

The Minimizing Vector Theorem in Symmetrized Max-Plus Algebra

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Assuming **ZF** and its consistency, we study some topological and geometrical properties of the symmetrized max-plus algebra in the absence of the axiom of choice in order to discuss the minimizing vector theorem for finite products of copies of the symmetrized max-plus algebra. Several relevant statements that follow from the axiom of countable choice restricted to sequences of subsets of the real line are shown. Among them, it is proved that if all simultaneously complete and connected subspaces of the plane are closed, then the real line is sequential. A brief discussion about semidenrites is included. Older known proofs in **ZFC** of several basic facts relevant to proximal and Chebyshev sets in metric spaces are replaced by new proofs in **ZF**. It is proved that a nonempty subset C of the symmetrized max-plus algebra is Chebyshev in this algebra if and only if C is simultaneously closed and connected. An application of it to a version of the minimizing vector theorem for finite products of the symmetrized max-plus algebra is shown. Open problems concerning some statements independent of **ZF** and other statements relevant to Chebyshev sets are posed.

Keywords: Symmetrized max-plus algebra, metric, complete metric, Cantor complete metric, proximal set, Chebyshev set, convexity, geometric convexity, minimizing vector theorem, semidendrite, ZF, axiom of countable choice for the real line, independence results.

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1. Introduction

The Max-Plus Algebra (shortly: MPA), known also as the max-plus dioid ([10]) or the tropical semiring ([22]), is the idempotent semiring (see Definition 1 in [17]) over the union of the set of all real numbers and one-point set consisting of

$\varepsilon = -\infty$, equipped with the maximum (as the addition) and the addition (as the multiplication). The zero and the unit of this semiring are respectively ε and 0.

Despite the remarkable analogy between these two basic operations in MPA and the conventional algebra, the principal difference reflects in missing of inverse elements related to the first operation: $\max$. A partial remedy is the Symmetrized Max-Plus Algebra (shortly: SMPA) which is an extension of MPA built by a method similar to that used in extending a given set of all natural numbers satisfying Peano's postulates to a set that can be called the set of all integers.

Compared to conventional linear algebra and linear system theory, the linear max-plus algebra is far from fully developed, even though many papers and textbooks were written on the topics of matrix calculations, among them, relevant to different types of decomposition, spectral theory, linear systems and linear optimization. Some important papers dealing with these subjects also were written in SMPA; for instance, let us refer to [21].

In this paper, we give some topological and geometrical properties of SMPA in order to start a deeper investigation of the minimizing vector theorem in products of SMPA and point out new open problems relevant to Chebyshev sets.

Although most mathematicians seem to assume **ZFC** as the axiomatic foundation, we realize that the axiom of choice (**AC**) is not needed to obtain the main results of our work. Therefore, the basic set-theoretic assumption of this study is **ZF**, formulated in [16], without any form of the axiom of choice. If it occurs necessary, we also use **AC** or another axiom which is a weaker form of **AC**; however, we clearly inform which statements in our article are independent of **ZF**. We use the notation of [12] for **ZF** and axioms relevant to the axiom of choice. All topological notions not defined here can be found in [5], [12] and [18]. The book [14] is the most useful source of information about validity and falsity in models of **ZF**.

Our paper is organized as follows. Section 2 concerns basic properties of d -proximal and d -Chebyshev sets in an arbitrary metric space (X, d) . New proofs in **ZF** to several facts on d -proximality are shown. The proofs are more subtle than the already known ones in **ZFC**. We give a brief introduction about calculation in MPA and the construction of max-plus algebra of pairs (MPAP) in Subsection 3.1. In Subsection 3.2, we give a short presentation of the construction of SMPA from MPAP and some remarks about computation in SMPA. In Section 4, we define equivalent Euclidean and inner metrics in the set $\mathbb{S}$ of all points of SMPA. Equipped with the natural topology induced by the inner metric, the space $\mathbb{S}$ is a semidendrite which is embeddable in the complex plane $\mathbb{C}$. Some properties of semidendrites in **ZF** are shown. Section 5 concerns finite products $\mathbb{S}^n$ of $\mathbb{S}$. Natural metrics in $\mathbb{S}^n$ are considered. A notion of a geometric segment in $\mathbb{S}^n$ is introduced and it is shown that geometric segments can be identified with suitable broken lines in $\mathbb{C}^n$. Section 6 is about completeness and Cantor-completeness of the metrics in $\mathbb{S}^n$ defined in Section 5. A few set-theoretic open problems are posed. New equivalents of the independent of **ZF** statement that $\mathbb{R}$ is sequential are found. In Section 7, we introduce the structure of a semimodule in $\mathbb{S}^n$. Section 8 is devoted to distinct concepts of convex sets in $\mathbb{S}^n$. The final Section 9 is

about the minimizing vector theorem in $\mathbb{S}^n$. We give here a complete, satisfactory characterization of Chebyshev sets in $\mathbb{S}$. We also investigate finite products of Chebyshev sets in $\mathbb{S}$. We pose open problems on the minimizing vector theorem. Our paper is a preparation to deeper investigations of Chebyshev sets in $\mathbb{S}^n$ in the future.

We shall use the set ω of all finite ordinal numbers of von Neumann as the set of all non-negative integers (see [16]).

2. Chebyshev sets in metric spaces – preliminaries

The aim of this section is to establish terminology and basic properties of Chebyshev sets in an arbitrary metric space. An extensive bibliography on Chebyshev sets in normed spaces which includes also survey articles is given in [6], so it is not necessary to repeat this bibliography here.

2.1. d -proximal and d -Chebyshev sets

Throughout this section, we assume that (X, d) is a metric space. As usual, for $x \in X$ and a positive real number r , we denote $B_d(x, r) = \{y \in X : d(x, y) < r\}$ and $\bar{B}_d(x, r) = \{y \in X : d(x, y) \leq r\}$. The symbol $\tau(d)$ stands for the topology induced by d . For simplicity, we denote by X the topological space $(X, \tau(d))$. If Y is a subset of X , we shall consider Y as a metric subspace of (X, d) and as a topological subspace of $(X, \tau(d))$. The following definition can be regarded as an adaptation of Definition 2.3 of [6] to arbitrary metric spaces. If it is not stated otherwise, $\mathbb{R}^n$ will stand for the n -dimensional Euclidean space equipped with the Euclidean metric induced by the standard inner product in $\mathbb{R}^n$.

Definition 2.1. Let $K \subseteq X$.

- (i) We define a set-valued mapping $P_{d,K} : X \rightarrow \mathcal{P}(X)$ by $P_{d,K}(x) = \{y \in K : d(x, y) = d(x, K)\}$ if $K \neq \emptyset$ and by $P_{d,K}(x) = \emptyset$ if $K = \emptyset$, where $x \in X$.
- (ii) We say that a d -nearest point to $x \in X$ in K or, equivalently, a best d -approximation of x in K is a point which belongs to the set $P_{d,K}(x)$.
- (iii) The set K is called d -proximal in X if $P_{d,K}(x) \neq \emptyset$ for all $x \in X$.
- (iv) The set K is called a d -Chebyshev set in X if $P_{d,K}(x)$ is a singleton for each $x \in X$.
- (v) If $x \in X$ is such that $P_{d,K}(x)$ is a singleton, we denote by $p_{d,K}(x)$ the unique point of $P_{d,K}(x)$ and we call $p_{d,K}(x)$ the d -projection of the point x onto K .
- (vi) If K is a d -Chebyshev set in X , then the mapping $p_{d,K} : X \rightarrow K$ is called the d -projection of X onto K . If this does not lead to misunderstanding, d -projections will be called *metric projections*.

Definition 2.2. Let Y be a subset of the metric space (X, d) and let $K \subseteq Y$. We call K d -proximal in Y if, for each $y \in Y$, the set $P_{d,K}(y)$ is nonempty. If, for each $y \in Y$, the set $P_{d,K}(y)$ is a singleton, we say that K is d -Chebyshev in Y .

A set which is d -Chebyshev in a subspace Y of the metric space (X, d) need not be d -Chebyshev in X . (Take a non-Chebyshev subset Y in $\mathbb{R}^2$. Then Y is Chebyshev in itself.)

Although it has been written a lot about Chebyshev sets in normed vector spaces, in general, it is not known much about the nature of d -Chebyshev sets for an arbitrary metric d ; however, the concepts of proximal and Chebyshev sets in metric spaces, especially in strictly convex metric spaces, were investigated in the past, for instance, in [1], [19] and [20]. Unfortunately, even in the case when d is induced by the inner product of a Hilbert space over the field $\mathbb{R}$, versions of the axiom of choice have been involved in proofs of some properties of d -Chebyshev sets, while it is more desirable to investigate whether a statement is provable in **ZF** or it can fail in a model of **ZF**. For instance, the statement that $\mathbb{R}^n$ is a Fréchet space was used in the proof on page 363 of [23] that every Chebyshev set in the n -dimensional Euclidean space $\mathbb{R}^n$ is closed. However, in view of Theorem 4.54 of [12], that $\mathbb{R}$ is Fréchet is an equivalent of **CC**($\mathbb{R}$) which asserts that every nonempty countable collection of nonempty subsets of $\mathbb{R}$ has a choice function (see Definition 2.9(1) of **CC**($\mathbb{R}$) in [12] and Form 94 in [14]). In Cohen's original model of **ZF** (see model $\mathcal{M}1$ in [14]), $\mathbb{R}$ is not Fréchet (see Form 73 of [14]). Therefore, the proof in [23] that Chebyshev sets in $\mathbb{R}^n$ are closed is not a proof in **ZF** but it is a too complicated proof in **ZF** + **CC**($\mathbb{R}$). If we replace the norm by the metric d in Proposition 2.9 of [6] and in its proof in [6], then we can see that the following more general fact than Proposition 2.9 of [6] has a trivially simple proof in **ZF** which is included below to show its simplicity:

Proposition 2.3. *Each d -proximal set in a nonempty metric space (X, d) is nonempty and closed in (X, d) . In particular, each d -Chebyshev set in the metric space (X, d) is closed in (X, d) .*

Proof. Let A be a d -proximal set in X . It is obvious that $A \neq \emptyset$ because $X \neq \emptyset$. Let $x \in \text{cl}_d(A)$ and $a \in P_{d,A}(x)$. Then $d(x, a) = d(x, A) = 0$, so $x \in A$ because $x = a$. Therefore, $A = \text{cl}_d(A)$. $\square$

Definition 2.4. The d -graph of a set $A \subseteq X$ is the set

$$\text{Gr}_d(A) = \bigcup_{x \in X} [\{x\} \times P_{d,A}(x)].$$

Let us notice that, in the terminology of set-valued analysis, $\text{Gr}_d(A)$ is the graph of the set-valued mapping $P_{d,A}$.

Proposition 2.5. *The d -graph of a closed subset of X is closed in $X \times X$.*

Proof. Let $K \subseteq X$. If $K = \emptyset$, then $\text{Gr}_d(K) = \emptyset$. Assume that $K \neq \emptyset$. Since the functions d and $d(\cdot, K)$ are both continuous, the set $A = \{(x, y) \in X \times X : d(x, y) = d(x, K)\}$ is closed in $X \times X$. Keeping this in mind, we infer from the equality $\text{Gr}_d(K) = (X \times K) \cap A$ that $\text{Gr}_d(K)$ is closed in $X \times X$ if K is closed in X . $\square$

Propositions 2.3 and 2.5 taken together imply the following theorem:

Theorem 2.6. *If K is a d -proximal subset of X then the d -graph of K is closed in $X \times X$. In particular, if K is a d -Chebyshev set in X , then the d -projection of X onto K has a closed graph.*

Theorem 2.6 is an extension and generalization of Corollary 2.19 from [6] and, for instance, of Theorem 4 from [19]. However, the proofs to Corollary 2.19 in [6] and to Theorem 4 in [19] are not proofs in **ZF** because, in **ZF**, only sequential closedness of Chebyshev sets was shown in [6] and [19], while it is known that a sequentially closed subset of a metric space may fail to be closed in a model of **ZF**. Namely, as in Theorem 4.55 of [12], we denote by $\mathbf{CC}(c\mathbb{R})$ the following statement: Every sequence of nonempty complete metric subspaces of $\mathbb{R}$ has a choice function. In view of Theorem 4.55 of [12], that $\mathbb{R}$ is sequential is an equivalent of $\mathbf{CC}(c\mathbb{R})$.

Remark 2.7. In Cohen's original model (see model $\mathcal{M}1$ in [14]), for each natural number $n \geq 1$, the space $\mathbb{R}^n$ is not sequential because it contains non-separable subspaces (see Form 92 of [14]).

It is known that $\mathbf{CC}(\mathbb{R})$ is strictly stronger than $\mathbf{CC}(c\mathbb{R})$ in **ZF** (see [12], 74–77).

Definition 2.8. Let K be a nonempty subset of X . We say that the mapping $P_{d,K}: X \rightarrow \mathcal{P}(X)$ is d -continuous at $x \in X$ if $P_{d,K}(x)$ is a singleton and the following condition is satisfied:

$$\forall \varepsilon \in (0; +\infty) \exists \delta \in (0; +\infty) \forall y, z \in X [(d(x, y) < \delta \wedge z \in P_{d,K}(y)) \Rightarrow d(z, p_{d,K}(x)) < \varepsilon].$$

Definition 2.8 is a modification of Definition 2.23 of [6]. If K is a nonempty d -Chebyshev set in X , then the metric projection $p_{d,K}$ of X onto K is continuous if and only if $P_{d,K}$ is d -continuous at each $x \in X$ in the sense of our Definition 2.8, while $p_{d,K}$ is sequentially continuous if and only if $P_{d,K}$ is continuous at each $x \in X$ in the sense of Definition 2.23 of [6] under the additional assumption that X is a normed vector space and d is induced by the norm of X . In general, given metric spaces (X, d) and (Y, ρ) , the sequential continuity of a mapping $f: X \rightarrow Y$ is not equivalent to the continuity of f in **ZF** (see Theorem 4.54 of [12] and Form 94 in [14] which fails in model $\mathcal{M}1$ of [14]). It can be deduced from the proof in **ZF** to Theorem 3.15 in [12] that the following theorem holds true in **ZF**:

Theorem 2.9. *If X is a separable metrizable space and Y is a metrizable space, then a mapping $f: X \rightarrow Y$ is continuous if and only if f is sequentially continuous.*

Remark 2.10. Let K be a d -Chebyshev set in a normed vector space X where d is induced by the norm of X . If X is finitely dimensional, the classical proofs of the continuity of the metric projection $p_{d,K}$ of X onto K given, for instance, in [6] and [23] have a gap in **ZF** because only sequential continuity of $p_{d,K}$ was shown there. To deduce from the sequential continuity of $p_{d,K}$ its continuity, we need Theorem 2.9. If X is infinitely dimensional, $p_{d,K}$ may fail to be continuous (see Example 2.54 in [6] and page 246 in [9]).

We are going to give a direct proof in **ZF** of the continuity of $p_{d,K}$ if K is a d -Chebyshev set in an arbitrary nonempty metric space (X, d) having the Heine-Borel property.

Definition 2.11. Let Y be a subset of the metric space (X, d) . We say that:

- (i) Y has the *Heine-Borel property with respect to d* or, equivalently, Y is *d -totally complete* if, for each $K \subseteq Y$, it is true that K is compact if K is simultaneously closed in Y and d -bounded;
- (ii) Y is *d -boundedly compact* if, for each positive real number r and for each $x \in X$, the set $Y \cap \bar{B}_d(x, r)$ is compact.

The Heine-Borel property, called also total compactness by some authors (see [19]), is a familiar notion in topology and in the theory of metric spaces. Definition 2.11 (ii) is an adaptation of Definition 2.25 of [6] to metric spaces, given here in order to notice that a subset Y of the metric space (X, d) is d -boundedly compact if and only if Y has the Heine-Borel property with respect to d .

Theorem 2.12. Let K be a d -boundedly compact subset of the metric space (X, d) and let $x \in X$ be such that $P_{d,K}(x)$ is a singleton. Then the mapping $P_{d,K}$ is d -continuous at x .

Proof. Suppose that $P_{d,K}$ is not d -continuous at x . There exists a positive real number ε such that, for each $n \in \omega \setminus \{0\}$, the set

$$F_n = \left[\bigcup \{P_{d,K}(y) : y \in B_d(x, \frac{1}{n})\} \right] \setminus B_d(p_{d,K}(x), \varepsilon)$$

is nonempty. Let $n \in \omega \setminus \{0\}$ and $y \in B_d(x, \frac{1}{n})$. For any $z \in P_{d,K}(y)$, we have: $d(x, z) \leq d(x, y) + d(y, z) \leq \frac{1}{n} + d(y, K) \leq \frac{1}{n} + d(y, x) + d(x, K) \leq \frac{2}{n} + d(x, K)$. This implies that the set F_n is d -bounded. Hence the set $C_n = \text{cl}_X(F_n)$ is d -bounded. Then $C_n \cap K$ is compact because K is d -boundedly compact. Since $F_n \subseteq C_n \cap K$, we obtain that $C_n \subseteq K$. Of course, the collection $\{C_n : n \in \omega \setminus \{0\}\}$ is centred.

Thus, it follows from the compactness of C_n that there exists $z_0 \in \bigcap_{n \in \omega \setminus \{0\}} C_n$. Then $z_0 \in K$. Let $g(t) = d(t, K)$ for $t \in X$. Let us fix a positive real number η . Since the function g is continuous, there exists $n_0 \in \omega \setminus \{0\}$ such that $\frac{1}{n_0} < \varepsilon$ and if $t \in B_d(x, \frac{1}{n_0})$, then $|g(t) - g(x)| < \eta$. Since $z_0 \in \text{cl}_X(F_n)$, there exists $z_1 \in F_n$ such that $d(z_0, z_1) < \frac{1}{n}$. There exists $y_1 \in B_d(x, \frac{1}{n})$ such that $z_1 \in P_{d,K}(y_1)$. Since $z_0 \in K$, we have that $d(x, K) \leq d(x, z_0)$. On the other hand,

$$d(x, z_0) \leq d(x, y_1) + d(y_1, z_1) + d(z_1, z_0) \leq \frac{2}{n} + d(y_1, K) \leq \frac{2}{n} + d(x, K) + \eta.$$

This, together with the arbitrariness of the numbers $\eta > 0$ and $n \geq n_0$, implies that $d(x, z_0) \leq d(x, K)$. In consequence, we obtain that $d(x, z_0) = d(x, K) = d(x, p_{d,K}(x))$, so $z_0 = p_{d,K}(x)$. However, $\varepsilon \leq d(p_{d,K}(x), z_1) \leq d(p_{d,K}(x), z_0) + d(z_0, z_1) \leq d(p_{d,K}(x), z_0) + \frac{1}{n}$ and thus $d(p_{d,K}(x), z_0) > 0$. The contradiction obtained completes the proof. $\square$

An application of our proof to Theorem 2.12 is that the following immediate consequence holds true in **ZF**:

Corollary 2.13. *If K is a d -boundedly compact d -Chebyshev set in X , then the d -projection of X onto K is continuous.*

2.2. Operations on d -Chebyshev sets

As in Subsection 2.1, let us assume that (X, d) is a nonempty metric space.

Proposition 2.14. *Let $\mathcal{C}$ be a nonempty collection of d -Chebyshev sets in X and let $K = \bigcup \mathcal{C}$. Suppose that K is d -boundedly compact and has the property that, for each pair x, y of points of K , there exists $C \in \mathcal{C}$ such that $x, y \in C$. Then K is a d -Chebyshev set in X .*

Proof. Let $x_0 \in X$. Take any $c \in K$. Put $r = d(x_0, c)$ and $E = \bar{B}_d(x_0, r) \cap K$. Then $d(x_0, K) = d(x_0, E)$. Since E is nonempty and compact, there exists $c_0 \in E$ such that $d(x_0, c_0) = d(x_0, E)$, so K is d -proximal. Suppose that $c_1, c_2 \in K$ and $d(x_0, c_1) = d(x_0, c_2) = d(x_0, K)$. There exists $C \in \mathcal{C}$ such that $c_1, c_2 \in C$. Let $i \in \{1, 2\}$ and let $a = p_{d,C}(x_0)$. Then $d(x_0, c_i) \leq d(x_0, a)$ because $a \in K$. Since $c_i \in C$, we have $d(x_0, a) \leq d(x_0, c_i)$. Hence $d(x_0, c_i) = d(x_0, a)$. This implies that $c_i = a$ because $P_{d,C}(x_0) = \{a\}$. $\square$

The following notions have appeared useful for investigations of Chebyshev sets, especially in strictly convex metric spaces (see, for instance, [1], [19] and [20]):

Definition 2.15. Let $Y \subseteq X$ and let $x, y \in Y$. Then:

- (i) (cf. Definition 2.1 of [20]) the d -segment in Y between points x and y is the set $[x, y]_{d,Y}$ defined by:

$$[x, y]_{d,Y} = \{z \in Y : d(x, z) + d(z, y) = d(x, y)\};$$

- (ii) if $Y = X$, then $[x, y]_d = [x, y]_{d,Y}$;
- (iii) (cf. Definition 2.6 of [20]) a set $A \subseteq Y$ is called d -convex in Y if, for each pair of points $x, y \in A$, the inclusion $[x, y]_{d,Y} \subseteq A$ holds;
- (iv) (cf. Definition 2.1 of [20]) a point $z \in X$ is d -between x and y in Y if $z \in [x, y]_{d,Y}$.

The following proposition, relevant to Corollary 1 of [1], follows immediately from Proposition 2.14.

Proposition 2.16. *If all d -segments in X are d -Chebyshev sets, then every nonempty d -boundedly compact d -convex set in X is d -Chebyshev in X .*

Proposition 2.17. *Suppose that A_1, A_2 and $A = A_1 \cup A_2$ are all d -Chebyshev sets in X . Then, for each $x \in X$, we have:*

$$p_{d,A}(x) = \begin{cases} p_{d,A_1}(x) & \text{if } d(x, p_{d,A_1}(x)) \leq d(x, p_{d,A_2}(x)), \\ p_{d,A_2}(x) & \text{otherwise.} \end{cases} \quad (1)$$

Proof. Let $x \in X$. Denote $a = p_{d,A}(x)$ and $a_i = p_{d,A_i}(x)$ for $i \in \{1, 2\}$. Of course, $d(x, a) \leq d(x, a_i)$ for $i \in \{1, 2\}$. Suppose that $a \in A_1$. Then $d(x, a_1) \leq d(x, a)$, so $d(x, a) = d(x, a_1)$. This implies that $a = a_1$ and $d(x, a_1) \leq d(x, a_2)$. If $a \in A_2 \setminus A_1$, then $a = a_2$ and $d(x, a_1) > d(x, a)$. $\square$

Using similar elementary arguments as in the proof to Proposition 2.17, one can show that the following proposition holds:

Proposition 2.18. *Let $n \in \omega \setminus \{0\}$. Assume that, for each $i \in n$, we are given a d -Chebyshev set A_i in X such that the set $A = \bigcup_{i \in n} A_i$ is d -Chebyshev in X . Let $x \in X$ and let $a = p_{d,A}(x)$. Then $a = p_{d,A_i}(x)$ for each $i \in n$ such that $a \in A_i$.*

Now, we assume that $n \in \omega \setminus \{0\}$ and (X_i, d_i) is a nonempty metric space for each $i \in n$, while $X = \prod_{i \in n} X_i$. Usually, the equivalent metrics ρ_k for $k \in 3$ are considered in X where, for each $x, y \in X$, we have:

$$\begin{aligned}\rho_0(x, y) &= \max\{d_i(x(i), y(i)) : i \in n\}, \\ \rho_1(x, y) &= \sqrt{\sum_{i \in n} d_i(x(i), y(i))^2}, \\ \rho_2(x, y) &= \sum_{i \in n} d_i(x(i), y(i)).\end{aligned}$$

Since the theorem below will be of essential importance, we include its simple proof for completeness:

Theorem 2.19. *Let $A_i \subseteq X_i$ for each $i \in n$ and let $A = \prod_{i \in n} A_i$, while $k \in \{1, 2\}$. Then the following conditions are satisfied:*

- (i) *For each $x \in X$, the equality $P_{\rho_k, A}(x) = \prod_{i \in n} P_{d_i, A_i}(x(i))$ holds.*
- (ii) *A is a ρ_k -Chebyshev set in X if and only if A_i is a d_i -Chebyshev set in X_i for each $i \in n$.*

Proof. To prove (i), assume first that $a \in P_{\rho_k, A}(x)$. Let $j \in n$. Then $a(j) \in A_j$. Consider any $y_j \in A_j$ and define $y \in X$ as follows: $y(i) = a(i)$ for each $i \in n \setminus \{j\}$, while $y(j) = y_j$. Since $y \in A$, we have $\rho_k(x, a) \leq \rho_k(x, y)$. This implies that $d_j(x(j), a(j)) \leq d_j(x(j), y_j)$, so $a(j) \in P_{d_j, A_j}(x(j))$ and, in consequence, $a \in \prod_{i \in n} P_{d_i, A_i}(x(i))$.

Now, assume that $b \in \prod_{i \in n} P_{d_i, A_i}(x(i))$. Then $b \in A$. Moreover, if $z \in A$, then, since $d_i(x(i), b(i)) \leq d_i(x(i), z(i))$ for each $i \in n$, we have that $\rho_k(x, b) \leq \rho_k(x, z)$. Hence $b \in P_{\rho_k, A}(x)$. This completes the proof to (i).

That (ii) also holds is an immediate consequence of (i). $\square$

Example 2.20 shows that Theorem 2.19 cannot be extended to the metric ρ_0 .

Example 2.20. Let $A_0 = A_1 = [-1, 1]$. For $x, y \in \mathbb{R}$, let $d_0(x, y) = d_1(x, y) = |x - y|$. Then, for $x, y \in \mathbb{R}^2$, we have $\rho_0(x, y) = \max\{|x(0) - y(0)|, |x(1) - y(1)|\}$. Now, let $x^* = (0, 2)$. Then $\{0\} = P_{d_0, A_0}(0)$ and $\{1\} = P_{d_1, A_1}(2)$, while, for $A = A_0 \times A_1$, the set $P_{\rho_0, A}(x^*)$ is infinite because $P_{\rho_0, A}(x^*) = [-1, 1] \times \{1\}$.

3. Calculations in dioids of interest

In this section, we recall the construction and calculations in algebraic structures of our interest. Readers can compare this material with [21].

3.1. Calculations in MPA and MPAP

The max as the addition operation and the addition as the multiplication operation are denoted respectively by $\oplus$ and $\otimes$. The union $\mathbb{R} \cup \{\varepsilon\}$ (where $\varepsilon = -\infty \notin \mathbb{R}$ is a fixed element) is denoted by $\mathbb{R}_\varepsilon$ and the semiring $(\mathbb{R}_\varepsilon, \oplus, \otimes)$ is denoted by $\mathbb{R}_{\max}$. So, for any $a, b \in \mathbb{R}_\varepsilon$,

$$a \oplus b = \max(a, b) \text{ and } a \otimes b = a + b. \quad (2)$$

The structure $\mathbb{R}_{\max}$ is a dioid (= an idempotent semiring) and a semifield (see Definition 1 in [17]). Notice that $\mathbb{R}_{\max}^n$ is naturally a semimodule over $\mathbb{R}_{\max}$.

Analogously to conventional algebra, scalars in linear combinations are written before vectors. The rules for the order of evaluation of the tropical operations are similar to those of conventional algebra, i.e. max-plus algebraic power has the highest priority, and the tropical multiplication has a higher priority than the tropical addition. All rules of calculation in $\mathbb{R}_{\max}$ can be illustrated in the following example:

Example 3.1.

$$\begin{aligned} 2 \oplus (3^5 \oplus 2^{-1}) \otimes 1 \oplus \varepsilon^2 &= 2 \oplus (5 \times 3 \oplus (-2)) \otimes 1 \oplus \varepsilon \\ &= 2 \oplus (15 \oplus (-2)) \otimes 1 = 2 \oplus (15 \otimes 1 \oplus (-2) \otimes 1) \\ &= 2 \oplus ((15 + 1) \oplus (-2 + 1)) = 2 \oplus (16 \oplus (-1)) = \max(2, 16, -1) = 16. \end{aligned}$$

There is no inverse related to max operation, which makes a simple equation $a \oplus x = b$ often insolvable. For this reason, we first consider the algebra of pairs as a tool in the construction of the Symmetrized Max-Plus Algebra.

We define $\mathcal{P}_\varepsilon = \mathbb{R}_\varepsilon \times \mathbb{R}_\varepsilon$ with the operations $\oplus$ and $\otimes$ based on (2):

$$(a, b) \oplus (c, d) = (a \oplus c, b \oplus d) \quad (3)$$

$$(a, b) \otimes (c, d) = (a \otimes c \oplus b \otimes d, a \otimes d \oplus b \otimes c) \quad (4)$$

It is easy to verify that $(\mathcal{P}_\varepsilon, \oplus, \otimes)$ is a dioid with $(\varepsilon, \varepsilon)$ and $(0, \varepsilon)$ as its zero and unit, respectively. Notice that $(\varepsilon, \varepsilon)$ is the absorbing element for $\otimes$. This structure is called the *Max-Plus Algebra of Pairs* (abbreviated to MPAP) and denoted by $\mathcal{P}_{\max}$. As $\mathbb{R}_{\max}$ is isomorphic to the subdioid of pairs (a, ε) , $a \in \mathbb{R}_\varepsilon$, then $\mathbb{R}_{\max}$ itself can be considered as a subdioid of $\mathcal{P}_{\max}$. If $u = (x, y) \in \mathcal{P}_\varepsilon$, then one defines the *max-plus norm* (or *absolute value*) of u as $|u|_\oplus = x \oplus y$ and we have two unary operators: $\ominus$ (the max-plus algebraic minus operator) and $(\cdot)^\bullet$ (the balance operator) such that $\ominus u = (y, x)$ and $u^\bullet = u \oplus (\ominus u) = (|u|_\oplus, |u|_\oplus)$. The properties of these operators can be expressed in the following proposition:

Proposition 3.2. (cf. [21]) For all u and v in $\mathcal{P}_\varepsilon$, we have:

- (i) $u^\bullet = (\ominus u)^\bullet = (u^\bullet)^\bullet$;
- (ii) $u \otimes v^\bullet = (u \otimes v)^\bullet = u^\bullet \otimes v = u^\bullet \otimes v^\bullet$;
- (iii) $\ominus(\ominus u) = u$, $\ominus(u \oplus v) = (\ominus u) \oplus (\ominus v)$ and $(\ominus u) \otimes v = \ominus(u \otimes v) = u \otimes (\ominus v)$.

Remark 3.3. The above can be interpreted: $(\cdot)^\bullet$ is a projection onto another subdoid of $\mathcal{P}_\varepsilon$ consisting of fixed points of $\ominus$ and $\ominus$ is an involution commuting with $\oplus$.

We shall write $u \ominus v$ instead of $u \oplus (\ominus v)$.

3.2. Construction of SMPA

For the readers' convenience, we recall in brief the construction of SMPA described in [21]. Since $u \ominus u = (|u|_\oplus, |u|_\oplus) = (\varepsilon, \varepsilon) \Rightarrow u = (\varepsilon, \varepsilon)$, then $\mathcal{P}_{\max}$ is, as $\mathbb{R}_{\max}$, without inverses with respect to $\oplus$. For this reason, the balance relation ∇ is introduced: For all $u = (a, b) \in \mathcal{P}_\varepsilon$ and $v = (c, d) \in \mathcal{P}_\varepsilon$, we have

$$u \nabla v \Leftrightarrow a \oplus d = b \oplus c. \quad (5)$$

This relation is reflexive and symmetric, but not transitive because, for real numbers x, y, z such that $x > z > y$, we have that

$$x = x \oplus x = x \oplus y \neq y \oplus z,$$

which gives $(x, y) \nabla (x, x)$ and $(x, x) \nabla (z, x)$, but (x, y) is not balanced with (z, x) . So we need to modify the relation ∇ somewhat to get an equivalence relation $\sim$ in $\mathcal{P}_\varepsilon$ associated to it such that it is possible, in a standard way, to extend the operations $\oplus$ and $\otimes$ of $\mathcal{P}_{\max}$ to the quotient set $\mathbb{S} = \mathcal{P}_\varepsilon / \sim$ to obtain the semiring $\mathcal{P}_{\max} / \sim$ which contains not inverses but symmetric elements which have some properties of inverses. Let the equivalence relation $\sim$ be defined as follows: for all $u, v \in \mathcal{P}_\varepsilon$ with $u = (a, b)$ and $v = (c, d)$ where $a, b, c, d \in \mathbb{R}_\varepsilon$, we have

$$u \sim v \Leftrightarrow \begin{cases} u \nabla v & \text{if } a \neq b \text{ and } c \neq d, \\ u = v & \text{otherwise.} \end{cases} \quad (6)$$

The structure $(\mathbb{S}, \oplus, \otimes) = \mathcal{P}_{\max} / \sim$ is called the *Symmetrized Max-Plus Algebra* (shortly: SMPA) and denoted by $\mathbb{S}_{\max}$. The semifield $\mathbb{R}_{\max}$ can be identified with a subdoid of $\mathbb{S}_{\max}$, and $\mathbb{S}_{\max}$ can be considered as a semimodule over $\mathbb{R}_{\max}$.

We have three types of equivalence classes of $\sim$ which are called max-plus positive, max-plus negative and balanced elements, respectively. Namely, for $u \in \mathcal{P}_\varepsilon$, let $\bar{u} = \{v \in \mathcal{P}_\varepsilon : u \sim v\}$. Then, for $a \in \mathbb{R}$, we get

$$\overline{(a, \varepsilon)} = \{(a, b) \in \mathcal{P}_{\max} : a > b\}, \quad (7)$$

$$\overline{(\varepsilon, a)} = \{(a, b) \in \mathcal{P}_{\max} : a < b\}. \quad (8)$$

$$\text{Moreover, for each } a \in \mathbb{R}_\varepsilon, \text{ we get } \overline{(a, a)} = \{(a, a)\}. \quad (9)$$

If $a \in \mathbb{R}$, the element $\oplus a = \overline{(a, \varepsilon)}$ of $\mathbb{S}$ is called *max-plus positive*, while the element $\ominus a = \overline{(\varepsilon, a)}$ of $\mathbb{S}$ is called *max-plus negative*. If $a \in \mathbb{R}_\varepsilon$, the element $a^\bullet = \overline{(a, a)}$ of $\mathbb{S}$ is called *balanced*. The element $\varepsilon^\bullet$ is the *zero element* of $\mathbb{S}_{\max}$. Let $\mathbb{S}^\oplus$ be the set of all elements of $\mathbb{S}$ which are either max-plus positive or equal to $\varepsilon^\bullet$. Let $\mathbb{S}^\ominus$ be the set of all elements of $\mathbb{S}$ which are either max-plus negative or equal to $\varepsilon^\bullet$. Finally, let $\mathbb{S}^\bullet$ be the set of all balanced elements of $\mathbb{S}$. The set $\mathbb{S}$ can be written as the union $\mathbb{S}^\oplus \cup \mathbb{S}^\ominus \cup \mathbb{S}^\bullet$. The sets $\mathbb{S}^\oplus, \mathbb{S}^\ominus$ and $\mathbb{S}^\bullet$ have only one common point $\varepsilon^\bullet$. Let us put $\oplus \varepsilon = \ominus \varepsilon = \varepsilon^\bullet$. We can identify $\mathbb{S}^\oplus$ with $\mathbb{R}_{\max}$ by assigning $\oplus a$ to $a \in \mathbb{R}_\varepsilon$. Similarly, $\mathbb{S}^\ominus$ can be identified with $\ominus \mathbb{R}_{\max} = \{\ominus a : a \in \mathbb{R}_\varepsilon\}$, and $\mathbb{S}^\bullet$ can be identified with $(\mathbb{R}_{\max})^\bullet = \{a^\bullet : a \in \mathbb{R}_\varepsilon\}$.

Definition 3.4. (cf. [21]) Let $a \in \mathbb{S}$. The *max-plus positive part* $a^\oplus$ and the *max-plus negative part* $a^\ominus$ of a are defined as follows:

- (i) if $a \in \mathbb{S}^\oplus$, then $a^\oplus = a$ and $a^\ominus = \varepsilon$;
- (ii) if $a \in \mathbb{S}^\ominus$ then $a^\oplus = \varepsilon$ and $a^\ominus = \ominus a$;
- (iii) if $a \in \mathbb{S}^\bullet$, then there exists $x \in \mathbb{R}_\varepsilon$ such that $a = \overline{(x, x)}$ and then $a^\oplus = a^\ominus = x$.

It is clear from this definition that, for each $a \in \mathbb{S}$, the following equality holds:

$$a = a^\oplus \ominus a^\ominus. \quad (10)$$

This decomposition is unique if we allow only $a = x \ominus y$ ($x, y \in \mathbb{S}^\oplus$) with either $x = \varepsilon$ or $y = \varepsilon$ or $x = y$.

4. Topology and geometry in SMPA

4.1. Metrics on SMPA

In this subsection, we introduce two distances on $\mathbb{S}$.

Proposition 4.1. (cf. [21]) *The formula*

$$|a| = a^\oplus \oplus a^\ominus \quad (11)$$

defines an absolute value on $\mathbb{S}_{\max}$ as a dioid (or a norm with $\mathbb{S}_{\max}$ understood as a semimodule over the semifield $\mathbb{R}_{\max}$).

Proof. Step 1. If $|a| = \varepsilon$, then

$$\max(a^\oplus, a^\ominus) = -\infty \Leftrightarrow a^\oplus = a^\ominus = -\infty \Leftrightarrow a = \varepsilon.$$

Step 2. Naturally, we have $|a \otimes b| = (a \otimes b)^\oplus \oplus (a \otimes b)^\ominus$.

We discuss three cases:

If $a, b \in \mathbb{S}^\oplus$, then $a = a^\oplus$ and $b = b^\oplus$. Hence $(a \otimes b)^\oplus = a^\oplus \otimes b^\oplus$, $(a \otimes b)^\ominus = \varepsilon$ and $|a| = a^\oplus$, $|b| = b^\oplus$, which gives $|a \otimes b| = a^\oplus \otimes b^\oplus = |a| \otimes |b|$.

If $a, b \in \mathbb{S}^\ominus$, then $a^\ominus = \ominus a$ and $a^\oplus = \varepsilon$, $b^\ominus = \ominus b$ and $b^\oplus = \varepsilon$, which leads to $(\ominus a) \otimes (\ominus b) = (a \otimes b)^\oplus$, then $(a \otimes b)^\ominus = \varepsilon$, $|a| = \ominus a$, $|b| = \ominus b$, which finally gives

$$|a \otimes b| = (a \otimes b)^\oplus = (\ominus a) \otimes (\ominus b) = |a| \otimes |b|.$$

If $a \in \mathbb{S}^\oplus$, $b \in \mathbb{S}^\ominus$, then $a = a^\oplus$, $b^\ominus = \ominus b$ and $b^\oplus = \varepsilon$, which yields the equation $a \otimes (\ominus b) = \ominus(a \otimes b) = (a \otimes b)^\ominus$, then, $(a \otimes b)^\oplus = \varepsilon$, $|a| = a$, $|b| = \ominus b$, which finally gives $|a \otimes b| = (a \otimes b)^\ominus = a \otimes (\ominus b) = |a| \otimes |b|$.

So in all three cases, the operation $|\cdot|$ is multiplicative.

Step 3. Similarly, we can prove the triangle inequality (equality, in fact). $\square$

Remark 4.2. The max-plus absolute value $|\cdot|$ does not produce a distance on $\mathbb{S}$ in the similar way as the usual absolute value produces a distance on $\mathbb{R}$ because, if we define $\rho(x, y) = |x \ominus y|$ for $x, y \in \mathbb{S}$, then, for $a \in \mathbb{S}$ with $a \neq \varepsilon^\bullet$, we get $\rho(a, a) = |a \ominus a| = |a^\bullet| = |a| \neq \varepsilon$.

Corollary 4.3. Each $a \in \mathbb{S}$ can be written as

$$a = \text{sgn}(a) |a| \quad (12)$$

where $\text{sgn}(a) \in \{\oplus, \ominus, \bullet\}$. If $a \neq \varepsilon^\bullet$, then this formula is unique.

Corollary 4.4. One can write the formulas for actions in $\mathbb{S}$:

$$a \oplus b = \text{sgn}(a \oplus b) (|a| \oplus |b|), \quad (13)$$

$$a \otimes b = \text{sgn}(a) \text{sgn}(b) (|a| \otimes |b|) \quad (14)$$

$$\text{where} \quad \text{sgn}(a \oplus b) = \begin{cases} \text{sgn}(a), & |a| > |b|, \\ \text{sgn}(b), & |b| > |a|, \\ \text{sgn}(a), & |a| = |b|, \text{sgn}(a) = \text{sgn}(b), \\ \bullet, & |a| = |b|, \text{sgn}(a) \neq \text{sgn}(b), \end{cases}$$

and the multiplication table for signs is obvious.

Definition 4.5. The *canonical embedding* of $\mathbb{S}$ into $\mathbb{C}$ is the mapping $\Phi: \mathbb{S} \rightarrow \mathbb{C}$ defined in the following way: for $e^{-\infty} = 0$, $\theta = \frac{-1+\sqrt{3}i}{2}$ and any $r \in \mathbb{R}_\varepsilon$, we put

$$\Phi(\oplus r) = \theta e^r, \quad \Phi(\ominus r) = \theta^2 e^r, \quad \Phi(r^\bullet) = e^r.$$

We denote the Euclidean metric in $\mathbb{R}^n$ by $d_{e,n}$ or by d_e when it is clear what n is. In particular, for the plane $\mathbb{C} = \mathbb{R}^2$, we have $d_e = d_{e,2}$. As we have already mentioned, $\mathbb{R}^n$ will be usually considered with its Euclidean metric d_e and with the natural topology induced by d_e .

Definition 4.6. The *natural* (or *usual*) topology of $\mathbb{S}$ is the smallest topology in $\mathbb{S}$ which makes the mapping $\Phi: \mathbb{S} \rightarrow \mathbb{C}$ continuous.

We shall consider two natural metrics in $\mathbb{S}$ that are defined below:

Definition 4.7. Let $d_1: \mathbb{S} \times \mathbb{S} \rightarrow \mathbb{R}$ be defined as follows:

$$d_1(a, b) = d_e(\Phi(a), \Phi(b))$$

for all $a, b \in \mathbb{S}$. The metric d_1 in $\mathbb{S}$ will be called the *Euclidean distance* in $\mathbb{S}$.

Definition 4.8. Let $d_2: \mathbb{S} \times \mathbb{S} \longrightarrow \mathbb{R}$ be defined as follows:

$$d_2(a, b) = \begin{cases} |\exp |a| - \exp |b|| & \text{if } \operatorname{sgn}(a) = \operatorname{sgn}(b) \\ \exp |a| + \exp |b| & \text{if } \operatorname{sgn}(a) \neq \operatorname{sgn}(b) \end{cases}$$

for all $a, b \in \mathbb{S}$. The metric d_2 in $\mathbb{S}$ will be called the *inner distance* in $\mathbb{S}$.

Remark 4.9. It is clear that both Euclidean and inner distances in $\mathbb{S}$ are equivalent and they induce the natural topology of $\mathbb{S}$; moreover, the metrics d_1 and d_2 are uniformly equivalent and equal on each of the lines $\mathbb{S}^\oplus$, $\mathbb{S}^\ominus$, $\mathbb{S}^\bullet$. As a topological space, we shall consider $\mathbb{S}$ only with its natural topology. The space $\mathbb{S}$ is homeomorphic with the subspace $\Phi(\mathbb{S})$ of $\mathbb{C} = \mathbb{R}^2$. The mapping Φ is an isometry of the metric space $(\mathbb{S}, d_1)$ onto the metric subspace $\Phi(\mathbb{S})$ of $(\mathbb{C}, d_e)$. The metric space $(\mathbb{S}, d_2)$ is isometric with:

- (1) the metric subspace $(\mathbb{R} \times \{0\}) \cup (\{0\} \times [0, +\infty))$ of the metric space $(\mathbb{R}^2, \rho_2)$ where $\rho_2(x, y) = |x(0) - y(0)| + |x(1) - y(1)|$ for all $x, y \in \mathbb{R}^2$;
- (2) the metric subspace $\{z \in \mathbb{R}^2 : z(1) = |z(0)| \vee (z(0) = 0 \wedge z(1) < 0)\}$ of the metric space $(\mathbb{R}^2, \rho_0)$ where $\rho_0(x, y) = \max(|x(0) - y(0)|, |x(1) - y(1)|)$ for all $x, y \in \mathbb{R}^2$.

Let X be a topological space. We recall that it is said that a point $c \in X$ **disconnects** X between points $a, b \in X$ if there exists a pair U, V of disjoint open sets in X such that $X \setminus \{c\} = U \cup V$ and $a \in U$, while $b \in V$. The space X is a *dendrite* if it is a continuum such that, for each pair a, b of distinct points of X , there exists a point $c \in X$ which disconnects X between points a and b . It is said that X is a *semicontinuum* if, for each pair a, b of points of X , there is a continuum C in X such that $a, b \in C$. Let us say that X is a *semidendrite* if it is a semicontinuum such that, for each pair a, b of distinct points of X , there exists a point $c \in X$ which disconnects X between a and b .

Proposition 4.10. *The space $\mathbb{S}$ is a semidendrite.*

Proof. It suffices to notice that $\mathbb{S}$ is homeomorphic with the semidendrite $\Phi(\mathbb{S})$ on the plane $\mathbb{C}$. $\square$

Basic properties of dendrites are nicely described in [2] and [18]. Some of the properties can be easily proved in **ZF** and they are also properties of semidendrites. For example, that every semidendrite is a Hausdorff space and that every semicontinuum contained in a semidendrite is a semidendrite are statements obviously true in **ZF**. However, the axiom of choice is heavily used, for instance, in an adaptation to metrizable semidendrites of the known proof for metrizable dendrites of the following theorem:

Theorem 4.11. *It holds true in **ZFC** that, for each pair a, b of distinct points of a metrizable semidendrite X , there exists in X exactly one arc with end-points a and b .*

Problem 4.12. *Can **ZFC** be replaced by **ZF** in Theorem 4.11?*

4.2. Continuity of the inner operations

In this subsection we study the continuity and the discontinuity of the inner operations in $\mathbb{S}_{\max}$ with respect to the usual topology in $\mathbb{S}$. The following proposition is obvious:

Proposition 4.13. *The operation $\otimes$ is continuous in the usual topology of $\mathbb{S}$.*

Proposition 4.14. *The operation $\oplus$ is discontinuous in the usual topology of $\mathbb{S}$.*

Proof. For $r \in \mathbb{R}$, we have discontinuity of $\oplus: \mathbb{S} \times \mathbb{S} \rightarrow \mathbb{S}$ at $(\oplus r, \ominus r)$, $(\ominus r, \oplus r)$, $(\oplus r, r^\bullet)$, $(r^\bullet, \oplus r)$, $(\ominus r, r^\bullet)$, and $(r^\bullet, \ominus r)$. For example:

$$\lim_{s \rightarrow r^-} \oplus(\oplus s, r^\bullet) = r^\bullet, \quad \lim_{s \rightarrow r^+} \oplus(\oplus s, r^\bullet) = \oplus r. \quad \square$$

Remark 4.15. Let τ be the quotient topology induced by the map $|\cdot|: \mathbb{S} \rightarrow \mathbb{R}_\varepsilon$ where $\mathbb{R}_\varepsilon$ is equipped with its standard topology. The topology τ is not T_0 ; however, both operations $\oplus$ and $\otimes$ from $(\mathbb{S}, \tau) \times (\mathbb{S}, \tau)$ to $(\mathbb{S}, \tau)$ are continuous.

5. Topology and geometry of finite products of SMPA

Let $n \in \omega \setminus \{0\}$. Then the set $\mathbb{S}^n$ of all functions $x: n \rightarrow \mathbb{S}$ is identified with the n -Cartesian product of $\mathbb{S}$. In particular, $\mathbb{S}^1$ is identified with $\mathbb{S}$. The *usual* or *natural topology* of $\mathbb{S}^n$ is the product topology of the space $\mathbb{S}^n$ where $\mathbb{S}$ is equipped with its usual topology (see Definition 4.6).

5.1. The main metrics in $\mathbb{S}^n$

We consider the Euclidean metric d_1 and the inner metric d_2 in $\mathbb{S}$. For $j \in \{1, 2\}$, it is most natural to associate with the distance function d_j the following metrics $\rho_{0,j}$, $\rho_{1,j}$ and $\rho_{2,j}$ in $\mathbb{S}^n$:

$$\begin{aligned} \rho_{0,j}(x, y) &= \max\{d_j(x(i), y(i)) : i \in n\}, \\ \rho_{1,j}(x, y) &= \sqrt{\sum_{i \in n} d_j(x(i), y(i))^2}, \\ \rho_{2,j}(x, y) &= \sum_{i \in n} d_j(x(i), y(i)) \end{aligned}$$

where $x, y \in \mathbb{S}^n$. Of course, the metrics $\rho_{k,j}$ are all equivalent and they induce the natural topology of $\mathbb{S}^n$. We shall pay special attention to the metrics $\rho_{1,1}$ and $\rho_{1,2}$.

Definition 5.1. The metric $\rho_{1,1}$ will be denoted by D_1 and called the *Euclidean metric* in $\mathbb{S}^n$, while the metric $\rho_{1,2}$ will be denoted by D_2 and called the *inner metric* in $\mathbb{S}^n$.

Definition 5.2. For the canonical embedding Φ of $\mathbb{S}$ into $\mathbb{C}$, let $\Phi_i = \Phi$ where $i \in n$. Then the product $\prod_{i \in n} \Phi_i$ will be denoted by $[^n\Phi]$ and called the *canonical embedding* of $\mathbb{S}^n$ into $\mathbb{C}^n$.

Remark 5.3. The mapping $[\cdot]^n\Phi$ is an isometry of the metric space $(\mathbb{S}^n, D_1)$ onto the metric subspace $[\cdot]^n\Phi(\mathbb{S}^n)$ of $(\mathbb{C}^n, d_{e,2n})$ where the set $\mathbb{C}^n$ is identified with $\mathbb{R}^{2n}$ in an obvious way.

Remark 5.4. For $x, y \in \mathbb{S}^n$, let $L_{x,y}$ be the shortest broken line in $\mathbb{C}^n$ which connects $[\cdot]^n\Phi(x)$ with $[\cdot]^n\Phi(y)$ and is contained in $[\cdot]^n\Phi(\mathbb{S}^n)$. Then $D_2(x, y)$ is the length in $\mathbb{C}^n$ of the curve $L_{x,y}$, while $[x, y]_{D_2} = [\cdot]^n\Phi^{-1}(L_{x,y})$.

5.2. Geometric segments as broken lines

For $x, y \in \mathbb{R}^n$ (or $x, y \in \mathbb{C}^n$, resp.), let $[x, y] = \{(1-t)x + ty : 0 \leq t \leq 1\}$ be the traditional segment with end-points x, y .

Definition 5.5. Let $x, y \in \mathbb{S}^n$. Then:

- (i) the D_2 -segment in $\mathbb{S}^n$ between x and y will be denoted by $[x, y]_g$ and called the *geometric segment* between x and y ;
- (ii) if the traditional segment $[[\cdot]^n\Phi(x), [\cdot]^n\Phi(y)]$ in $\mathbb{C}^n$ is a subset of $[\cdot]^n\Phi(\mathbb{S}^n)$, then $[\cdot]^n\Phi^{-1}([[\cdot]^n\Phi(x), [\cdot]^n\Phi(y)])$ will be called the *traditional segment* in $\mathbb{S}^n$ between x and y .

For an arbitrary geometric segment $[a, b]_g$, let us demonstrate a method of finding a possibly small finite collection $\mathcal{C}$ of traditional segments such that $[a, b]_g = \bigcup \mathcal{C}$.

Definition 5.6. For distinct elements $u, v \in \{\oplus, \ominus, \bullet\}$, we define an embedding $\Psi_{(u,v)}: \mathbb{S}^u \cup \mathbb{S}^v \rightarrow \mathbb{R}$ as follows: for each $r \in \mathbb{R}$, we put $\bullet r = r^\bullet$, $\Psi_{(u,v)}(ur) = e^r$, while $\Psi_{(u,v)}(vr) = -e^r$. We put $\Psi_{(u,v)}(\varepsilon^\bullet) = 0$. Moreover, let $\Psi_{(u,u)}: \mathbb{S}^u \rightarrow [0; +\infty)$ be defined by the equality $\Psi_{(u,u)}(x) = \Psi_{(u,v)}(x)$ for each $x \in \mathbb{S}^u$ and for each $v \in \{\oplus, \ominus, \bullet\} \setminus \{u\}$.

Now, let us fix $n \in \omega \setminus \{0\}$ and let $a, b \in \mathbb{S}^n$ be given. Assume that $a \neq b$. For each $j \in n$ we choose an ordered pair $(u(j), v(j)) \in \{\oplus, \ominus, \bullet\} \times \{\oplus, \ominus, \bullet\}$ such that $a(j) \in \mathbb{S}^{u(j)}$ and $b(j) \in \mathbb{S}^{v(j)}$. Let $\Psi = \prod_{j \in n} \Psi_{(u(j), v(j))}$. To simplify notation, put $\Psi_j = \Psi_{(u(j), v(j))}$ for $j \in n$. Consider the traditional segment $[\Psi(a), \Psi(b)]$ in $\mathbb{R}^n$. Let J be the set of all $j \in n$ which have the property that there exists a unique $t_j \in [0, 1]$ such that $(1-t_j)\Psi_j(a(j)) + t_j\Psi_j(b(j)) = 0$ and let $K \subseteq J$ be a maximal subset of J such that, for all $l, p \in K$, $t_l \neq t_p$ if $l \neq p$. Suppose that the set K is nonempty and has exactly k elements. Let $j_0, \dots, j_{k-1}$ be the injective sequence of all elements of K such that $t_{j_l} \leq t_{j_p}$ if $l, p \in k$ and $l \leq p$. Put $x_p = (1-t_{j_p})\Psi(a) + t_{j_p}\Psi(b)$ for each $p \in k$. We get the broken line $L = [\Psi(a), x_0] \cup \bigcup_{p \in k-1} [x_p, x_{p+1}] \cup [x_{k-1}, \Psi(b)]$. Then $\Psi^{-1}(L)$ is the geometric segment $[a, b]_g$, while the length $|L|$ of L is equal to $D_2(a, b)$. Now, it is easily seen that every geometric segment in $\mathbb{S}^n$ is a union of at most $n+1$ traditional segments. The following example shows how to use this method in practice.

Example 5.7. Let us consider points $a, b \in \mathbb{S}^3$ where $a = (\oplus 0, \ominus \ln 3, [\ln 2]^\bullet)$ and $b = (\ominus 0, 0^\bullet, \oplus 0)$. To follow the method described above, we put $u(0) = \oplus$, $v(0) = \ominus$, $u(1) = \ominus$, $v(1) = \bullet$, $u(2) = \bullet$, $v(2) = \oplus$ and $\Psi = \prod_{j \in 3} \Psi_{(u(j), v(j))}$. For

$t \in [0, 1]$, we have $(1-t)\Psi(a) + t\Psi(b) = (1-2t, 3-4t, 2-3t)$. Then $K = \{0, 1, 2\}$, while $t_0 = \frac{1}{2}$, $t_1 = \frac{3}{4}$, $t_2 = \frac{2}{3}$. In consequence, $x_0 = (0, 1, \frac{1}{2})$, $x_1 = (-\frac{1}{3}, \frac{1}{3}, 0)$, $x_2 = (-\frac{1}{2}, 0, -\frac{1}{4})$. For the broken line $L = [\Psi(a), x_0] \cup [x_0, x_1] \cup [x_1, x_2] \cup [x_2, \Psi(b)]$, we have $|L| = D_2(a, b) = \sqrt{29}$. In this case, $[a, b]_g$ is a union of four traditional segments in $\mathbb{S}^3$.

Remark 5.8. Let $u, v \in \{\oplus, \ominus, \bullet\}$ be distinct. Suppose that $a \in \mathbb{S}^u \setminus \{\varepsilon^\bullet\}$, while $b \in \mathbb{S}^v \setminus \{\varepsilon^\bullet\}$. Then $[a, b]_{d_1} = \{a, b\}$.

6. Completeness in $\mathbb{S}^n$

According to [11] and [15], to consider completeness in metric spaces in the absence of the axiom of choice, it is necessary to make a distinction between the notions of metric completeness and Cantor completeness.

6.1. Completeness and Cantor completeness of metrics

To make our terminology clear, we recall the following definitions with an additional notation which is more suitable to our aims:

Definition 6.1. Let (X, d) be a metric space and let $Y \subseteq X$. It is said that:

- (i) the set Y is **complete** in (X, d) or, equivalently, **d -complete** in X if each d -Cauchy sequence of points of Y converges in (X, d) to a point from Y ;
- (ii) the set Y is **Cantor complete** in (X, d) or, equivalently, Y is **Cantor d -complete** in X if, for each sequence (F_n) of nonempty d -closed in Y sets such that $F_{n+1} \subseteq F_n$ for each $n \in \omega \setminus \{0\}$ and $\lim_{n \rightarrow +\infty} \text{diam}_d(F_n) = 0$, we have $\bigcap_{n \in \omega \setminus \{0\}} F_n \neq \emptyset$ where $\text{diam}_d(F_n)$ denotes the diameter of F_n in the metric space (X, d) .

All Cantor complete subspaces of the metric space (X, d) are closed in (X, d) but, in some models of **ZF**, complete subsets of metric spaces need not be closed (see [11], Proposition 6 of [15] and Theorem 4.55 of [12]).

Let us recall that **CC** is the axiom of countable choice which states that every nonempty countable collection of nonempty sets has a choice function (see Definition 2.5 of [12] and Form 8 in [14]). In the light of Theorem 7 of [15], that every complete metric space is Cantor complete is an equivalent of **CC**.

6.2. Completeness in $\mathbb{S}$

Theorem 6.2. Let $i \in \{1, 2\}$. Then $(\mathbb{S}, d_i)$ is a Cantor complete metric space. Therefore, all closed subspaces of $\mathbb{S}$ are d_i -complete. That all d_i -complete subspaces of $\mathbb{S}$ are closed in $\mathbb{S}$ is an equivalent of **CC**($c\mathbb{R}$).

Proof. The subspace $\Phi(\mathbb{S})$ of the metric space $(\mathbb{C}, d_e)$ is Cantor complete. This, together with Remark 4.9, implies that the metric spaces $(\mathbb{S}, d_1)$ and $(\mathbb{S}, d_2)$ are

Cantor complete. Now, suppose that $\mathbf{CC}(c\mathbb{R})$ fails. Then, by Theorem 4.55 of [12], $\mathbb{R}$ is not sequential. Let X be a subset of $\mathbb{R}_+$ such that X is sequentially closed but not closed in $\mathbb{R}$. Let us consider X as a metric subspace of the complete metric space $(\mathbb{R}, d)$ where $d(x, y) = |x - y|$ for $x, y \in \mathbb{R}$. Then the metric d is complete on X . Since the metric space X is isometric with a metric subspace of the metric subspace $\Phi(\mathbb{S}^\oplus)$ of the semidendrite $\Phi(\mathbb{S})$, it follows that, for $i \in \{1, 2\}$, not all complete subspaces of $(\mathbb{S}, d_i)$ are closed in $\mathbb{S}$.

Finally, let us assume that $\mathbf{CC}(c\mathbb{R})$ holds. Then, by Theorem 4.55 of [12], all complete subspaces of $\mathbb{R}$ are closed in $\mathbb{R}$. This implies that all complete subspaces of $(\mathbb{S}, d_i)$ are closed in $\mathbb{S}$ where $i \in \{1, 2\}$. $\square$

Remark 6.3. For $i \in \{1, 2\}$, it is independent of $\mathbf{ZF}$ that all d_i -complete subspaces of $\mathbb{S}$ are closed in $\mathbb{S}$. In the light of Remark 2.7, for instance, in model $\mathcal{M}1$ of [14], there exist metric subspaces of $(\mathbb{S}, d_i)$ that are simultaneously complete and not closed.

Remark 6.4. There is a mistake in Condition 7 of Theorem 4.54 of [12]. Namely, Condition 7 of Theorem 4.54 on page 74 of [12] is the following statement:

(*) *Each second countable topological space is separable.*

It is worth noticing that (*) is equivalent $\mathbf{CC}$. To show that (*) implies $\mathbf{CC}$, one can use the method of the proof to Theorem 4.64 in [12]. It is well known by topologists and set theorists that $\mathbf{CC}$ implies (*). Condition 7 of Theorem 4.54 of [12] should be replaced by the following statement:

(**) *Every second countable T_0 -space is separable.*

We know from a private communication of S. da Silva with E. Wajch that Horst Herrlich was aware of his mistake after [12] had appeared in print.

6.3. Completeness of the main metrics of $\mathbb{S}^n$

One can deduce from Theorem 6.2 and from the definitions of $\rho_{k,i}$ (see Subsection 5.1) that the following theorem holds:

Theorem 6.5. *Let $i \in \{1, 2\}$ and $k \in 3$. Then, for each $n \in \omega \setminus \{0\}$, the metric space $(\mathbb{S}^n, \rho_{k,i})$ is Cantor complete, so all closed subspaces of $\mathbb{S}^n$ are $\rho_{k,i}$ -complete.*

Problem 6.6. *Let $i \in \{1, 2\}, k \in 3$. Consider any $n \in \omega \setminus \{0\}$ with $n > 1$. Does $\mathbf{CC}(c\mathbb{R})$ imply in $\mathbf{ZF}$ that all $\rho_{k,i}$ -complete subspaces of $\mathbb{S}^n$ are closed in $\mathbb{S}^n$?*

A satisfactory solution to Problem 6.6 is unknown to us and it may be complicated. However, the following proposition follows easily from known facts:

Proposition 6.7. *Let $i \in \{1, 2\}, k \in 3$ and let $n \in \omega \setminus \{0\}$. If $\mathbf{CC}(\mathbb{R})$ holds, then each $\rho_{k,i}$ -complete subspace of $\mathbb{S}^n$ is closed in $\mathbb{S}^n$.*

Proof. Suppose that $\mathbf{CC}(\mathbb{R})$ holds. Let $X \subseteq \mathbb{S}^n$. In view of Theorem 4.54 of [12], the subspace X of $\mathbb{S}^n$ is separable. If X is not closed in $\mathbb{S}^n$, it follows from the separability of X and from the first-countability of $\mathbb{S}^n$ that there exists a sequence

of points of X which converges in $\mathbb{S}^n$ to a point from $\mathbb{S}^n \setminus X$. Then X is not $\rho_{k,i}$ -complete. $\square$

Definition 6.8. For a positive integer n , let us say that a set $A \subseteq \mathbb{R}^n$ is *well placed* if, for each open subset V of $\mathbb{R}^n$ such that $V \cap A \neq \emptyset$, there exist $p, q \in V \cap \mathbb{Q}^n$ such that $[p, q] \cap A \neq \emptyset$ where $[p, q]$ is the standard segment in $\mathbb{R}^n$ with end-points p and q .

The following theorem leads to a partial solution to Problem 6.6:

Theorem 6.9. *For each positive integer $n > 1$, the following conditions are all equivalent in **ZF**:*

- (i) **CC**($c\mathbb{R}$);
- (ii) *each well placed complete subset of $\mathbb{R}^n$ is closed in $\mathbb{R}^n$;*
- (iii) *each well placed, simultaneously connected and complete subset of $\mathbb{R}^n$ is closed in $\mathbb{R}^n$;*
- (iv) *each well placed, simultaneously connected and complete subspace of the plane $\mathbb{R}^2$ is closed in $\mathbb{R}^2$.*

Proof. It is clear that the implications (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) hold. Now assume (i) and use some ideas of the proof that (7) implies (5) in Theorem 4.55 in [12]. Consider a well placed, nonempty complete subset X of $\mathbb{R}^n$. Let $J = \{(p, q) \in \mathbb{Q}^n \times \mathbb{Q}^n : [p, q] \cap X \neq \emptyset\}$. For each pair $(p, q) \in J$, let $Y(p, q) = \{t \in [0, 1] : (1 - t)p + tq \in X\}$. Then $Y(p, q)$ is a complete subspace of $\mathbb{R}$. Under the assumption that **CC**($c\mathbb{R}$) holds, there exists $f \in \prod_{(p, q) \in J} Y(p, q)$. Since X is well placed, the set $\{f(p, q) : (p, q) \in J\}$ is dense in X . Hence X is separable. Since every separable complete subset of a metric space is closed, the set X is closed in $\mathbb{R}^n$. In consequence, (i) $\Rightarrow$ (ii).

Now, suppose that **CC**($c\mathbb{R}$) fails. In view of Theorem 4.55 of [12], there exists a sequentially closed subset A of $\mathbb{R}$ such that A is not closed in $\mathbb{R}$. Let $I = [0, 1]$ be the unit interval of $\mathbb{R}$ and let $X = (A \times I) \cup (\mathbb{R} \times \{1\})$. Of course X is well placed and connected in $\mathbb{R}^2$. Let $(z_n)_{n \in \omega \setminus \{0\}}$ be a Cauchy sequence in X and let x_n, y_n be real numbers such that z_n is the ordered pair (x_n, y_n) for $n \in \omega \setminus \{0\}$. Put $C = \{a \in A : (\{a\} \times I) \cap \{z_n : n \in \omega \setminus \{0\}\} \neq \emptyset\}$. Suppose that C is infinite. Then there exists an injective subsequence (z_{n_k}) of (z_n) such that $x_{n_k} \in A$. The sequence (x_{n_k}) is a Cauchy sequence of points of A . Let x_0 be the limit of (x_{n_k}) in $\mathbb{R}$. Since A is sequentially closed, we have $x_0 \in A$. Suppose that the segment $\{x_0\} \times I$ does not contain accumulation points of the sequence (z_{n_k}) . Then, by the compactness of $\{x_0\} \times I$, there exists an open set V in $\mathbb{R}^2$ such that $(\{x_0\} \times I) \subseteq V$ and $V \cap \{z_{n_k} : k \in \omega \setminus \{0\}\} = \emptyset$. This is impossible because x_0 is the limit of (x_{n_k}) .

The contradiction obtained shows that the sequence (z_{n_k}) has an accumulation point $z_0 \in \{x_0\} \times I$. Then z_0 is the limit of (z_n) in X . Now, suppose that C is finite. Then the subspace $Y = (\mathbb{R} \times \{1\}) \cup \bigcup_{a \in C} (\{a\} \times I)$ is closed in $\mathbb{R}^2$ and, moreover, there exists $n_0 \in \omega \setminus \{0\}$ such that $z_n \in Y$ for each natural number

$n > n_0$. Since Y is complete and closed in $\mathbb{R}^2$, the limit of (z_n) in $\mathbb{R}^2$ belongs to Y . In consequence, X is complete. Of course, X is not closed in $\mathbb{R}^2$. $\square$

Corollary 6.10. *Let $n > 1$ be a positive integer. Let $i \in \{1, 2\}$ and $k \in 3$. If all simultaneously $\rho_{k,i}$ -complete and connected subspaces of $\mathbb{S}^n$ are closed in $\mathbb{S}^n$, then $\text{CC}(c\mathbb{R})$ holds.*

Problem 6.6 can be regarded as equivalent to the following one:

Problem 6.11. *Let $n \in \omega \setminus \{0\}$ and $n > 1$. If $\mathbf{M}$ is a model of $\mathbf{ZF}$ such that $\text{CC}(c\mathbb{R})$ holds in $\mathbf{M}$, is it true in $\mathbf{M}$ that every complete subspace of $\mathbb{R}^n$ is closed in $\mathbb{R}^n$?*

7. Product semimodules from SMPA

Let $n \in \omega \setminus \{0\}$. For all $x, y \in \mathbb{S}^n$, one can define $x \oplus y \in \mathbb{S}^n$ and $x \otimes y \in \mathbb{S}^n$ in an obvious way:

$$(x \oplus y)(i) = x(i) \oplus y(i) \quad (15)$$

$$(x \otimes y)(i) = x(i) \otimes y(i) \quad (16)$$

where, for each $i \in n$, $x(i) \oplus y(i)$ and $x(i) \otimes y(i)$ are defined as in Corollary 4.4. Then $(\mathbb{S}^n, \oplus, \otimes)$ is a semiring.

7.1. $\mathbb{S}$ as a semimodule

By a scalar we mean an element of $\mathbb{R}_\varepsilon$. For $a \in \mathbb{S}$, $\lambda \in \mathbb{R}_\varepsilon$, we define

$$\lambda \otimes a = \text{sgn}(a)(\lambda + |a|). \quad (17)$$

The operation $\otimes: \mathbb{R}_\varepsilon \times \mathbb{S} \rightarrow \mathbb{S}$ is well defined, and we can consider it as an outer operation from $\mathbb{R}_\varepsilon \times \mathbb{S}$ to $\mathbb{S}$ or the restriction to $\mathbb{R}_\varepsilon \times \mathbb{S}$ of the inner operation $\otimes$ in $\mathbb{S}_{\max}$ because $\mathbb{R}_{\max} \subset \mathbb{S}_{\max}$. Considering $\otimes$ as the outer operation, we can look at $\mathbb{S}_{\max}$ as at a semimodule (“semi-vector-space”) over the semifield $\mathbb{R}_{\max}$.

7.2. $\mathbb{S}^n$ as a semimodule

Let $n \in \omega \setminus \{0\}$. We consider $\mathbb{S}^n$ as the product semimodule with the inner operation $\oplus$ defined by equation (15) and the outer operation $\otimes: \mathbb{R}_\varepsilon \times \mathbb{S}^n \rightarrow \mathbb{S}^n$ defined as follows:

$$(\lambda \otimes a)(i) = \lambda \otimes a(i), \quad (18)$$

where $\lambda \in \mathbb{R}_\varepsilon$, $a \in \mathbb{S}^n$ and $i \in n$, while $\lambda \otimes a(i)$ is as in (17).

We deduce the following two corollaries from Propositions 4.13 and 4.14, respectively:

Corollary 7.1. *The operation $\otimes: \mathbb{S}^n \times \mathbb{S}^n \rightarrow \mathbb{S}^n$ is continuous in the usual topology of $\mathbb{S}^n$.*

Corollary 7.2. *The operation $\oplus: \mathbb{S}^n \times \mathbb{S}^n \rightarrow \mathbb{S}^n$ is discontinuous in the usual topology of $\mathbb{S}^n$.*

8. Convexities in $\mathbb{S}^n$

If, for each pair of points x, y of a space X , a segment $[x, y]$ with end-points x and y is defined, then a set $A \subseteq X$ is called convex if, for all $x, y \in A$, we have $[x, y] \subseteq A$ (see, for instance, [13]). In particular, for $x, y \in \mathbb{S}^n$, we can consider distinct kinds of segments with end-points x and y , among them, traditional segments, geometric segments and semimodule segments. These kinds of segments in $\mathbb{S}^n$ lead to distinct kinds of convexity in $\mathbb{S}^n$. Let us turn our attention to traditional, semimodule and geometric convexity in $\mathbb{S}^n$. One can also investigate $\rho_{k,j}$ -convex sets in $\mathbb{S}^n$ for each $k \in 3$ and $j \in \{1, 2\}$.

8.1. Traditionally convex sets in $\mathbb{S}^n$

Let $n \in \omega \setminus \{0\}$. A convex set in $\mathbb{C}^n$ is a set $A \subseteq \mathbb{C}^n$ such that, for each pair $x, y \in A$, the traditional segment $[x, y]$ in the affine space $\mathbb{C}^n$ is contained in A .

Definition 8.1. A set $C \subseteq \mathbb{S}^n$ is called *traditionally convex* if $[^n\Phi](C)$ is convex in $\mathbb{C}^n$.

Remark 8.2. Let us notice that a nonempty set $C \subseteq \mathbb{S}$ is simultaneously compact and traditionally convex in $\mathbb{S}$ if and only if C is a traditional segment in $\mathbb{S}$.

All traditionally convex subsets of $\mathbb{S}^n$ are connected.

8.2. Semimodule convex sets in $\mathbb{S}^n$

The semimodule structure of $\mathbb{S}^n$ allows us to think about semimodule convexity of subsets of $\mathbb{S}^n$. In analogy to conventional algebra, we define a *semimodule segment* $[a, b]_{sm}$ with end-points $a \in \mathbb{S}^n$ and $b \in \mathbb{S}^n$ as follows:

$$[a, b]_{sm} = \{(\lambda \otimes a) \oplus (\gamma \otimes b) : \lambda, \gamma \in \mathbb{R}_\varepsilon, \text{ with } \lambda \oplus \gamma = 0\}. \quad (19)$$

Definition 8.3. A set $A \subseteq \mathbb{S}^n$ is said to be *semimodule convex* if, for each pair a, b of points of A , the semimodule segment $[a, b]_{sm}$ is contained in A .

Convexity in arbitrary idempotent semimodules was introduced by Cohen, Gaubert and Quadrat [3]. Some analogues of famous theorems of functional analysis like Minkowski's theorem, separation theorem, and some notions of geometry like simplices and convex polytopes were considered in Max-Plus Algebra (for instance, one can refer to [4], [7], [8]).

Remark 8.4. We have the following semimodule segments in $\mathbb{S}$ for $r, s \in \mathbb{R}_+$ with $r < s$:

$$[\oplus r, \ominus r]_{sm} = \{\oplus r, \ominus r, r^\bullet\}, \quad (20)$$

$$[\oplus r, r^\bullet]_{sm} = \{\oplus r, r^\bullet\}, \quad (21)$$

$$[\ominus r, \oplus s]_{sm} = \{\ominus r, r^\bullet\} \cup \{\oplus t : r < t \leq s\} \quad (22)$$

$$[r^\bullet, \oplus s]_{sm} = \{r^\bullet\} \cup \{\oplus t : r < t \leq s\} \quad (23)$$

$$[\ominus r, s^\bullet]_{sm} = \{\ominus r\} \cup \{t^\bullet : r \leq t \leq s\} \quad (24)$$

Other cases of semimodule segments in $\mathbb{S}$ are similar or obvious.

Notice that the intersection of two semimodule segments may not be a semimodule segment. Also, if $[a, b]_{sm}$ and $[c, d]_{sm}$ are semimodule segments in $\mathbb{S}$ such that $[a, b]_{sm} \cap [c, d]_{sm} \neq \emptyset$, then, in general, the union $[a, b]_{sm} \cup [c, d]_{sm}$ need not be a semimodule segment. However, if $c \in [a, b]_{sm}$, then $[a, b]_{sm} = [a, c]_{sm} \cup [c, b]_{sm}$. All disconnected semimodule segments in $\mathbb{S}$ have either two or three connected components.

Example 8.5. Consider the semimodule segment in $\mathbb{S}$ between $a = \oplus 1$ and $b = \ominus 0$. Then $[a, b]_{sm} = \{b\} \cup \{0^\bullet\} \cup \{\oplus t : t \in (0, 1]\}$, so the segment $[a, b]_{sm}$ has three connected components. Moreover, this segment is not closed in $\mathbb{S}$. For $i \in \{1, 2\}$, both $\ominus 0$ and $0^\bullet$ are d_i -nearest points to $\varepsilon^\bullet$ in $[a, b]_{sm}$. More precisely, $P_{d_i, [a, b]_{sm}}(\varepsilon^\bullet) = \{\ominus 0, 0^\bullet\}$ for $i \in \{1, 2\}$.

Example 8.6. A semimodule segment in $\mathbb{S}^2$ can have five connected components. For example, consider

$$\begin{aligned} C &= [(\oplus 0, \oplus 1), (\oplus 1, \oplus 0)]_{sm} \\ &= \{(\oplus 0, \oplus 1), (\oplus 1, \oplus 0), (\oplus 1, 0^\bullet), (0^\bullet, \oplus 1)\} \cup \{(\ominus t, \ominus s) : \max(t, s) = 1, t, s > 0\}. \end{aligned}$$

Then C has exactly four isolated points and exactly one connected component of C is of the form $[\Phi^2]^{-1}(L)$ where $L \subseteq \mathbb{C}$ is a broken line with its end-points deleted.

8.3. Geometric convexity in $\mathbb{S}^n$

Definition 8.7. A set $A \subseteq \mathbb{S}^n$ is said to be *geometrically convex* if, for each pair a, b of points of A , the geometric segment $[a, b]_g$ is contained in A .

The notions of a geometrically convex set and a D_2 -convex set in $\mathbb{S}^n$ coincide. Of course, each traditionally convex subset of $\mathbb{S}^n$ is geometrically convex but not every geometrically convex set in $\mathbb{S}^n$ is traditionally convex. Since geometric segments in $\mathbb{S}^n$ are connected sets, each geometrically convex subset of $\mathbb{S}^n$ is connected. In the case of $\mathbb{S}$, we can state the following propositions:

Proposition 8.8. *The geometric segment $[a, b]_g$ between points $a, b \in \mathbb{S}$ is the intersection of all connected subsets of $\mathbb{S}$ that contain both a and b .*

Proposition 8.9. *A set $A \subseteq \mathbb{S}$ is geometrically convex if and only if A is connected.*

The proofs to Propositions 8.8 and 8.9 are so easy that we omit them. Of course, if $n > 1$ then, for distinct $a, b \in \mathbb{S}^n$, the intersection of all connected subsets of $\mathbb{S}^n$ that contain both a and b is the two-point set $\{a, b\}$.

Let us recall that, if X is a topological space, $a, b \in X$, while C is a continuum in X such that $a, b \in C$, then C is called *irreducible between a and b* if, for each continuum $D \subseteq C$ such that $a, b \in D$, the equality $D = C$ holds. The axiom of choice is used in the standard proof of the theorem of **ZFC** asserting that, for

each pair of distinct points of a Hausdorff continuum X there exists a continuum C in X such that C is irreducible between a and b .

Problem 8.10. *Is there a model $\mathbf{M}$ of $\mathbf{ZF}$ such that there exists a Hausdorff continuum X in $\mathbf{M}$ which has the property that, for a pair a, b of distinct points of X , there does not exist in $\mathbf{M}$ a subcontinuum C of X such that C is irreducible between a and b ?*

Remark 8.11. Suppose that we work inside a model $\mathbf{M}$ of $\mathbf{ZFC}$. Let X be a semidendrite in $\mathbf{M}$ and let $a, b \in X$. We can generalize properties of dendrites given on page 125 in [18] to semidendrites. Namely, we can prove that there exists in $\mathbf{M}$ a unique continuum $C \subseteq X$ such that $a, b \in C$ and C is irreducible between a and b . Let us denote this continuum C by $[a, b]_{c-g}$ and call it the *c-geometric segment* of X between a and b . Now, assume, in addition, that X is metrizable in $\mathbf{M}$ and that $a \neq b$. Then $[a, b]_{c-g}$ is an arc with end-points a and b (see Lecture VIII of [18]). Notice that $[a, b]_{c-g}$ is the intersection of all connected subspaces of X that contain both a and b , so $[a, b]_{c-g}$ is the smallest (with respect to inclusion) among all connected subsets of X that contain both a and b . Let us call a set $A \subseteq X$ *c-geometrically convex* in X if, for each pair a, b of distinct points of A , the inclusion $[a, b]_{c-g} \subseteq A$ holds. Then a set $A \subseteq X$ is connected if and only if it is c-geometrically convex in X . If $X = \mathbb{S}$, then $\mathbf{M}$ can be any model of $\mathbf{ZF}$.

Remark 8.12. For a pair a, b of points of $\mathbb{R}^2$, let $\mathcal{C}(a, b)$ be the collection of all connected subsets of $\mathbb{R}^2$ which contain both a and b . Then each continuum $C \in \mathcal{C}(a, b)$ such that C is irreducible between a and b is a minimal (with respect to inclusion) element of $\mathcal{C}(a, b)$. Since $\bigcap \{C : C \in \mathcal{C}(a, b)\} = \{a, b\}$, to define a c-geometric segment in $\mathbb{R}^2$ between a and b as the intersection $\bigcap \{C : C \in \mathcal{C}(a, b)\}$ does not lead to anything reasonable. However, it may be still interesting to investigate in deep all topological spaces X which have the property that, for each pair a, b of points of X , the intersection of all connected subsets of X that contain both a and b is connected in X .

Remark 8.13. Let $[a, b]_g$ and $[c, d]_g$ be geometric segments in $\mathbb{S}$. Then the intersection $[a, b]_g \cap [c, d]_g$ is a geometric segment in $\mathbb{S}$. Also, if $c \in [a, b]_g$, then $[a, b]_g = [a, c]_g \cup [c, b]_g$. However, if $[a, b]_g \cap [c, d]_g \neq \emptyset$, then $[a, b]_g \cup [c, d]_g$ is a dendrite which need not be a geometric segment in $\mathbb{S}$.

9. The minimizing vector theorem in $\mathbb{S}^n$

In what follows, all Hilbert spaces are assumed to be over the field $\mathbb{R}$. Given a Hilbert space H , we consider H as a metric space equipped with the metric ρ_H of H which is induced by the inner product of H , i. e. $\rho_H(x, y) = \|x - y\|$ where $\|\cdot\|$ is the norm defined by the inner product of H . The minimizing vector theorem, also called the Hilbert projection theorem, is a famous result in convex analysis saying that it holds true in $\mathbf{ZF} + \mathbf{CC}$ that, for every point $x \in H$ in a Hilbert space H and every nonempty closed convex subset C of H , there exists a unique point $y \in C$ for which $\|x - y\|$ is the distance of x from C . In particular, this is true in every model $\mathbf{M}$ of $\mathbf{ZF} + \mathbf{CC}$ for any closed affine subspace C of

H provided H is in $\mathbf{M}$ and we work on H inside $\mathbf{M}$. In the case when C is a closed affine subspace of H , a necessary and sufficient condition for $y \in C$ to be the nearest point to x in C is that the vector $x - y$ be orthogonal to C . A result of **ZF** is the following version of the vector minimizing theorem:

Theorem 9.1. *For every Hilbert space H , for each nonempty, simultaneously convex and Cantor complete subset C of H and for each $x \in H$, there exists a unique $y \in C$ such that $\|x - y\|$ is the distance of x from C .*

As we have already mentioned in subsection 6.1, according to Theorem 7 of [15], in a model of **ZF**, a complete subset of a metric space need not be Cantor complete. Therefore, one may ask whether it holds true in every model $\mathbf{M}$ of **ZF** that all simultaneously complete and convex subsets of a Hilbert space H in $\mathbf{M}$ are closed in H . A partial answer to this question is given in the following proposition:

Proposition 9.2. *Let H be a finitely dimensional Hilbert space and let C be a nonempty convex complete subset of H . Then C is closed in H .*

Proof. Let Y be the intersection of all affine subspaces of H that contain C . Since H is finitely dimensional, it is known that $\text{int}_Y(C) \neq \emptyset$. Let $a \in \text{int}_Y(C)$ and $b \in \text{cl}_Y(C)$. Then the interval $I = [a, b] = \{(1 - t)a + tb : t \in \mathbb{R}, 0 \leq t \leq 1\}$ is contained in $\text{cl}_Y(C)$, while $I \setminus \{b\}$ is a subset of $\text{int}_Y(C)$. Of course, there exists a sequence (x_n) of points of $I \setminus \{b\}$ which converges in H to b . Since (x_n) is a Cauchy sequence of points of C , we have $b \in C$ because C is complete. Hence $C = \text{cl}_Y(C) = \text{cl}_H(C)$. $\square$

At this moment, we do not know a satisfactory solution to the following problem:

Problem 9.3. *Is it true in **ZF** that, for an arbitrary infinitely dimensional Hilbert space H , all simultaneously convex and complete subsets of H are Cantor complete in H ?*

We recommend [6] as one of the newest survey articles on the minimizing vector theorem. We are going to state some simple versions of this theorem for $\mathbb{S}^n$.

9.1. A characterization of Chebyshev sets in $\mathbb{S}$

For $i \in \{1, 2\}$, the following theorem gives a complete characterization of d_i -Chebyshev sets in $\mathbb{S}$:

Theorem 9.4. *For each nonempty closed subset C of $\mathbb{S}$, the following conditions are equivalent:*

- (i) C is connected;
- (ii) C is a d_1 -Chebyshev set;
- (iii) C is a d_2 -Chebyshev set.

Proof. Assume that C is connected and $x \in \mathbb{S}$. Consider the following two cases (a) and (b):

(a) $\varepsilon^\bullet \notin C$. Then C is contained in one of the lines $\mathbb{S}^\oplus$, $\mathbb{S}^\ominus$, $\mathbb{S}^\bullet$, where connectedness is equivalent with traditional convexity, so $P_{d_1,C}(x) = P_{d_2,C}(x)$ and $P_{d_1,C}(x)$ is a singleton.

(b) $\varepsilon^\bullet \in C$. Then C is star-like; hence, it is easily seen that $P_{d_1,C}(x) = P_{d_2,C}(x)$ and $P_{d_1,C}(x)$ is a singleton which lies on the same line as x .

In consequence, (i) implies both (ii) and (iii). On the other hand, if C were disconnected, there would exist a connected component K of $\mathbb{S} \setminus C$, distinct points $a_1, a_2 \in C \cap \text{cl}_{\mathbb{S}}(K)$ and points $y_1, y_2 \in K$ such that $a_1, a_2 \in P_{d_i,C}(y_i)$ for $i \in \{1, 2\}$. Therefore, each of (ii) and (iii) implies (i). $\square$

Corollary 9.5. *If C is a nonempty, simultaneously connected and closed subset of $\mathbb{S}$, then $p_{d_1,C} = p_{d_2,C}$.*

Proposition 9.6. *For each nonempty closed set $C \subseteq \mathbb{S}$, the properties that C is geometrically convex in $\mathbb{S}$ is equivalent to each one of the conditions (i)–(iii) of Theorem 9.4.*

Let us have a brief look at semimodule convex closed subsets of $\mathbb{S}$.

Example 9.7. The semimodule segment $A = [\oplus 0, \ominus 0]_{sm} = \{\oplus 0, \ominus 0, 0^\bullet\}$ is a closed, semimodule convex set in $\mathbb{S}$ with three connected components. For each $i \in \{1, 2\}$, we have $P_{d_i,A}(\varepsilon^\bullet) = A$.

Proposition 9.8. *Let C be a nonempty, simultaneously semimodule convex and closed subset of $\mathbb{S}$. Then C is d_i -proximal and, for each $x \in \mathbb{S}$, the set $P_{d_i,C}(x)$ consists of at most three points.*

Proof. This is clear since, on each of the lines $\mathbb{S}^\oplus$, $\mathbb{S}^\ominus$, $\mathbb{S}^\bullet$, semimodule convexity agrees with the traditional convexity. $\square$

9.2. Chebyshev sets in $\mathbb{S}^n$

Definition 9.9. A set $A \subseteq \mathbb{S}^n$ is said to be *box semimodule convex* if it is a Cartesian product $\prod_{i \in n} A_i$ of semimodule convex subsets A_i of $\mathbb{S}$.

We can generalize Proposition 9.8 as follows:

Proposition 9.10. *Let $j, k \in \{1, 2\}$. Suppose that A is a nonempty, closed, box semimodule convex subset of $\mathbb{S}^n$. Then A is $\rho_{k,j}$ -proximal and, moreover, for each $x \in \mathbb{S}^n$, the set $P_{\rho_{k,j},A}(x)$ consists of at most 3^n points.*

Proof. Since closed balls in $(\mathbb{S}^n, \rho_{k,j})$ are compact, each nonempty closed set in $\mathbb{S}^n$ is $\rho_{k,j}$ -proximal. That $P_{\rho_{k,j},A}(x)$ consists of at most 3^n points follows directly from Theorem 2.19 (i) and from Proposition 9.8. $\square$

Example 9.11. In view of Example 9.7, the set $A = \{\oplus 0, \ominus 0, 0^\bullet\}^n$ in $\mathbb{S}^n$ is box semimodule convex. Let $x_0 \in \mathbb{S}^n$ be defined by: $x_0(i) = \varepsilon^\bullet$ for each $i \in n$. Then, if $k \in 3$ and $j \in \{1, 2\}$, the set $P_{\rho_{k,j},A}(x_0)$ consists of exactly 3^n points.

Theorem 9.12. *Let $j, k \in \{1, 2\}$. Suppose that $A = \prod_{i \in n} A_i$ where each A_i is nonempty and closed in $\mathbb{S}$. Then A is $\rho_{k,j}$ -Chebyshev in $\mathbb{S}^n$ if and only if A is connected.*

Proof. It suffices to apply Theorems 2.19 (ii) and 9.4. □

The following problem is still unsolved:

Problem 9.13. *Let $i \in \{1, 2\}$ and A be a nonempty closed subset of $\mathbb{S}^n$ where $n > 1$. Is it true that A is D_i -Chebyshev if and only if A is geometrically convex in $\mathbb{S}^n$?*

We leave Problem 9.13 for future investigations in another work.

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