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Convex Decompositions of Convex Open Sets with Polytopes or Finite Sets Removed

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We provide convex decompositions for the convex open sets with polytopes or finite sets removed, some of which are minimal in a certain sense. The valence of a function $f\colon O \to \mathbb{R}^n$, whose restrictions to all convex subsets of $O \subseteq \mathbb{R}^n$ are injective, cannot exceed the number of convex components of such decompositions. It is therefore worth to investigate the smallest number of convex subsets of O needed to cover O. While the convex decompositions of the mentioned open complements are the main issue of this paper, a few remarks on this smallest number are provided by the end of the paper in the last section.

Keywords: Convex decompositions, CIP-functions, valence of a function.

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1. Introduction

The coverings by convex subsets of a set $O \subseteq \mathbb{R}^n$, equally called convex decompositions of O, were studied before by Breen [2] , McKinney [8], Valentine [16] or Lawrence, Hare and Kenell [7]. For example Valentine [16] proved that the 3-convex closed subsets of the plane $\mathbb{R}^2$ are decomposable in unions of three or fewer closed convex sets. On the other hand, Breen [2] proved that the 3-convex open subsets of the plane are decomposable in unions of four or fewer convex sets. Recall that a subset S of $\mathbb{R}^d$ is said to be *3-convex* if for every three distinct points in S, at least one of the segments determined by these points lies in S. A characterization of the sets which are decomposable into k convex subsets is given by Lawrence, Hare and Kenell [7] with no restrictions on the set.

In this paper we provide convex decompositions for the convex open sets with some compact subsets removed. We focus our attention on finite sets and polytopes to play the role of the compact sets. We also observe here that such decompositions of a set $O \subseteq \mathbb{R}^n$, assumed to be open all over the paper, provide upper bounds for the cardinalities of the inverse images of any function $f\colon O \to \mathbb{R}^n$ whose restrictions to the convex subsets of O are all injective. Such maps are said to satisfy the *convex injectivity property* (CIP) and we call them *CIP-functions.* More precisely, the number of solutions of any equation $f(x) = y$, where $f\colon O \to \mathbb{R}^n$ is

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a CIP-function and $y \in \mathbb{R}^n$, does not exceed the number of convex subsets of O needed to cover O. Note that every function satisfying the requirements of some global injectivity criterion, except the convexity assumption on its domain, is a CIP function. In other words, every global injectivity criterion is potentially a source of CIP functions if we get rid of the convexity assumption on the domain of the function involved in the criterion.

The paper is organized as follows: In the next section we state the convex decomposition results on the convex open sets with finite sets or polytopes removed. We provide there finitely many convex (open) sets which cover such complements. One result concerns the open sets with codimension one smooth boundaries. The later ones cannot be covered by finitely many convex sets. The next three sections are devoted to the proofs of the decompositions results stated in the second section. In the last section we provide a few remarks on the smallest number of convex open sets needed to cover the open sets $O \subseteq \mathbb{R}^n$, called the *convex decomposition number* of O, as upper bound for the valence of CIP-functions $f\colon O \to \mathbb{R}^n$.

2. Convex decomposition results

Theorem 2.1. *A convex open subset of $\mathbb{R}^n$ with k points removed can be covered by $k+1$ convex sets.*

In particular, a punctured open convex subset of $\mathbb{R}^n$ can be covered by two convex sets and this is a first step for the proof of Theorem 2.1 (see Lemma 3.1).

Theorem 2.2. *If $D \subseteq \mathbb{R}^n$ is a convex open set and $p \in D$, then $D \setminus \{p\}$ can be covered by $n+1$, but not fewer, convex open sets.*

To state the next result on the first class of open subsets of $\mathbb{R}^n$ we need to recall that the facets of an r-dimensional polyhedral subset of $\mathbb{R}^n$ are its proper $(r-1)$-dimensional faces (see [18, p. 110]). Note that the compact polyhedra concide with the class of compact convex sets which can be represented as the convex hulls of finite sets and they are called polytopes (see [18, p. 105,114, Theorem 3.2.5]).

Theorem 2.3. *Let P be a compact convex n-dimensional subset of $\mathbb{R}^n$. If P is a polytope with s facets, then $\mathbb{R}^n \setminus P$ can be covered by s, but not fewer, open convex sets. Conversely, if for some neighbourhood C of P the set $C \setminus P$ can be covered by s convex (possibly non-open) sets, then P is a polytope with at most s facets.*

Theorem 2.4. *A non-convex connected and bounded open set in $\mathbb{R}^2$ with a smooth boundary cannot be covered by finitely many convex sets.*

3. The proofs of Theorems 2.1 and 2.2

We shall first prove the following

Lemma 3.1. *If $D \subseteq \mathbb{R}^n$ is a convex open set and $p \in D$, then the non- convex punctured set $D \setminus \{p\}$ can be covered by two convex sets.*

Proof. We proceed by induction with respect to n. For $n = 1$ the open convex set $D \subseteq \mathbb{R}$ is an open interval and the statement is obviously true in this case. We next assume the statement true for $n = k$, assume $D \subseteq \mathbb{R}^{k+1}$, consider a linear functional $\varphi\colon \mathbb{R}^{k+1} \to \mathbb{R}$ and observe that $D \setminus \{p\}$ is covered by the union

$$\big(D \cap \varphi^{-1}(-\infty, \varphi(p))\big) \cup \big[\big(D \cap \varphi^{-1}(\varphi(p))\big) \setminus \{p\}\big] \cup \big(D \cap \varphi^{-1}(\varphi(p), \infty)\big).$$

Also $D \cap \varphi^{-1}(-\infty, \varphi(p))$ and $D \cap \varphi^{-1}(\varphi(p), \infty)$ are convex open subsets of D and $(D \cap \varphi^{-1}(\varphi(p))) \setminus \{p\}$ can be covered by two of its convex subsets, due to the induction hypothesis, say $(D \cap \varphi^{-1}(\varphi(p))) \setminus \{p\} = C_1 \cup C_2$, where C_1, C_2 are convex subsets of $(D \cap \varphi^{-1}(\varphi(p))) \setminus \{p\}$. Thus

$$\begin{aligned} D \setminus \{p\} &= (D \cap \varphi^{-1}(-\infty, \varphi(p))) \cup (C_1 \cup C_2) \cup (D \cap \varphi^{-1}(\varphi(p), \infty)) \\ &= [(D \cap \varphi^{-1}(-\infty, \varphi(p))) \cup C_1] \cup [C_2 \cup (D \cap \varphi^{-1}(\varphi(p), \infty))]. \end{aligned}$$

We shall only show now that $(D \cap \varphi^{-1}(-\infty, \varphi(p))) \cup C_1$ is convex, as the convexity of $C_2 \cup (D \cap \varphi^{-1}(\varphi(p), \infty))$ can be proved similarly. In this respect we consider $x, y \in (D \cap \varphi^{-1}(-\infty, \varphi(p))) \cup C_1$ and show that for every $t \in [0, 1]$ we have $(1-t)x + ty \in (D \cap \varphi^{-1}(-\infty, \varphi(p))) \cup C_1$. If either $x, y \in D \cap \varphi^{-1}(-\infty, \varphi(p))$ or $x, y \in C_1$, then there is nothing to be proved, as the sets $D \cap \varphi^{-1}(-\infty, \varphi(p))$ and C_1 are both convex. Thus, we only need to treat the case $x \in D \cap \varphi^{-1}(-\infty, \varphi(p))$ and $y \in C_1$. Since $C_1, D \cap \varphi^{-1}(-\infty, \varphi(p)) \subseteq D$ and D is convex, it follows that $(1-t)x + ty \in D$, for every $t \in [0, 1]$. On the other hand, taking into account that $x \in \varphi^{-1}(-\infty, \varphi(p))$ and $y \in C_1 \subseteq \varphi^{-1}(\varphi(p))$, it follows that for $t \in [0, 1)$ we have

$$\begin{aligned} \varphi((1-t)x + ty) &= (1-t)\varphi(x) + t\varphi(y) = (1-t)\varphi(x) + t\varphi(p) \\ &< (1-t)\varphi(p) + t\varphi(p) = \varphi(p), \end{aligned}$$

i.e. $(1-t)x + ty \in \varphi^{-1}(-\infty, \varphi(p))$. Thus, for every $t \in [0, 1]$,

$$(1-t)x + ty \in \big(D \cap \varphi^{-1}(-\infty, \varphi(p))\big) \cup C_1. \qquad \square$$

Remark 3.2. If A is an a-dimensional affine variety of $\mathbb{R}^n$, then the non-convex set $\mathbb{R}^n \setminus A$ can be covered by two of its convex subsets: $\mathbb{R}^n \setminus A = \pi^{-1}(C_1) \cup \pi^{-1}(C_2)$, where C_1, C_2 is a convex cover of some punctured space $B \setminus \{p\}$ where B is some $(n-a)$-dimensional affine variety orthogonal to A and $\pi\colon \mathbb{R}^n \setminus A \to B \setminus \{p\}$ is the orthogonal projection of $\mathbb{R}^n \setminus A$ on $B \setminus \{p\}$. The convexity of C_1, C_2 and the liniarity of π ensure the convexity of $\pi^{-1}(C_1)$ and $\pi^{-1}(C_2)$.

Proof of Theorem 2.1. Since the case $n = 1$ is rather obvious, we shall only treat here the case $n \geq 2$. Assume that $\{f_1, \ldots, f_k\}$ are the k points removed. Note that the union

$$\bigcup_{1 \leq i,j \leq k} f_{ij}^{\perp} \cap S^{n-1}$$

cannot cover the whole sphere S^{n-1}, where

$$f_{ij} := \frac{f_i - f_j}{\|f_i - f_j\|} \text{ and } f_{ij}^{\perp} = \{x \in \mathbb{R}^n \mid \langle x, f_{ij} \rangle = 0\}.$$

Indeed, the topological dimension of S^{n-1} is $n-1$ and the topological dimension of the union is $n-2$. Consider a vector

$$v \in S^{n-1} \setminus \bigcup_{1 \le i,j \le q} f_{ij}^{\perp} \cap S^{n-1},$$

and observe that the restriction $\varphi|_F$ is one-to-one, where F stands for the set $\{f_1, \ldots, f_k\}$ and $\varphi\colon \mathbb{R}^n \to \mathbb{R}$ is the linear functional defined by $\varphi(x) = \langle x, v\rangle$. We may therefore assume that $\varphi(f_1) < \cdots < \varphi(f_k)$ which leads to the following decomposition

$$D \setminus F = \bigcup_{i=1}^{k+1} \big(D \cap \varphi^{-1}(\varphi(f_{i-1}), \varphi(f_i))\big) \cup \left[\big(\varphi^{-1}(\varphi(f_i)) \cap D\big) \setminus \{f_i\}\right],$$

where $\varphi(f_0) := -\infty$ and $\varphi(f_{k+1}) := \infty$. According to Lemma 3.1, the non-convex set $(\varphi^{-1}(\varphi(f_i)) \cap D) \setminus \{f_i\}$ can be covered by two of its convex subsets, namely $(\varphi^{-1}(\varphi(f_i)) \cap D) \setminus \{f_i\} = C_i \cup K_i$. Thus

$$\begin{aligned} D \setminus F &= \bigcup_{i=1}^{k+1} \big(D \cap \varphi^{-1}(\varphi(f_{i-1}), \varphi(f_i))\big) \cup (C_i \cup K_i) \\ &= \bigcup_{i=1}^{k+1} C_{i-1} \cup \big(D \cap \varphi^{-1}(\varphi(f_{i-1}), \varphi(f_i))\big) \cup K_i \end{aligned}$$

where $C_0 = K_{k+1} = \emptyset$. Thus, the statement is now completely proved, as one can easily show that the sets $C_{i-1} \cup (D \cap \varphi^{-1}(\varphi(f_{i-1}), \varphi(f_i))) \cup K_i$, $1 \le i \le k+1$ are convex.

Proof of Theorem 2.2. Let C_i denote the convex open subset

$$\{x \in D \setminus \{p\} \mid \langle x - p, e_i\rangle > 0\}$$

of $D \setminus \{p\}$, for $i = 1, \ldots, n+1$, where $e_1, \ldots, e_n$ is the standard basis of $\mathbb{R}^n$ and $e_{n+1} := -e_1 - \cdots - e_n$. Observe that

$$D \setminus \{p\} = C_1 \cup \cdots \cup C_n \cup C_{n+1},$$

as the inequalities $\langle x - p, e_i\rangle \le 0$, for some $x \in D \setminus \{p\}$ and $i = 1, \ldots, n+1$ would imply $\langle x - p, e_{n+1}\rangle > 0$. Indeed we have

$$\begin{aligned} \langle x - p, e_{n+1}\rangle \le 0 \quad &\Leftrightarrow \ \langle x - p, e_1 + \cdots + e_n\rangle \ge 0 \\ &\Leftrightarrow \ \langle x - p, e_1\rangle + \cdots + \langle x - p, e_n\rangle \ge 0 \\ &\Leftrightarrow \ \langle x - p, e_1\rangle + \cdots + \langle x - p, e_n\rangle = 0 \\ &\Leftrightarrow \ \langle x - p, e_i\rangle, \ \text{for } i = 1, \ldots, n \\ &\Leftrightarrow \ x - p = 0, \ \text{impossible since } x \in D \setminus \{p\}. \end{aligned}$$

We now assume that $D \setminus \{p\}$ admits a finite cover by $m < n$ open convex subsets, say $D = \{V_1, \ldots, V_m\}$. Consider some nonzero linear functionals $\psi_i\colon \mathbb{R}^n \to \mathbb{R}$,

$\psi_i(x) = \langle x, v_i \rangle$ such that $\psi_i(p) \leq \psi_i(x)$ for all $x \in V_i$. Since V_i is open, it follows that $\psi_i(p) < \psi_i(x)$ for all $x \in V_i$, as every possible point $q \in V_i$ with the property $\psi_i(q) = \psi_i(p)$ would imply, due to the openness of V_i , the existence of some other point $q \in V_i$ such that $\psi_i(q) < \psi_i(p)$, impossible since $\psi_i\big|_{V_i} \geq \psi_i(p)$. Thus

$$\mathcal{D}' = \{V_1', \ldots, V_m'\}$$

is another covering of $D \setminus \{p\}$ by the m open convex subsets

$$V_i' := \{x \in D \setminus \{p\} : \langle x - p, v_i \rangle > 0\}, \quad i = 1, \ldots, m$$

such that $V_i \subseteq V_i'$ for $i = 1, \ldots, m$. The intersection

$$U := \{v \in \mathbb{R}^n \setminus \{p\} : \langle x - p, v_i \rangle = 0\}$$

of the punctured hyperplanes $\psi_i^{-1}(\psi_i(p)) \setminus p$, $i = 1, \ldots, m$ has dimension n–r, where $r = \dim \operatorname{Span}\{v_1, \ldots, v_m\}$. Since $r \leq m \leq n-1$, it follows that the intersection $U \cap (D \setminus \{p\})$ is not empty and obviously not contained in $\bigcup_{i=1}^m V_i'$, a contradiction with the quality of the collection $\{V_1', \ldots, V_m'\}$ to cover $D \setminus \{p\}$.

4. The proof of Theorem 2.3

We first recall that the polytope P can be equally represented as the convex closure of a finite set of extreme points (vertices) $v_1, \ldots, v_r$ as well as the intersection

$$\bigcap_{i=1}^{s} \varphi_i^{-1}(-\infty, a_i],$$

where $\varphi_i \colon \mathbb{R}^n \to \mathbb{R}$ are some linearly independent functionals and where a_i ($i = 1, \ldots, k$) are some real constants (see e.g. [18, Th. 3.2.5]). Recall also that the facets of P are the $(n-1)$-dimensional faces of P and s is the number of facets of P in this case. Thus

$$C \setminus P = C \setminus \bigcap_{i=1}^{s} \varphi_i^{-1}(-\infty, a_i] = \bigcup_{i=1}^{s} \big(C \setminus \varphi_i^{-1}(-\infty, a_i]\big) = \bigcup_{i=1}^{s} C \cap \varphi_i^{-1}(a_i, +\infty),$$

and the last above representation provides a covering by s convex open sets of $C \setminus P$, as $C \cap \varphi_i^{-1}(a_i, +\infty)$ are convex open subsets of C. Note that the facets of P are, according with [18, Th. 3.2.1 (iii)], the sets

$$\varphi^{-1}(a_1) \cap P, \ldots, \varphi^{-1}(a_s) \cap P.$$

We shall only prove now the converse part of the statement, as the proof of the fact that $\mathbb{R}^n \setminus P$ cannot be covered by a smaller number of convex open sets can be similarly done.

If $C \setminus P$ can be covered by finitely many convex sets $X_1, ..., X_s$, then for $i = 1, ..., s$ there is a non-zero affine functional ψ_i such that $\psi_i\big|_P \leq 0$ and $\psi_i\big|_{X_i} \geq 0$. The set $Q = \bigcap_{i=1}^{s} \psi_i^{-1}(-\infty, 0]$ is a polyhedron containing P. Since the interior of P is non-empty, the relation $Q \supsetneqq P$ would imply that the set $Q \setminus P$ contains an open subset of C, which is impossible, since $Q \setminus P$ is contained in the union of the hyperplanes $\varphi_i^{-1}(0)$, as we shall show, and no such an open set is covered by finitely many affine hyperplanes $\varphi_i^{-1}(0)$. Hence $P = Q$ and we are done, as Q has at most s facets. In order to show that $Q \setminus P \subseteq \cap_{i=1}^{s} \psi_i^{-1}(0)$, we first observe that

$$Q \setminus P \subseteq C \setminus P \subseteq \bigcup_{i=1}^{s} X_i \subseteq \bigcup_{i=1}^{s} \psi_i^{-1}[0, \infty).$$

Thus, for $q \in Q \setminus P$ there exists $1 \leq j \leq s$ such that $\varphi_j(q) \geq 0$. On the other hand $\varphi_j(q) \leq 0$ as $q \in Q \setminus P \subseteq Q = \cap_{i=1}^{s} \psi_i^{-1}(-\infty, 0]$ and the fact $\varphi_j(q) = 0$ is now completely proved.

Corollary 4.1. *If C is a convex open subset of $\mathbb{R}^n$ and $P \subseteq C$ is an arbitrary polytope with s facetes, then $C \setminus P$ can be covered by $s + 2$ convex sets.*

Indeed $\mathbb{R}^n \setminus \mathrm{aff}(P)$ can be, according to Remark 3.2, covered by 2 convex sets and $\mathrm{aff}(P) \setminus P$ can be covered by s convex sets, where $\mathrm{aff}(X)$ stands for the affine hull of $X \subseteq \mathbb{R}^n$.

5. The proof of Theorem 2.4

Proposition 5.1. *If $\mathfrak{D}$ is a cover of the open set $D \subseteq \mathbb{R}^n$ by convex subsets of D, then for each $x \in \mathrm{bd}(D)$ there exists a sequence $\{D_n\} \subset \mathfrak{D}$ and $x_n \in \mathrm{bd}(D_n)$ such that $x_n \to x$, as $n \to \infty$.*

Proof. Consider a sequence $\{x'_n\} \subset D$ such that $x'_n \to x$ as $n \to \infty$. For each $n \geq 1$, there exists some $D_n \in \mathfrak{D}$ such that $x'_n \in D_n$, as $\mathfrak{D}$ is a cover of D. We next denote by x_n the unique projection of x on its closure $\mathrm{cl}(D_n)$ (see [18, Th. 2.4.1, p. 65]), which is still convex [18, Th. 2.3.5, p. 62], i.e. the point $x_n \in \mathrm{bd}(D_n)$ with the property

$$||x_n - x|| = \inf\{||y - x|| : y \in \mathrm{cl}(D_n)\}$$

and observe that $||x_n - x|| \leq ||x'_n - x|| \to 0$ as $n \to \infty$. This shows that $x_n \to x$ as $n \to \infty$. $\square$

Remark 5.2. The proof of Proposition 5.1 actually shows that for each $x \in \mathrm{bd}(D)$ there exists a sequence $\{D_n\} \subset \mathfrak{D}$ such that $d(x, D_n) \to 0$ as $n \to \infty$, where $d(a, A)$ stands for $\inf\{||a - a'|| : a' \in A\}$.

Corollary 5.3. *If $\mathfrak{D}$ is a finite cover of the open set $D \subseteq \mathbb{R}^n$ by convex subsets of D, then for each $x \in \mathrm{bd}(D)$ there exists $C \in \mathfrak{D}$ such that $x \in \mathrm{bd}(C)$.*

Proof. Assume that some $x \in \mathrm{bd}(D)$ does not belong to any $\mathrm{bd}(C)$, $C \in \mathfrak{D}$, namely $d(x, C) \geq c > 0$ for every $C \in \mathfrak{D}$ and some $c > 0$. This contradicts Remark 5.2. $\square$

Proof of Theorem 2.4. We first observe that each component of the boundary of D is a simple closed curve. We parametrize every component of the boundary of D in such a way that the normal vector $\vec{n}$ of the Frenet reference system $\{\vec{t}, \vec{n}\}$ points towards the domain D. Assume that D is not convex, i.e. its boundary is either a connected simple closed curve having points of positive and negative curvature, or the boundary consists of several simple closed curves, possibly of positive/negative curvature (see [4, Proposition 1, p. 397]). In the first case we assume that $c\colon [a,b] \to \mathrm{bd}(D) \subset \mathbb{R}^2$ is the parametrization of $\mathrm{bd}(D)$. Since the local canonical equations of the curve c are

$$\begin{cases} x = s & + \text{ higher order terms} \\ y = k\,\frac{s^2}{2} & + \text{ higher order terms} \end{cases}, \quad s \in (-\varepsilon, \varepsilon),$$

where k stands for the curvature of c, it follows that, around a point p at which $k < 0$, the curve c lies locally in the half-plane determined by its tangent line at p towards which $-\vec{n}$ is oriented to. Indeed, $\vec{n} \cdot \left(s\,\vec{t} + k\frac{s^2}{2}\,\vec{n}\right) = k\frac{s^2}{2} < 0$, for all $s \in (-\varepsilon, \varepsilon)$. We shall show that every convex subset C of D with the property that $p \in C \cap \mathrm{bd}(D)$, lies in the opposite half-plane, i.e. $\vec{n} \cdot \vec{pq} \geq 0$ for all $q \in C$. In this respect we consider a point $q = (q_1, q_2) \in C$ close enough to p to have $q_1 \in (-\varepsilon, \varepsilon)$. Observe that $\vec{pq} = q_1\,\vec{t} + (\vec{n} \cdot \vec{pq})\,\vec{n}$. The parametric equations of the segment $[pq]$ with respect to the reference system $(p, \vec{t}, \vec{n})$ are

$$\begin{cases} x = q_1 t \\ y = (\vec{n} \cdot \vec{pq})t \end{cases}, \; t \in [0,1] \quad \text{or} \quad y = \frac{\vec{n} \cdot \vec{pq}}{q_1}\,x.$$

The inclusion $[qp] \subseteq \mathrm{cl}(C)$ can be characterized by

$$\frac{\vec{n} \cdot \vec{pq}}{q_1} s \geq k\frac{s^2}{2} \quad \text{i.e.} \quad \vec{n} \cdot \vec{pq} \geq k q_1 \frac{s}{2},$$

for s in some interval with one endpoint at zero. Passing to the limit in this inequality as $s \to 0$, we obtain $\vec{n} \cdot \vec{pq} \geq 0$.

In other words, a small enough arc α of the boundary $\mathrm{bd}(D)$ around p and each convex subset, say C, of D whose boundary touches the boundary of D at p, lies in opposite half-planes determined by the tangent line of $\mathrm{bd}(D)$ at p and $\alpha \cap C = \{p\}$.

If $\mathfrak{C} = \{C_1, \ldots, C_p\}$ would be a finite cover of D by convex subsets, then

$$\alpha \subset \mathrm{bd}(D) \subseteq \mathrm{cl}(D) = \mathrm{cl}(C_1) \cup \cdots \cup \mathrm{cl}(C_p),$$

i.e.
$$\alpha = (\alpha \cap \mathrm{cl}(C_1)) \cup \cdots (\alpha \cup \mathrm{cl}(C_p)),$$

impossible since each set $\alpha \cap \mathrm{cl}(C_1), \ldots, \alpha \cup \mathrm{cl}(C_p)$ is finite and α is infinite uncountable. Consequently, every collection of convex subsets of D which cover the whole set D is infinite in this case. The case of several simple closed curves composing the boundary of D, possibly of positive/negative curvature, can be treated similarly.

Remark 5.4. One can also prove that a connected non-convex and bounded open subset O of $\mathbb{R}^3$ with smooth boundary cannot be covered by finitely many convex sets. Indeed, the second fundamental form of the boundary is not positive semi-definite at some of its point p ([3]), i.e. one principal curvature of the boundary, oriented by its inward unit normal vector field, is negative at p. The normal section $\pi \cap O$ of O through p is an open subset of the plane π through p normal to the boundary of O and parallel to the corresponding principal direction. By using a similar argument with that in the proof of Theorem 2.4, the connected component of $\pi \cap O$ whose boundary contains p cannot be covered by finitely many convex sets, as its curvature is negative at p. Therefore the set O itself cannot be covered by finitely many convex sets.

6. Final remarks

As we mentioned in the first section, the valence of any CIP-function $f\colon O \to \mathbb{R}^n$ cannot exceed any number, and in particular the smallest one, of convex subsets of O needed to cover O. It is therefore worth to investigate the *convex decomposition number* of O, denoted by $\operatorname{convdec}(O)$, which is defined as the smallest cardinal number k such that O can be covered by k convex subsets of O. Note that $\operatorname{convdec}(O) \le \aleph_0$, as $\mathbb{R}^n$ is a second-countable space and each of its open subset is, for instance, a countable union of open balls. A convex decomposition of O by $\operatorname{convdec}(O)$ convex subsets of O is said to be minimal. If we restrict ourselves to convex open subsets of O, we then obtain a greater number $\operatorname{Convdec}(O)$, defined as the smallest cardinal number k such that O can be covered by k convex open subsets of O. In other words, if $f\colon O \to \mathbb{R}^n$ is a CIP-function, then

$$\operatorname{Val}(f) \le \operatorname{convdec}(O) \le \operatorname{Convdec}(O), \tag{1}$$

where $\operatorname{Val}(f) := \sup\{\operatorname{card} f^{-1}(y) : y \in \mathbb{R}^n\}$ stands for the valence of f, as defined in [10]. With these notations the results of the second section can be stated in the following way:

Theorem 6.1. *If $D \subseteq \mathbb{R}^n$ is a convex open set and $F \subset D$ is a finite set, then*

$$\operatorname{convdec}(D \setminus F) \le \operatorname{card}(F) + 1.$$

Theorem 6.2. *If $D \subseteq \mathbb{R}^n$ is a convex open set and $p \in D$, then*

$$\operatorname{Convdec}(D \setminus \{p\}) = n + 1.$$

Theorem 6.3. *Let P be a compact convex n-dimensional subset of $\mathbb{R}^n$. If P is a polyhedron with s facets, then*

$$\operatorname{convdec}(\mathbb{R}^n \setminus P) = \operatorname{Convdec}(\mathbb{R}^n \setminus P) = s. \tag{2}$$

Conversely, if for some neighbourhood C of P the number $\operatorname{Convdec}(C \setminus P)$ *is finite, then P is a polyhedron with at most s facets, where s stands here for* $\operatorname{Convdec}(C \setminus P)$ *as well.*

Theorem 6.4. *If $D \subseteq \mathbb{R}^2$ is a bounded connected and open set with smooth boundary, then either D is convex or* $\operatorname{convdec}(D) = \aleph_0$.

Remark 6.5. If $D \subseteq \mathbb{R}^n$ is a convex open set and $p \in D$, then

$$\mathrm{Val}(f) \leq \mathrm{convdec}(D \setminus \{p\}) = 2$$

for every CIP-function $f\colon D \setminus \{p\} \to \mathbb{R}^n$. This upper bound is sharp for $n = 2$ as the valence of the CIP-functions $g\colon D^2 \setminus \{0\} \to \mathbb{C}, g(z) = z^2$ and $1/g$ is two.

The left hand side inequality of (1) works for a larger class of maps namely for MCIP-functions. We say that $f\colon D \to \mathbb{R}^n$ satisfies the *minimal convex injectivity property*, or shortly f is an MCIP-function, if the restriction of f to every convex component of a certain minimal convex decomposition of D, is injective. We provide below a few MCIP-functions which are not CIP-functions and which realize equality in the left hand side inequality of (1). We only justify the first example of the following list and leave the discussion of the further examples to the reader.

Example 6.6. Let P_m be the regular polygon $\mathrm{conv}\{\varepsilon_0, \varepsilon_1, \dots, \varepsilon_{m-1}\}$, whose vertices $\varepsilon_i = e^{\frac{2\pi i}{m}}$, $i = 0, \dots, m-1$ are the roots of order m of unity. The functions $f\colon \mathbb{C} \setminus P_m \to \mathbb{C}$, $f(z) = z^m$ and $\frac{1}{f}$ are MCIP on $\mathbb{C} \setminus P$ but none of them is a CIP-function. In fact the functions f and $\frac{1}{f}$ realize equality in the left hand side inequality of (1) as

$$\mathrm{Val}(f) = \mathrm{Val}\left(\frac{1}{f}\right) = \mathrm{convdec}\left(\mathbb{C} \setminus P_m\right) = m,$$

and this shows the sharpness of the left hand side inequality in (1) for MCIP-functions. Indeed, a minimal convex decomposition of the open set $\mathbb{C} \setminus P_m$ is $\{C_1, \dots, C_m\}$, where

$$C_j := \left\{ z \in \mathbb{C} \setminus P_m \mid \frac{2\pi(j-1)}{m} \leq \arg(z) < \frac{2\pi j}{m} \right\},\ j = 1, \dots, m.$$

The restrictions of f and $\frac{1}{f}$ to every C_j are injective and this shows that f and $\frac{1}{f}$ are MCIP-functions, but none of them is a CIP-function, as their restrictions to every convex subset of $\mathbb{C} \setminus P_m$ of the form

$$\{z \in \mathbb{C} \setminus P_m \mid \varphi_j(z) > 0\},$$

where $\varphi_j(z) = i\psi_j(z)$ and

$$\psi_j(z) = z(\bar{\varepsilon}_{j-1} - \bar{\varepsilon}_j) + \bar{z}(\varepsilon_j - \varepsilon_{j-1}) + (\bar{\varepsilon}_{j-1}\varepsilon_j - \varepsilon_{j-1}\bar{\varepsilon}_j),$$

for $j = 1, \dots, m$, are obviously not injective.

Example 6.7. The sharpness of the left hand side inequality in (1) can be extended, via the function f of the previous item, to

$$\mathbb{R}^3 \setminus \mathrm{conv}(P_m \cup \{s, n\}),$$

where $s = (0, 0, -1)$, $n = (0, 0, 1) \in \mathbb{R}^3$. Indeed, an MCIP-function, which is not a CIP function and realizes equality in (1), is

$$s \circ (f \times \mathrm{id})\colon \mathbb{R}^3 \setminus \mathrm{conv}(P_m \cup \{s, n\}) \to \mathbb{R}^3,$$

where $\mathbb{R}^3$ is naturally identified with $\mathbb{C} \times \mathbb{R}$, $(f \times \mathrm{id})(z, t) = (z^m, t)$, $s\colon \mathbb{R}^3 \to \mathbb{R}^3$, $s(z, t) = (z, |t|)$. More precisely

$$\mathrm{Val}(s \circ (f \times \mathrm{id})) = \mathrm{convdec}\,(\mathrm{conv}(P_m \cup \{s, n\})) = 2m.$$

Example 6.8. Consider the function $g\colon D^3 \setminus \{0\} \to \mathbb{R}^3$, $g(z, t) = (z^2, t)$, where

$$D^3 = \{(z, t) \in \mathbb{C} \times \mathbb{R} : |z|^2 + |t|^2 < 1\},$$

Note that g is not a local diffeomorphism as its critical set is $(D^3 \setminus \{0\}) \cap Oz$. The restriction $g\big|_{D^3 \setminus Oz}$ is a non-injective MCIP-local diffeomorphism and the function

$$h\colon D^3 \setminus Oz \to \mathbb{R}^3,\ h(z, t) = \left(\frac{1}{z^2}, t\right)$$

is a non-injective MCIP-local diffeomorphism too.

Note that $f\colon O \to \mathbb{R}^n$ is a CIP-function if and only if its restriction to every convex open subset of O is injective, as every segment $[xy] \subseteq O$ is actually contained in a convex open subset of O. Thus, by removing the convexity requirement in some global injectivity criterion on convex open sets, every function which is subject to the remaining hypothesis of the criterion is a CIP-function. Such a criteria for C^1 -smooth maps $f\colon D \to \mathbb{R}^n$ on convex open sets $D \subseteq \mathbb{R}^n$ is due to Gale-Nikaidô [5] and it ensures the global injectivity of f under the hypothesis

$$\langle (df)_x(y), y\rangle > 0,\ \forall\ x \in D,\ y \in \mathbb{R}^n \setminus \{0\}. \tag{3}$$

The requirement (3) ensures the *convex injectivity property* of $f\colon O \to \mathbb{R}^n$ if we get rid of the convexity requirement on O.

For $n = 2$, the condition (3) is equivalent to

$$\mathrm{Re}(f_z) > |f_{\bar z}|,\ \text{ on } D. \tag{4}$$

If we increase significantly the regularity of f up to holomorphy, i.e. $f_{\bar z} \equiv 0$, then the condition (4) is equivalent to

$$\mathrm{Re}(f') > 0. \tag{5}$$

and it ensures the *convex injectivity property* of f or the global injectivity when the domain D of f is convex. The latter situation corresponds to the Alexander-Noshiro-Warschawski univalence criterion [1, 11, 17]. If we get rid of the convexity assumption in this injectivity criterion, there exists non- injective regular functions, i.e. holomorphic with nonvanishing derivaive, satisfying (5).

Remark 6.9. If $D_1, D_2 \subseteq \mathbb{R}^n$ are convex open domains with nonempty intersection such that their union is not a convex set, then

$$\operatorname{convdec}(D_1 \cup D_2) = \operatorname{Convdec}(D_1 \cup D_2) = 2$$

and the upper estimate (1) is sharp for $n = 2$. Indeed, according to Tims [15], there exists a holomorphic function $g\colon D_1 \cup D_2 \to \mathbb{R}^2$ such that $\operatorname{Re}(g'(z)) > 0$ for all $z \in D_1 \cup D_2$ and

$$\operatorname{Val}(g) = \operatorname{convdec}(D_1 \cup D_2) = \operatorname{Convdec}(D_1 \cup D_2) = 2\,.$$

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