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Polar Convexity and Critical Points of Polynomials

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A set A, in the extended complex plane, is called convex with respect to a pole u, if for any $x, y \in A$ the arc on the unique circle through x, y, and u, that connects x and y but does not contain u, is in A. If the pole u is taken at infinity, this notion reduces to the usual convexity. Polar convexity is connected with the classical Gauss-Lucas' and Laguerre's theorems for complex polynomials. If a set is convex with respect to u and contains the zeros of a polynomial, then it contains the zeros of its polar derivative with respect to u. A set may be convex with respect to more than one pole. The main goal of this article is to find the relationships between a set in the extended complex plane and its poles.

Keywords: Zeros and critical points of polynomials, Gauss-Lucas' Theorem, Laguerre's Theorem, polar derivative, pole of a set, polar convexity.

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1. Introduction

A *circular domain* (open or closed) is the interior or exterior of a circle, or a half-plane determined by a line in the complex plane. The critical points of a polynomial are the zeros of its derivative. The classical Gauss-Lucas' theorem says that the critical points of a polynomial of degree $n \geq 2$ are located in the convex hull $\operatorname{conv}\{z_1, \ldots, z_n\}$ of its zeros $\{z_1, \ldots, z_n\}$. In other words, every closed half plane in the complex plane that contains the zeros, also contains the critical points.

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The polar derivative of a polynomial with respect to a pole u, see equation (12) for the definition, extends the notion of a derivative and provides more flexibility by allowing one to change the pole. In fact, when the pole u approaches infinity, the zeros of the polar derivative converge to the critical points of the polynomial. An analogue of the Gauss-Lucas' theorem is a result by Laguerre. It states that if a circular domain contains the zeros of a polynomial, but not the pole u, then it contains all zeros of the polar derivative with respect to u, see Theorem 3.9. Laguerre's result implies the Gauss-Lucas' theorem since one can take the pole u to be outside of a closed half plane, containing the zeros of the polynomial, then let u approach infinity and use the fact that the zeros of the polar derivative are continuous with respect to u.

The starting point of this work is Proposition 3.8. It is a simple observation that extends the Gauss-Lucas' theorem to polar derivatives in a way that strengthens the Laguerre's theorem. It states that if $z_1, \dots, z_n$ are points in the extended complex plane and are the zeros of a polynomial, then the zeros of its polar derivative with respect to a pole u are in the set $\mathrm{conv}_u\{z_1, \dots, z_n\}$, defined by (3). It is a fact that if D is a circular domain that contains $z_1, \dots, z_n$ but not u, then D contains $\mathrm{conv}_u\{z_1, \dots, z_n\}$.

The set $\mathrm{conv}_u\{z_1, \dots, z_n\}$ is the convex hull of the points $z_1, \dots, z_n$ with respect to the pole u. A set A is called *convex with respect to a pole* u, if for any $x, y \in A$ the arc on the unique circle through x, y, and u, that connects x and y but does not contain u, is in A. If a set is convex with respect to u and contains the zeros of a polynomial, then it contains the zeros of its polar derivative with respect to u. A set may be convex with respect to several poles. The main goal of this article is to study the relationships between a set in the extended complex plane and its poles.

2. Notation

Let $\mathbb{C}$ be the complex plane and let $\mathbb{C}^* := \mathbb{C} \cup \{\infty\}$ be the extended complex plane. It is well-known, that for any three distinct points z_1, z_2, and u in the extended complex plane $\mathbb{C}^*$, the unique circle C defined by z_1, z_2 and u can be parametrized as follows

$$C = \Big\{ z \in \mathbb{C} : z = u + \frac{1}{\frac{1-t}{z_1-u} + \frac{t}{z_2-u}}, \ \text{ where } t \in \mathbb{R} \cup \{\infty\} \Big\}.$$

In particular, when $t \in [0,1]$ the point

$$z = u + \frac{1}{\frac{1-t}{z_1-u} + \frac{t}{z_2-u}} \tag{1}$$

goes over the arc of C starting at z_1 and ending at z_2, not containing u; when $t \in [-\infty, 0]$ it goes over the arc of C starting at u and ending at z_1, not containing z_2; and when $t \in [1, \infty]$ the point z goes over the arc of C starting at z_2 and ending at u, not containing z_1. Thus, throughout this work, we use the notation

$$\mathrm{arc}_u[z_1, z_2] := \Big\{u + \frac{1}{\frac{1-t}{z_1-u} + \frac{t}{z_2-u}} : t \in [0,1]\Big\},$$

for distinct $z_1, z_2, u \in \mathbb{C}$ and call it *the arc between* z_1 *and* z_2 *with respect to* u. The right-hand side of (1) can be viewed as a Möbius transformation with respect to any of z_1, z_2, or u:

$$\begin{aligned} u + \frac{1}{\frac{1-t}{z_1-u} + \frac{t}{z_2-u}} &= \frac{(z_2-(1-t)u)z_1 - tuz_2}{tz_1 - u + (1-t)z_2} = \frac{(z_1-tu)z_2 - (1-t)uz_1}{(1-t)z_2 + tz_1 - u} \\ &= \frac{((1-t)z_1 + tz_2)u - z_1z_2}{u - tz_1 - (1-t)z_2}. \end{aligned}$$

This allows one to extend the definition by continuity as follows. For distinct $z_1, z_2 \in \mathbb{C}$, let

$$\mathrm{arc}_\infty[z_1, z_2] := \{(1-t)z_1 + tz_2 : t \in [0,1]\},$$

be the segment between z_1 and z_2. For distinct $u, z_2 \in \mathbb{C}$, let

$$\mathrm{arc}_u[\infty, z_2] := \Big\{\frac{z_2 - (1-t)u}{t} : t \in [0,1]\Big\},$$

be the ray in $\mathbb{C}^*$ with origin z_2 in the direction opposite from u (traversed from ∞ to z_2 as $t \in [0,1]$). And for distinct $u, z_1 \in \mathbb{C}$, let

$$\mathrm{arc}_u[z_1, \infty] := \Big\{\frac{z_1 - tu}{1-t} : t \in [0,1]\Big\},$$

be the ray in $\mathbb{C}^*$ with origin z_1 in the direction opposite from u. Finally, if any of the points u, z_1, z_2 in $\mathbb{C}^*$ are equal, with common value $v \in \mathbb{C}^*$, then define

$$\mathrm{arc}_u[z_1, z_2] := \{v\}. \tag{2}$$

One readily shows that the set $\mathrm{arc}_u[z_1, z_2]$ converges to $\mathrm{arc}_\infty[z_1, z_2]$ as u approaches infinity. We also use the following variations of the notation

$$\mathrm{arc}_u\langle z_1, z_2\rangle := \Big\{u + \frac{1}{\frac{1-t}{z_1-u} + \frac{t}{z_2-u}} : t \in \langle 0,1\rangle\Big\},$$

where the symbol '$\langle$' stands for '(' or '[' and the symbol '$\rangle$' stands for ')' or ']'.

Next, $D[c;r]$ denotes the closed disk with centre $c \in \mathbb{C}$ and radius r, while $D(c;r)$ denotes the open such disk. By $C(z_1, u, z_2)$ we denote the circle through three distinct points z_1,u, z_2 in $\mathbb{C}$, oriented from z_1 through u to z_2. By $D[z_1, u, z_2]$ we denote the closed circular domain, having the circle $C(z_1, u, z_2)$ as its boundary, and lying to the left of it. Similarly, $D(z_1, u, z_2)$ denotes the open such circular domain.

Let $\mathrm{arc}[z_1, u, z_2]$ be the arc on the unique circle, determined by the three distinct points z_1, u, z_2, that starts at z_1, goes through u and ends in z_2. We use the variations $\mathrm{arc}\langle z_1, u, z_2\rangle$ to indicate whether the end points are included or not. For example, we say that $\mathrm{arc}(z_1, u, z_2)$ is the *open arc* from z_1 through u to z_2, where the points z_1 and z_2 are not included.

Let $z_1, z_2 \in \mathbb{C}$ be distinct. Denote by $\mathrm{ray}(z_1, z_2)$ the ray with origin z_1, passing through z_2. Denote by $\mathrm{ray}(z; v)$ the ray with origin z in the direction of vector v. Denote by $\mathrm{line}[z_1, z_2]$ the line through z_1 and z_2. Such lines are assumed oriented from z_1 to z_2, so that we can talk about the left half plane and right half plane determined by it.

By A^c we denote the *complement* of a set A in $\mathbb{C}^*$.

3. Main definition and motivation

3.1. Main definitions

Given any points $z_1, \dots, z_n \in \mathbb{C}^*$ and a $u \in \mathbb{C}$ distinct from them, define the set

$$\mathrm{conv}_u\{z_1, \dots, z_n\} := \Big\{u + \frac{1}{\sum_{i=1}^n \frac{t_i}{z_i - u}} \in \mathbb{C} : t_i \geq 0 \text{ with } \sum_{i=1}^n t_i = 1\Big\}. \tag{3}$$

Call this set the *convex hull of* $z_1, \dots, z_n$ *with respect to the pole* u. The expression

$$u + \frac{1}{\sum_{i=1}^n \frac{t_i}{z_i - u}} \quad \text{for some } \; t_i \geq 0 \text{ with } \sum_{i=1}^n t_i = 1 \tag{4}$$

is called a *convex combination with pole* u. It is easy to see that

$$\text{since } u \notin \{z_1, \dots, z_n\}, \text{ we have } u \notin \mathrm{conv}_u\{z_1, \dots, z_n\}. \tag{5}$$

The set $\mathrm{conv}_u\{z_1, \dots, z_n\}$ should be considered a subset of $\mathbb{C}^*$. That is, it may contain ∞. For example, if $z_1 = \infty$, then let $t_1 = 1$ and $t_2 = \cdots = t_n = 0$. Another case is when $u \in \mathrm{conv}\{z_1, \dots, z_n\}$. Then, $0 \in \mathrm{conv}\{z_1 - u, \dots, z_n - u\}$, implying that $0 \in \mathrm{conv}\{1/(z_1 - u), \dots, 1/(z_n - u)\}$. Hence, there are $t_i \geq 0$ with $\sum_{i=1}^n t_i = 1$, such that $\sum_{i=1}^n t_i/(z_i - u) = 0$.

For $z_1, \dots, z_n \in \mathbb{C}$ and $u = \infty$, define

$$\mathrm{conv}_\infty\{z_1, \dots, z_n\} := \mathrm{conv}\{z_1, \dots, z_n\}$$

and note that (5) extends to the case $u = \infty$ as well.

It is easy to see that for $u \in \mathbb{C}$, the Möbius transformation

$$T(z) := \frac{1}{z - u} \quad \text{with} \quad T^{-1}(z) = u + \frac{1}{z} \tag{6}$$

satisfies

$$T(\mathrm{conv}_u\{z_1,\dots,z_n\}) = \mathrm{conv}_\infty\{T(z_1),\dots,T(z_n)\}. \tag{7}$$

Indeed, this follows from the fact that

$$\text{if } z = u + \frac{1}{\sum_{i=1}^n \frac{t_i}{z_i - u}}, \text{ then } T(z) = \sum_{i=1}^n t_i T(z_i). \tag{8}$$

Lemma 3.1. *Let $z_1,\dots,z_n \in \mathbb{C}^*$ and let $u \in \mathbb{C}^*$ be distinct from them. Let $\{u_m\} \subset \mathbb{C}$ be a sequence converging to u, such that $u_m \notin \{z_1,\dots,z_n\}$ for all m. If $v_m \in \mathrm{conv}_{u_m}\{z_1,\dots,z_n\}$ are such that the sequence $\{v_m\}$ converges to $v \in \mathbb{C}^*$, then $v \in \mathrm{conv}_u\{z_1,\dots,z_n\}$.*

Conversely, if $v \in \mathrm{conv}_u\{z_1,\dots,z_n\}$, then there is a sequence $\{v_m\} \subset \mathbb{C}$, converging to v, such that $v_m \in \mathrm{conv}_{u_m}\{z_1,\dots,z_n\}$ for every m.

Proof. Let $v_m = u_m + \frac{1}{\sum_{i=1}^n \frac{t_i^m}{z_i - u_m}}$ for some $t_i^m \geq 0$ with $\sum_{i=1}^n t_i^m = 1$.

Without loss of generality, the sequence $\{t_i^m\}$ converges to t_i as m approaches infinity, for all $i = 1,\dots,n$. Clearly, $t_i \geq 0$ with $\sum_{i=1}^n t_i = 1$.

If $u \in \mathbb{C}$, then just taking the limit in the displayed equation concludes the case. If $u = \infty$, then $z_1,\dots,z_n \in \mathbb{C}$. Using that $\sum_{i=1}^n t_i^m = 1$, one can show

$$v_m = u_m + \frac{1}{\sum_{i=1}^n \frac{t_i^m}{z_i - u_m}} = \frac{u_m^{n-1}(\sum_{i=1}^n t_i^m z_i) + \text{lower order terms in } u}{u_m^{n-1} + \text{lower order terms in } u}.$$

The result follows after taking the limit as m approaches infinity.

The converse follows easily by going backwards in the displayed equations. □

The basic idea of polar convexity is somewhat old and explained well in [1, p. 53–54]. In fact, we have the following result, given in [1, Problem 104].

Proposition 3.2. *Let $u, z_1,\dots,z_n \in \mathbb{C}^*$ with u different from $z_1,\dots,z_n$. The set $\mathrm{conv}_u\{z_1,\dots,z_n\}$ is the intersection of all closed circular domains that contain the points $z_1,\dots,z_n$ and have u on the their boundary, with the point u removed. Moreover, if $a, b \in \mathrm{conv}_u\{z_1,\dots,z_n\}$, then $\mathrm{arc}_u[a,b]$ is in $\mathrm{conv}_u\{z_1,\dots,z_n\}$.*

Definition 3.3. A set $A \subset \mathbb{C}^*$ is called *convex with respect to the pole $u \in \mathbb{C}^*$* if for any $z_1, z_2, \in A$, we have $\mathrm{arc}_u[z_1,z_2] \subset A$.

Sometimes we say that A is *convex with respect to u* or that it is *u-convex* for short. For example, every circular domain D is convex with respect to any $u \in \mathrm{cl}D^c$. Convex sets with respect to a pole have analogous properties to convex sets. If A is convex with respect to u, then since $T(A)$, defined by (6), is convex in the usual sense, one can define notions from convex analysis, see [3], such as relative interior of A, relative boundary, extreme points of A with respect to u, etc. in an intuitive way. Thus, if A is convex with respect to u and $\{z_1, \dots, z_n\} \subset A$, then

$$\operatorname{conv}_u\{z_1, \dots, z_n\} \subset A. \tag{9}$$

In other words, if u is different from $\{z_1, \dots, z_n\}$, then the smallest u-convex set containing those points is $\operatorname{conv}_u\{z_1, \dots, z_n\}$. But if $u \in \{z_1, \dots, z_n\}$, then the smallest u-convex set containing $\{z_1, \dots, z_n\}$ is

$$\{u\} \cup \operatorname{conv}_u\{z_i : z_i \neq u \text{ for } i = 1, \dots, n\}, \tag{10}$$

where we used (2). Thus, we extend definition (3) as follows.

Definition 3.4. For any $u, z_1, \dots, z_n \in \mathbb{C}^*$, define the set $\operatorname{conv}_u\{z_1, \dots, z_n\}$ to be the one given by (3), when $u \notin \{z_1, \dots, z_n\}$; and to be the one given by (10), when $u \in \{z_1, \dots, z_n\}$.

Care needs to be exercised, since the set $\operatorname{conv}_u\{z_1, ..., z_n\}$ does not behave in a continuous way when $u \notin \{z_1, \dots, z_n\}$ converges to a point in $\{z_1, \dots, z_n\}$. Relationship (7) holds in the extended sense as well and (5) can now be restated as

$$u \in \operatorname{conv}_u\{z_1, \dots, z_n\}, \text{ if and only if } u \in \{z_1, \dots, z_n\}. \tag{11}$$

As explained in the next subsection, of bigger interest is the relationship between a set and the set of all its poles.

Definition 3.5. Denote by $\mathcal{P}(A)$ the set of all poles of A. That is, $u \in \mathcal{P}(A)$, if and only if A is convex with respect to u.

Lemma 3.6. *For any $A \subsetneqq \mathbb{C}^*$, we have $(\operatorname{int} A) \cap \mathcal{P}(A) = \emptyset$.*

Proof. Suppose that $u \in (\operatorname{int} A) \cap \mathcal{P}(A)$. Since $A \neq \mathbb{C}^*$, there is a circle C through u, that is not entirely contained in A. Since $u \in \operatorname{int} A$, there are points $z_1, z_2 \in (\operatorname{int} A) \cap C$, such that $\operatorname{arc}[z_1, u, z_2] \subset A$. Therefore, the complement arc in C, $\operatorname{arc}_u[z_1, z_2]$, is not entirely contained in A, a contradiction with the fact that u is a pole of A. □

Non-degenerate Möbius transformations $T(z) = (az + b)/(cz + d)$ are one-to-one and send circles onto circles (lines are considered circles through infinity), hence $\operatorname{arc}_u[z_1, z_2]$ is mapped onto the arc $\operatorname{arc}_{T(u)}[T(z_1), T(z_2)]$. Thus, if A is convex with respect to u, then $T(A)$ is convex with respect to $T(u)$. The Carathéodory's theorem [3, Theorem 17.1], and (7), show that every $v \in \operatorname{conv}_u\{z_1, \dots, z_n\}$ is a convex combination with pole u of at most three points among $\{z_1, \dots, z_n\}$.

In detail, if
$$v = u + \frac{1}{\frac{t_1}{z_1 - u} + \frac{t_2}{z_2 - u} + \frac{t_3}{z_3 - u}},$$

then
$$v = u + \frac{1}{\frac{t_1}{z_1 - u} + \frac{t_2 + t_3}{w - u}}, \quad \text{where} \quad w := u + \frac{1}{\frac{t_2/(t_2+t_3)}{z_2 - u} + \frac{t_3/(t_2+t_3)}{z_3 - u}}.$$

This shows $v \in \mathrm{arc}_u[z_1, w]$ and $w \in \mathrm{arc}_u[z_2, z_3]$. Thus, if $T(z)$ is a non-degenerate Möbius transformation, we conclude that $T(v) \in \mathrm{arc}_{T(u)}[T(z_1), T(w)]$, where $T(w)$ is on $\mathrm{arc}_{T(u)}[T(z_2), T(z_3)]$. In other words

$$T(\mathrm{conv}_u\{z_1, \ldots, z_n\}) \subset \mathrm{conv}_{T(u)}\{T(z_1), \ldots, T(z_n)\}.$$

The former set contains the points $T(z_1), \ldots, T(z_n)$ and is polar with respect to $T(u)$, while the latter set is the smallest with these properties. This shows the following lemma.

Lemma 3.7. *For any $u, z_1, \ldots, z_n \in \mathbb{C}^*$ and a non-degenerate Möbius transformation $T(z)$, one has*

$$T(\mathrm{conv}_u\{z_1, \ldots, z_n\}) = \mathrm{conv}_{T(u)}\{T(z_1), \ldots, T(z_n)\}.$$

3.2. Relationship with critical points of polynomials

Polar convexity allows one to understand better the relationship between the zeros of a polynomial and its critical points. The famous Gauss-Lucas' theorem says that the critical points of a polynomial

$$\hat{p}(z) := (z - \hat{z}_1) \cdots (z - \hat{z}_n)$$

are located in $\mathrm{conv}\{\hat{z}_1, \ldots, \hat{z}_n\}$.

A notion more flexible than the derivative is that of a polar derivative. Denote by $\overline{\mathcal{P}}_n$, where $n \geq 0$, the set of all polynomials of degree at most n. (The degree of a polynomial that is a non-zero constant is zero and the degree of 0 is defined to be -1.) Let $\overline{\mathcal{P}}_{-1} := \{0\}$. For any $u \in \mathbb{C}^*$ and $n \geq 0$, the linear operator

$$\mathcal{D}_u : \overline{\mathcal{P}}_n \to \overline{\mathcal{P}}_{n-1}$$

defined by
$$\mathcal{D}_u(p; z) := \begin{cases} np(z) - (z-u)p'(z) & \text{if } u \neq \infty, \\ p'(z) & \text{if } u = \infty, \end{cases} \tag{12}$$

is called the *polar derivative of $p(z)$ with pole u*. Finally, if $0 \in \overline{\mathcal{P}}_{-1}$, define $\mathcal{D}_u(0; z) := 0 \in \overline{\mathcal{P}}_{-1}$. It is obvious that

$$\lim_{u \to \infty} \frac{1}{u} \mathcal{D}_u(p; z) = p'(z), \tag{13}$$

hence the zeros of $\mathcal{D}_u(p; z)$ approach the zeros of $p'(z)$ as u approaches infinity. Note that if $p(z) \in \overline{\mathcal{P}}_n$, for $n \geq 1$, is a constant polynomial (different from zero), then $\mathcal{D}_u(p; z) \in \overline{\mathcal{P}}_{n-1}$ is a non-zero constant, whenever $u \neq \infty$. Similarly, for $p(z) \in \overline{\mathcal{P}}_n$, for $n \geq 1$, we have that $\mathcal{D}_u(p; z) \equiv 0$, if and only if $p(z)$ is a constant multiple of $(z-u)^n$.

If the polynomial $p(z) \in \overline{\mathcal{P}}_n$ has actual degree $m \in \{0, 1, \ldots, n\}$, then it is convenient to consider ∞ as a zero of multiplicity $n - m$. Constant polynomials in $\overline{\mathcal{P}}_n$ have a single zero, namely ∞, of multiplicity n. The polynomial $(z-u)^m \in \overline{\mathcal{P}}_n$, for $1 \leq m < n$, has two distinct zeros: u and ∞.

The Möbius transformation (6) defines the map

$$T[\cdot] : \overline{\mathcal{P}}_n \to \overline{\mathcal{P}}_n, \quad \text{by} \quad T[p](z) := (z-u)^n p(T(z)). \tag{14}$$

It sends a polynomial $p(z) \in \overline{\mathcal{P}}_n$ with zeros $\{z_1, \dots, z_m, \infty, \dots, \infty\}$ to the polynomial $T[p](z)$ with zeros $\{T^{-1}(z_1), \dots, T^{-1}(z_m), T^{-1}(\infty), \dots, T^{-1}(\infty)\}$. If in the last n-tuple, ℓ of the numbers are infinity, then the degree of $T[p](z)$ is $n-\ell$. The connection between $\mathcal{D}$ and T is

$$\mathcal{D}_u(T[p]; z) = T[p'](z). \tag{15}$$

The verification of these claims is a straightforward calculation.

A result extending the Gauss-Lucas' theorem holds for polar derivatives.

Proposition 3.8. *Let $p(z) \in \overline{\mathcal{P}}_n$ be a polynomial of degree m and assume zeros $z_1, ..., z_n \in \mathbb{C}^*$. Let $u \in \mathbb{C}^*$. If $\mathcal{D}_u(p; z) \not\equiv 0$, then all its zeros are in* $\operatorname{conv}_u\{z_1, ..., z_n\}$.

Proof. Suppose $u = \infty$. In this case we are in the realm of the Gauss-Lucas' theorem, but we need to show that it formally holds for polynomials in $\overline{\mathcal{P}}_n$, with $n \geq 1$. Indeed, let $p(z) \in \overline{\mathcal{P}}_n$ be a polynomial of degree m. Since $\mathcal{D}_u(p; z) \not\equiv 0$, we have $m \geq 1$. So, let $z_1, \dots, z_m \in \mathbb{C}$ be its finite zeros; and let $z_{m+1} = \dots = z_n := \infty$. Then, $p'(z) \in \overline{\mathcal{P}}_{n-1}$ has $m-1$ finite zeros and ∞ is a zero of multiplicity $n-m$. Thus, all zeros of $p'(z)$ are in $\operatorname{conv}_\infty\{z_1, \dots, z_n\}$, using the extended Definition 3.4.

Suppose now $u \neq \infty$. The zeros of $\hat{p}(z) := T^{-1}[p](z)$ are $\hat{z}_i := T(z_i)$, for $i = 1, ..., n$. Let $\hat{v}_1, \dots, \hat{v}_{n-1}$ be the critical points of $\hat{p}(z)$, that is, the zeros of $\hat{p}'(z) \in \overline{\mathcal{P}}_{n-1}$. By the first part of this proof, they are contained in $\operatorname{conv}_\infty\{\hat{z}_1, \dots, \hat{z}_n\}$.

Applying (15) to $\hat{p}(z)$ and using the fact that $T[\hat{p}](z) = p(z)$, we obtain

$$\mathcal{D}_u(p; z) = \mathcal{D}_u(T[\hat{p}]; z) = T[\hat{p}'](z).$$

Since the zeros of the latter polynomial are $v_i := T^{-1}(\hat{v}_i)$ for $i = 1, \dots, n-1$, we obtain that $\{v_i : i = 1, \dots, n-1\}$ are the zeros of $\mathcal{D}_u(p; z) \in \overline{\mathcal{P}}_{n-1}$. Using (7), which holds in the extended sense of Definition 3.4, we obtain that v_i, for $i = 1, \dots, n-1$, are contained in

$$T^{-1}(\operatorname{conv}_\infty\{\hat{z}_1, \dots, \hat{z}_n\}) = \operatorname{conv}_u\{z_1, \dots, z_n\}.$$

This concludes the proof. □

Proposition 3.8, albeit being an elementary consequence of the Gauss-Lucas' theorem, strengthens another fundamental result in the geometric theory of polynomials, see [2, p. 98], stating that:

Theorem 3.9. (Laguerre) *Let $p(z)$ be a polynomial of degree $n \geq 2$ and let $u \in \mathbb{C}$. A circular domain containing the zeros of $p(z)$, but not the point u, contains all zeros of the polar derivative $\mathcal{D}_u(p; z)$.*

To see that Proposition 3.8 strengthens Laguerre's theorem, note that if D is a circular domain not containing u, then D is convex with respect to u. Hence, if D contains the zeros of $p(z)$, then $\mathrm{conv}_u\{z_1, \ldots, z_n\} \subset D$. All that justifies our interest in the set of poles of a given set A as the next self-evident result states.

Theorem 3.10. *Let $A \subset \mathbb{C}^*$ be any set and let $\mathcal{P}(A) \subset \mathbb{C}^*$ be the set of its poles. Then, for any natural n, any $z_1, \ldots, z_n \in A$, and any $u \in \mathcal{P}(A)$, the zeros of $\mathcal{D}_u(p; z)$ are in A, where $p(z) := (z - z_1) \cdots (z - z_n)$, provided that $\mathcal{D}_u(p; z) \not\equiv 0$.*

4. Properties of $\mathrm{conv}_u\{z_1, \ldots, z_n\}$

The next result shows a duality property of polar convexity.

Theorem 4.1. *Let $u, v, z_1, \ldots, z_n$ be any points in $\mathbb{C}^*$, then*

$$v \in \mathrm{conv}_u\{z_1, \ldots, z_n\} \;\; \textit{and} \;\; v \notin \{z_1, \ldots, z_n\} \;\; \textit{implies} \;\; u \in \mathrm{conv}_v\{z_1, \ldots, z_n\}.$$

Proof. Suppose for concreteness that $z_1, \ldots, z_{n'} \in \mathbb{C}$ and $z_{n'+1} = \cdots = z_n = \infty$. Let $v \in \mathrm{conv}_u\{z_1, \ldots, z_n\}$ and $v \notin \{z_1, \ldots, z_n\}$. This implies that the points $\{z_1, \ldots, z_n\}$ are not all equal. We consider several cases.

Case 1a. Suppose that $u, v \in \mathbb{C}$ and $u \notin \{z_1, \ldots, z_n\}$. Then, by (11), we see that $u \notin \mathrm{conv}_u\{z_1, \ldots, z_n\}$ and hence $v \neq u$. By definition, v can be expressed as

$$v = u + \frac{1}{\sum_{i=1}^{n} \frac{\mu_i}{z_i - u}},$$

for some $\mu_i \geq 0$, $i = 1, \ldots, n$ with $\sum_{i=1}^{n} \mu_i = 1$. Since $v \neq u$, we calculate

$$\begin{aligned} \frac{1}{v-u} &= \sum_{i=1}^{n} \frac{\mu_i}{z_i - u} = \sum_{i=1}^{n'} \frac{\mu_i}{z_i - u} = \sum_{i=1}^{n'} \frac{\mu_i(v-u)}{(z_i - u)(v-u)} = \sum_{i=1}^{n'} \frac{\mu_i(v - z_i + z_i - u)}{(z_i - u)(v-u)} \\ &= \sum_{i=1}^{n'} \frac{\mu_i(v - z_i)}{(z_i - u)(v-u)} + \sum_{i=1}^{n'} \frac{\mu_i(z_i - u)}{(z_i - u)(v-u)} = \sum_{i=1}^{n'} \frac{\mu_i(v - z_i)}{(z_i - u)(v-u)} + \frac{\sum_{i=1}^{n'} \mu_i}{v-u}. \end{aligned}$$

This shows that

$$\frac{1}{v-u} \sum_{i=1}^{n'} \frac{\mu_i(v - z_i)}{(z_i - u)} = \frac{1 - \sum_{i=1}^{n'} \mu_i}{v-u} \quad \text{or} \quad \sum_{i=1}^{n'} \frac{\mu_i(v - z_i)}{(z_i - u)} = 1 - \sum_{i=1}^{n'} \mu_i.$$

Multiplying the numerator and denominator of each term with the conjugate of its denominator results in

$$\sum_{i=1}^{n'} \frac{\mu_i(v - z_i)\big((\overline{z_i - v}) - (\overline{u - v})\big)}{|u - z_i|^2} = 1 - \sum_{i=1}^{n'} \mu_i.$$

So, we have

$$\sum_{j=1}^{n'} \frac{\mu_j (v - z_j)(\overline{z_j - v})}{|u - z_j|^2} = \Big(1 - \sum_{i=1}^{n'} \mu_i\Big) + \sum_{i=1}^{n'} \frac{\mu_i (v - z_i)(\overline{u - v})}{|u - z_i|^2},$$

or equivalently

$$-\sum_{j=1}^{n'} \frac{\mu_j |z_j - v|^2}{|u - z_j|^2} = \Big(1 - \sum_{i=1}^{n'} \mu_i\Big) + (\overline{u - v}) \sum_{i=1}^{n'} \frac{\mu_i (v - z_i)}{|u - z_i|^2}.$$

Conjugating both sides and solving for u, using that $v \notin \{z_1, \dots, z_n\}$, gives:

$$\begin{aligned} u &= v + \frac{\big(1 - \sum_{i=1}^{n'} \mu_i\big) + \sum_{j=1}^{n'} \frac{\mu_j |z_j - v|^2}{|u - z_j|^2}}{\sum_{i=1}^{n'} \frac{\mu_i (\overline{z_i - v})}{|u - z_i|^2}} \\ &= v + \frac{\big(1 - \sum_{i=1}^{n'} \mu_i\big) + \sum_{j=1}^{n'} \frac{\mu_j |z_j - v|^2}{|u - z_j|^2}}{\sum_{i=1}^{n'} \frac{\mu_i |z_i - v|^2}{|u - z_i|^2} \frac{1}{z_i - v}} = v + \frac{1}{\sum_{i=1}^{n} \frac{t_i}{z_i - v}}, \end{aligned}$$

where

$$t_i := \begin{cases} \dfrac{(\mu_i |z_i - v|^2)/(|u - z_i|^2)}{(1 - \sum_{i=1}^{n'} \mu_i) + \sum_{j=1}^{n'} (\mu_j |z_j - v|^2)/(|u - z_j|^2)} & \text{for } i = 1, \dots, n', \\ \dfrac{(1 - \sum_{i=1}^{n'} \mu_i)/(n - n')}{(1 - \sum_{i=1}^{n'} \mu_i) + \sum_{j=1}^{n'} (\mu_j |z_j - v|^2)/(|u - z_j|^2)} & \text{for } i = n' + 1, \dots, n. \end{cases} \tag{16}$$

This shows that $u \in \operatorname{conv}_v\{z_1, \dots, z_n\}$. The opposite direction, in the case when $v \notin \{z_1, \dots, z_n\}$, is symmetric.

Case 1b. Suppose that one of $u, v \in \mathbb{C}^*$ is infinity. One of them must be finite. Let $T(z)$ be a Möbius transformation sending both u and v to finite points. Then, by the previous case, $T(u) \in \operatorname{conv}_{T(v)}\{T(z_1), \dots, T(z_n)\}$ and reversing the transformation concludes the case.

Case 2. Suppose that $u \in \{z_1, \dots, z_n\}$. Then

$$\operatorname{conv}_u\{z_1, \dots, z_n\} = \{u\} \cup \operatorname{conv}_u\{z_i : z_i \neq u \text{ for } i = 1, \dots, n\}.$$

If $v = u$, then we are done for trivial reasons. Supposing $v \neq u$, implies that $v \in \operatorname{conv}_u\{z_i : z_i \neq u \text{ for } i = 1, \dots, n\}$. If $v \in \{z_i : z_i \neq u \text{ for } i = 1, \dots, n\}$, then

$$\operatorname{conv}_v\{z_1, \dots, z_n\} = \{v\} \cup \operatorname{conv}_v\{z_i : z_i \neq v \text{ for } i = 1, \dots, n\}$$

and we are done, since the last set contains u.
Now suppose finally, that $v \notin \{z_i : z_i \neq u \text{ for } i = 1, \dots, n\}$. Then, by Case 1, we see that $u \in \operatorname{conv}_v\big(\{z_i : z_i \neq u \text{ for } i = 1, \dots, n\}\big) \subset \operatorname{conv}_v\{z_1, \dots, z_n\}$. □

Corollary 4.2. (Weak duality) *Let $u, v, z_1, \dots, z_n$ be points in $\mathbb{C}^*$, satisfying $u, v \notin \{z_1, \dots, z_n\}$, then*

$$v \in \mathrm{conv}_u\{z_1, \dots, z_n\}, \quad \textit{if and only if} \quad u \in \mathrm{conv}_v\{z_1, \dots, z_n\}.$$

Corollary 4.3. *Let $u, z_1, \dots, z_n$ be any points in $\mathbb{C}^*$. The set $\mathrm{conv}_u\{z_1, \dots, z_n\}$ is unbounded, if and only if $\infty \in \{z_1, \dots, z_n\}$ or $u \in \mathrm{conv}_\infty\{z_1, \dots, z_n\}$.*

Corollary 4.4. *Let $u, v, z_1, \dots, z_n$ be points in $\mathbb{C}^*$, with $u, v \notin \{z_1, \dots, z_n\}$. Then, v is on the relative boundary of $\mathrm{conv}_u\{z_1, \dots, z_n\}$, if and only if u is on the relative boundary of $\mathrm{conv}_v\{z_1, \dots, z_n\}$.*

Proof. If v is in $\mathrm{conv}_u\{z_1, \dots, z_n\}$, then $v \neq u$ or else (11) is violated. Since $v \notin \mathrm{conv}_u\{z_1, \dots, z_n\}$ is equivalent to $u \notin \mathrm{conv}_v\{z_1, \dots, z_n\}$, it is sufficient to show that v is in the relative interior of $\mathrm{conv}_u\{z_1, \dots, z_n\}$, if and only if u is in the relative interior of $\mathrm{conv}_v\{z_1, \dots, z_n\}$.

Suppose that $u, v \in \mathbb{C}$. Then, v is in the relative interior of $\mathrm{conv}_u\{z_1, \dots, z_n\}$, if and only if $T(v)$ is in the relative interior of $\mathrm{conv}_\infty\{T(z_1), \dots, T(z_n)\}$, where $T(z) = 1/(z-u)$. By [3, Theorem 6.9], this happens, if and only if $T(v) = \sum_{i=1}^n m_i T(z_i)$, for some $m_i > 0$, $i = 1, \dots, n$ with $\sum_{i=1}^n m_i = 1$. This is equivalent to $v = u + 1/(\sum_{i=1}^n m_i/(z_i - u))$. Using (16), in the proof of Theorem 4.1 Case 1.a, one can see that $t_i > 0$ for all $i = 1, \dots, n$, implying that u is in the relative interior of $\mathrm{conv}_v\{z_1, \dots, z_n\}$.

If $v \in \mathbb{C}$ and $u = \infty$, then all $\{z_1, \dots, z_n\}$ are finite. If v is in the relative interior of $\mathrm{conv}_\infty\{z_1, \dots, z_n\}$, then either the latter set has a non-empty interior, or all points $\{z_1, \dots, z_n\}$ are on a segment. In the first case, the image of $\mathrm{conv}_\infty\{z_1, \dots, z_n\}$ under $T(z) := v + 1/(z-v)$ has a non-empty interior and ∞ is in it. In the second case, that image is a straight line with a missing open segment. Thus, ∞ is in its relative interior. This shows, that ∞ is in the relative interior of $\mathrm{conv}_v\{z_1, \dots, z_n\}$

The case $v = \infty$ and $u \in \mathbb{C}$, is handled analogously. □

Proposition 4.5. *Let $u, v, z_1, \dots, z_n$ be any points in $\mathbb{C}^*$, not on a circle, with $u \neq v$. Then, neither of the sets $\mathrm{conv}_u\{z_1, \dots, z_n\}$ and $\mathrm{conv}_v\{z_1, \dots, z_n\}$ contains the other.*

Proof. We only show that $\mathrm{conv}_u\{z_1, \dots, z_n\} \nsubseteq \mathrm{conv}_v\{z_1, \dots, z_n\}$, with the opposite non-inclusion being similar. We need to exhibit a $w \in \mathrm{conv}_u\{z_1, \dots, z_n\}$ that is not in $\mathrm{conv}_v\{z_1, \dots, z_n\}$. Note first, that the points $z_1, \dots, z_n$ cannot all be equal and the set $\{z_1, \dots, z_n\} \setminus \{u, v\}$ contains at least two distinct points.

Case 1. Suppose that $u \notin \{z_1, \dots, z_n\}$ and $v \notin \{z_1, \dots, z_n\}$. Then by (11), $v \notin \mathrm{conv}_v\{z_1, \dots, z_n\}$. If $v \in \mathrm{conv}_u\{z_1, \dots, z_n\}$, then we are done by setting $w := v$ above. So, suppose $v \notin \mathrm{conv}_u\{z_1, \dots, z_n\}$. Consider the points: $\hat{u} := T(u)$, $\hat{v} := T(v)$, and $\hat{z}_i := T(z_i)$, $i = 1, \dots, n$, where T is defined by (6). That is, $\hat{u} = \infty$, and

$$T(\mathrm{conv}_u\{z_1, \dots, z_n\}) = \mathrm{conv}_\infty\{\hat{z}_1, \dots, \hat{z}_n\}.$$

The premise of this case, implies that $\infty \notin \{\hat{z}_1, \dots, \hat{z}_n\}$, hence the latter set is the ordinary convex hull of $\{\hat{z}_1, \dots, \hat{z}_n\}$, and the finite point $\hat{v}$ is not in it. Since the points $\{\hat{z}_1, \dots, \hat{z}_n\}$ and $\hat{v}$ are not on a straight line, there is a closed half plane, with boundary passing through (at least) two distinct points from $\hat{z}_1, \dots, \hat{z}_n$, containing the convex hull of $\hat{z}_1, \dots, \hat{z}_n$, but not containing $\hat{v}$. Suppose the two points are $\hat{z}_i$ and $\hat{z}_j$. Choose any point $\hat{w}$ on the segment $(\hat{z}_i, \hat{z}_j)$, such that $\hat{w} \notin \{\hat{v}\} \cup \{\hat{z}_1, \dots, \hat{z}_n\}$. The set $\mathrm{conv}_{\hat{w}}\{\hat{z}_1, \dots, \hat{z}_n\}$, is in the same closed half plane as well. In other words, we have $\hat{w} \in \mathrm{conv}_\infty\{\hat{z}_1, \dots, \hat{z}_n\}$ and $\hat{v} \notin \mathrm{conv}_{\hat{w}}\{\hat{z}_1, \dots, \hat{z}_n\}$. Reversing the transformation T, and letting $w := T^{-1}(\hat{w})$, we have $w \in \mathrm{conv}_u\{z_1, \dots, z_n\}$ and $v \notin \mathrm{conv}_w\{z_1, \dots, z_n\}$. By Theorem 4.1, we have $w \notin \mathrm{conv}_v\{z_1, \dots, z_n\}$, concluding the argument. For reference, we note that $w \neq v$.

Case 2. Suppose that $u \notin \{z_1, \dots, z_n\}$ and $v \in \{z_1, \dots, z_n\}$. Then,

$$\mathrm{conv}_v\{z_1, \dots, z_n\} = \{v\} \cup \mathrm{conv}_v\{z_i : z_i \neq v \text{ for } i = 1, \dots, n\}.$$

Apply Case 1 to the points u, v, and $\{z_i : z_i \neq v \text{ for } i = 1, \dots, n\}$, clearly not on a circle, to conclude that there is a point $w \neq v$, such that

$$w \in \mathrm{conv}_u\{z_i : z_i \neq v \text{ for } i = 1, \dots, n\} \subset \mathrm{conv}_u\{z_1, \dots, z_n\}$$

that is not in $\mathrm{conv}_v\{z_i : z_i \neq v \text{ for } i = 1, \dots, n\}$. Since $w \neq v$, we get $w \notin \mathrm{conv}_v\{z_1, \dots, z_n\}$.

Case 3. Suppose that $u \in \{z_1, \dots, z_n\}$ and $v \notin \{z_1, \dots, z_n\}$. This case is similar to Case 1, so we use the same notation and summarize the differences. We have $v \notin \mathrm{conv}_v\{z_1, \dots, z_n\}$ and may assume $v \notin \mathrm{conv}_u\{z_1, \dots, z_n\}$. For the transformation $T_v(z) := 1/(z - v)$ with $\hat{v} = \infty$, we have

$$T_v(\mathrm{conv}_v\{z_1, \dots, z_n\}) = \mathrm{conv}_\infty\{\hat{z}_1, \dots, \hat{z}_n\},$$

the convex hull of $\{\hat{z}_1, \dots, \hat{z}_n\}$. There is a closed half plane, with boundary passing through (at least) two distinct points from $\hat{z}_i, \hat{z}_j$, containing the convex hull and $\hat{u}$ being in the interior of the half plane. Thus, the open arc $\mathrm{arc}_{\hat{u}}(\hat{z}_i, \hat{z}_j)$ is contained in the complement of the closed half plane. Any point $\hat{w}$ on it is in $\mathrm{conv}_{\hat{u}}\{\hat{z}_1, \dots, \hat{z}_n\}$ but not in $\mathrm{conv}_\infty\{\hat{z}_1, \dots, \hat{z}_n\}$. Reversing the transformation, we get $w \in \mathrm{conv}_u\{z_1, \dots, z_n\}$ and $w \notin \mathrm{conv}_v\{z_1, \dots, z_n\}$. Moreover, $w \neq v$.

Case 4. Suppose that $u \in \{z_1, \dots, z_n\}$ and $v \in \{z_1, \dots, z_n\}$. In that case we must have $n \geq 4$. Proceed as in Cases 2 and 3. □

The condition in Proposition 4.5 that the point are not on a circle is necessary. Indeed, if the points are on a circle, with $z_1 \neq z_2$ being adjacent, and with $u \neq v$ between z_1 and z_2, then $\mathrm{conv}_u\{z_1, \dots, z_n\} = \mathrm{conv}_v\{z_1, \dots, z_n\} = \mathrm{arc}_v[z_1, z_2]$.

Proposition 4.6. *Let $v, z_1, \dots, z_n$ be any points in $\mathbb{C}^*$.*

(a) *If $v, z_1, \dots, z_n$ are on the same circle C and $\{z_1, \dots, z_n\} \setminus \{v\}$ contains at least two points, then*

$$\bigcap_{\substack{u \in \mathrm{conv}_v\{z_1,\dots,z_n\} \\ u \notin \{z_1,\dots,z_n\}}} \mathrm{conv}_u\{z_1, \dots, z_n\} = \mathrm{cl}\big(C \setminus \mathrm{conv}_v\{z_1, \dots, z_n\}\big) \cup \{z_1, \dots, z_n\}.$$

(b) *If $v, z_1, \dots, z_n$ are not on the same circle, then*

$$\bigcap_{\substack{u \in \operatorname{conv}_v\{z_1,\dots,z_n\} \\ u \notin \{z_1,\dots,z_n\}}} \operatorname{conv}_u\{z_1, \dots, z_n\} = \{v\} \cup \{z_1, \dots, z_n\}.$$

Proof. (a) Without loss of generality, suppose that, the points $z_1, \dots, z_n$ are ordered clock wise on the circle C, with v being between z_1 and z_n. Then,

$$\operatorname{conv}_v\{z_1, \dots, z_n\} = \operatorname{arc}_v[z_1, z_n].$$

For any $u \in \operatorname{conv}_v\{z_1, \dots, z_n\}$, different from $z_1, \dots, z_n$, if u is between z_i and z_{i+1}, then $\operatorname{conv}_u\{z_1, \dots, z_n\} = \operatorname{arc}_u[z_i, z_{i+1}]$ contains $\operatorname{arc}[z_1, v, z_n]$ but does not contain $\operatorname{arc}(z_i, u, z_{i+1})$. The result follows from here.

(b) Using Theorem 4.1, it is easy to see that the right-hand side is contained in the intersection on the left-hand side. To show the opposite inclusion, take $x \notin \{z_1, \dots, z_n\}$, such that

$$x \in \bigcap_{\substack{u \in \operatorname{conv}_v\{z_1,\dots,z_n\} \\ u \notin \{z_1,\dots,z_n\}}} \operatorname{conv}_u\{z_1, \dots, z_n\}.$$

That is, for any $w \in \operatorname{conv}_v\{z_1, \dots, z_n\}$, such that $w \notin \{z_1, \dots, z_n\}$, we have $x \in \operatorname{conv}_w\{z_1, \dots, z_n\}$. In other words, for any $w \in \operatorname{conv}_v\{z_1, \dots, z_n\}$, such that $w \notin \{z_1, \dots, z_n\}$, we have $w \in \operatorname{conv}_x\{z_1, \dots, z_n\}$. Equivalently,

$$\operatorname{conv}_v\{z_1, \dots, z_n\} \subset \operatorname{conv}_x\{z_1, \dots, z_n\}.$$

According to Proposition 4.5, this happens only if $x = v$. □

Example 4.7. (a) $\mathcal{P}(\emptyset) = \mathbb{C}^*$ and $\mathcal{P}(\mathbb{C}^*) = \mathbb{C}^*$.

(b) Suppose the distinct points z_1, u_1, z_2, u_2 are co-circular, say positioned clockwise in this order on the circle. Then

$$\mathcal{P}(\operatorname{arc}[z_1, u_1, z_2]) = \operatorname{arc}[z_2, u_2, z_1].$$

(c) Proposition 4.5 shows that if $u, z_1, \dots, z_n$ are not on a circle, then

$$\mathcal{P}(\operatorname{conv}_u\{z_1, \dots, z_n\}) = \{u\}.$$

(d) For any closed circular domain D, we have $\mathcal{P}(D) = \operatorname{cl}(\mathbb{C}^* \setminus D)$.

(e) Let D_1 and D_2 be two closed discs with $\operatorname{int}(D_1 \cap D_2) \neq \emptyset$, and such that none contains the other. Then,

$$\mathcal{P}(D_1 \cap D_2) = \operatorname{cl}(D_1 \cup D_2)^c,$$
$$\mathcal{P}\big(\operatorname{cl}(D_1 \cup D_2)^c\big) = D_1 \cap D_2, \quad \text{and} \quad \mathcal{P}(\operatorname{cl}(D_1 \setminus D_2)) = \operatorname{cl}(D_2 \setminus D_1).$$

(f) The last example readily generalizes to n closed circular domains $D_1, \dots, D_n$, such that $D_i \cap D_j$ has non-empty interior and $D_i \not\subset D_j$ for all distinct $i, j = 1, \dots, n$. For any zero-one vector $x \in \mathbb{R}^n$, define the set

$$A_x := \{z \in \mathbb{C}^* : z \in D_i \text{ if } x_i = 1 \text{ and } z \in \operatorname{cl}(D_i^c) \text{ if } x_i = 0 \text{ for } i = 1, \dots, n\}.$$

Then, we always have $A_{e-x} \subseteq \mathcal{P}(A_x)$, where $e \in \mathbb{R}^n$ is the all-one vector.

5. Sets convex with respect to several poles

Recall that for any $A \subsetneqq \mathbb{C}^*$, we have $(\operatorname{int} A) \cap \mathcal{P}(A) = \emptyset$. Another simple observation is that for any $A, B \subset \mathbb{C}^*$, we have $\mathcal{P}(A) \cap \mathcal{P}(B) \subset \mathcal{P}(A \cap B)$. The intersection of two sets, convex with respect to u, is also convex with respect to u.

For any set $A \subseteq \mathbb{C}^*$ and $u \notin \operatorname{int} A$, denote by $\operatorname{conv}_u(A)$ the smallest set, with respect to inclusion, containing A and convex with respect to u. Analogous to Proposition 3.2, $\operatorname{conv}_u(A)$ is the intersection of all circular domains that contain A and have u on its boundary. To see that just apply the transformation (6) and use standard results from convex analysis.

Next, we consider sets that are convex with respect to more than one pole. For any $A, U \subset \mathbb{C}^*$, such that $(\operatorname{int} A) \cap U = \emptyset$, let $\operatorname{conv}_U(A)$ denote the smallest set, containing A and convex with respect to every $u \in U$. The minimality in the definition implies, that if $U \subseteq V$, then $\operatorname{conv}_U(A) \subseteq \operatorname{conv}_V(A)$. Analogously, if $A \subseteq B$, then $\operatorname{conv}_U(A) \subseteq \operatorname{conv}_U(B)$. Finally, if $T(z)$ is a non-degenerate Möbius transformation, then similarly to Lemma 3.7, one establishes $T(\operatorname{conv}_U(A)) = \operatorname{conv}_{T(U)}(T(A))$.

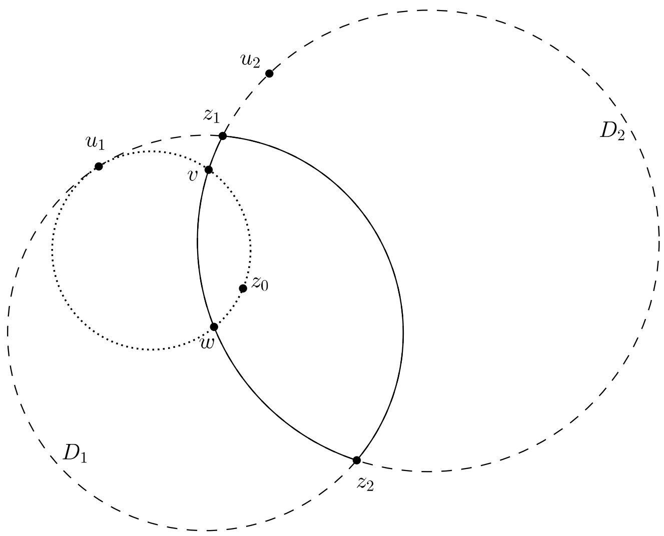

Figure 5.1: Illustrating the proof of Lemma 5.1.

Lemma 5.1. *Let z_1, z_2, u_1, u_2 be distinct points in $\mathbb{C}^*$ not on a circle. Let D_1 be the closed circular domain, such that $z_1, z_2, u_1 \in \partial D_1$ and $u_2 \notin D_1$. Similarly, let D_2 be the closed circular domain, such that $z_1, z_2, u_2 \in \partial D_2$ and $u_1 \notin D_2$. Then*

$$\operatorname{conv}_{\{u_1,u_2\}}\{z_1, z_2\} = D_1 \cap D_2 = \operatorname{conv}_{u_2}\{\operatorname{conv}_{u_1}\{z_1, z_2\}\}.$$

If z_1, z_2, u_1, u_2 are co-circular, then $\operatorname{conv}_{\{u_1,u_2\}}\{z_1, z_2\}$ *is that circle or just the arc* $\operatorname{arc}_{u_1}[z_1, z_2]$, *depending on whether $\{z_1, z_2\}$ and $\{u_1, u_2\}$ interlace or not.*

Proof. We only consider the case when both D_1 and D_2 are bounded, as shown on Figure 5.1. The other possibilities are analogous. The boundary of $D_1 \cap D_2$ is the union of the arcs $\operatorname{arc}_{u_1}[z_1, z_2]$ and $\operatorname{arc}_{u_2}[z_1, z_2]$ and is in $\operatorname{conv}_{\{u_1,u_2\}}\{z_1, z_2\}$.

Let z_0 be an arbitrary point in the interior of $D_1 \cap D_2$. There is a circle through u_1 and z_0, intersecting $\mathrm{arc}_{u_2}[z_1, z_2]$ in two points, call them v and w. (One can choose that circle to be tangent to the boundary of D_1 and contained in it.) Since v,w are in $\mathrm{conv}_{\{u_1,u_2\}}\{z_1, z_2\}$, we conclude that $z_0 \in \mathrm{arc}_{u_1}[v, w]$ is also there. This shows that $\mathrm{conv}_{\{u_1,u_2\}}\{z_1, z_2\} \supseteq D_1 \cap D_2$. The opposite inclusion and the iterative representation follow readily. $\square$

The next result shows how to construct the set $\mathrm{conv}_{\{u_1,\dots,u_m\}}\{z_1, \dots, z_n\}$ inductively.

Theorem 5.2. *Let $\{u_1, \dots, u_m\}$ and $\{z_1, \dots, z_n\}$ be non-intersecting sets in $\mathbb{C}^*$. Then*

$$\mathrm{conv}_{\{u_1,\dots,u_m\}}\{z_1, \dots, z_n\} = \mathrm{conv}_{u_m}\big\{\mathrm{conv}_{\{u_1,\dots,u_{m-1}\}}\{z_1, \dots, z_n\}\big\}. \tag{17}$$

Proof. One inclusion is easy. Indeed, we have

$$\mathrm{conv}_{\{u_1,\dots,u_m\}}\{z_1, \dots, z_n\} \supseteq \mathrm{conv}_{\{u_1,\dots,u_{m-1}\}}\{z_1, \dots, z_n\}$$

and since the former set is convex with respect to u_m, the minimality in the definition implies

$$\mathrm{conv}_{\{u_1,\dots,u_m\}}\{z_1, \dots, z_n\} \supseteq \mathrm{conv}_{u_m}\big\{\mathrm{conv}_{\{u_1,\dots,u_{m-1}\}}\{z_1, \dots, z_n\}\big\}.$$

To show the opposite inclusion, all we have to do is demonstrate that the latter set is convex with respect to u_1, and by symmetry it is convex with respect to all $u_1, \dots, u_{m-1}$. The minimality property of $\mathrm{conv}_{\{u_1,\dots,u_m\}}\{z_1, \dots, z_n\}$, then confirms the opposite inclusion.

The proposition is trivial when $m = 1$. It is also easy to see in the case $m = 2$ and $n = 2$. Note that the set $\mathrm{conv}_{\{u_1,\dots,u_m\}}\{z_1, \dots, z_n\}$ contains all crescents $\mathrm{conv}_{\{u_i,u_j\}}\{z_s, z_t\}$.

The rest is divided into several cases. Without loss of generality, we may assume that $u_m = \infty$, which means that all $z_1, \dots, z_n$ are finite. Before we continue with the case study, we consider a simplifying step.

Claim. If the proposition holds for $m = 2$ and $n-1$, then for z_n, we may assume that $z_n \notin \mathrm{conv}_{\{u_1,\infty\}}\{z_1, \dots, z_{n-1}\}$.

Proof. Since, $\mathrm{conv}_{\{u_1,\infty\}}\{z_1, \dots, z_n\}$ is the smallest set containing $z_1, \dots, z_n$ and convex with respect to u_1 and ∞, if $z_n \in \mathrm{conv}_{\{u_1,\infty\}}\{z_1, \dots, z_{n-1}\}$, we have

$$\begin{aligned} &\mathrm{conv}_{\{u_1,\infty\}}\{z_1, \dots, z_{n-1}, z_n\} \subseteq \mathrm{conv}_{\{u_1,\infty\}}\{z_1, \dots, z_{n-1}\} \\ &= \mathrm{conv}_{\infty}\big\{\mathrm{conv}_{u_1}\{z_1, \dots, z_{n-1}\}\big\} \subseteq \mathrm{conv}_{\infty}\big\{\mathrm{conv}_{u_1}\{z_1, \dots, z_{n-1}, z_n\}\big\}. \end{aligned}$$

The opposite inclusion follows from general principles, as explained in the beginning of the proof. We are done with the claim. $\square$

Note that both sets $\mathrm{conv}_{\infty}\{z_1, \dots, z_{n-1}\}$ and $\mathrm{conv}_{u_1}\{z_1, \dots, z_{n-1}\}$ are contained in $\mathrm{conv}_{\{u_1,\infty\}}\{z_1, \dots, z_{n-1}\}$, as well as every crescent $\mathrm{conv}_{\{u_1,\infty\}}\{z_i, z_j\}$.

Case 1. Let $m = 2$ and $n = 3$. Special case, when z_1, z_2, z_3 are on one line are handled using the claim. So, suppose that z_1, z_2, z_3 form a non-degenerate triangle. The lines line$[z_1, z_2]$, line$[z_1, z_3]$, and line$[z_2, z_3]$ divide the plane into seven regions. We distinguish several different sub-cases, depicted on Figure 5.2.

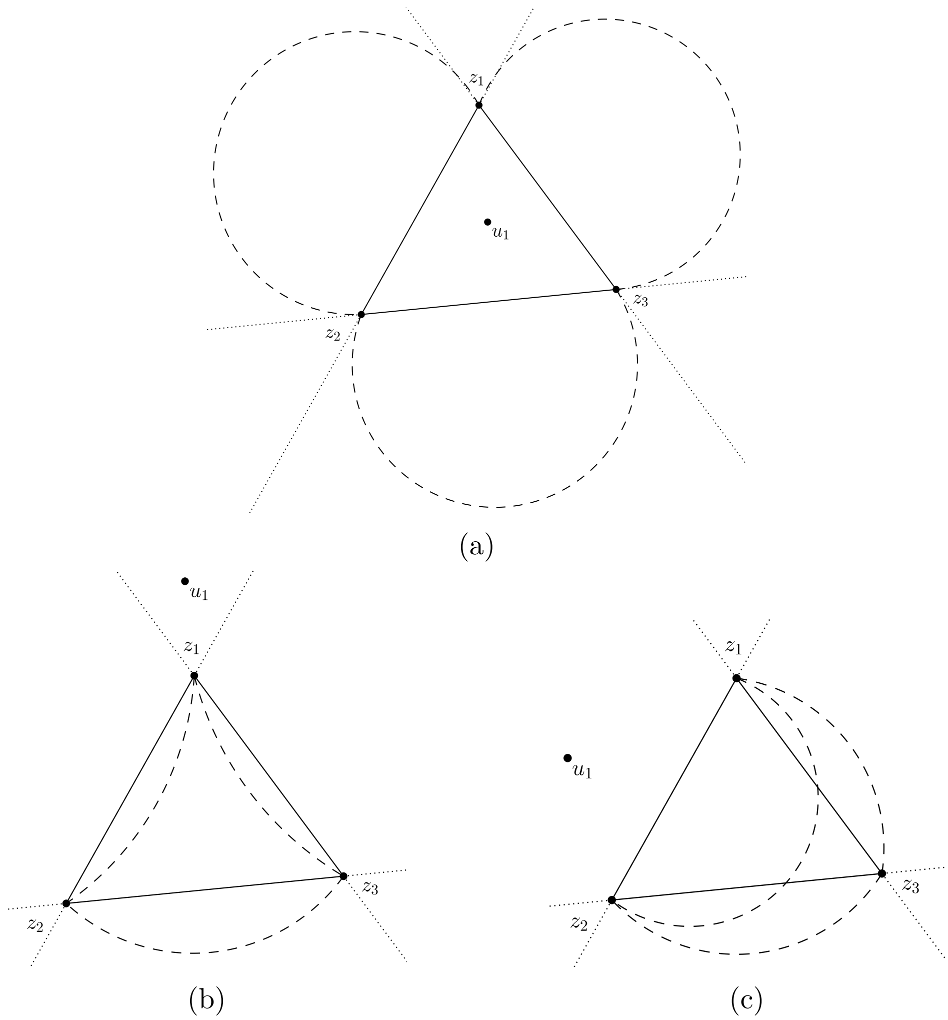

Figure 5.2: Illustrating Case 1 in the proof of Theorem 5.2.

Case 1a. Suppose that u_1 is in the triangle with vertices z_1, z_2, z_3, see Figure 5.2a, then,

$$\operatorname{conv}_{u_1}\{z_1, z_2, z_3\} = D[z_1, u_1, z_2] \cap D[z_3, u_1, z_1] \cap D[z_2, u_1, z_3]. \tag{18}$$

(This is the set outside of the dashed arcs on the figure.) If u_1 is in the interior of the triangle, then $\operatorname{conv}_\infty\{\operatorname{conv}_{u_1}\{z_1, z_2, z_3\}\} = \mathbb{C}^*$, which is convex with respect to u_1. While, if u_1 is on one of its edges, say $u_1 \in (z_1, z_2)$, then

$$\operatorname{conv}_\infty\{\operatorname{conv}_{u_1}\{z_1, z_2, z_3\}\} = D[z_1, u_1, z_2].$$

That is the closed half-plane to the left of line$[z_1, z_2]$. It is convex with respect to u_1.

Case 1b. Suppose that u_1 is in the sector with vertex z_1 bounded by the rays $\mathrm{ray}[z_1; z_1 - z_2]$ and $\mathrm{ray}[z_1; z_1 - z_3]$, see Figure 5.2b. Then,

$$\mathrm{conv}_\infty\{\mathrm{conv}_{u_1}\{z_1, z_2, z_3\}\} = \\ = \mathrm{conv}_{u_1}\{z_1, z_2, z_3\} \cup \mathrm{conv}_{\{u_1,\infty\}}\{z_1, z_2\} \cup \mathrm{conv}_{\{u_1,\infty\}}\{z_1, z_3\}.$$

(Note that if u_1 is on the $\mathrm{ray}[z_1; z_1 - z_3]$, then the formula still holds, where by Lemma 5.1 we have $\mathrm{conv}_{\{u_1,\infty\}}\{z_1, z_3\} = [z_1, z_3]$, since the points z_1, z_3 and $\{u_1, \infty\}$ do not interlace. Similarly, if u_1 is on the $\mathrm{ray}[z_1; z_1 - z_2]$.) This set is convex with respect to u_1, since it can be represented as an intersection of a closed disk and two half-planes, not containing u_1 in their interiors.

Case 1c. Suppose that u_1 is in the region bounded by the segment $[z_1, z_2]$ and the rays $\mathrm{ray}[z_1; z_1 - z_3]$ and $\mathrm{ray}[z_2; z_2 - z_3]$, see Figure 5.2c. If u_1 is inside of the disk $D[z_1, z_2, z_3]$, then z_3 is in the crescent $\mathrm{conv}_{\{u_1,\infty\}}\{z_1, z_2\}$ and we are done by the claim. So, suppose in addition, that u_1 is outside of the disk $D[z_1, z_2, z_3]$. Then

$$\mathrm{conv}_\infty\{\mathrm{conv}_{u_1}\{z_1, z_2, z_3\}\} = \mathrm{conv}_{u_1}\{z_1, z_2, z_3\} \cup \mathrm{conv}_{\{u_1,\infty\}}\{z_1, z_2\}.$$

This set is convex with respect to u_1, since it can be represented as an intersection of two closed disks and one half-plane, not containing u_1 in their interiors.

Case 2. Let $m = 2$ and $n = 4$. The claim shows that, without loss of generality, every z_1, z_2, z_3, z_4 is an extreme point, with respect to ∞, of $\mathrm{conv}_\infty\{z_1, \ldots, z_4\}$ and is an extreme point, with respect to u_1, of $\mathrm{conv}_{u_1}\{z_1, \ldots, z_4\}$. The claim also implies that z_4 is not in the sector with vertex z_2 and bounded by the rays $\mathrm{ray}[z_2; z_2 - z_1]$ and $\mathrm{ray}[z_2; z_2 - z_3]$, since then $z_2 \in \mathrm{conv}_\infty\{z_1, z_3, z_4\}$. Analogously, z_4 is not in the sector with vertex z_3 and bounded by the rays $\mathrm{ray}[z_3; z_3 - z_1]$ and $\mathrm{ray}[z_3; z_3 - z_2]$; and z_4 is not in the sector with vertex z_1 and bounded by the rays $\mathrm{ray}[z_1; z_1 - z_2]$ and $\mathrm{ray}[z_1; z_1 - z_3]$. The case when u_1 is in the quadrilateral with vertices $z_1, \ldots, z_4$ is handled analogously to Case 1a.

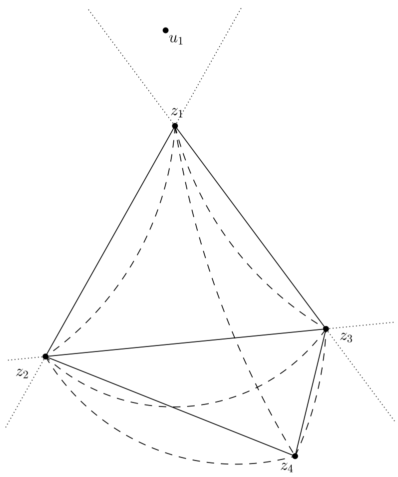

Figure 5.3: Illustrating the situation at the beginning of Case 2a in the proof of Theorem 5.2.

Case 2a. Suppose the points u_1 and z_1, z_2, z_3 are positioned as in Figure 5.2b. By the claim, we may assume that $z_4 \notin \mathrm{conv}_{\{u_1,\infty\}}\{z_1, z_2, z_3\}$.

If z_4 is in the region bounded by the segment $[z_2, z_3]$ and the rays ray$[z_2; z_2 - z_1]$ and ray$[z_3; z_3 - z_1]$, see Figure 5.3, then

$$\begin{aligned}\operatorname{conv}_\infty\{\operatorname{conv}_{u_1}\{z_1, \dots, z_4\}\} = \\ = \operatorname{conv}_\infty\{z_1, \dots, z_4\} \cup \operatorname{conv}_{\{u_1,\infty\}}\{z_2, z_4\} \cup \operatorname{conv}_{\{u_1,\infty\}}\{z_3, z_4\}.\end{aligned}$$

This set is convex with respect to u_1, since it is the intersection of two closed half-planes (not containing u_1 in their interior), namely with boundaries line$[z_1, z_2]$ and line$[z_1, z_3]$ with two disks $D[u_1, z_2, z_4]$ and $D[z_3, u_1, z_4]$.

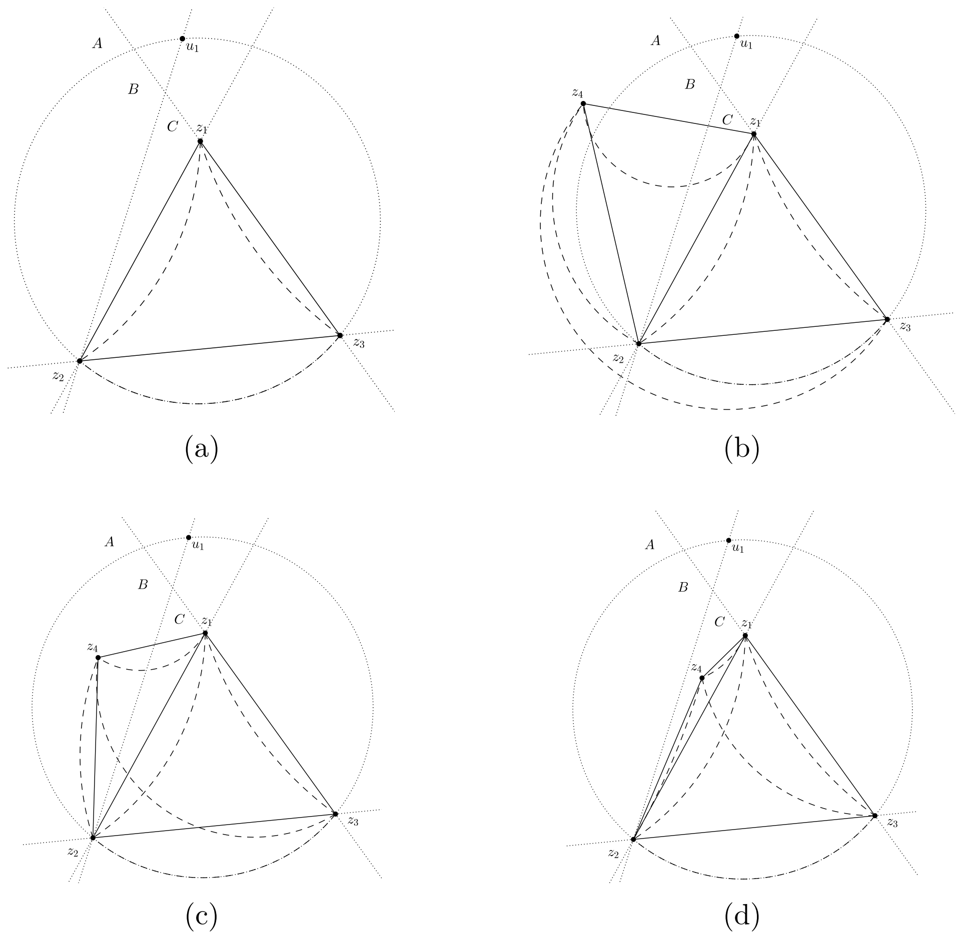

Figure 5.4: Illustrating Case 2a in the proof of Theorem 5.2.

Suppose now that z_4 is in the region bounded by the segment $[z_1, z_2]$ and the rays ray$[z_1; z_1 - z_3]$ and ray$[z_2; z_2 - z_3]$. (The case when z_4 is in the region bounded by the segment $[z_1, z_3]$ and the rays ray$[z_1; z_1 - z_2]$ and ray$[z_3; z_3 - z_2]$ is handled analogously.) The line line$[u_1, z_2]$ and the circle $C(u_1, z_2, z_3)$ divide that region into three parts, as illustrated on Figure 5.4a. We consider each part separately.

Case 2a.A. Suppose that z_4 is in part A, see Figure 5.4b. In that case z_2 is in $\operatorname{conv}_{u_1}\{z_1, z_3, z_4\}$ and we are done by the claim.

Case 2a.B. Suppose that z_4 is in part B, see Figure 5.4c. Then

$$\begin{gathered}\operatorname{conv}_\infty\{\operatorname{conv}_{u_1}\{z_1,\ldots,z_4\}\} = \\ = \operatorname{conv}_\infty\{z_1,\ldots,z_4\} \cup \operatorname{conv}_{\{u_1,\infty\}}\{z_2,z_3\} \cup \operatorname{conv}_{\{u_1,\infty\}}\{z_2,z_4\}.\end{gathered}$$

This set is convex with respect to u_1, since it is the intersection of two closed half-planes (not containing u_1 in their interior), namely with boundaries line$[z_1,z_3]$ and line$[z_1,z_4]$ with two disks $D[u_1,z_2,z_3]$ and $D[u_1,z_4,z_2]$.

Case 2a.C. Suppose that z_4 is in part C, see Figure 5.4d. Then

$$\operatorname{conv}_\infty\{\operatorname{conv}_{u_1}\{z_1,\ldots,z_4\}\} = \operatorname{conv}_\infty\{z_1,\ldots,z_4\} \cup \operatorname{conv}_{\{u_1,\infty\}}\{z_2,z_3\}.$$

This set is convex with respect to u_1, since it is the intersection of three closed half-planes (not containing u_1 in their interior), namely with boundaries line$[z_1,z_3]$, line$[z_1,z_4]$, and line$[z_2,z_4]$ with the disk $D[u_1,z_2,z_3]$.

Case 2b. Suppose the points u_1 and z_1,z_2,z_3 are positioned as in Figure 5.2c. By the claim, we may assume that $z_4 \notin \operatorname{conv}_{\{u_1,\infty\}}\{z_1,z_2,z_3\}$.

Case 2b.i. Suppose that z_4 is in the region bounded by the segment $[z_1,z_3]$ and the rays ray$[z_1;z_1-z_2]$ and ray$[z_3;z_3-z_2]$. (The case when z_4 is in the region bounded by the segment $[z_2,z_3]$ and the rays ray$[z_2;z_2-z_1]$ and ray$[z_3;z_3-z_1]$ is handled analogously.)

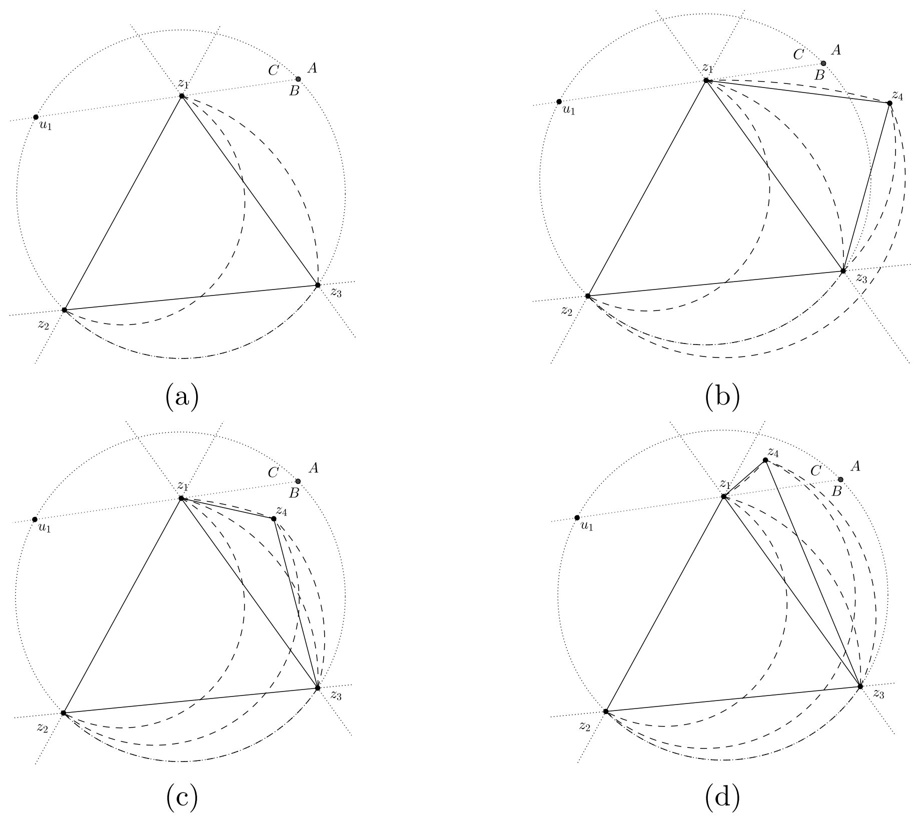

Figure 5.5: Illustrating Case 2b.i.α in the proof of Theorem 5.2.

Case 2b.i.α. Suppose, in addition, that z_1 is in the disk $D[u_1, z_2, z_3]$. Then, the line line$[u_1, z_1]$ and the circle $C(u_1, z_2, z_3)$ divide that region into three parts, as illustrated on Figure 5.5a. (The line line$[u_1, z_1]$, stops when it intersects the circle $C(u_1, z_2, z_3)$.) We consider each part separately.

Case 2b.i.α.A. Suppose that z_4 is in part A, see Figure 5.5b. In that case z_3 is in $\mathrm{conv}_{u_1}\{z_1, z_2, z_4\}$ and we are done by the claim.

Case 2b.i.α.B. Suppose that z_4 is in part B, see Figure 5.5c. Then

$$\mathrm{conv}_\infty\{\mathrm{conv}_{u_1}\{z_1, \ldots, z_4\}\} =$$
$$= \mathrm{conv}_\infty\{z_1, ..., z_4\} \cup \mathrm{conv}_{\{u_1,\infty\}}\{z_1, z_4\} \cup \mathrm{conv}_{\{u_1,\infty\}}\{z_2, z_3\} \cup \mathrm{conv}_{\{u_1,\infty\}}\{z_3, z_4\}.$$

This set is convex with respect to u_1, since it is the intersection of one closed half-planes (not containing u_1 in their interior), namely with boundary line$[z_1, z_2]$ with three disks $D[z_1, u_1, z_4]$, $D[u_1, z_2, z_3]$, and $D[u_1, z_3, z_4]$.

Case 2b.i.α.C. Suppose that z_4 is in part C, see Figure 5.5d. Then

$$\mathrm{conv}_\infty\{\mathrm{conv}_{u_1}\{z_1, \ldots, z_4\}\} =$$
$$= \mathrm{conv}_\infty\{z_1, \ldots, z_4\} \cup \mathrm{conv}_{\{u_1,\infty\}}\{z_2, z_3\} \cup \mathrm{conv}_{\{u_1,\infty\}}\{z_3, z_4\}.$$

This set is convex with respect to u_1, since it is the intersection of two closed half-planes (not containing u_1 in their interior), namely with boundaries line$[z_1, z_2]$, line$[z_1, z_4]$ with two disks $D[u_1, z_2, z_3]$ and $D[u_1, z_3, z_4]$.

Case 2b.i.β. We are still in the case, where z_4 is in the region bounded by the segment $[z_1, z_3]$ and the rays ray$[z_1; z_1 - z_2]$ and ray$[z_3; z_3 - z_2]$. Suppose, in addition, that z_1 is outside of the disk $D[u_1, z_2, z_3]$. Then, the line line$[u_1, z_1]$ and the circle $C(u_1, z_2, z_1)$ divide that region into three parts, as illustrated on Figure 5.6a. We consider each part separately.

Case 2b.i.β.A. Suppose that z_4 is in part A, see Figure 5.6b. In that case

$$\mathrm{conv}_\infty\{\mathrm{conv}_{u_1}\{z_1, \ldots, z_4\}\} = \mathrm{conv}_{\{u_1,\infty\}}\{z_1, z_2\},$$

which is convex with respect to u_1.

Case 2b.i.β.B. Suppose that z_4 is in part B, see Figure 5.6c. In that case

$$\mathrm{conv}_\infty\{\mathrm{conv}_{u_1}\{z_1, \ldots, z_4\}\} =$$
$$= \mathrm{conv}_\infty\{z_1, \ldots, z_4\} \cup \mathrm{conv}_{\{u_1,\infty\}}\{z_1, z_4\} \cup \mathrm{conv}_{\{u_1,\infty\}}\{z_2, z_4\}.$$

This set is convex with respect to u_1, since it is the intersection of a closed hyperplane with boundary line$[z_1, z_2]$ (not containing u_1 in its interior) and the disks $D[u_1, z_4, z_1]$, $D[u_1, z_2, z_4]$.

Case 2b.i.β.C. Suppose that z_4 is in part C, see Figure 5.6d. In that case

$$\mathrm{conv}_\infty\{\mathrm{conv}_{u_1}\{z_1, \ldots, z_4\}\} = \mathrm{conv}_\infty\{z_1, \ldots, z_4\} \cup \mathrm{conv}_{\{u_1,\infty\}}\{z_2, z_4\}.$$

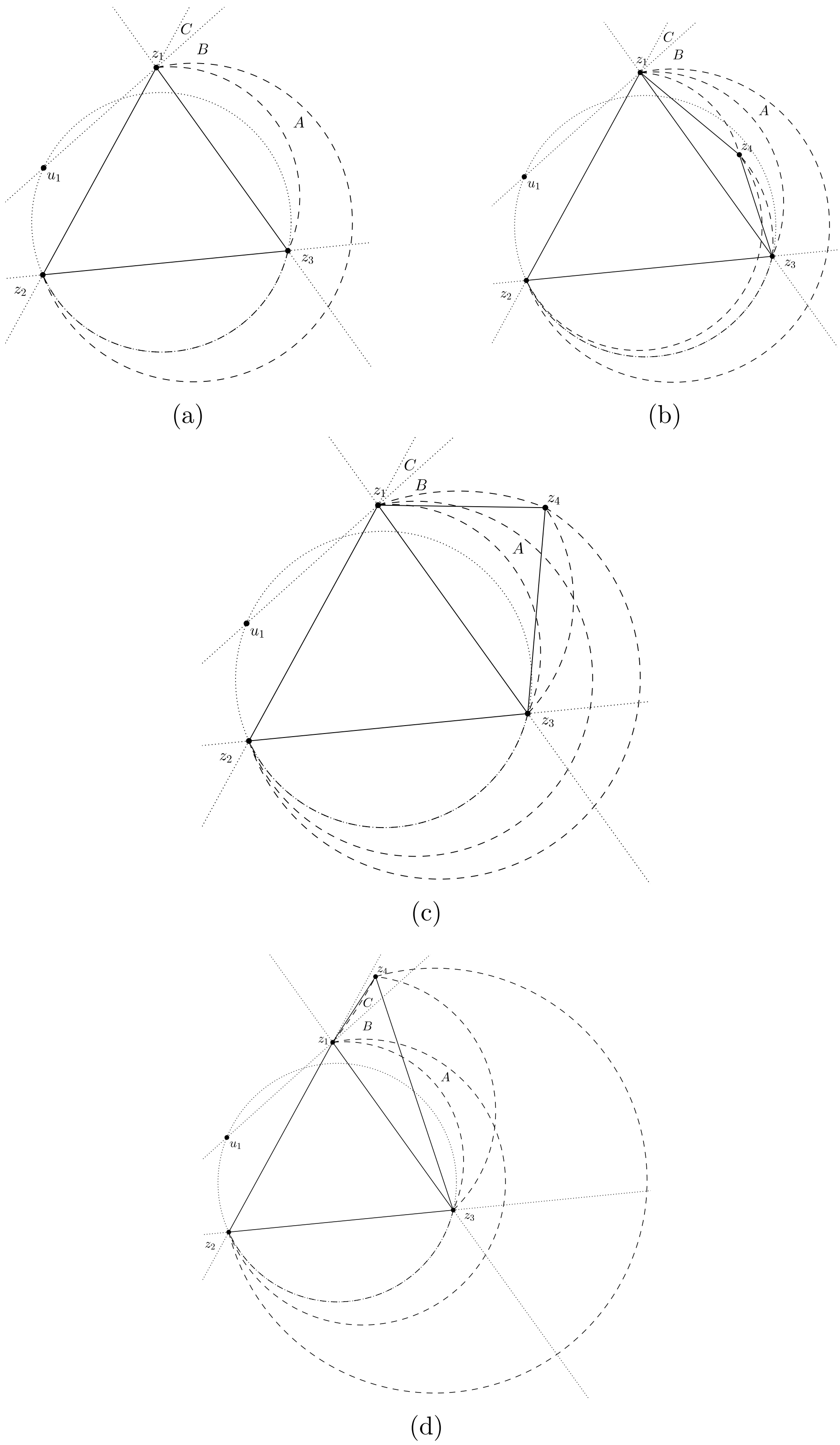

Figure 5.6: Illustrating Case 2b.i.β in the proof of Theorem 5.2.

This set is convex with respect to u_1, since it is the intersection of the closed hyperplanes with boundary line$[z_1, z_2]$ and line$[z_1, z_4]$ (not containing u_1 in its interior) and the disk $D[u_1, z_2, z_4]$.

Case 2b.ii. Suppose that z_4 is in the region bounded by the segment $[z_1, z_2]$ and the rays ray$[z_1; z_1 - z_3]$ and ray$[z_2; z_2 - z_3]$, see Figure 5.7. This case falls into those considered above, after one relabels the points z_1, z_2, z_3, z_4 as z_3, z_2, z_4, z_1.

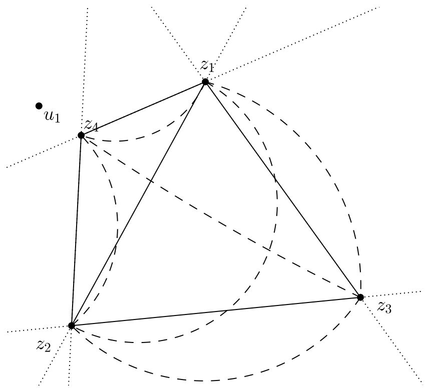

Figure 5.7: Illustrating Case 2b.ii in the proof of Theorem 5.2.

Case 3. Let $m = 2$ and $n \geq 5$. Recall, that the goal is to show that the set $\mathrm{conv}_\infty\big\{\mathrm{conv}_{u_1}\{z_1, \dots, z_n\}\big\}$ is convex with respect to u_1. Suppose that this holds for $m = 2$ poles and $n - 1$ points. By the claim, we may assume that $z_1, \dots, z_n$ are all extreme points of $\mathrm{conv}_\infty\{z_1, \dots, z_n\}$. Let a, b be any points in $\mathrm{conv}_\infty\big\{\mathrm{conv}_{u_1}\{z_1, \dots, z_n\}\big\}$. It is not difficult to see, that there is a subset of $\{z_1, \dots, z_n\}$, consisting of $n - 1$ points, say $\{z_1, \dots, z_{n-1}\}$, such that $a, b \in \mathrm{conv}_\infty\{z_1, \dots, z_{n-1}\}$. (Here we use the condition $n \geq 5$.) By the induction hypothesis

$$\mathrm{arc}_{u_1}[a, b] \subset \mathrm{conv}_\infty\big\{\mathrm{conv}_{u_1}\{z_1, \dots, z_{n-1}\}\big\} \subset \mathrm{conv}_\infty\big\{\mathrm{conv}_{u_1}\{z_1, \dots, z_n\}\big\}.$$

Case 4. Finally, consider the general case, $m \geq 2$ and $n \geq 2$. The goal is to show that the set $\mathrm{conv}_\infty\big\{\mathrm{conv}_{\{u_1,\dots,u_{m-1}\}}\{z_1, \dots, z_n\}\big\}$ is convex with respect to u_1. Let a, b be any points in the convex hull of $\mathrm{conv}_{\{u_1,\dots,u_{m-1}\}}\{z_1, \dots, z_n\}$. Then, there are six points

$$v_1, \dots, v_6 \in \mathrm{conv}_{\{u_1,\dots,u_{m-1}\}}\{z_1, \dots, z_n\},$$

such that $a \in \mathrm{conv}_\infty\{v_1, v_2, v_3\}$ and $b \in \mathrm{conv}_\infty\{v_4, v_5, v_6\}$. (By Carathéodory's theorem in dimension two, see [3, Theorem 17.1].) Now, Case 3 shows that

$$\mathrm{arc}_{u_1}[a, b] \subset \mathrm{conv}_\infty\big\{\mathrm{conv}_{u_1}\{v_1, \dots, v_6\}\big\}$$

and we are done, since

$$\mathrm{conv}_\infty\big\{\mathrm{conv}_{u_1}\{v_1, \dots, v_6\}\big\} \subset \mathrm{conv}_\infty\big\{\mathrm{conv}_{\{u_1,\dots,u_{m-1}\}}\{z_1, \dots, z_n\}\big\}.$$

The proof of the theorem is complete. □

The next corollary can be used as the basis of an algorithm for constructing polar convex hulls with several poles.

Corollary 5.3. *Let* $\{u_1, \ldots, u_m\}$ *and* $\{z_1, \ldots, z_n\}$ *be non-intersecting sets in* $\mathbb{C}^*$. *Then* $\mathrm{conv}_{\{u_1,\ldots,u_m\}}\{z_1, \ldots, z_n\} = \mathrm{conv}_{u_m}\{\mathrm{conv}_{u_{m-1}}\{\cdots \mathrm{conv}_{u_1}\{z_1, \ldots, z_n\}\}\}$. *The result does not depend on the order in which the convex hulls are taken.*

Corollary 5.4. *Let* $\{u_1, \ldots, u_m\}$ *and* $\{z_1, \ldots, z_n\}$ *be non-intersecting sets in* $\mathbb{C}^*$. *Then the set* $\mathrm{conv}_{\{u_1,\ldots,u_m\}}\{z_1, \ldots, z_n\}$ *has empty interior, if and only if the points* $\{u_1, \ldots, u_m\}$ *and* $\{z_1, \ldots, z_n\}$ *are on a circle.*

Proof. Apply the Möbius transformation $T(z) = 1/(z - u_m)$ to both sides of (17) to obtain

$$T(\mathrm{conv}_{\{u_1,\ldots,u_m\}}\{z_1, \ldots, z_n\}) = \mathrm{conv}_{\infty}\big\{\mathrm{conv}_{\{T(u_1),\ldots,T(u_{m-1})\}}\{T(z_1), \ldots, T(z_n)\}\big\}.$$

The latter set has empty interior, if and only if $\mathrm{conv}_{\{T(u_1),\ldots,T(u_{m-1})\}}\{T(z_1), \ldots, T(z_n)\}$ is contained in a line. This implies that $T(u_1), \ldots, T(u_{m-1})$ are also on that line. □

Theorem 5.5. *Let* $\{u_1, \ldots, u_m\}$ *and* $\{z_1, \ldots, z_n\}$ *be non-intersecting sets in* $\mathbb{C}^*$, *such that* $\mathrm{conv}_{\{u_1,\ldots,u_m\}}\{z_1, \ldots, z_n\}$ *has non-empty interior. Then, the boundary of* $\mathrm{conv}_{\{u_1,\ldots,u_m\}}\{z_1, \ldots, z_n\}$ *is a simple curve, made of finitely many circular arcs. Each arc has the following properties:*

(a) *it connects two points from* $\{z_1, \ldots, z_n\}$ *and is a part of the boundary of a closed circular domain* D, *containing all* $\{z_1, \ldots, z_n\}$;

(b) *the boundary of* D *contains a pole* u_i *but* $u_j \notin \mathrm{int} D$ *for all* $j \neq i$;

(c) *the set* $\mathrm{conv}_{\{u_1,\ldots,u_m\}}\{z_1, \ldots, z_n\}$ *is equal to the intersection of all such closed circular domains* D.

Proof. The proof is by induction on m. The case $m = 1$ is easily seen by applying the Möbius transformation $T(z) = 1/(z - u_1)$ to the set $\mathrm{conv}_{\{u_1\}}\{z_1, \ldots, z_n\}$, turning it convex, for which the corollary is trivial, then reversing the transformation. Assuming the theorem holds for $m - 1$, apply the Möbius transformation $T(z) = 1/(z - u_m)$ to both sides of (17) to obtain

$$T(\mathrm{conv}_{\{u_1,\ldots,u_m\}}\{z_1, \ldots, z_n\}) = \mathrm{conv}_{\infty}\big\{\mathrm{conv}_{\{T(u_1),\ldots,T(u_{m-1})\}}\{T(z_1), \ldots, T(z_n)\}\big\}.$$

The induction hypothesis applies to $\mathrm{conv}_{\{T(u_1),\ldots,T(u_{m-1})\}}\{T(z_1), \ldots, T(z_n)\}$. To simplify the notation, we omit T and show $\mathrm{conv}_{\infty}\big\{\mathrm{conv}_{\{u_1,\ldots,u_{m-1}\}}\{z_1, \ldots, z_n\}\big\}$ satisfies the description in the statement. (Then, one reverses the Möbius transformation to complete the argument.) Since the latter set has non-empty interior, the points $\{u_1, \ldots, u_{m-1}\}$ and $\{z_1, \ldots, z_n\}$ are not all collinear.

Case 1. Suppose that $n = 2$. If $m = 2$, then $\mathrm{conv}_{u_1}\{z_1, z_2\} = \mathrm{arc}_{u_1}[z_1, z_2]$ and its convex hull is a circular segment that satisfies the conditions of the statement. So, suppose $m \geq 3$, then there are poles that are not on the line $\mathrm{line}[z_1, z_2]$. If all poles are on one side of $\mathrm{line}[z_1, z_2]$, then $\mathrm{conv}_{\{u_1,\ldots,u_{m-1}\}}\{z_1, z_2\}$ is the lens formed by the two outermost arcs $\mathrm{arc}_{u_i}[z_1, z_2]$, $i = 1, \ldots, m-1$. Its convex hull is a circular segment again. If there are poles on each side of line$[z_1, z_2]$, then $\mathrm{conv}_{\{u_1,\ldots,u_{m-1}\}}\{z_1, z_2\}$ is a convex lens.

Case 2. Suppose that $n \geq 3$. According to the induction hypothesis, the set $\mathrm{conv}_{\{u_1,\ldots,u_{m-1}\}}\{z_1, \ldots, z_n\}$ is equal to the intersection of all closed circular domains D such that a) $\{z_1, \ldots, z_n\} \subset D$; b) some u_i is in ∂D and $u_j \notin \mathrm{int}D$ for all $j \neq i$; and c) $\mathrm{arc}_{u_i}[z_j, z_k] = \partial D \cap \partial(\mathrm{conv}_{\{u_1,\ldots,u_{m-1}\}}\{z_1, \ldots, z_n\})$ for some z_j and z_k.

We now focus on the convex hull of $\mathrm{conv}_{\{u_1,\ldots,u_{m-1}\}}\{z_1, \ldots, z_n\}$. It is the intersection of all closed half-planes whose boundary has a point in common with the boundary of $\mathrm{conv}_{\{u_1,\ldots,u_{m-1}\}}\{z_1, \ldots, z_n\}$. Let H be such a closed half-plane.

Case 2a. Suppose that the boundary of H is tangential to the boundary of $\mathrm{conv}_{\{u_1,\ldots,u_{m-1}\}}\{z_1, \ldots, z_n\}$ at a point w. By the induction hypothesis w is on an arc $\mathrm{arc}_{u_i}[z_1, z_2]$, which is a part of the boundary of a closed circular domain D, where z_1 and z_2 are distinct and on ∂D.

Suppose that D is unbounded. Let z_3 be distinct from z_1 and z_2. Consider the lines line$[u_i, z_1]$ and line$[u_i, z_2]$. The situation is illustrated on Figure 5.8. Point z_3 cannot be on the same side of the line line$[u_i, z_1]$ as $\mathrm{arc}_{u_i}[z_1, z_2]$, since then $\mathrm{arc}_{u_i}[z_1, z_3]$ is not contained entirely in the domain H. (If $w = z_1$, then no matter where z_3 is, $\mathrm{arc}_{u_i}[z_1, z_3]$ is not contained entirely in H.) Similarly with line$[u_i, z_2]$. Hence, z_3 needs to be in the sector with vertex u_i opposite from $\mathrm{arc}_{u_i}[z_1, z_2]$. Then, the segments $[z_1, z_3]$ and $[z_2, z_3]$ are in $\mathrm{conv}_{\{u_1,\ldots,u_m\}}\{z_1, \ldots, z_n\}$. At least one of these segments contains a point that, together with w are on the same side with respect to one of the lines line$[u_i, z_1]$ and line$[u_i, z_2]$. This is a contradiction.

This shows that D is bounded, hence convex. Since $\mathrm{conv}_{\{u_1,\ldots,u_{m-1}\}}\{z_1, \ldots, z_n\}$ is contained in D, the half-plane H does not contribute to the convex hull of that set. We can discard it.

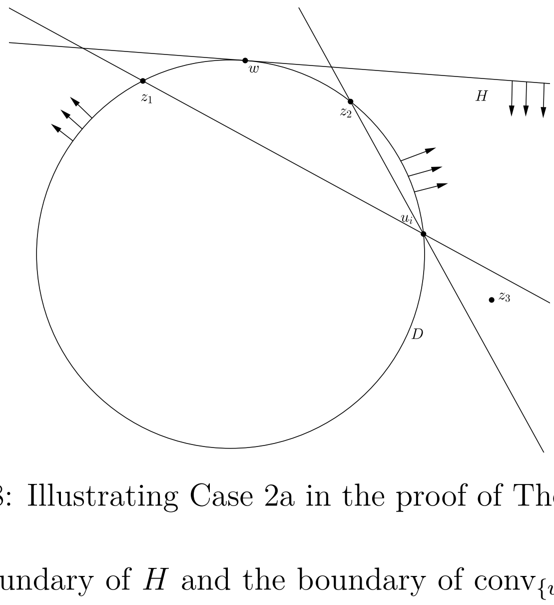

Figure 5.8: Illustrating Case 2a in the proof of Theorem 5.5.

Case 2b. If the boundary of H and the boundary of $\mathrm{conv}_{\{u_1,\ldots,u_{m-1}\}}\{z_1, \ldots, z_n\}$ are not tangential, then one can rotate H around the common boundary point, until the boundaries become tangential (in which case we discard H by the previous case), or the two boundaries touch at more than one point. These points must be sharp (or else we are in the previous case) and by the induction hypothesis are in the set $\{z_1, \ldots, z_n\}$. Say z_1 and z_2 are two of them. The half plane H cannot contain a pole u_j, for $j \in \{1, \ldots, m-1\}$, in its interior or else the arc $\mathrm{arc}_{u_j}[z_1, z_2]$ is in $\mathrm{conv}_{\{u_1,\ldots,u_{m-1}\}}\{z_1, \ldots, z_n\}$ but not in H. Thus, the closed circular domain H satisfies properties a) and b) in the statement of the theorem and $\mathrm{conv}_{\{u_1,\ldots,u_m\}}\{z_1, \ldots, z_n\}$ is equal to the intersection of all such closed circular domains. The induction is complete. $\square$

The proof of Theorem 5.2, Case 1, gives a description of $\mathrm{conv}_{\{u_1,u_2\}}\{z_1, z_2, z_3\}$ as intersection of closed circular domains. The next corollary generalizes that representation, giving a geometric description of $\mathrm{conv}_{\{u_1,\ldots,u_m\}}\{z_1, \ldots, z_n\}$. It also extends Lemma 5.1 to more than one pole.

Corollary 5.6. *Let $\{u_1, \ldots, u_m\}$ and $\{z_1, \ldots, z_n\}$ be non-intersecting sets in $\mathbb{C}^*$, such that $\mathrm{conv}_{\{u_1,\ldots,u_m\}}\{z_1, \ldots, z_n\}$ has non-empty interior. For each $i = 1, \ldots, m$, define the set of closed circular domains*

$$\mathcal{L}_i := \big\{D : u_i \in \partial D \text{ and } u_j \notin \mathrm{int}D \text{ for all } j \neq i, j = 1, \ldots, m,$$
$$\{z_1, \ldots, z_n\} \subset D \text{ with } |\{z_1, \ldots, z_n\} \cap \partial D| \geq 2\big\}.$$

Then
$$\mathrm{conv}_{\{u_1,\ldots,u_m\}}\{z_1, \ldots, z_n\} = \bigcap_{i=1}^{m} \bigcap_{D \in \mathcal{L}_i} D. \tag{19}$$

If $\mathcal{L}_i = \emptyset$, for some i, then we use the convention $\bigcap_{D\in\mathcal{L}_i} D = \mathbb{C}^*$.

Proof. The set on the right-hand side of (19) is convex with respect to every u_i, $i = 1, \ldots, m$ and that it contains $\{z_1, \ldots, z_n\}$. Therefore, it contains the smallest set with those properties, showing that

$$\mathrm{conv}_{\{u_1,\ldots,u_m\}}\{z_1, \ldots, z_n\} \subseteq \bigcap_{i=1}^{m} \bigcap_{D \in \mathcal{L}_i} D.$$

The opposite inclusion follows from Theorem 5.4. $\square$

Corollary 5.7. *Let $\{u_1, \ldots, u_m\}$ and $\{z_1, \ldots, z_n\}$ be non-intersecting sets in $\mathbb{C}^*$, such that $\mathrm{conv}_{\{u_1,\ldots,u_m\}}\{z_1, \ldots, z_n\}$ has non-empty interior. Then*

$$\mathrm{conv}_{\{z_1,\ldots,z_n\}}\{u_1, \ldots, u_m\} \subseteq \mathcal{P}\big(\mathrm{conv}_{\{u_1,\ldots,u_m\}}\{z_1, \ldots, z_n\}\big) \subseteq \bigcap_{i=1}^{m} \bigcap_{D \in \mathcal{L}_i^m} \mathrm{cl}(D^c). \tag{20}$$

Proof. For any $D \in \mathcal{L}_i$, we have $\{u_1, \ldots, u_m\} \subset \mathrm{cl}(D^c)$ and $\{z_1, \ldots, z_n\} \cap D^c = \emptyset$. Hence, $\mathrm{cl}(D^c)$ is convex with respect to all $z_1, \ldots, z_n$, implying that

$$\mathrm{conv}_{\{z_1,\ldots,z_n\}}\{u_1, \ldots, u_m\} \subset \mathrm{cl}(D^c).$$

In other words, for any $v \in \mathrm{conv}_{\{z_1,\ldots,z_n\}}\{u_1, \ldots, u_m\}$, we have $v \notin \mathrm{int}\, D$. This implies that v is a pole of $\mathrm{conv}_{\{u_1,\ldots,u_m\}}\{z_1, \ldots, z_n\}$ or the first inclusion in (20).

Suppose now that v is a pole of $\mathrm{conv}_{\{u_1,\ldots,u_m\}}\{z_1,\ldots,z_n\}$. This means that for any two points a and b from that set, the arc $\mathrm{arc}_v[a,b]$ is in it as well. Let $D \in \mathcal{L}_i$ and let z_k and z_ℓ be two distinct points on the boundary of D. Since

$$\mathrm{arc}_v[z_k, z_\ell] \subset \mathrm{conv}_{\{u_1,\ldots,u_m\}}\{z_1,\ldots,z_n\} \subset D,$$

we conclude that $v \in \mathrm{cl}(D^c)$. $\square$

The first part of the next corollary follows from the definitions, while the second part uses inclusion (20).

Corollary 5.8. *If* $z_n \in \mathrm{conv}_{\{u_1,\ldots,u_m\}}\{z_1,\ldots,z_{n-1}\}$, *then*

$$\mathrm{conv}_{\{u_1,\ldots,u_m\}}\{z_1,\ldots,z_{n-1},z_n\} = \mathrm{conv}_{\{u_1,\ldots,u_m\}}\{z_1,\ldots,z_{n-1}\}.$$

If $u_m \in \mathrm{conv}_{\{z_1,\ldots,z_n\}}\{u_1,\ldots,u_{m-1}\}$, *then*

$$\mathrm{conv}_{\{u_1,\ldots,u_m\}}\{z_1,\ldots,z_n\} = \mathrm{conv}_{\{u_1,\ldots,u_{m-1}\}}\{z_1,\ldots,z_n\}.$$

Theorem 5.9. *For any* $A \subset \mathbb{C}^*$, *we have* $A \subseteq \mathcal{P}(\mathcal{P}(A))$.

Proof. We first take care of a few trivial cases. If A is empty, then $\mathcal{P}(A) = \mathbb{C}^*$ and $\mathcal{P}(\mathcal{P}(A)) = \mathcal{P}(\mathbb{C}^*) = \mathbb{C}^*$. Hence, $A \subseteq \mathcal{P}(\mathcal{P}(A))$. If A contains one point only, then $\mathcal{P}(A) = \mathbb{C}^*$ and the result holds again. If $\mathcal{P}(A)$ is empty, then $\mathcal{P}(\mathcal{P}(A)) = \mathbb{C}^*$ contains A. If $\mathcal{P}(A)$ contains only one point, then gain $\mathcal{P}(\mathcal{P}(A)) = \mathbb{C}^*$.

Thus, suppose that A and $\mathcal{P}(A)$ both contain at least two points. Fix $z_1 \in A$ and distinct $u_1, u_2 \in \mathcal{P}(A)$ arbitrarily. We have to show that z_1 is a pole of $\mathcal{P}(A)$, that is, $\mathrm{arc}_{z_1}[u_1,u_2] \subseteq \mathcal{P}(A)$. In other words, we need to show that for every $v \in \mathrm{arc}_{z_1}[u_1,u_2]$ and any $z_2, z_3 \in A$, we have $\mathrm{arc}_v[z_2,z_3] \subseteq A$. The case when $\{u_1,u_2\}$ and $\{z_1,z_2,z_3\}$ are all on a circle can be verified by considering several trivial cases. We omit the details and assume that the five points are not on a circle. Using (20), we obtain

$$v \in \mathrm{arc}_{z_1}[u_1,u_2] \subseteq \mathrm{conv}_{\{z_1,z_2,z_3\}}\{u_1,u_2\} \subseteq \mathcal{P}(\mathrm{conv}_{\{u_1,u_2\}}\{z_1,z_2,z_3\}).$$

Since u_1 and u_2 are poles of A and $z_1, z_2, z_3 \in A$, we conclude that

$$\mathrm{conv}_{\{u_1,u_2\}}\{z_1,z_2,z_3\} \subseteq A$$

and the proof is complete. $\square$

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