_X denotes the open unit ball in X. Therefore, for every $n \in \mathbb{N}$ and the bounded set $\widetilde{B} + B_Y$, where B_Y is the open unit ball in Y, there exists $x_n \in U_n$ such that

$$F(x_n) \cap (\widetilde{B} + B_Y - K) = \emptyset. \tag{8}$$

Since $x_n \in \frac{1}{2^n} D$, $2^k x_n \in D$ for every $k \leq n$. Moreover, $\rho_X(2^k x_n, 0) \leq 2^k \rho_X(x_n, 0) \leq \frac{1}{2^n}$, which means that the sequence $(2^n x_n)_{n\in\mathbb{N}}$ is a null-sequence in X (ρ_X denotes here the metric in X). The set A is not null-finite, so there exists $a \in X$ such that the set $\Omega := \{n \in \mathbb{N} : a + 2^n x_n \in A\}$ is not finite. Then $a \in \operatorname{cl} A \subset D$. Choose $k \in \mathbb{N}$ so large that $2^{-k+1} a \in -D$ (such number k exists as $-D$ is a neighborhood of 0). By (6) we have

$$F(a + 2^n x_n) \cap (\widetilde{B} - K) \neq \emptyset, \quad n \in \Omega,$$

and hence, using the fact that $\widetilde{B} \subset 2^n \widetilde{B}$, also

$$F(a + 2^n x_n) \cap (2^n \widetilde{B} - K) \neq \emptyset, \quad n \in \Omega. \tag{9}$$

Observe now that for every number $n \in \Omega$, $n > k$, by Lemma 2.1 we have

$$\begin{aligned} F(2^{n-k} x_n) + K &= F\left(\frac{-2^{-k+1} a}{2} + \frac{2^{-k+1} a + 2^{n-k+1} x_n}{2}\right) + K \\ &\supset \frac{1}{2} F(-2^{-k+1} a) + \frac{1}{2} F(2^{-k+1} a + 2^{n-k+1} x_n). \end{aligned}$$

Hence, using Lemma 2.4 and the fact that K is a cone, and multiplying both sides by 2^k, we get

$$2^n F(x_n) + K \supset 2^{k-1} F(-2^{-k+1} a) + F(a + 2^n x_n). \tag{10}$$

Since the set $C := 2^{k-1} F(-2^{-k+1} a)$ is bounded, we can take $n_0 \in \Omega$, $n_0 > k$, such that $2^{-n_0} C \subset B_Y$. By (9) there exist $b \in \widetilde{B}$ and $k \in K$ such that

$$2^{n_0} b - k \in F(a + 2^{n_0} x_{n_0}).$$

From here, using (10) we have

$$2^{n_0} b - k + C \subset 2^{n_0} F(x_{n_0}) + K,$$

and hence

$$b + 2^{-n_0}C \subset F(x_{n_0}) + K. \tag{11}$$

On the other hand

$$b + 2^{-n_0}C \subset \widetilde{B} + B_Y. \tag{12}$$

Condition (11) is contrary to (12) because, in view of (8),

$$(F(x_{n_0}) + K) \cap (\widetilde{B} + B_Y) = \emptyset.$$

Thus F is K-continuous on D. If, moreover, $F\colon D \to \mathcal{C}(Y)$, then, by Lemma 2.1, F is also $\operatorname{cl} K$-convex. This finishes the proof. □

Now, using a similar method as in the proof of Theorem 3.1, we will prove an analogous result for K-midconcave set-valued maps. Note, however, that this result can not be obtained as a consequence of Theorem 3.1. Namely, the fact known for single-valued functions: *"f is concave if and only if $-f$ is convex"*, does not hold for set-valued maps. Therefore, even if some properties of K-convex and K-concave set-valued maps are analogous, they have to be proved separately.

Theorem 3.2. *Let X and Y be real linear metric spaces with invariant metrics. Assume that D is an open convex subset of X, $A \subset D$ is a set which is not null-finite and $\operatorname{cl} A \subset D$, and K is a convex cone in Y. If a set-valued map $F\colon D \to \mathcal{BC}(Y)$ is K-midconcave and K-lower bounded on A, then F is K-continuous on D. If, moreover, all values of F belong to $\mathcal{CC}(Y)$, then F is $\operatorname{cl} K$-concave.*

Proof. Let F be K-midconcave and K-lower bounded on A. In view of Remark 2.3, without loss of generality we may assume that $0 \in D$.

Since F is K-lower bounded on A, there exists a bounded set $B \subset Y$ such that

$$F(x) \subset B + K \subset \widetilde{B} + K \quad \text{for each} \quad x \in A, \tag{13}$$

where $\widetilde{B}$ is given by (5).

For the proof by contradiction suppose that F is not K-continuous on D. Then, in view of Lemma 2.2, F is not K-lower bounded on any neighbourhood U_n given by (7). Thus, for every $n \in \mathbb{N}$ and the bounded set $\widetilde{B} + B_Y + F(0)$ there exists $x_n \in U_n$ such that

$$F(x_n) \not\subset \widetilde{B} + B_Y + F(0) + K. \tag{14}$$

Since the sequence $(2^n x_n)_{n\in\mathbb{N}}$ is a null-sequence in X and the set A is not null-finite, there exists $a \in X$ such that the set $\Omega := \{n \in \mathbb{N} : a + 2^n x_n \in A\}$ is not finite. Then $a \in \operatorname{cl} A \subset D$. Choose $k \in \mathbb{N}$ so large that $2^{-k+1}a \in -D$. According to (13) we have

$$F(a + 2^n x_n) \subset \widetilde{B} + K \subset 2^n \widetilde{B} + K, \quad n \in \Omega. \tag{15}$$

By the K-midconcavity of F, for every number $n \in \Omega$, $n > k$,

$$F(2^{n-k}x_n) \subset \frac{1}{2}F(-2^{-k+1}a) + \frac{1}{2}F\left(\frac{a+2^n x_n}{2^{k-1}}\right) + K$$

and, in view of Lemma 2.1,

$$F\left(\frac{a+2^n x_n}{2^{k-1}}\right) \subset \frac{1}{2^{k-1}}F(a+2^n x_n) + \left(1 - \frac{1}{2^{k-1}}\right)F(0) + K.$$

Hence, since K is a cone,

$$F(2^{n-k}x_n) \subset \frac{1}{2}F(-2^{-k+1}a) + \frac{1}{2^k}F(a+2^n x_n) + \left(\frac{1}{2} - \frac{1}{2^k}\right)F(0) + K.$$

Consequently, in view of Lemma 2.1,

$$\begin{aligned} F(x_n) &\subset \frac{1}{2^{n-k}}F(2^{n-k}x_n) + \left(1 - \frac{1}{2^{n-k}}\right)F(0) + K \\ &\subset 2^{k-n-1}F(-2^{-k+1}a) + 2^{-n}F(a+2^n x_n) + (1 - 2^{-n} - 2^{k-n-1})F(0) + K. \end{aligned}$$

Now, setting $C := 2^{k-1}F(-2^{-k+1}a) + (-2^{k-1} - 1)F(0)$, according to (15) we get

$$F(x_n) \subset 2^{-n}C + 2^{-n}F(a+2^n x_n) + F(0) + K \subset 2^{-n}C + \widetilde{B} + F(0) + K. \quad (16)$$

Since C is bounded, we can take $n_0 \in \Omega$, $n_0 > k$, such that $2^{-n_0}C \subset B_Y$. Then, by (16),

$$F(x_{n_0}) \subset B_Y + \widetilde{B} + F(0) + K$$

which contradicts (14).

Thus F is K-continuous on D. By Lemma 2.1, if $F\colon D \to \mathcal{CC}(Y)$, then it is also $\mathrm{cl}\,K$-convex. This ends the proof. □

Remark 3.3. In the above theorem the assumption that F is K-lower bounded on A can not be replaced by the weaker assumption that F is partially K-lower bounded on A. For instance, if $a\colon \mathbb{R} \to \mathbb{R}$ is a discontinuous additive function and $K = [0, \infty)$, then the set-valued map $F\colon \mathbb{R} \to \mathcal{BC}(\mathbb{R})$ defined by

$$F(x) = [-|a(x)|,\, 0\,], \quad x \in \mathbb{R},$$

is K-midconcave and partially K-lower bounded on the whole $\mathbb{R}$, but it is not K-continuous. Clearly, F is not K-lower bounded on any “large” set.

Since additive set-valued maps with convex values are midconvex, from Theorem 3.1 (for $K = \{0\}$) and Lemma 2.5, we obtain the continuity and linearity of such maps under the assumption that they are partially bounded on a ”large” set.

Theorem 3.4. *Let X and Y be real linear metric spaces with invariant metrics. Assume that L is an open convex cone in X, $A \subset L$ is a set which is not null-finite and $\operatorname{cl} A \subset L$. If $F \colon L \to \mathcal{BC}(Y)$ is an additive set-valued map partially bounded on A, then it is continuous on L. If, moreover, all values of F are in $\mathcal{CC}(Y)$, then F is linear.*

The above theorem generalizes Theorem 5.1 from [16], where an analogous result is proved for additive set-valued maps bounded on a set with a nonempty interior.

In the same way as in the proofs of Corollaries 11.3 and 11.4 in [2], we can show that each Borel non-Haar meager set $A \subset D$ (universally measurable non-Haar null set $A \subset D$, respectively) contains a subset B such that $\operatorname{cl} B \subset D$ and B is also a Borel non-Haar-meager set (a universally measurable non-Haar-null set). Therefore, using Theorem 1.2 and Theorem 3.1, we obtain the following set-valued counterparts of the classical results of A. Ostrowski [18] and M.R. Mehdi [15].

Corollary 3.5. *Let X be a real linear Polish space and Y be a real linear metric space with an invariant metric. Let D be an open convex subset of X and K be a convex cone in Y. Assume that $A \subset D$ is either a non-Haar-null universally measurable set or a non-Haar-meager Borel set. If $F \colon D \to \mathcal{B}(Y)$ is a K-midconvex set-valued map partially K-upper bounded on A, then F is K-continuous on D.*

Analogous results, under the assumptions that $X = \mathbb{R}^n$ and A is a set of positive Lebesgue measure, or X is a real topological vector space and A is a second category set with the Baire property, were obtained in Theorems 3.5 and 3.6 in [16].

Similarly, using Theorem 3.2, we get a counterpart of Theorems 4.8 and 4.9 in [16].

Corollary 3.6. *Let X be a real linear Polish space and Y be a real linear metric space with an invariant metric. Let D be an open convex subset of X and K be a convex cone in Y. Assume that $A \subset D$ is a universally measurable set which is not Haar-null (or a Borel set which is not Haar-meager). If $F \colon D \to \mathcal{BC}(Y)$ is a K-midconcave set-valued map K-lower bounded on A, then F is K-continuous on D.*

Corollaries 3.5 and 3.6 answer Problem 1.1 in the case of K-midconvex (K-midconcave) set-valued maps. The solution of Problem 1.1 in the case of additive set-valued maps we obtain by Theorems 1.2 and 3.4.

Corollary 3.7. *Let X be a real linear Polish space and Y be a real linear metric space with an invariant metric. Let L be an open convex cone in X. Assume that $A \subset L$ is either a non-Haar-null universally measurable set or a non-Haar-meager Borel set. If $F \colon D \to \mathcal{BC}(Y)$ is an additive set-valued map partially bounded on A, then F is continuous on L.*

References

[1] A. Averna, T. Cardinali: Sui concetti di K-convessità (K-concavità) e di K-convessità* (K-concavità*), Riv. Mat. Univ. Parma (4) 16 (1990) 311–330.

[2] T. Banakh, E. Jabłońska: Null-finite sets in metric groups and their applications, Israel J. Math., to appear.

[3] K. Baron, R. Ger: Problem (P239), in: The 21st International Symposium on Functional Equations, August 6–13 (1983), Konolfingen, Switzerland, Aequationes Math. 26 (1984) 225–294.

[4] J. Benoist, N. Popovici: Generalized convex set-valued maps, J. Math. Anal. Appl. 288 (2003) 161–166.

[5] J. M. Borwein: Multivalued convexity and optimization: A unified approach to inequality and equality constraints, Math. Programming 13 (1977) 183–199.

[6] J. P. R. Christensen: On sets of Haar measure zero in abelian Polish groups, Israel J. Math. 13 (1972) 255–260.

[7] U. B. Darji: On Haar meager sets, Topology Appl. 160 (2013) 2396–2400.

[8] M. Doležal, V. Vlasăk: Haar meager sets, their hulls, and relationship to compact sets, J. Math. Anal. Appl. 446(1) (2017) 852–863.

[9] M. Doležal, V. Vlasăk, B. Vejnar, M. Rmoutil: Haar meager sets revisited, J. Math. Anal. Appl. 440(2) (2016) 922–939.

[10] M. Elekes, D. Nagy: Haar null and Haar meager sets: a survey and new results, arXiv: 1508.02053 (2015).

[11] M. Elekes, Z. Vidnyánszky: Haar null sets without G_δ hulls, Israel J. Math. 209 (2015) 199–214.

[12] A. Gilanyi, C. Gonzales, K. Nikodem, Zs. Páles: Bernstein-Doetsch type theorems with Tabor type error terms for set-valued maps, Set-Valued Var. Anal. 25 (2017) 441–462.

[13] E. Jabłońska: Some analogies between Haar meager sets and Haar null sets in abelian Polish groups, J. Math. Anal. Appl. 421 (2015) 1479–1486.

[14] E. Matoušková, M. Zelený: A note on intersections of non-Haar null sets, Colloq. Math. 96 (2003) 1–4.

[15] M. R. Mehdi: On convex functions, J. London Math. Soc. 39 (1964) 321–326.

[16] K. Nikodem: K-convex and K-concave set-valued functions, Zeszyty Nauk. Politechniki Łódzkiej Mat. 559, Rozprawy Mat. 114, Łódź (1989).

[17] K. Nikodem: Continuity properties of midconvex set-valued maps, Aequationes Math. 62 (2001) 175–183.

[18] A. Ostrowski: Über die Funktionalgleichung der Exponentialfunktion und verwandte Funktionalgleichungen, Jber. Deutsch. Math. Vergng. 38 (1929) 54–62.

[19] F. Papalini: La K-midpoint* convessità (concavità) e la K-semicontinuità inferiore (superiore) di una multifunzione, Riv. Mat. Univ. Parma (4) 16 (1990) 149–159.

[20] L. Thibault: Continuity of measurable convex multifunctions, in: Multifunctions and Integrands, Lecture Notes in Math. 1091, Springer, Berlin et al. (1983) 216–224.