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Sets with the Unique Footpoint Property and φ-Convex Subsets of Riemannian Manifolds

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The concepts of φ-convexity and prox-regularity for sets are studied in the framework of finite-dimensional Riemannian manifolds. We prove that the two concepts coincide and these sets are contained in the class of sets with the unique footpoint property. Then by using a characterization of elements of proximal normal cone, we show that under certain conditions a set with the unique footpoint property is a φ-convex set.

Keywords: Phi-convex sets, prox-regular sets, sets with the unique footpoint property, nonsmooth analysis, Riemannian manifolds.

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1. Introduction

In his fundamental paper [11], Federer introduced the notion of sets with positive reach. A closed subset of $\mathbb{R}^n$ is called a *set with positive reach* if there exists $r > 0$ such that every $x \in \mathbb{R}^n$ with $d_S(x) < r$ has a unique nearest point $P_S(x)$ to S, where d_S is the distance function associated to S. Then various motivations and problems lead authors to introduce many classes of suitable sets related to this notion under distinct names: prox-regular, φ-convex, proximally smooth, etc. All these concepts are actually known to coincide, see [8].

A closed subset $S \subset \mathbb{R}^n$ is called a *set with the unique footpoint property* (UFP-set, for short) if there exists a neighborhood U of S such that for every $x \in U$ there exists a unique point $P_S(x) \in S$ closest to x. All convex sets and all sets with C^2-boundary belong to this class. This class of sets also contains the class of sets with positive reach, however, they do not coincide, see [18, Example 1].

In [15] the notion of UFP-set, which was called EFP-set, and some of Federer's results were generalized to the setting of Riemannian manifolds. It was also proved that the metric projection to a UFP-set S is locally Lipschitz in a neighborhood of S.

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By using this result the author generalized several concepts on convex subsets of Riemannian manifolds to UFP-sets, see [14]. In Riemannian geometry the interest in these sets was stimulated by the fact that due to curvature effects outer and inner parallel sets of locally convex sets are only UFP-sets in general. Moreover, it was shown in [3] that the notion of UFP-set is independent of the Riemannian structure. Note that unlike a Hilbert space a manifold in general does not have a linear structure and to deal with the concepts of metric projection and distance function from sets in manifolds different techniques are needed. Moreover, these notions are not of local type and can not be studied by local techniques.

The basic tool to deal with the notions such as prox-regularity and φ-convexity is nonsmooth analysis and hence to study these concepts on Riemannian manifolds one should first establish this theory on manifolds. In the past few years, a number of results have been obtained on numerous aspects of nonsmooth analysis and their applications on Riemannian manifolds; see, for instance, [1, 16].

The notion of φ-convexity (as titled p-convexity) in linear spaces was introduced in [5] and in [9] the author introduced several important properties of p-convex sets in infinite-dimensional Hilbert spaces. It was shown that certain properties which hold globally for convex sets are still valid locally for φ-convex sets. For example a closed subset of a Hilbert space is convex if and only if its corresponding metric projection is globally nonempty and single-valued. On the other hand, it was proved in [5] that the metric projection into a φ-convex subset S of a Hilbert space H is locally nonempty unique and Lipschitz continuous. In [4] the concept of φ-convexity for the sets was generalized to infinite-dimensional manifolds. It was proved that if S is a φ-convex subset of an infinite-dimensional Hadamard manifold M, then there exists a neighborhood U of S in M such that the metric projection $P_S\colon U \to S$ is single valued and locally Lipschitz. Moreover, it was shown that under the same assumptions on S and M, there exists a neighborhood U of S in M such that d_S^2 is C^1 with locally Lipschitz gradient.

A nonempty subset of a Hilbert space is called *prox-regular* if it can be supported at any point of its boundary by a ball of locally fixed radius in all possible normal directions. These sets were studied in [19] and it was proved in [20] that the prox-regularity is a necessary and sufficient condition for the existence and Lipschitz continuity of metric projection. In [13] the notion of prox-regularity was extended to the framework of finite-dimensional Riemannian manifolds and was characterized as the hypomonotonicity of the truncated limiting normal cone.

The aim of this paper is to study UFP-sets and their relation to φ-convex sets in complete Riemannian manifolds. Moreover, we show that in a Riemannian manifold, the two classes of φ-convex sets and prox-regular sets coincide and they are contained in the class of UFP-sets. Furthermore, we show that under certain assumptions, a UFP-set is φ-convex. To this end, we obtain some results regarding the differentiability of the distance function to a closed set and we characterize the elements of the proximal normal cone to a set in a complete Riemannian manifold.

The rest of the paper is organized as follows. In Section 2 we present some basic constructions and preliminaries in Riemannian manifolds and nonsmooth analysis, widely used in the sequel. In Section 3 we show that on a Riemannian manifold the notions of φ-convexity and prox-regularity of sets coincide. Section 4 is devoted to UFP-sets and the main results of this paper is presented in this section.

2. Preliminaries and notations

In this section we present some notations and results of Riemannian manifolds and nonsmooth analysis which we need in what follows; see, e.g., [6, 10, 22]. Throughout this paper, (M, g) is a finite-dimensional connected Riemannian manifold endowed with a Riemannian metric $g_x = \langle ., . \rangle_x$ on each tangent space T_xM and the Riemannian connection ∇ on M associated with the Riemannian metric. As usual, we denote by $d(x, y)$ the *Riemannian distance* from x to y for every $x, y \in M$. The open and closed *metric ball* centered at x with radius r will be denoted by $B(x, r)$ and $\overline{B}(x, r)$, respectively. The map $\exp_x \colon U_x \to M$ will stand for the *exponential map* at x, where U_x is an open subset of the tangent space T_xM containing $0_x \in T_xM$. The exponential map at x maps straight lines of T_xM passing through $0_x \in T_xM$ into geodesics of M passing through x.

Recall that a Riemannian manifold M is *complete* if for all $x \in M$, any geodesic $\gamma(t)$ starting from x is defined for all $t \in \mathbb{R}$. By the Hopf-Rinow theorem, if M is complete and connected, then (M, d) is a complete metric space and for any $x, y \in M$, there is at least one minimizing geodesic joining x to y. Moreover, for all $x \in M$, the exponential map at x is defined on all of T_xM; see [10]. A *Hadamard manifold* is a complete, simply-connected Riemannian manifold with nonpositive sectional curvature. In a Hadamard manifold H the exponential map $\exp_x \colon T_xH \to H$ is a diffeomorphism for any $x \in H$ and any two points of H can be joined by a unique minimizing geodesic.

Let $\gamma \colon I \to M$ be a smooth curve in M. For every $a, b \in I$, we denote the *parallel transport* along γ from $\gamma(a)$ to $\gamma(b)$ by $L^{\gamma}_{\gamma(a)\gamma(b)}$. When γ is a unique minimizing geodesic joining x to y, we denote L^{γ}_{xy} by L_{xy}. In what follows L_{xy} will be used whenever it is well-defined.

A subset $A \subset M$ is called *convex* if for any $x, y \in A$ there is a unique minimizing geodesic with unit speed joining x to y and it is contained in A. It is known that for every $x \in M$ there is a largest $r(x) > 0$ such that, for $0 < r \leq r(x)$, each open ball contained in $B(x, r)$ is convex and any geodesic segment contained in $B(x, r)$ is a minimizing geodesic joining its end points. Moreover, the function $x \mapsto r(x)$ from M to $\mathbb{R}^+ \cup \{+\infty\}$ is continuous; see [22]. If A is convex, then for every $x, y \in A$, both $\exp_x^{-1} y$ and $\exp_y^{-1} x$ are defined and

$$\| \exp_x^{-1} y\| = d(x, y) = \| \exp_y^{-1} x\|.$$

Moreover, $L_{xy} (\exp_x^{-1} y) = -\exp_y^{-1} x$.

For a fixed point $z \in M$, we define the function $\varphi \colon M \to \mathbb{R}$ by $\varphi(x) = d^2(x, z)$. The function φ is C^∞ on $M \setminus C_z$, where C_z is the cut locus of z. It is well-known that if x belongs to a convex neighborhood of z, then $\nabla \varphi(x) = -2 \exp_x^{-1} z$. Note that there is no simple formula for the Hessian of φ on arbitrary Riemannian manifolds. But it is possible to estimate it; for more details see [2]. Recall that the Hessian of a C^2 function ψ on M is defined by

$$\text{Hess}\, \psi(x)(v, w) := \langle \nabla_X \nabla\psi, Y \rangle (x),$$

for every $x \in M$ and $v, w \in T_xM$ where X, Y are any vector fields such that $X(x) = v$ and $Y(x) = w$ and $\nabla\psi$ denotes the gradient of ψ.

The following result is a local version of [2, Proposition 2.2] for Riemannian manifolds which do not have any global bound on its curvatures.

Proposition 2.1. *Let M be a Riemannian manifold and $z \in M$. Assume that $R > 0$ and $k_0 > 0$ are given such that $\mid k \mid \leq k_0$ for every sectional curvature k on $B(z, R)$. Then for every $r > 0$ with $r < \min\left\{r(z), R, \frac{\pi}{2\sqrt{k_0}}\right\}$, the function $\varphi(x) := d^2(x, z)$ is smooth on $B(z, r)$ and its Hessian satisfies*

$$0 \leq \text{Hess}\, \varphi(x)(v)^2 \leq \left(2 + \frac{2}{3}\, k_0\, d^2(x, z)\right) \|v\|^2,$$

for every $x \in B(z, r)$ and $v \in T_xM$.

A nonempty set $A \subset M$ is called *locally convex* if for any $x \in \overline{A}$ there exists an $\varepsilon = \varepsilon(x) > 0$ such that $A \cap B(x, \varepsilon)$ is convex. If A is a closed locally convex subset of a Riemannian manifold M, then there is an open neighborhood W of A such that on W the metric projection to A, denoted by P_A, which is defined by

$$P_A(x) = \left\{ y \in A : d(x, y) = \inf_{z \in A}\, d(x, z) \right\} \quad \text{for every } x \in M,$$

is single-valued and as a map is locally Lipschitz on W; see [24].

Let S be a closed subset of a Riemannian manifold M and $x \in S$. The *proximal normal cone* to S at x, denoted by $N_S^P(x)$, is defined as $N^P_{\exp_x^{-1}(S\cap U)}(0_x)$, where U is a normal neighborhood of x and $N^P_{\exp_x^{-1}(S\cap U)}(0_x)$ has the usual meaning as a proximal normal cone to a subset of a Hilbert space. It is easy to check that $\xi \in N_S^P(x)$ if and only if there is $\sigma > 0$ such that

$$\langle \xi, \exp_x^{-1} y \rangle \leq \sigma\, d^2(x, y)$$

for every y in S in a neighborhood of x.

Recall that for a lower semicontinuous function $f \colon M \to (-\infty, +\infty]$ and $x \in M$, the *proximal subdifferential* of f at x, denoted by $\partial_p f(x) \subseteq T_pM$, is defined as $\partial_p\left(f \mathrm{o} \exp_x\right)(0)$ for $x \in \mathrm{dom}\, f$ and $\partial_p f(x) = \emptyset$ otherwise; see [4, 6] for more details. It is worth mentioning that $\xi \in \partial_p f(x)$ if and only if there exists a C^2 function g on a neighborhood of x such that $dg(x) = \xi$ and $f - g$ attains a local minimum at x.

Let $f\colon M \to \mathbb{R}$ be a locally Lipschitz function on M. Recall that the *Clarke generalized gradient* of f at x is

$$\partial_c f(x) = \operatorname{co}\left\{ \lim_{i\to\infty} \nabla f(x_i)\colon x_i \to x, x_i \in \mathcal{D}(f) \right\}.$$

The following theorem, which is called Lebourg's mean value theorem, has been proved in [12].

Theorem 2.2. *Let M be a Riemannian manifold and $x, y \in M$. Assume that $\gamma\colon [0,1] \to M$ is a smooth path joining x to y and f is a Lipschitz function around $\gamma[0,1]$. Then there exist $\delta \in (0,1)$ and $v \in \partial_c f(\gamma(\delta))$ such that*

$$f(y) - f(x) = \langle v, \dot{\gamma}(\delta) \rangle.$$

3. φ-convex subsets of Riemannian manifolds

In this section we introduce the notion of φ-convexity for sets in Riemannian manifolds and we show that these sets and prox-regular sets are in fact the same. The concept of φ-convex sets in the framework of infinite-dimensional Hadamard manifolds were introduced in [4] and some properties of these sets were studied there.

Definition 3.1. Let $S \subseteq M$ be closed and $\varphi\colon S \to [0, \infty)$ be a continuous function. The set S is said to be *φ-convex* if for every $x \in S$ and $v \in N_S^P(x)$

$$\langle v, \exp_x^{-1} y \rangle \leq \varphi(x)\, \|v\|\, d^2(x,y), \tag{1}$$

for every $y \in U \cap S$ where U is a convex neighborhood of x.

Note that this definition is independent of the choice of any convex neighborhood of x. Therefore, in the special case in which M is a Hadamard manifold, our definition of φ-convexity coincides with the one in [4]. Moreover, for every x in the interior of S the proximal normal cone to S at x is trivial. Hence in the definition of φ-convexity it suffices to consider the existence of a continuous function on the boundary of S.

Example 3.2. Every locally convex subset of a Riemannian manifold is φ-convex by considering $\varphi \equiv 0$. □

In [13] another class of subsets of Riemannian manifolds was introduced which contains convex sets. These sets are called prox-regular sets. In what follows, we show that actually these two classes of sets are equal.

Definition 3.3. Let S be a closed subset of a Riemannian manifold M and $\bar{x} \in S$. The set S is said to be *prox-regular* at $\bar{x}$ if there exist $\epsilon > 0$ and $\sigma > 0$ such that $B(\bar{x}, \epsilon)$ is convex and for every $x \in S \cap B(\bar{x}, \epsilon)$ and $v \in N_S^P(x)$ with $\|v\| < \epsilon$,

$$\langle v, \exp_x^{-1} y \rangle \leq \sigma\, d^2(x,y) \quad \text{for every } y \in S \cap B(\bar{x}, \epsilon). \tag{2}$$

Moreover, S is called *prox-regular* if it is prox-regular at each point of S.

The arguments in the following theorem mainly follow those for the case of Hilbert spaces; see [7, Proposition 6.2].

Theorem 3.4. *Let $S \subseteq M$ be closed. Then there exists a continuous function $\varphi\colon S \to [0, \infty)$ such that S is φ-convex if and only if S is prox-regular.*

Proof. Let $\bar{x} \in S$ and $\epsilon > 0$ be such that $B(\bar{x}, \epsilon)$ is convex with compact closure. Then for every $x \in S \cap B(\bar{x}, \epsilon)$ and $v \in N_S^P(x)$,

$$\langle v, \exp_x^{-1} y\rangle \leq \varphi(x)\ \|v\|\ d^2(x, y),$$

for every $y \in S \cap B(\bar{x}, \epsilon)$. Let c be an upper bound for φ on $S \cap B(\bar{x}, \epsilon)$. Therefore, for every $x \in S \cap B(\bar{x}, \epsilon)$ and $v \in N_S^P(x)$ with $\|v\| < \epsilon$ we have

$$\langle v, \exp_x^{-1} y\rangle \leq c\epsilon\ d^2(x, y) \ \text{ for every } y \in S \cap B(\bar{x}, \epsilon),$$

hence it suffices to set $\sigma = c\epsilon$.

Let us prove the converse statement. For every $x_\alpha \in S$, there exist constants $\epsilon_\alpha > 0$ and $\rho_\alpha > 0$ such that $U_\alpha := B(x_\alpha, \epsilon_\alpha)$ is convex and for every $x \in S \cap B(x_\alpha, \epsilon_\alpha)$ and $v \in N_S^P(x)$ with $\|v\| < \epsilon_\alpha$,

$$\langle v, \exp_x^{-1} y\rangle \leq \rho_\alpha d^2(x, y) \ \text{ for every } y \in S \cap B(x_\alpha, \epsilon_\alpha).$$

For every α, we define $\sigma_\alpha := \frac{\rho_\alpha}{\epsilon_\alpha}$ and we consider the open cover $\mathcal{U} := \{U_\alpha, S^c\}$ of M where S^c denotes the complement of S. Let $\{\varphi_\alpha, \psi\}$ be the partition of unity subordinate to the cover $\mathcal{U}$, that is, the functions ψ and φ_α are continuous on M for every α such that

(i) supp $\psi \subseteq S^c$ and supp $\varphi_\alpha \subseteq U_\alpha$ for every α;
(ii) $0 \leq \psi \leq 1$ and $0 \leq \varphi_\alpha \leq 1$ for every α;
(iii) The family $\{\text{supp } \varphi_\alpha\}_\alpha$ is locally finite;
(iv) $\sum_\alpha \varphi_\alpha(x) + \psi(x) = 1$.

We now define $\varphi(x) := \sum_\alpha \sigma_\alpha \varphi_\alpha(x)$ for every $x \in S$. Since supp $\psi \subseteq S^c$, we have $\psi(x) = 0$. The function φ is continuous on S and for every $x \in S$ there exist $i = 1, \cdots, n$ such that

$$\varphi(x) = \sum_{i=1}^{n} \sigma_i \varphi_i(x),$$

that is, $x \in B(x_i, \epsilon_i)$ for every $i = 1, \cdots, n$. Then for every i and $v \in N_S^P(x)$,

$$\langle \frac{\epsilon_i v}{\|v\|}, \exp_x^{-1} y\rangle \leq \rho_i d^2(x, y) \ \text{ for every } y \in S \cap B(x_i, \epsilon_i).$$

This implies that $\langle v, \exp_x^{-1} y\rangle \leq \sigma_i \|v\| d^2(x, y)$ for every $y \in S \cap B(x_i, \epsilon_i)$.

Let $r > 0$ be such that $B(x, r)$ is convex and $B(x, r) \subseteq B(x_i, \epsilon_i)$ for every i. Then we have $\langle v, \exp_x^{-1} y\rangle \leq \sigma_i \|v\| d^2(x, y)$ for every $y \in S \cap B(x, r)$ and $i = 1, \cdots, n$.

Suppose that $\sigma_j = \min\{\sigma_i : i = 1, \cdots, n\}$, hence we have $\varphi(x) \geq \sigma_j$. On the other hand, for every $y \in S \cap B(x, r)$

$$\langle v, \exp_x^{-1} y\rangle \leq \sigma_j \|v\| d^2(x, y) \leq \varphi(x) \|v\| d^2(x, y).$$

Since $x \in S$ is arbitrary, we conclude that S is φ-convex. $\square$

4. Sets with the unique footpoint property

In this section, we study the class of UFP-sets and we show that for a UFP-set S, under certain conditions, there exists a continuous function φ such that S is φ-convex. Throughout this section the manifold M is assumed to be complete.

Let S be a nonempty closed subset of M. We recall the definition of a UFP-set from [3]. The set $\mathrm{Unp}(S)$ is defined as follows

$$\mathrm{Unp}(S) := \{y \in M : \text{ there exists a unique } x \in S \text{ such that } d_S(y) = d(x, y)\},$$

and we consider the projection map $P_S \colon \mathrm{Unp}(S) \to S$. For every $x \in S$ we define

$$\mathrm{reach}(S, x) := \sup\{r \geq 0 : B(x, r) \subseteq \mathrm{Unp}(S)\},$$

and

$$\mathrm{reach}(S) := \inf_{x \in S} \mathrm{reach}(S, x).$$

It can be shown that $\mathrm{reach}(S, x) = d(x, \partial \mathrm{Unp}(S))$, where $\partial \mathrm{Unp}(S)$ denotes the boundary of $\mathrm{Unp}(S)$ and we consider $d(x, \emptyset) = +\infty$. This implies that the function $x \mapsto \mathrm{reach}(S, x)$ is continuous on S. It is worth mentioning that if $\mathrm{reach}(S, x_0) = +\infty$ for some $x_0 \in S$, then $\mathrm{reach}(S, x) = +\infty$ for all $x \in S$.

Using these notations, we now define UFP-sets as follows:

Definition 4.1. Let $S \subseteq M$ be nonempty and closed. The set S is called a UFP-*set* if for every $x \in S$, $\mathrm{reach}(S, x) > 0$.

Similar to the case of Euclidean spaces, see for instance [11, 21, 23], we introduce sets with positive reach on a Riemannian manifold. A nonempty closed set $S \subseteq M$ with the property that $\mathrm{reach}(S) > 0$ is a set with positive reach.

Clearly, a set with positive reach is a UFP-set. The converse does not necessarily hold, for instance see [17, Example 3] in $\mathbb{R}^2$. Note that if a UFP-set S is compact, then it is a set with positive reach.

The following proposition is a direct consequence of Theorem 3.4 and [13, Theorem 3.12] and shows that the class of UFP-sets contains the class of φ-convex sets or prox-regular sets. Note that by a result in [20] a closed set $S \subseteq \mathbb{R}^n$ is a UFP-set if and only if it is prox-regular at each of its (boundary) points.

Proposition 4.2. *Every φ-convex subset of a Riemannian manifold M is a* UFP-*set.*

Note that every φ-convex set is not necessarily a set with positive reach. In the following we present an example in the plane; see [18, Example 4] for more details.

Example 4.3. Suppose that for every $n \geq 1$, D_n is the unit disk in the plane centered at $(2n - 2^{1-n}, 0)$ and S is the union of these disks. Let a_n, b_n be the points $(2n + 1 - 2^{1-n}, 0)$ and $(2n - 1 - 2^{1-n}, 0)$ for $n \geq 1$, respectively. The distance between D_n and D_{n+1} is 2^{-n}, then S is φ-convex with $\varphi(x) = 2^{-n}$ for every $x \in \partial D_n$. On the other hand, for every $n \geq 2$, $\mathrm{reach}(S, a_n) = \mathrm{reach}(S, b_n) = 2^{-(n+1)}$. This implies that $\mathrm{reach}(S) = 0$ and hence S is not a set with positive reach.

In what follows, we will answer the question whether a UFP-set is φ-convex for a continuous function φ. We show that this holds under some conditions. To this end, we need to characterize the proximal normal cone to a set in a Riemannian manifold and prove some auxiliary results.

In the following theorem, we characterize the elements of $N_S^P(x)$.

Theorem 4.4. *Let S be a nonempty closed subset of M. Suppose that $x \in S$ and $B(x,r)$ is a convex neighborhood of x. Then*

$$N_S^P(x) = \left\{\lambda \exp_x^{-1} y : \quad x \in P_S(y), \ y \in B(x,r) \text{ and } \lambda \geq 0\right\}. \tag{3}$$

Proof. We denote the set in the right of (3) by $\mathcal{A}$. Let $v \in \mathcal{A}$ and $v \neq 0$, then there exist $\lambda > 0$ and $y_0 \in B(x,r) \setminus S$ such that $v = \lambda \exp_x^{-1} y_0$ and $x \in P_S(y_0)$. We define $\varphi(z) := d^2(z, y_0)$ for every $z \in M$, hence φ is smooth on $B(x,r)$ and $0 \in \partial_P(\varphi + I_S)(x)$. Consequently, we have $2\exp_x^{-1} y_0 \in N_S^P(x)$. It follows that $v \in N_S^P(x)$.

Conversely, we suppose that $v \in N_S^P(x)$ and $v \neq 0$. Then there exist $\sigma > 0$ and $R > 0$ such that $B(x,R)$ is a convex ball with compact closure and

$$\langle v, \exp_x^{-1} y\rangle \leq \sigma\, d^2(x,y), \tag{4}$$

for every $y \in B(x,R)$. Let $k_0 > 0$ be such that $|k| \leq k_0$ for all sectional curvatures k on $B(x,R)$. We choose $r > 0$ such that

$$r < \min\left\{R, \frac{\pi}{4\sqrt{K_0}}\right\},$$

hence $B(x,r)$ is convex with compact closure. Moreover, we choose $\epsilon > 0$ small enough such that $\epsilon < \frac{r}{2\|v\|}$ and put $y_0 := \exp_x(\epsilon v)$ and $\alpha(t) := \exp_x(tv)$ for every $t \in [0, \epsilon]$. Then $y_0 \in B(x, r/2)$ and α is contained in $B(x, r/2)$.

We claim that $x \in P_S(y_0)$. In fact, for every $s \notin B(x,r)$ we have $d(s, y_0) \geq r/2 > d(x, y_0)$, hence to prove the claim it is enough to show that $d(s,y_0) \geq d(x,y_0)$ for every $s \in B(x,r) \cap S$.

Suppose that $s \in B(x,r) \cap S$ and for every $z \in M$ we define $\varphi(z) := d^2(y_0, z)$ and we put $U := B(x,R)$. Let γ be the minimal geodesic joining x and s, hence it is entirely in U. Using the Taylor expansion, there exists $t_0 \in (0,1)$ such that

$$d^2(y_0, s) = d^2(y_0, x) - 2\langle \exp_x^{-1} y_0, \exp_x^{-1} s\rangle + \frac{1}{2} \operatorname{Hess} \varphi(x_0)(v_0)^2, \tag{5}$$

where $x_0 = \gamma(t_0) = \exp_x(t_0 \exp_x^{-1} s)$ and $v_0 = \dot{\gamma}(t_0)$.

Now we show that there exists a constant $c > 0$ such that

$$\|H(z)\| \geq c \ \text{ for every } z \in B(x,r), \tag{6}$$

where $H(z) := \operatorname{Hess} \varphi(z)$ and

$$\|H(z)\| = \sup\left\{|H(z)(v)^2| : v \in T_z M, \|v\| = 1\right\}.$$

Indeed, suppose that $z \in \bar{B}(x,r)$. If $z = y_0$, then $\|H(y_0)\| = 1$. Otherwise, $d(z, y_0) > 0$ and

$$d(z, y_0) < 2r < \frac{\pi}{2\sqrt{k_0}}.$$

Hence according to [2, p. 254], for some $w \in T_zM$ we have

$$\operatorname{Hess}\varphi(z)(w)^2 \geq 2\sqrt{k_0}d(z,y_0)\frac{\cos\left(\sqrt{k_0}d(z,y_0)\right)}{\sin\left(\sqrt{k_0}d(z,y_0)\right)}\|w\|^2,$$

it follows that

$$\|H(z)\| \geq 2\sqrt{k_0}d(z,y_0)\cot\left(\sqrt{k_0}d(z,y_0)\right). \tag{7}$$

Since $0 < \sqrt{k_0}d(z,y_0) < \frac{\pi}{2}$, the right hand side of (7) is positive. We set

$$c(z) := 2\sqrt{k_0}d(z,y_0)\cot\left(\sqrt{k_0}d(z,y_0)\right),$$

then $c(z) > 0$ for every $z \in \overline{B}(x,r) \setminus \{y_0\}$. Consequently, $\|H(z)\| > 0$ for every $z \in \overline{B}(x,r)$, then the continuity of $\|H(z)\|$ on the compact set $\overline{B}(x,r)$ implies that there exists $z_0 \in \overline{B}(x,r)$ such that $\|H(z)\| \geq \|H(z_0)\|$ for every $z \in \overline{B}(x,r)$. We put $c := \|H(z_0)\|$, hence c is the desired positive constant and the proof of the claim is complete. Then (5) and (6) imply that

$$d^2(y_0,s) \geq d^2(y_0,x) - 2\langle \exp_x^{-1} y_0, \exp_x^{-1} s\rangle + \frac{c}{2}\|\exp_x^{-1} s\|^2,$$

for every $s \in B(x,r) \cap S$. On the other hand, (4) implies that

$$-2\langle \exp_x^{-1} y_0, \exp_x^{-1} s\rangle \geq -2\epsilon\sigma\|\exp_x^{-1} s\|^2,$$

for every $s \in U \cap S$ and hence we have

$$d^2(y_0,s) \geq d^2(y_0,x) + \left(\frac{c}{2} - 2\epsilon\sigma\right)d^2(x,s).$$

We choose $\epsilon > 0$ small enough such that $\epsilon < \frac{c}{4\sigma}$, thus we have

$$d^2(y_0,s) \geq d^2(y_0,x),$$

for every $s \in B(x,r) \cap S$. This implies that by choosing $\epsilon < \min\left\{\frac{c}{4\sigma}, \frac{r}{2\|v\|}\right\}$, the point $y_0 = \exp_x(\epsilon v)$ has the property that $x \in P_S(y_0)$. It follows that $v \in \mathcal{A}$. $\square$

The previous theorem establishes that whenever y is in a convex neighborhood of x and $x \in P_S(y)$, then $\exp_x^{-1} y \in N_S^P(x)$. In the following example we show that the converse is not necessarily true.

Example 4.5. Consider the sphere S^2 and let S be the upper hemisphere. Suppose that x is the point $(-1,0,0)$ and $y = (\frac{\sqrt{2}}{2}, 0, -\frac{\sqrt{2}}{2})$, then $\exp_x^{-1} y \in N_S^P(x)$ but $x \notin P_S(y)$. $\square$

Proposition 4.6. *Let $S \subseteq M$ be a nonempty closed set. Suppose that $x \in S$ and $y \notin S$. If $\exp_x^{-1} y \in N_S^P(x)$, then there exists a point y_0 on the minimal geodesic joining x to y such that $x \in P_S(y_0)$.*

Proof. We put $v := \exp_x^{-1} y$, then by Theorem 4.4, there exist $\lambda \geq 0$ and a point y_0 such that $x \in P_S(y_0)$ and $v = \lambda \exp_x^{-1} y_0$. According to the proof of Theorem 4.4, we have $y_0 = \exp_x(\epsilon v)$ for some $\epsilon > 0$. We can choose ϵ small enough such that $\epsilon < 1$, therefore y_0 is the desired point. $\square$

The following proposition presents some properties of projection and distance function on complete Riemannian manifolds.

Proposition 4.7. *Let S be a nonempty closed subset of M. Then the following statements are equivalent:*

(a) $x \in P_S(y)$;

(b) $x \in P_S(y_t)$ *for every* $t \in [0,1]$, *where* $y_t = \gamma(t)$ *and* γ *is a minimal geodesic joining* x *to* y;

(c) $d_S(y_t) = td(x,y)$ *for every* $t \in [0,1]$.

Proof. The proof is trivial. $\square$

Corollary 4.8. *If $x \in P_S(y)$, then $P_S(\gamma(t)) = \{x\}$ for every $t \in [0,1)$ where γ is a minimal geodesic joining x to y.*

Now we proceed to obtain some results on the differentiability of the distance function to a set.

Lemma 4.9. *Let $S \subseteq M$, $x \notin S$ and d_S be differentiable at x. Then $x \in \mathrm{Unp}(S)$ and*

$$\nabla d_S(x) = \frac{1}{d_S(x)} \dot{\gamma}(1) = \frac{1}{d_S(x)} L_{P_S(x)x} \dot{\gamma}(0),$$

where γ is the minimizing geodesic joining $P_S(x)$ to x on $[0,1]$.

Proof. Since d_S is Lipschitz on M with the Lipschitz constant one, we have $\|\nabla d_S(x)\| \leq 1$. Let $\bar{x} \in P_S(x)$, then by Proposition 4.7,

$$d_S(\gamma(t)) = td(x,\bar{x}) \text{ for every } t \in [0,1],$$

where γ is a minimizing geodesic joining $\bar{x}$ to x on $[0,1]$. Thus we have

$$\langle \nabla d_S(x), \dot{\gamma}(1) \rangle = d(x,\bar{x}) = \|\dot{\gamma}(1)\|,$$

which implies that

$$\nabla d_S(x) = \frac{1}{d_S(x)} \dot{\gamma}(1).$$

Now we suppose that $y \in P_S(x)$, then

$$\nabla d_S(x) = \frac{1}{d_S(x)} \dot{\xi}(1),$$

where ξ is a minimizing geodesic joining y to x on $[0,1]$. It follows that $\dot{\xi}(1) = \dot{\gamma}(1)$ and so $\gamma(1-t)$ and $\xi(1-t)$ are geodesics starting at x with the same velocity. Then they coincide on $[0,1]$ and hence $\bar{x} = y$ and $x \in \mathrm{Unp}(S)$. Moreover, this shows that the minimizing geodesic joining x to $P_S(x)$ is unique. $\square$

Corollary 4.10. *Let M be a Hadamard manifold, $S \subseteq M$ be nonempty and closed and $x \notin S$. If d_S is differentiable at x, then $x \in \mathrm{Unp}(S)$ and*

$$\nabla d_S(x) = -\frac{1}{d_S(x)} \exp_x^{-1} P_S(x).$$

Lemma 4.11. *The projection map $P_S \colon \mathrm{Unp}(S) \to S$ is continuous.*

Proof. The proof is similar to the proof of [11, Theorem 4.8]. $\square$

For convenience, let us denote $\mathrm{Unp}(S)$ by U in the sequel. In the following, we show that the distance function d_S is continuously differentiable on the set $V \cap \mathrm{int}(U \setminus S)$, where V is a neighborhood of S. To this end, we need to prove the following lemma.

Lemma 4.12. *Let W be an open subset of M and $f \colon W \to \mathbb{R}$ be Lipschitz. Moreover, suppose that there exists a continuous vector field g on W such that*

$$\nabla f(x) = g(x) \quad \text{for every } x \in \mathcal{D}(f),$$

where $\mathcal{D}(f)$ is the set of all points in which f is differentiable. Then

$$\nabla f(x) = g(x) \quad \text{for every } x \in W.$$

Proof. For $x \in W$ and every convergent sequence $\{x_i\}$ to x in $\mathcal{D}(f)$ we have

$$\lim_i \nabla f(x_i) = \lim_i g(x_i) = g(x).$$

Therefore $\partial_c f(x) = \{g(x)\}$ for every $x \in W$.

Let $\bar{x} \in W$ and $v \in T_{\bar{x}}M$. We consider $r > 0$ such that $B(x,r)$ is a convex ball contained in W. Moreover, we choose $t > 0$ such that $t < \frac{r}{\|v\|}$ and set $y := \exp_{\bar{x}} tv$. Then according to Theorem 2.2, there exist $t_0 \in (0,1)$ and $v \in \partial_c f\left(\gamma(t_0)\right)$ with

$$f(y) - f(\bar{x}) = \langle v, \dot{\gamma}(t_0) \rangle,$$

where γ is the minimizing geodesic joining $\bar{x}$ to y and so contained in $B(x,r)$. Therefore

$$f\left(\exp_{\bar{x}} tv\right) - f(\bar{x}) = \langle L_{z\bar{x}} g(z), tv \rangle,$$

where z is a point on the minimizing geodesic joining $\bar{x}$ to the point $y = \exp_{\bar{x}} tv$. Then $z \to \bar{x}$ as $t \to 0$ and we have

$$\lim_{t \to 0} \frac{f\left(\exp_{\bar{x}} tv\right) - f(\bar{x})}{t} = \lim_{z \to \bar{x}} \langle L_{z\bar{x}} g(z), v \rangle.$$

Hence the continuity of g implies that

$$\lim_{t\to 0} \frac{f(c(t)) - f(c(0))}{t} = \langle g(\bar{x}), v\rangle,$$

where $c(t) = \exp_{\bar{x}} tv$. It follows that f is differentiable at $\bar{x}$ and

$$df(\bar{x})(v) = \langle g(\bar{x}), v\rangle,$$

where $df(\bar{x})$ is the differential of f at $\bar{x}$. Since $\bar{x}$ and v were arbitrary, this implies that $\nabla f(x) = g(x)$ for every $x \in W$. □

Theorem 4.13. *Let S be a nonempty closed subset of M. Then there exists a neighborhood V of S such that the distance function d_S is continuously differentiable on the set $V \cap \operatorname{int}(U \setminus S)$.*

Proof. We put $V := \bigcup_{z\in S} B\left(z, \frac{r(z)}{2}\right)$ and $W := V \cap \operatorname{int}(U \setminus S)$.

Since d_S is Lipschitz, according to Lemma 4.9 we have

$$\nabla d_S(x) = \frac{1}{d_S(x)} L_{P_S(x)x}\dot{\gamma}(0) \quad \text{a.e. on } W, \tag{8}$$

where γ is the unique minimizing geodesic joining $P_S(x)$ to x on $[0,1]$. Moreover, for every $x \in W$ there exists $z \in S$ such that $x \in B\left(z, \frac{r(z)}{2}\right)$. This implies that

$$d(P_S(x), z) \le d(P_S(x), x) + d(x, z) < r(z).$$

Thus (8) can be written as $\nabla d_S(x) = -\frac{1}{d_S(x)} \exp_x^{-1} P_S(x)$ a.e. on W.

For every $x \in W$ we define $g(x) := -\frac{1}{d_S(x)} \exp_x^{-1} P_S(x)$.

Then, using Lemma 4.11, one has the the continuity of P_S on U which implies that the function g is continuous on W. Hence by Lemma 4.12 we conclude that $\nabla f(x) = g(x)$ for every $x \in W$. This completes the proof of the theorem. □

Note that in general the set $\operatorname{int}(U)$ is not necessarily a neighborhood of S. In fact, this is true when S is a UFP-set. Therefore, Theorem 4.13 implies that for a UFP-set S there exists a neighborhood W of S such that the distance function d_S is continuously differentiable on $W \setminus S$.

Remark 4.14. It is worth mentioning that if the set S has the property

(P) for every $x \in \operatorname{int}(U \setminus S)$ there exists a convex set containing both x and $P_S(x)$,

then according to the proof of Theorem 4.13, d_S is continuously differentiable on the set $\operatorname{int}(U \setminus S)$.

Example 4.15. Let S be the closed upper hemisphere in S^2. Then $\mathrm{Unp}(S) = S^2 - \{q\}$ where q is the south pole. For every $x \in \mathrm{int}(U \setminus S)$, we have x is in the convex ball $B\left(P_S(x), \frac{\pi}{2}\right)$. Therefore, the set S satisfies (P) and hence d_S is continuously differentiable on the lower hemisphere except the south pole. □

Corollary 4.16. *Let M be a Hadamard manifold and $S \subseteq M$ be a nonempty closed set. Then d_S is continuously differentiable on* $\mathrm{int}(U \setminus S)$.

For the proof of the main result of this section, we need to verify the following lemma. Recall that U denotes the set $\mathrm{Unp}(S)$.

Lemma 4.17. *Suppose that $x \in S$ and $v \in T_xM$ with $\|v\| = 1$ and define*

$$\tau := \sup\left\{t \geq 0 : P_S\left(\exp_x tv\right) = x\right\}.$$

If $0 < \tau < \frac{r(x)}{2}$ (or $0 < \tau < r(x)$ and S satisfies (P)), then $\exp_x \tau v \notin \mathrm{int}(U)$, where U denotes $\mathrm{Unp}(S)$.

Proof. We put $y := \exp_x \tau v$ and on the contrary, we suppose that $y \in \mathrm{int}(U)$. By the definition of τ, there exists a sequence $\{t_i\}$ such that $P_S\left(\exp_x t_i v\right) = x$ for every i and $t_i \to \tau$. Then for each i, $\exp_x t_i v \in U$ and hence the continuity of P_S on U implies that

$$P_S(y) = \lim_{i \to \infty} P_S\left(\exp_x t_i v\right) = x.$$

Since by the assumption $\tau < r(x)$, it follows that $d_S(y) = \tau$. Thus $y \in \mathrm{int}(U \setminus S)$ and according to Lemma 4.9, $\nabla d_S(y) = \frac{1}{\tau} L_{xy}(\tau v) = L_{xy} v$. We now consider the following differential equation

$$\dot{c}(t) = \nabla d_S(c(t)), \quad c(0) = y.$$

Since $y \in V \cap \mathrm{int}(U \setminus S)$, where $V = \bigcup_{z \in S} B\left(z, \frac{r(z)}{2}\right)$ and by Theorem 4.13, ∇d_S is continuous on $V \cap \mathrm{int}(U \setminus S)$, it follows that there exist $r > 0$ and a solution $c\colon (-r, r) \to \mathrm{int}(U \setminus S)$ such that $c(0) = y$. In the case where we assume $0 < \tau < r(x)$ and S satisfies (P), by using Remark 4.14 such a solution can be obtained. Therefore for every $s \in (-r, r)$, we have

$$\frac{d}{ds} d_S\left(c(s)\right) = \|\nabla d_S(c(s))\|^2 = 1.$$

Hence we have $d_S\left(c(s)\right) = s + d_S(y)$ for every $s \in (-r, r)$. This implies that for every t_1, t_2 such that $-r < t_1 < t_2 < r$,

$$\int_{t_1}^{t_2} \|\dot{c}(s)\|\, ds = t_2 - t_1.$$

On the other hand we have

$$d\left(c(t_1), c(t_2)\right) \geq d_S\left(c(t_1)\right) - d_S\left(c(t_2)\right) = t_2 - t_1.$$

It follows that the solution c is minimizing and hence it is a geodesic. Moreover, the property $\dot{c}(0) = L_{xy} v$ implies that

$$c(s) = \exp_x(\tau + s)v \;\text{ for every } s \in [0, r).$$

We choose r small enough such that $\tau + r < r(x)$. Then for every $s \in [0, r)$,

$$d_S\left(c(s)\right) = s + \tau = d\left(x, c(s)\right),$$

that is, $P_S\left(\exp_x(\tau + s)v\right) = x$ for every $s \in [0, r)$. This is a contradiction to τ being the supremum. □

Theorem 4.18. *Let $S \subseteq M$ be a* UFP*-set and assume that the following assumption holds: For every $x \in \partial S$ and $v \in N_S^P(x)$ with $\|v\| = 1$ we have $\tau_v < \frac{r(x)}{2}$ (or, $\tau_v < r(x)$ and S satisfies (P)), where*

$$\tau_v := \sup\left\{t \geq 0 : P_S\left(\exp_x(tv)\right) = x\right\}.$$

Then there exists a continuous function φ on ∂S such that the set S is φ-convex.

Proof. Let $x \in \partial S$ and $v \in N_S^P(x)$. Without loss of generality, we assume that $N_S^P(x) \neq \{0\}$ and $\|v\| = 1$. We claim that $\tau_v > 0$. In fact, according to Theorem 4.4, there exist $\lambda > 0$ and a point y near x such that $x = P_S(y)$ and $v = \lambda \exp_x^{-1} y$. Therefore $\tau_v \geq \frac{1}{\lambda} > 0$.

We first suppose that $\tau_v < \infty$ and put $y_0 := \exp_x(\tau_v v)$. Then using Lemma 4.17, we deduce that $y_0 \notin \operatorname{int}(U)$. This implies that $B(x, \tau_v) \nsubseteq U$ and hence $\operatorname{reach}(S, x) \leq \tau_v$. We choose $t > 0$ such that $t \in T_v$, where

$$T_v := \left\{t \geq 0 : P_S\left(\exp_x(tv)\right) = x\right\}.$$

We put $y_t := \exp_x(tv)$, hence $P_S(y_t) = x$ and $y_t \in B(x, r(x))$. Then we fix $z_0 \in S \cap B(x, r(x))$. By the Taylor expansion, there exists $s_0 \in (0, 1)$ such that

$$d^2(y_t, z_0) = d^2(y_t, x) - 2\langle \exp_x^{-1} y_t, \exp_x^{-1} z_0\rangle + \frac{1}{2}\langle \operatorname{Hess} d_{y_t}^2\left(\gamma(s_0)\right)\dot\gamma(s_0), \dot\gamma(s_0)\rangle, \quad (9)$$

where γ is the minimizing geodesic joining x to z_0 on $[0, 1]$. To proceed, we claim that there exists a continuous function c on ∂S such that for each $x \in \partial S$,

$$\operatorname{Hess} d_y^2(z)(v)^2 \leq c(x)\|v\|^2, \quad (10)$$

for every $y, z \in B(x, r(x))$ and $v \in T_z M$.

To prove the claim, let $x \in M$. We first suppose that $r(x) < \infty$ and let $R > 0$ be arbitrary. Put $A := \overline{B}(x, R)$, hence A is compact and there exists $\bar r > 0$ such that $r(z) \leq \bar r$ for every $z \in A$. This implies that for every $z \in A$

$$B\left(z, r(z)\right) \subseteq B\left(x, R + \bar r\right).$$

Since M is complete, there exists $k_0 > 0$ such that for every $z \in A$ and every sectional curvature k on $B\left(z, r(z)\right)$ we have $\mid k \mid \leq k_0$. Similarly, it follows that for every $x_\alpha \in M$, $r\left(x_\alpha\right) < \infty$ and there exists $k_\alpha > 0$ such that for every $z \in U_\alpha := B\left(x_\alpha, R\right)$ and every sectional curvature k on $B\left(z, r(z)\right)$ we have $\mid k \mid \leq k_\alpha$. Let $\{\psi_\alpha\}$ be the partition of unity subordinate to the cover $\{U_\alpha\}$ and define

$$k(x) := \sum_\alpha \psi_\alpha(x) k_\alpha, \quad \text{for every } x \in \partial S.$$

Now let $x \in \partial S$, then there exist finitely many points x_i, $i = 1, \cdots, n$ such that $B(x, r(x)) \subseteq B(x_i, R)$ and $|\ k\ | \leq k_i$ on $B(x, r(x))$ for $i = 1, \cdots, n$. This implies that $|\ k\ | \leq k(x)$ on $B(x, r(x))$ for every $x \in \partial S$. Using Proposition 2.1, we conclude that

$$\text{Hess } d_y^2(z)(v)^2 \leq \left(2 + \frac{2}{3}\, k(x)\, d^2(y, z)\right) \|v\|^2,$$

for every $y, z \in B(x, r(x))$ and $v \in T_z M$. Therefore, we have

$$\text{Hess } d_y^2(z)(v)^2 \leq \left(2 + \frac{8}{3}\, k(x)\, r(x)^2\right) \|v\|^2,$$

for every $y, z \in B(x, r(x))$ and $v \in T_z M$. Then the function c defined by

$$c(x) := 2 + \frac{8}{3}\, k(x)\, r(x)^2 \ \text{ for every } x \in \partial S,$$

is the desired function. In the case where $r(x) = +\infty$, by choosing $\bar{r} > 0$ and using $B(z, \bar{r})$ instead of $B(z, r(z))$ in the above argument, we obtain that

$$\text{Hess } d_y^2(z)(v)^2 \leq \left(2 + \frac{8}{3}\, k(x)\, \bar{r}^2\right) \|v\|^2.$$

This completes the proof of the claim.

Now by using (9) and (10) we derive that

$$d^2(y_t, z_0) \leq d^2(y_t, x) - 2\langle \exp_x^{-1} y_t, \exp_x^{-1} z_0 \rangle + \frac{1}{2} c(x)\, d^2(x, z_0).$$

Since $P_S(y_t) = x$, we conclude that $\langle \exp_x^{-1} y_t, \exp_x^{-1} z_0 \rangle \leq \dfrac{c(x)}{4}\, d^2(x, z_0)$.

It follows that $\langle v, \exp_x^{-1} z_0 \rangle \leq \dfrac{c(x)}{4t} d^2(x, z_0)$ for every $t \in T_v$.

Then we have $\langle v, \exp_x^{-1} z_0 \rangle \leq \dfrac{c(x)}{4\tau_v} d^2(x, z_0)$ and for every $z \in S \cap B(x, r(x))$

$$\langle v, \exp_x^{-1} z \rangle \leq \frac{c(x)}{4\ \text{reach}(S, x)} d^2(x, z).$$

Thus for every $v \in N_S^P(x)$ and $z \in S \cap B(x, r(x))$ we have

$$\langle v, \exp_x^{-1} z \rangle \leq \frac{c(x)}{4\ \text{reach}(S, x)} \|v\| d^2(x, z).$$

Hence it suffices to define, for every $x \in \partial S$,

$$\varphi(x) := \frac{c(x)}{4\ \text{reach}(S, x)}.$$

In the case where $\tau_v = +\infty$, it follows that $\tau_w = +\infty$ for every $w \in N_S^P(y)$ and every $y \in S$, then $\operatorname{reach}(S, y) = +\infty$ for every $y \in S$. We choose $t > 0$ and put $y_t := \exp_x(tv)$. Along the same lines as above, we have

$$\langle v, \exp_x^{-1} z\rangle \leq \frac{c(x)}{4t} d^2(x, z) \quad \text{for every } z \in S.$$

Therefore as $t \to \infty$, we conclude that

$$\langle v, \exp_x^{-1} z\rangle \leq 0 \quad \text{for every } z \in S,$$

then S is φ-convex with $\varphi \equiv 0$ on ∂S. □

Example 4.19. Let $M = S^2$ and A be a subset of M which is obtained by removing the sector $0 \leq \theta < \frac{\pi}{6}$ from S^2 where θ is the angle from negative z-axis. Consider for example $x = \left(0, -\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$ and $v \in N_A^P(x)$ with $\|v\| = 1$, then $v = \left(0, -\frac{\sqrt{3}}{2}, \frac{1}{2},\right)$. In this case, we have

$$\operatorname{Unp}(A) = S^2 \setminus \{q\} \quad \text{and} \quad \operatorname{reach}(A) = \frac{\pi}{6},$$

where q is the south pole. Moreover, $r(x) = \frac{\pi}{2}$ and $\tau_v = \frac{\pi}{6}$, hence the assumptions of Theorem 4.18 holds and so A is φ-convex with $\varphi(x) = \frac{3}{\pi}$ for every $x \in \partial A$. □

Corollary 4.20. *Let M be a Hadamard manifold and $S \subseteq M$ be a* UFP*-set. Then there exists a continuous function φ on ∂S such that the set S is φ-convex.*

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