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Some Strong Laws of Large Numbers for Double Arrays of Random Sets with Gap Topology

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The aim of this paper is to state some strong laws of large numbers for a double array of independent (or pairwise independent) random sets with the gap topology under various settings. Our results improve some previously reported results. Some illustrative examples are provided.

Keywords: Random set, gap topology, double array, compactly uniformly integrable in the Cesàro sense.

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1. Introduction

In recent decades, the strong laws of large numbers (SLLN) for sequences of unbounded random sets gave rise to applications in several fields such as optimization and control, stochastic and integral geometry, mathematical economics, etc. In 1981, Z. Artstein and S. Hart [1] obtained a strong law of large numbers for independent identically distributed random variables whose values are closed subsets of $\mathbb{R}^d$ and applied it to a problem of optimal allocations. Later, F. Hiai [13] and C. Hess [10] independently proved similar results for random sets in an infinite dimensional Banach space with respect to the Mosco convergence. Moreover, C. Hess [11] obtained a convergence result on the Wijsman topology of SLLN for sequences of pairwise independent identically distributed random sets in a separable Banach space. In particular, C. Hess was able to show that the SLLN for

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the slice topology follows from the SLLN for the Wijsman topology, provided the space has a strongly separable dual (see [11, Proposition 3.9]). The SLLN for a double array of independent (or pairwise independent) random sets was introduced by C. Castaing, N. V. Quang and D. X. Giap [6, 7] with Mosco convergence, Wijsman convergence and Slice convergence.

On the hyperspace, there are many topologies in which the Hausdorff metric is the strongest. However, when the closed sets are unbounded, the Hausdorff distance between them is usually equal to infinity. Therefore, it is not well-suited to the case of random sets with unbounded values. For the space of all closed (possibly unbounded) subsets of a Banach space $\mathfrak{X}$, denoted by $c(\mathfrak{X})$, at first the Painlevé-Kuratowski convergence was introduced; it corresponds to the Fell topology, but it is only suitable in a finite dimensional space. The Mosco convergence is an interesting infinite dimensional extension of the Painlevé-Kuratowski convergence and it is suited to reflexive spaces. The Mosco convergence was also introduced for studying variational inequalities. One may ask for a topology on $c(\mathfrak{X})$ that involves more explicitly the norm of the space $\mathfrak{X}$ by means of a distance between sets which is useful to study the rate of convergence. In order to fulfill this requirement, one considers the case where $c(\mathfrak{X})$ is endowed with the Wijsman topology. Indeed, when $\mathfrak{X}$ is separable, it is metrizable and separable. Unlike the Mosco topology, it has interesting properties even in the non-reflexive case. The Wijsman topology is also useful for studying the convergence of best approximations.

On the other hand, in the special case of random sets with convex values, it is possible to prove the SLLN with respect to a topology τ, the so-called *slice topology*, which is stronger than both the Mosco topology and the Wijsman topology, and even in non-reflexive Banach spaces τ has "nice" properties, such as metrizability, second countability and variational inequalities. The slice topology on $c(\mathfrak{X})$ is the weak (or initial) topology determined by the family of gap functionals $\{D(C,.) : C$ is a nonempty slice of a ball$\}$. However, in the practical applications of probability theory, we need to consider sets C that are bounded and convex. In this context, the best candidate seems to be the gap topology given by the weak (or initial) topology corresponding to the gap functionals $\{D(C,.) : C$ is a nonempty bounded convex set$\}$.

The SLLN for sequences of pairwise independent and identically distributed random sets with the gap topology established by Terán [23]. In this paper, combining Terán's method in [23] with our techniques for building a double array of selections to prove the "liminf" part of Painlevé-Kuratowski convergence, we establish some SLLN for double arrays of independent (or pairwise independent) random sets with the gap topology under various settings. Some illustrative examples are provided.

This paper is organized as following: In Section 2, we introduce some basic notions: hyperspace of a Banach space, random set, Painlevé-Kuratowski convergence, Wijsman convergence, Slice convergence, convergence in the gap topology. In Section 3, we prove the strong law of large numbers for a double array of pairwise independent random sets, compactly uniformly integrable in the Cesàro sense.

Finally, Section 4 is concerned with the strong law of large numbers for a double array of independent (or pairwise independent) closed valued random variables in a real separable Banach space with or without any geometric property. The type of convergence considered here is convergence in the gap topology.

2. Preliminaries

Throughout this paper, let $(\Omega, \mathcal{A}, \mathbf{P})$ be a probability space, $(\mathfrak{X}, \|.\|)$ be a separable Banach space. Let $\mathfrak{X}^*$ be the dual space of $\mathfrak{X}$ and S^* be its unit sphere. In the present paper, $\mathbb{R}$ (resp. $\mathbb{Q}$) (resp. $\mathbb{N}$) will denote the set of all real numbers (resp. rational numbers) (resp. positive integers).

Let $c(\mathfrak{X})$ (resp. $k(\mathfrak{X})$) be the family of all nonempty closed (resp. compact) subsets of $\mathfrak{X}$. For any $A, B \in c(\mathfrak{X})$, $\mathrm{cl}A$, $\mathrm{co}A$, $\overline{\mathrm{co}}A$ denote the *norm-closure*, the *convex hull* and the *closed convex hull* of A respectively; the *distance function* $d(., A)$ of A, the *support function* $s(., A)$ of A, the *norm* $\|A\|$ of A and the *gap* $D(A, B)$ between two sets A and B are defined by

$$d(x, A) = \inf\{\|x - y\| : y \in A\},\ x \in \mathfrak{X},$$
$$s(x^*, A) = \sup\{\langle x^*, y\rangle : y \in A\},\ x^* \in \mathfrak{X}^*,$$
$$\|A\| = \sup\{\|x\| : x \in A\},$$
$$D(A, B) = \inf\{\|x - y\| : x \in A, y \in B\},$$

and also, for any $r > 0$, we put $A^r = \{x \in \mathfrak{X} : d(x, A) \le r\}$.

Let $U^- := \{C \in c(\mathfrak{X}) : C \cap U \neq \emptyset\}$, where $U \subset \mathfrak{X}$. Let $\mathcal{B}_{c(\mathfrak{X})}$ be the σ-field on $c(\mathfrak{X})$ generated by the sets U^-, taken for all open subsets U of $\mathfrak{X}$. We recall that $\mathcal{B}_{c(\mathfrak{X})}$ is the *Effrös σ-field.* A mapping $F : \Omega \to c(\mathfrak{X})$ is said to be *a random set* if it is $(\mathcal{A}, \mathcal{B}_{c(\mathfrak{X})})$- measurable; namely, for every open subset U of $\mathfrak{X}$, we have $F^{-1}(U^-) \in \mathcal{A}$. Given a random set F, we define a sub-σ-field of $\mathcal{A}$ by setting $\mathcal{A}_F = \{F^{-1}(\mathcal{U}) : \mathcal{U} \in \mathcal{B}_{c(\mathfrak{X})}\}$.

For each $\mathcal{F}$-measurable random set F, the family of all $\mathcal{F}$-measurable random elements f with $f(\omega) \in F(\omega)$ a.s., is denoted by $S^0_F(\mathcal{F})$, where $\mathcal{F}$ is a sub-σ-field of $\mathcal{A}$. Shorter notation for $S^0_F(\mathcal{A})$ is S^0_F.

For $1 \le p < \infty$, $\mathbf{L}^p(\Omega, \mathcal{A}, \mathbf{P}, \mathfrak{X}) = \mathbf{L}^p(\mathfrak{X})$ denotes the space of $\mathcal{A}$-measurable functions $f : \Omega \to \mathfrak{X}$ such that the norm $\|f\|_p = (E\|f\|^p)^{\frac{1}{p}}$ is finite.

Set $S^p_F(\mathcal{F}) = \{f \in \mathbf{L}^p(\Omega, \mathcal{F}, \mathbf{P}, \mathfrak{X}) : f(\omega) \in F(\omega),\ \text{a.s.}\}$. If $\mathcal{F} = \mathcal{A}$, then $S^p_F(\mathcal{F})$ is denoted by S^p_F.

For any sub-σ-field $\mathcal{F}$ of $\mathcal{A}$ and any $\mathcal{F}$-measurable random set F, the *expectation* of F over Ω, with respect to $\mathcal{F}$, is defined by $E(F, \mathcal{F}) = \{Ef : f \in S^1_F(\mathcal{F})\}$, where $Ef = \int_\Omega f d\mathbf{P}$ is the usual Bochner integral of f. We denote $E(F, \mathcal{A})$ for short by EF. We note that EF is not always closed.

For notational convenience, for $a, b \in \mathbb{R}$, $\max\{a, b\}$ is denoted by $a \vee b$ and $\min\{a, b\}$ is denoted by $a \wedge b$. The logarithms are to the base 2, for $a \in \mathbb{R}$, $\log(a \vee 1)$ will be denoted by $\log^+ a$.

Let $\mathbb{N}^d = \{\mathbf{n} = (n_1, \ldots, n_d) : n_i \in \mathbb{N}, 1 \le i \le d\}$ be a set of d-dimentional positive integer lattice points ($d \ge 1$). Denote $\mathbf{n}_{\max} = \max\{n_1, \ldots, n_d\}$.

Let us denote by s the strong topology of $\mathfrak{X}$. The array $\{x_{\mathbf{n}} : \mathbf{n} \in \mathbb{N}^d\}$ of $\mathfrak{X}$ is said to *converge to an element* $x \in \mathfrak{X}$ as $\mathbf{n}_{\max} \to \infty$, denoted by $s\text{-}\lim_{\mathbf{n}_{\max}\to\infty} x_{\mathbf{n}} = x$ if for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $\|x_{\mathbf{n}} - x\| < \varepsilon$ for $\mathbf{n}_{\max} \ge N$.

Let $\{A_{\mathbf{n}} : \mathbf{n} \in \mathbb{N}^d\}$ be an array in $c(\mathfrak{X})$. We put

$$s\text{-}\liminf_{\mathbf{n}_{\max}\to\infty} A_{\mathbf{n}} = \{x \in \mathfrak{X} : x = s\text{-}\lim_{\mathbf{n}_{\max}\to\infty} x_{\mathbf{n}}, x_{\mathbf{n}} \in A_{\mathbf{n}}, \mathbf{n} \in \mathbb{N}^d\},$$

$$s\text{-}\limsup_{\mathbf{n}_{\max}\to\infty} A_{\mathbf{n}} = \{x \in \mathfrak{X} : x = s\text{-}\lim_{\mathbf{k}_{\max}\to\infty} x_{\mathbf{k}}, x_{\mathbf{k}} \in A_{\mathbf{n}_{\mathbf{k}}}, \mathbf{k} \in \mathbb{N}^d\},$$

where $\{A_{\mathbf{n}_{\mathbf{k}}} : \mathbf{k} \in \mathbb{N}^d\}$ is a sub-array of $\{A_{\mathbf{n}} : \mathbf{n} \in \mathbb{N}^d\}$.

An array $\{A_{\mathbf{n}} : \mathbf{n} \in \mathbb{N}^d\}$ converges to A, in the *sense of Painlevé-Kuratowski*, relative to the topology s, as $\mathbf{n}_{\max} \to \infty$, if $s\text{-}\liminf_{\mathbf{n}_{\max}\to\infty} A_{\mathbf{n}} = s\text{-}\limsup_{\mathbf{n}_{\max}\to\infty} A_{\mathbf{n}} = A$. Clearly, this is true if and only if the following inclusions hold:

$$s\text{-}\limsup_{\mathbf{n}_{\max}\to\infty} A_{\mathbf{n}} \subset A \subset s\text{-}\liminf_{\mathbf{n}_{\max}\to\infty} A_{\mathbf{n}}.$$

The *Wijsman convergence* on $c(\mathfrak{X})$ is the pointwise convergence of distance functions. This means that an array $\{A_{\mathbf{n}} : \mathbf{n} \in \mathbb{N}^d\} \subset c(\mathfrak{X})$ converges to A as $\mathbf{n}_{\max} \to \infty$ with respect to Wijsman convergence, if for every $x \in \mathfrak{X}$, one has $d(x, A) = \lim_{\mathbf{n}_{\max}\to\infty} d(x, A_{\mathbf{n}})$.

An array $\{A_{\mathbf{n}} : \mathbf{n} \in \mathbb{N}^d\} \subset c(\mathfrak{X})$ *converges in the slice topology* to A as $\mathbf{n}_{\max} \to \infty$ if $D(A_{\mathbf{n}}, C) \to D(A, C)$ as $\mathbf{n}_{\max} \to \infty$ for all nonempty slices C of balls, and it *converges in the gap topology* to A as $\mathbf{n}_{\max} \to \infty$ if $D(A_{\mathbf{n}}, C) \to D(A, C)$ as $\mathbf{n}_{\max} \to \infty$ for all nonempty bounded closed convex subsets C of $\mathfrak{X}$.

A double array of random elements $\{f_{mn} : m \ge 1, n \ge 1\}$ is said to be *uniformly integrable in the Cesàro sense* if

$$\lim_{x\to\infty} \sup_{m,n\ge 1} (mn)^{-1} \sum_{i=1}^{m} \sum_{j=1}^{n} E\Big(\|f_{ij}\| \mathbf{1}_{(\|f_{ij}(.)\| > x)}\Big) = 0.$$

A double array of random elements $\{f_{mn} : m \ge 1, n \ge 1\}$ is said to be *compactly uniformly integrable in the Cesàro sense* (Cesàro CUI for short) if for every $\varepsilon > 0$, there exists a compact subset K of $\mathfrak{X}$ such that

$$\sup_{m,n\ge 1} (mn)^{-1} \sum_{i=1}^{m} \sum_{j=1}^{n} E\Big(\|f_{ij}\| \mathbf{1}_{(f_{ij}(.) \notin K)}\Big) < \varepsilon.$$

In general, compactly uniform integrability in the Cesàro sense is stronger than the uniform integrability in the Cesàro sense, but they are equivalent in the real or the identically distributed case. For more details, see [24].

Let A be a subset of Ω, the complement of the set A with respect to Ω is denoted by A^c.

As in [20], a double array of random sets $\{F_{mn} : m \geq 1, n \geq 1\}$ is said to be *compactly uniformly integrable in the Cesàro sense* (Cesàro CUI for short) if for every $\varepsilon > 0$, there exists a compact subset K of $\mathfrak{X}$ such that

$$\sup_{m,n\geq 1} (mn)^{-1} \sum_{i=1}^{m} \sum_{j=1}^{n} E\Big(\|F_{ij}\| \mathbf{1}_{(F_{ij}(.)\subset K)^c}\Big) < \varepsilon.$$

Now we proceed to state our main results.

3. SLLN for a double array of random sets compactly uniformly integrable in the Cesàro sense

In this section, we introduce and prove a SLLN for a double array of pairwise independent, Cesàro CUI random sets.

At first, we give some lemmas which will be used later.

Lemma 3.1. *Assume that $\{F_{mn} : m \geq 1, n \geq 1\}$ is a double array of Cesàro CUI random sets. Then*

(1) *$\{f_{mn} : m \geq 1, n \geq 1\}$ is a double array of Cesàro CUI random elements, where $f_{mn} \in S^0_{F_{mn}}$, for all $m \geq 1$, $n \geq 1$.*

(2) *$\{s(x^*, F_{mn}) : m \geq 1, n \geq 1\}$ is a double array of uniformly integrable in the Cesàro sense random variables, for every $x^* \in S^*$.*

Proof. Since $\{F_{mn} : m \geq 1, n \geq 1\}$ is a double array of Cesàro CUI random sets, for all $\varepsilon > 0$, there exists a compact subset K of $\mathfrak{X}$ such that

$$\sup_{m,n\geq 1} (mn)^{-1} \sum_{i=1}^{m} \sum_{j=1}^{n} E\Big(\|F_{ij}\| \mathbf{1}_{(F_{ij}(.)\subset K)^c}\Big) < \varepsilon,$$

where $(F_{ij}(.) \subset K)^c = \{\omega \in \Omega : F_{ij}(\omega) \subset K\}^c = \Omega \setminus \{\omega \in \Omega : F_{ij}(\omega) \subset K\}$.

For (1), since $f_{ij} \in S^0_{F_{ij}}$, $i \geq 1, j \geq 1$, we obtain

$$(f_{ij}(.) \notin K) \subset (F_{ij}(.) \subset K)^c \text{ for all } i \geq 1,\ j \geq 1.$$

This implies $\|f_{ij}\| \mathbf{1}_{(f_{ij}(.)\notin K)} \leq \|F_{ij}\| \mathbf{1}_{(F_{ij}(.)\subset K)^c}$ for all $i \geq 1,\ j \geq 1$.

Hence $\displaystyle \frac{1}{mn} \sum_{i=1}^{m} \sum_{j=1}^{n} E\Big(\|f_{ij}\| \mathbf{1}_{(f_{ij}(.)\notin K)}\Big) \leq \frac{1}{mn} \sum_{i=1}^{m} \sum_{j=1}^{n} E\Big(\|F_{ij}\| \mathbf{1}_{(F_{ij}(.)\subset K)^c}\Big).$

Thus

$$\sup_{m,n\geq 1} \frac{1}{mn} \sum_{i=1}^{m} \sum_{j=1}^{n} E\Big(\|f_{ij}\| \mathbf{1}_{(f_{ij}(.)\notin K)}\Big) \leq \sup_{m,n\geq 1} \frac{1}{mn} \sum_{i=1}^{m} \sum_{j=1}^{n} E\Big(\|F_{ij}\| \mathbf{1}_{(F_{ij}(.)\subset K)^c}\Big) < \varepsilon.$$

For (2), since K is the compact subset of $\mathfrak{X}$, it is bounded and so there exists a positive constant M such that $K \subset \{x \in \mathfrak{X} : \|x\| \leq M\} =: I$. Therefore, if $F_{ij}(.) \subset K$ then $|s(x^*, F_{ij})| \leq |s(x^*, I)| \leq \|I\| = M$ for all $x^* \in S^*$ and $i \geq 1$, $j \geq 1$.

Thus, for each $a > M$, we get

$$(|s(x^*, F_{ij})| > a) \subset (F_{ij}(.) \subset K)^c \text{ for all } i \geq 1, j \geq 1 \text{ and } x^* \in S^*.$$

This implies,

$$|s(x^*, F_{ij})|\mathbf{1}_{(|s(x^*,F_{ij})|>a)} \leq \|F_{ij}\|\mathbf{1}_{(F_{ij}(.)\subset K)^c} \text{ for all } i \geq 1, j \geq 1 \text{ and } x^* \in S^*.$$

Hence, for all $x^* \in S^*$,

$$\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n} E\Big(|s(x^*, F_{ij})|\mathbf{1}_{(|s(x^*,F_{ij})|>a)}\Big) \leq \frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n} E\Big(\|F_{ij}\|\mathbf{1}_{(F_{ij}(.)\subset K)^c}\Big).$$

Therefore, we obtain

$$\sup_{m,n\geq 1}\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n} E\Big(|s(x^*, F_{ij})|\mathbf{1}_{(|s(x^*,F_{ij})|>a)}\Big)$$
$$\leq \sup_{m,n\geq 1}\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n} E\Big(\|F_{ij}\|\mathbf{1}_{(F_{ij}(.)\subset K)^c}\Big) < \varepsilon. \qquad \square$$

Lemma 3.2. ([13, Lemma 3.1])

(1) *For each random set F with $S^1_F \neq \emptyset$, $\overline{\text{co}}EF = \overline{\text{co}}E(F, \mathcal{A}_F)$.*

(2) *Let F, G be two identically distributed random sets. For each $f \in S^1_F(\mathcal{A}_F)$, there exists a $g \in S^1_G(\mathcal{A}_G)$ such that f and g are identically distributed.*

(3) *For identically distributed random sets F, G with $S^1_F \neq \emptyset$, $E(F, \mathcal{A}_F) = E(G, \mathcal{A}_G)$.*

Lemma 3.3. ([6, Proposition 3.5]) *Let $\{x_{mn} : m \geq 1, n \geq 1\}$ be a double array of elements in a Banach space $\mathfrak{X}$ such that*

(1) *For each $m \geq 1$, $\displaystyle\frac{1}{n}\sum_{j=1}^{n} x_{mj} \to x$ as $n \to \infty$,*

(2) *For each $n \geq 1$, $\displaystyle\frac{1}{m}\sum_{i=1}^{m} x_{in} \to x$ as $m \to \infty$,*

(3) $\displaystyle\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n} x_{ij} \to x$ *as $m \wedge n \to \infty$, where x is a member of $\mathfrak{X}$. Then,*

$$\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n} x_{ij} \to x \quad \text{as} \quad m \vee n \to \infty.$$

Lemma 3.4. *Let A be a subset of $\mathfrak{X}$, B be the closed unit ball of a fixed arbitrary equivalent norm of $\mathfrak{X}$ and let $r > 0$. If A is a compact set then*

$$\overline{\text{co}}A + r.B = \overline{\text{co}}(A + r.B). \tag{1}$$

Proof. For any $x \in (\overline{\text{co}}A + r.B)$, there exists $x_1 \in \overline{\text{co}}A$ and $y_1 \in r.B$ such that $x = x_1 + y_1$. We have,

$$\begin{aligned}\langle x^*, x\rangle &= \langle x^*, x_1 + y_1\rangle = \langle x^*, x_1\rangle + \langle x^*, y_1\rangle \\ &\leq s(x^*, A) + s(x^*, r.B) = s(x^*, A + r.B), \text{ for all } x^* \in \mathfrak{X}^*.\end{aligned}$$

We recall that $x \in \overline{\text{co}}(A+r.B)$ if and only if $\langle x^*, x\rangle \leq s(x^*, A+r.B)$ for all $x^* \in \mathfrak{X}^*$. Therefore, $x \in \overline{\text{co}}(A + r.B)$. On the other hand, since $A + r.B \subset \overline{\text{co}}A + r.B$, to prove $\overline{\text{co}}(A + r.B) \subset \overline{\text{co}}A + r.B$, we need only show that $\overline{\text{co}}A + r.B$ is a closed convex set.

Clearly, $\overline{\text{co}}A + r.B$ is a convex set.

Now, let $\{x_n : n \geq 1\}$ be a sequence in $\overline{\text{co}}A + r.B$ such that $x_n \to x$ as $n \to \infty$. Then, for each $n \geq 1$, there exists $x_n^1 \in \overline{\text{co}}A$ and $y_n^1 \in r.B$ such that $x_n = x_n^1 + y_n^1$. Moreover, since $\mathfrak{X}$ is complete and A is a compact subset of $\mathfrak{X}$, then $\overline{\text{co}}A$ is also a compact set. Therefore, there exists a subsequence $\{x_{n_k}^1 : k \geq 1\}$ of $\{x_n^1 : n \geq 1\}$ such that $x_{n_k}^1 \to x_1 \in \overline{\text{co}}A$ as $k \to \infty$. At this point $y_{n_k}^1 \to y_1$ as $k \to \infty$. Hence, $y_1 \in r.B$ (by $r.B$ being a closed set).

We get, $x_{n_k} = x_{n_k}^1 + y_{n_k}^1 \to x_1 + y_1 \in \overline{\text{co}}A + r.B$ as $k \to \infty$. Therefore, $x = x_1 + y_1 \in \overline{\text{co}}A + r.B$ and we deduce that $\overline{\text{co}}A + r.B$ is a closed set. Hence, the proof is complete. □

The hypothesis of compactness of the set A in the above lemma cannot be dropped. The following example will show that Lemma 3.4 can be fail without the assumption of the compactness of the set A.

Example 3.5. Let $\mathfrak{X}$ be a non-reflexive Banach space. Let B be the closed unit ball with respect to the norm on $\mathfrak{X}$ and let r be a fixed positive real number. By applying James' Theorem (see [17, 15]), we can choose an $x^* \in \mathfrak{X}^*$ which does not attain its norm on the closed unit ball B, i.e., there is no $x \in B$ with $\langle x^*, x\rangle = \|x^*\|$. Clearly, $\|x^*\| > 0$. We put $A = \{x \in \mathfrak{X} : \langle x^*, x\rangle \geq r\|x^*\|\}$. Since x^* is a continuous linear functional, then the set A is closed and convex, and so $\overline{\text{co}}A = A$. By the definition of supremum, there exists a sequence $\{x_n : n \geq 1\} \subset B$ such that $\langle x^*, x_n\rangle \to \|x^*\|$ as $n \to \infty$. Therefore, $\langle x^*, x_n\rangle > 0$ for all n sufficiently large, and so without loss of generality, we can assume that $\langle x^*, x_n\rangle > 0$ for all n. Put $y_n = \frac{r\|x^*\|}{\langle x^*, x_n\rangle}.x_n$ for every $n \geq 1$. We get that

$$\langle x^*, y_n\rangle = \langle x^*, \frac{r\|x^*\|}{\langle x^*, x_n\rangle}.x_n\rangle = \frac{r\|x^*\|}{\langle x^*, x_n\rangle}.\langle x^*, x_n\rangle = r\|x^*\|,$$

which implies $y_n \in A$. Moreover,

$$\|y_n + (-rx_n)\| = \left\|rx_n\left(\frac{\|x^*\|}{\langle x^*, x_n\rangle} - 1\right)\right\| = r.\left|\frac{\|x^*\|}{\langle x^*, x_n\rangle} - 1\right|.\|x_n\| \to 0 \text{ as } n \to \infty.$$

This implies that 0 is a limit point of $A + rB$, because $y_n + (-rx_n) \in A + rB$ for every $n \geq 1$. If $0 \in A+rB$, i.e., there exist $x_0 \in A$ and $y_0 \in B$ satisfying $x_0 + ry_0 = 0$, then $x_0 = -ry_0 \in rB$. This yields $x_0 \in A \cap (rB)$. Since $x_0 \in A$, $\langle x^*, x_0\rangle \geq r\|x^*\|$. Since $x_0 \in rB$ (i.e., $\frac{x_0}{r} \in B$), we have $|\langle x^*, x_0\rangle| \leq \|x^*\|.\|x_0\| \leq r\|x^*\|$. Therefore, $\langle x^*, x_0\rangle = r\|x^*\|$ is equivalent to $\langle x^*, \frac{x_0}{r}\rangle = \|x^*\|$. This contradicts the choice of x^*. Hence, $0 \notin A + rB$. This shows that the set $A + rB$ (is equal to $\overline{\text{co}}A + rB$) is not closed, and so the conclusion (1) is not true. □

The Beer-Tamaki's lemma for the multidimensional case with the convergence as $\mathbf{n}_{\min} \to \infty$ was proved in [9, Lemma 4.5]. Similarly, we can get this result with the convergence as $\mathbf{n}_{\max} \to \infty$.

Lemma 3.6. *Let $A \in c(\mathfrak{X})$ and $\{A_{\mathbf{n}} : \mathbf{n} \in \mathbb{N}^d\}$ be a d-dimensional array in $c(\mathfrak{X})$ $(d \geq 1)$. If $(A_{\mathbf{n}})^r \to A^r$ as $\mathbf{n}_{\max} \to \infty$ in the sense of Painlevé-Kuratowski, relative to the topology s, for every $r > 0$, then $A_{\mathbf{n}} \to A$ as $\mathbf{n}_{\max} \to \infty$ in the Wijsman topology.*

For the sake of completeness, we need the following useful lemma.

Lemma 3.7. *Suppose that F, $F_{\mathbf{n}}$, $\mathbf{n} \in \mathbb{N}^d$ are the random sets in the separable Banach space $\mathfrak{X}$. If $F(\omega) \subset s\text{-}\liminf_{\mathbf{n}_{\max}\to\infty} F_{\mathbf{n}}(\omega)$ a.s., then*

$$\limsup_{\mathbf{n}_{\max}\to\infty} d(x, F_{\mathbf{n}}(\omega)) \leq d(x, F(\omega)) \text{ for every } x \in \mathfrak{X}, \text{ a.s.}$$

Proof. Since $\mathfrak{X}$ is separable, there is a countable dense subset D of $\mathfrak{X}$. From $F(\omega) \subset s\text{-}\liminf_{\mathbf{n}_{\max}\to\infty} F_{\mathbf{n}}(\omega)$ a.s., there exists a negligible $N \in \mathcal{A}$ such that for every

$$\omega \in \Omega \setminus N: \qquad F(\omega) \subset s\text{-}\liminf_{\mathbf{n}_{\max}\to\infty} F_{\mathbf{n}}(\omega). \tag{2}$$

Fix $\omega \in \Omega \setminus N$. Then, for any $x \in D$ and $p \in \mathbb{N}$, there exists $y \in F(\omega)$ satisfying

$$\|x - y\| \leq d(x, F(\omega)) + \frac{1}{p}.$$

By (2), for each $\mathbf{n} \in \mathbb{N}^d$, there exists $f_{\mathbf{n}} \in F_{\mathbf{n}}(\omega)$ such that $f_{\mathbf{n}} \to y$ as $\mathbf{n}_{\max} \to \infty$. Therefore,

$$\limsup_{\mathbf{n}_{\max}\to\infty} d(x, F_{\mathbf{n}}(\omega)) \leq \lim_{\mathbf{n}_{\max}\to\infty} \|x - f_{\mathbf{n}}(\omega)\| = \|x - y\| \leq d(x, F(\omega)) + \frac{1}{p}.$$

Letting $p \to \infty$, we get
$$\limsup_{\mathbf{n}_{\max}\to\infty} d(x, F_{\mathbf{n}}(\omega)) \leq d(x, F(\omega)). \tag{3}$$

Next, we recall that the distance function $d(., C) : \mathfrak{X} \to \mathbb{R}$ $(C \subset \mathfrak{X})$ is 1-Lipschitz; i.e., for every $x, y \in \mathfrak{X}$,

$$|d(x, C) - d(y, C)| \leq d(x, y) = \|x - y\|. \tag{4}$$

For any x in $\mathfrak{X}$, there exists a sequence $\{x_k : k \geq 1\} \subset D$ satisfying $\lim_{k\to\infty} x_k = x$.

Then, for each $\mathbf{n} \in \mathbb{N}^d$ and $k \geq 1$,

$$\begin{aligned} d(x, F_{\mathbf{n}}(\omega)) - d(x, F(\omega)) &\leq |d(x, F_{\mathbf{n}}(\omega)) - d(x_k, F_{\mathbf{n}}(\omega))| + \\ &+ |d(x_k, F_{\mathbf{n}}(\omega)) - d(x_k, F(\omega))| + |d(x_k, F(\omega)) - d(x, F(\omega))| \\ &\leq 2d(x, x_k) + |d(x_k, F_{\mathbf{n}}(\omega)) - d(x_k, F(\omega))| \text{ (by (4))}. \end{aligned}$$

Letting $\mathbf{n}_{\max} \to \infty$, we have

$$\limsup_{\mathbf{n}_{\max}\to\infty} [d(x, F_{\mathbf{n}}(\omega)) - d(x, F(\omega))] \leq 2d(x, x_k) \ \ \text{(by(3))}. \tag{5}$$

By taking $k \to \infty$ in (5), we obtain

$$\limsup_{\mathbf{n}_{\max}\to\infty} [d(x, F_{\mathbf{n}}(\omega)) - d(x, F(\omega))] \leq 0,$$

which yields the desired conclusion. □

The theorem below will establish the SLLN for a double array of pairwise independent, Cesàro CUI random sets with respect to the gap topology based on Terán's method [23] and the technique of Castaing, Quang and Giap [6].

Theorem 3.8. *Let $1 \leq p \leq 2$ and $\{F_{mn} : m \geq 1, n \geq 1\}$ be a double array of pairwise independent, Cesàro CUI random sets and satisfying the following two conditions:*

(i) $\displaystyle\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{E\|F_{mn}\|^p}{(mn)^p} < \infty,$

(ii) *there exists $X \in k(\mathfrak{X})$ such that*

$$X \subset s\text{-}\liminf_{m\vee n\to\infty} (\mathrm{cl}E(F_{mn}, \mathcal{A}_{F_{mn}})), \tag{6}$$

$$\limsup_{m\vee n\to\infty} s(x^*, \mathrm{cl}EF_{mn}) \leq s(x^*, X), \quad x^* \in \mathfrak{X}^*. \tag{7}$$

Then $\displaystyle\frac{1}{mn}\mathrm{cl}\sum_{i=1}^{m}\sum_{j=1}^{n} F_{ij}(\omega) \to \overline{\mathrm{co}}X$ *a.s. in the gap topology as $m \vee n \to \infty$.*

Proof. For $x \in \overline{\mathrm{co}}X$ and $\varepsilon > 0$, select $x_1, \ldots, x_k \in X$ such that $\|\frac{1}{k}\sum_{i=1}^{k} x_i - x\| < \varepsilon$. We choose an array $\{x_{ij} : 1 \leq i \leq k, 1 \leq j \leq k\}$ of members of X such that

$$x_{ij} = \begin{cases} x_{i+j-1} & \text{if } \ i + j \leq k + 1 \\ x_{i+j-1-k} & \text{if } \ i + j > k + 1. \end{cases}$$

It is easy to check that $\displaystyle\frac{1}{k}\sum_{i=1}^{k} x_i = \frac{1}{k^2}\sum_{i=1}^{k}\sum_{j=1}^{k} x_{ij}$, and for each $s = 1, \ldots, k$,

$$\frac{1}{k}\sum_{i=1}^{k} x_i = \frac{1}{k}\sum_{i=1}^{k} x_{is}, \quad \frac{1}{k}\sum_{i=1}^{k} x_i = \frac{1}{k}\sum_{j=1}^{k} x_{sj}.$$

By (6), for each $x_{ij} \in X$, $i,j = 1,\dots,k$, there exists $z^{ij}_{mn} \in \mathrm{cl}E(F_{mn}|\mathcal{A}_{F_{mn}})$ such that $\|z^{ij}_{mn} - x_{ij}\| \to 0$ as $m \vee n \to \infty$. Therefore, there exists $y^{ij}_{mn} \in E(F_{mn}|\mathcal{A}_{F_{mn}})$ such that $\|y^{ij}_{mn} - z^{ij}_{mn}\| \to 0$ as $m \vee n \to \infty$.

Hence, there exists a double array $\{f_{ij} : i \geq 1, j \geq 1\}$ of $f_{ij} \in S^1_{F_{ij}}(\mathcal{A}_{F_{ij}})$ such that for each $i = 1,\dots,k$ and $j = 1,\dots,k$

$$\|Ef_{(s-1)k+i,(t-1)k+j} - x_{ij}\| \to 0 \text{ as } s \vee t \to \infty. \tag{8}$$

Put $y_{mn} = Ef_{mn}$, for all $m \geq 1, n \geq 1$. If $m = (s-1)k + p$, $n = (t-1)k + q$, where $1 \leq p \leq k$, $1 \leq q \leq k$, then

$$\Big\|\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n} f_{ij}(\omega) - \frac{1}{k}\sum_{i=1}^{k} x_i\Big\| = \Big\|\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n} f_{ij}(\omega) - \frac{1}{k^2}\sum_{i=1}^{k}\sum_{j=1}^{k} x_{ij}\Big\|$$

$$\leq \Big\|\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}(f_{ij}(\omega) - y_{ij})\Big\| + \tag{9}$$

$$+ \Big\|\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n} y_{ij} - \frac{1}{k^2}\sum_{i=1}^{k}\sum_{j=1}^{k} x_{ij}\Big\|. \tag{10}$$

For (9), since $\{F_{mn} : m \geq 1,\ n \geq 1\}$ is a double array of pairwise independent, Cesàro CUI random sets with $\sum\limits_{m=1}^{\infty}\sum\limits_{n=1}^{\infty}(mn)^{-p}E\|F_{mn}\|^p < \infty$, by Lemma 3.1 we get that $\{f_{mn} : m \geq 1, n \geq 1\}$ is a double array of pairwise independent, Cesàro CUI random elements satisfying

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(mn)^{-p}E\|f_{mn}\|^p \leq \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(mn)^{-p}E\|F_{mn}\|^p < \infty.$$

Hence, by applying the SLLN for the double array $\{f_{mn} : m \geq 1, n \geq 1\}$ of pairwise independent, Cesàro CUI random elements (see [20, Theorem 3.2]), we get a.s. as $m \vee n \to \infty$

$$\Big\|\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}(f_{ij}(\omega) - y_{ij})\Big\| = \Big\|\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}(f_{ij}(\omega) - Ef_{ij})\Big\| \to 0. \tag{11}$$

For (10), we have

$$\Big\|\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n} y_{ij} - \frac{1}{k^2}\sum_{i=1}^{k}\sum_{i=1}^{k} x_{ij}\Big\| \leq \frac{st}{mn}\sum_{i=1}^{k}\sum_{j=1}^{k}\frac{1}{st}\sum_{l=1}^{s}\sum_{r=1}^{t}\|y_{(l-1)k+i,(r-1)k+j} - x_{ij}\|$$

$$+ \frac{s}{mn}\sum_{i=1}^{k}\sum_{j=1}^{k}\frac{1}{s}\sum_{l=1}^{s}\|y_{(l-1)k+i,(t-1)k+j}\| + \frac{t}{mn}\sum_{i=1}^{k}\sum_{j=1}^{k}\frac{1}{t}\sum_{r=1}^{t}\|y_{(s-1)k+i,(r-1)k+j}\|$$

$$+ (\frac{st}{mn} - \frac{1}{k^2})\Big\|\sum_{i=1}^{k}\sum_{j=1}^{k} x_{ij}\Big\|.$$

By (8) and by the same argument as in [6, Theorem 3.8], we obtain

$$\Big\|\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n} y_{ij} - \frac{1}{k^2}\sum_{i=1}^{k}\sum_{i=1}^{k} x_{ij}\Big\| \to 0 \ \text{ as } m \wedge n \to \infty,$$

$$\Big\|\frac{1}{m}\sum_{i=1}^{m} y_{in} - \frac{1}{k}\sum_{i=1}^{k} x_i\Big\| \to 0 \ \text{ as } m \to \infty, \ \text{for each } n \geq 1,$$

$$\Big\|\frac{1}{n}\sum_{j=1}^{n} y_{mj} - \frac{1}{k}\sum_{i=1}^{k} x_i\Big\| \to 0 \ \text{ as } n \to \infty, \ \text{for each } m \geq 1.$$

Combining this with (11) and by Lemma 3.3, we get

$$\Big\|\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n} f_{ij}(\omega) - \frac{1}{k}\sum_{i=1}^{k} x_i\Big\| \to 0 \ \text{ a.s. as } m \vee n \to \infty.$$

This implies $\frac{1}{k}\sum_{i=1}^{k} x_i \in s\text{-}\liminf_{m\vee n\to\infty} \frac{1}{mn}\mathrm{cl}\sum_{i=1}^{m}\sum_{j=1}^{n} F_{ij}(\omega)$ a.s.

Thus, $\overline{\mathrm{co}}X \subset s\text{-}\liminf_{m\vee n\to\infty} \frac{1}{mn}\mathrm{cl}\sum_{i=1}^{m}\sum_{j=1}^{n} F_{ij}(\omega)$ a.s. Therefore, there is a null subset N_1 of Ω so that, for every $\omega \in \Omega \setminus N_1$ and $x \in \overline{\mathrm{co}}X$, we can choose

$$x_{mn} \in \frac{1}{mn}\mathrm{cl}\sum_{i=1}^{m}\sum_{j=1}^{n} F_{ij}(\omega) \text{ for } m \geq 1, \ n \geq 1$$

such that $\lim_{m\vee n\to\infty} x_{mn} = x$. Therefore, for any $y \in r.B$, $\lim_{m\vee n\to\infty}(x_{mn} + y) = x + y$, where B is the closed unit ball with respect to the fixed arbitrary equivalent norm of $\mathfrak{X}$ and $r > 0$. Hence, we obtain

$$\overline{\mathrm{co}}X + r.B \subset s\text{-}\liminf_{m\vee n\to\infty}\Big(\frac{1}{mn}\mathrm{cl}\sum_{i=1}^{m}\sum_{j=1}^{n} F_{ij}(\omega) + r.B\Big).$$

Since $X \in k(\mathfrak{X})$, by Lemma 3.4 we have $\overline{\mathrm{co}}X + r.B = \overline{\mathrm{co}}(X + r.B)$. Combining this with the separation theorem,

$$\overline{\mathrm{co}}X + r.B = \bigcap_{x^*\in S^*}\{x : \langle x^*, x\rangle \leq s(x^*, \overline{\mathrm{co}}X + r.B)\}.$$

Since $\mathfrak{X}$ is separable, there exists a countable Mackey dense subset D^* of S^* with

$$\begin{aligned}\overline{\mathrm{co}}X + r.B &= \bigcap_{x^*\in D^*}\{x : \langle x^*, x\rangle \leq s(x^*, \overline{\mathrm{co}}X + r.B)\}\\ &= \bigcap_{x^*\in D^*}\{x : \langle x^*, x\rangle \leq s(x^*, \overline{\mathrm{co}}X) + s(x^*, r.B)\}.\end{aligned}$$

Next, by (6) and Lemma 3.2, we get

$$\begin{aligned} s(x^*, X) &\le s(x^*, s\text{-}\liminf_{m\vee n\to\infty}(\mathrm{cl}E(F_{mn}, \mathcal{A}_{F_{mn}}))) \le \liminf_{m\vee n\to\infty} s(x^*, \mathrm{cl}E(F_{mn}, \mathcal{A}_{F_{mn}})) \\ &= \liminf_{m\vee n\to\infty} s(x^*, \overline{\mathrm{co}}E(F_{mn}, \mathcal{A}_{F_{mn}})) = \liminf_{m\vee n\to\infty} s(x^*, \overline{\mathrm{co}}EF_{mn}) \\ &= \liminf_{m\vee n\to\infty} s(x^*, \mathrm{cl}EF_{mn}). \end{aligned}$$

Combining this with (7), we obtain

$$E(s(x^*, F_{mn})) = s(x^*, \mathrm{cl}EF_{mn}) \to s(x^*, X) = s(x^*, \overline{\mathrm{co}}X) < \infty \quad \text{as } m \vee n \to \infty,$$

which yields

$$\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n} E(s(x^*, F_{ij})) \to s(x^*, \overline{\mathrm{co}}X) \quad \text{as } m \vee n \to \infty. \tag{12}$$

Next, since $\{F_{mn} : m \ge 1, n \ge 1\}$ is a double array of pairwise independent, Cesàro CUI random sets and $\sum\limits_{m=1}^{\infty}\sum\limits_{n=1}^{\infty}(mn)^{-p}E\|F_{mn}\|^p < \infty$, by Lemma 3.1 it follows that $\{s(x^*, F_{mn}) : m \ge 1,\ n \ge 1\}$ is a double array of pairwise independent, uniformly integrable in the Cesàro sense random variables satisfying

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(mn)^{-p}E|s(x^*, F_{mn})|^p \le \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(mn)^{-p}E\|F_{mn}\|^p < \infty.$$

Hence, by applying the SLLN for the double array $\{s(x^*, F_{mn}) : m \ge n \ge 1\}$ of pairwise independent, uniformly integrable in the Cesàro sense random variables (see [20, Theorem 3.1]), we get

$$\Big|\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}\Big(s(x^*, F_{ij}(\omega)) - Es(x^*, F_{ij})\Big)\Big| \to 0 \quad \text{a.s. as } m \vee n \to \infty.$$

By (12), we obtain

$$\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n} s(x^*, F_{ij}(\omega)) \to s(x^*, \overline{\mathrm{co}}X) \quad \text{a.s. as } m \vee n \to \infty.$$

Therefore, there is a null subset N_2 of Ω such that for every $\omega \in \Omega \setminus N_2$ and as $m \vee n \to \infty$,

$$s\Big(x^*, \frac{1}{mn}\mathrm{cl}\sum_{i=1}^{m}\sum_{j=1}^{n} F_{ij}(\omega)\Big) = \frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n} s(x^*, F_{ij}(\omega)) \to s(x^*, \overline{\mathrm{co}}X).$$

Fix any $\omega \in \Omega \setminus N_2$. If $z \in s\text{-}\limsup\limits_{m\vee n\to\infty}\Big(\frac{1}{mn}\mathrm{cl}\sum\limits_{i=1}^{m}\sum\limits_{j=1}^{n} F_{ij}(\omega) + r.B\Big)$, there exists $x_{st} \in \frac{1}{m_s n_t}\mathrm{cl}\sum\limits_{i=1}^{m_s}\sum\limits_{j=1}^{n_t} F_{ij}(\omega)$ and $y_{st} \in r.B$ for $s, t \ge 1$ such that $z = \lim\limits_{s\vee t\to\infty}(x_{st} + y_{st})$.

Therefore,

$$\begin{aligned}
\langle x^*, z\rangle &= \langle x^*, \lim_{s\vee t\to\infty}(x_{st}+y_{st})\rangle = \lim_{s\vee t\to\infty}\langle x^*, x_{st}+y_{st}\rangle \\
&= \lim_{s\vee t\to\infty}\Big(\langle x^*, x_{st}\rangle + \langle x^*, y_{st}\rangle\Big) \le \limsup_{s\vee t\to\infty}\langle x^*, x_{st}\rangle + \limsup_{s\vee t\to\infty}\langle x^*, y_{st}\rangle \\
&\le \limsup_{s\vee t\to\infty} s\Big(x^*, \frac{1}{m_s n_t}\mathrm{cl}\sum_{i=1}^{m_s}\sum_{j=1}^{n_t}F_{ij}(\omega)\Big) + \limsup_{s\vee t\to\infty} s(x^*, r.B) \\
&= s(x^*, \overline{\mathrm{co}}X) + s(x^*, r.B) \text{ for every } x^*\in D^*.
\end{aligned}$$

This implies $z\in\overline{\mathrm{co}}X + r.B$. Hence

$$s\text{-}\limsup_{m\vee n\to\infty}\Big(\frac{1}{mn}\mathrm{cl}\sum_{i=1}^{m}\sum_{j=1}^{n}F_{ij}(\omega)+r.B\Big)\subset\overline{\mathrm{co}}X + r.B \text{ for all } \omega\in\Omega\setminus N_2.$$

Put $N=N_1\cup N_2$. Then N has probability zero and for every $\omega\in\Omega\setminus N$,

$$\frac{1}{mn}\mathrm{cl}\sum_{i=1}^{m}\sum_{j=1}^{n}F_{ij}(\omega)+r.B\to\overline{\mathrm{co}}X + r.B,$$

in the Painlevé-Kuratowski sense, relative to the topology s, as $m\vee n\to\infty$, for all $r>0$ and for B being the unit ball of any equivalent norm, in which the choice of N_1 and N_2 does not depend on B and r.

Therefore, by Lemma 3.6 and by the equality $A^r = A + r.B$,

$$\frac{1}{mn}\mathrm{cl}\sum_{i=1}^{m}\sum_{j=1}^{n}F_{ij}(\omega)\to\overline{\mathrm{co}}X \quad \text{as } m\vee n\to\infty,$$

in all the Wijsman topologies generated by equivalent norms, except on the set $\Omega\setminus N$. Combining this with [3, Theorem 5.2] and [4, Theorem 3.1], we obtain this convergence for the gap topology in the convex case.

Thus, in the convex case, we achieve convergence in the slice or the gap topology. It remains to prove that this convergence extends to the non-convex case. Let A be a non-empty bounded closed convex set. We must prove that

$$D\left(\frac{1}{mn}\mathrm{cl}\sum_{i=1}^{m}\sum_{j=1}^{n}F_{ij}(\omega), A\right)\to D(\overline{\mathrm{co}}X, A) \text{ a.s. as } m\vee n\to\infty$$

except on a null set simultaneously valid for all such A.

By the definition of the gap functional, we get

$$D(\overline{\mathrm{co}}X, A) = \inf_{x\in\overline{\mathrm{co}}X}\inf_{y\in A}\|x-y\| = \inf_{y\in A} d(y, \overline{\mathrm{co}}X).$$

Therefore,
$$D(\overline{\mathrm{co}}X, A)\le d(y, \overline{\mathrm{co}}X) \text{ for all } y\in A. \tag{13}$$

Moreover, for any $\varepsilon' > 0$, we can find $y \in A$ such that

$$|D(\overline{\text{co}}X, A) - d(y, \overline{\text{co}}X)| \leq \varepsilon'. \tag{14}$$

Furthermore, by (13), we get

$$\begin{aligned} D\Big(\frac{1}{mn}\text{cl}\sum_{i=1}^{m}\sum_{j=1}^{n}\overline{\text{co}}F_{ij}(\omega), A\Big) &\leq D\Big(\frac{1}{mn}\text{cl}\sum_{i=1}^{m}\sum_{j=1}^{n}F_{ij}(\omega), A\Big) \\ &\leq d\Big(y, \frac{1}{mn}\text{cl}\sum_{i=1}^{m}\sum_{j=1}^{n}F_{ij}(\omega)\Big). \end{aligned} \tag{15}$$

For any $\omega \in \Omega \setminus N$, by virtue of our intermediate results for the gap and Wijsman topologies, we obtain

$$D\Big(\frac{1}{mn}\text{cl}\sum_{i=1}^{m}\sum_{j=1}^{n}\overline{\text{co}}F_{ij}(\omega), A\Big) \to D(\overline{\text{co}}X, A) \quad \text{as } m \vee n \to \infty, \tag{16}$$

and $$\lim_{m\vee n\to\infty} d\Big(y, \frac{1}{mn}\text{cl}\sum_{i=1}^{m}\sum_{j=1}^{n}F_{ij}(\omega)\Big) = d(y, \overline{\text{co}}X) \quad \text{for all } y \in A. \tag{17}$$

Combining (14), (15), (16) and (17), we conclude that

$$\limsup_{m\vee n\to\infty}\Big|D\Big(\frac{1}{mn}\text{cl}\sum_{i=1}^{m}\sum_{j=1}^{n}F_{ij}(\omega), A\Big) - D(\overline{\text{co}}X, A)\Big| \leq \varepsilon'.$$

This with the arbitrariness of $\varepsilon' > 0$ yields the desired result irrespective of the nonempty closed bounded convex set A. □

The example below shows that condition (ii) cannot be obtained from other conditions.

Example 3.9. *Considering $\mathfrak{X} = \mathbb{R}$, $p > 1$ and two random sets $F_1, F_2 : \Omega \to c(\mathfrak{X})$ defined by $F_1(\omega) = [0,1]$ and $F_2(\omega) = [-1,0]$, for all $\omega \in \Omega$, we have $EF_1 = [0,1]$ and $EF_2 = [-1,0]$.*

We define the random sets $F_{mn} : \Omega \to c(\mathfrak{X})$ by

$$F_{mn}(\omega) = \begin{cases} F_1(\omega) & \text{if} \quad m+n \text{ is odd} \\ F_2(\omega) & \text{if} \quad m+n \text{ is even,} \end{cases}$$

for every $\omega \in \Omega$, $m \geq 1, n \geq 1$.

Clearly, $\mathcal{A}_{F_{mn}} = \{\emptyset, \Omega\}$. Therefore we infer that $\{F_{mn} : m \geq 1, n \geq 1\}$ is a double array of independent random sets and for all $m \geq 1$, $n \geq 1$,

$$EF_{mn} = \begin{cases} [0,1] & \text{if} \quad m+n \text{ is odd} \\ [-1,0] & \text{if} \quad m+n \text{ is even} \end{cases}$$

and $E\|F_{mn}\| = 1$.

Moreover, we consider the compact set $K = [-1, 1]$. We easily see for any $\varepsilon > 0$

$$\sup_{m,n\geq 1} (mn)^{-1} \sum_{i=1}^{m} \sum_{j=1}^{n} E\{\|F_{ij}\| \mathbf{1}_{((F_{ij}(.)\subset K)^c)}\} = 0 < \varepsilon$$

and $\sum\limits_{m=1}^{\infty} \sum\limits_{n=1}^{\infty} (mn)^{-p} E\|F_{mn}\|^p = \sum\limits_{m=1}^{\infty} \sum\limits_{n=1}^{\infty} (mn)^{-p} < \infty$. Furthermore, $\mathrm{cl}EF_{mn} = EF_{mn}$ for every $m \geq 1$, $n \geq 1$, and so $s\text{-}\liminf\limits_{m\vee n\to\infty} \mathrm{cl}EF_{mn} = \{0\}$. Therefore, if there is a compact set X satisfying (6), then $X = \{0\}$.

At this point, considering $x^* \in \mathfrak{X}^*$ defined by $\langle x^*, x\rangle = x$ for every $x \in \mathfrak{X}$, we get

$$\limsup_{m\vee n\to\infty} s(x^*, \mathrm{cl}EF_{mn}) = 1, \quad \text{and} \quad s(x^*, X) = 0,$$

and therefore (7) does not hold. In consequence, such a double array of random sets $\{F_{mn} : m \geq 1, n \geq 1\}$ is pairwise independent, Cesàro CUI and satisfies $\sum\limits_{m=1}^{\infty} \sum\limits_{n=1}^{\infty} (mn)^{-p} E\|F_{mn}\|^p < \infty$, but the condition (ii) does not hold. □

The following example shows that in the Theorem 3.8 condition (ii) cannot be dispensed with.

Example 3.10. Let $\mathfrak{X} = \mathbb{R}$, $1 < p \leq 2$ and $\{A_n : n \geq 1\} \subset \mathcal{A}$ be a family of pairwise independent events such that $\mathbf{P}(\bigcap_{n=1}^{\infty} A_n) > 0$.

Consider two sequence of random sets $\{F_n : n \geq 1\}$ and $\{G_n : n \geq 1\}$ defined by

$$F_n(\omega) = \begin{cases} [0,1] & \text{if } \omega \in A_{2n} \\ [-1,0] & \text{if } \omega \in (A_{2n})^c, \end{cases} \qquad G_n(\omega) = \begin{cases} [-1,0] & \text{if } \omega \in A_{2n+1} \\ [0,1] & \text{if } \omega \in (A_{2n+1})^c, \end{cases}$$

for every $n \geq 1$. We construct a double array of random sets F_{mn} for $m, n \geq 1$ as follows:

$$F_{mn}(\omega) = \begin{cases} F_n(\omega) & \text{if } m = 1 \\ G_n(\omega) & \text{if } m = 2 \\ \{0\} & \text{if } m \geq 3, \end{cases}$$

for every $\omega \in \Omega$. It is easy to see that $\{F_{mn} : m \geq 1,\ n \geq 1\}$ is a double array of pairwise independent random sets. Moreover, for every $\varepsilon > 0$, choose the compact subset $K = [-1, 1]$ of $\mathfrak{X}$ such that

$$\sup_{m,n\geq 1} (mn)^{-1} \sum_{i=1}^{m} \sum_{j=1}^{n} E\left(\|F_{ij}\| \mathbf{1}_{(F_{ij}(.)\subset K)^c}\right) = 0 < \varepsilon,$$

and so the double array $\{F_{mn} : m \geq 1, n \geq 1\}$ is Cesàro CUI. On the other hand, since

$$E\|F_{mn}\|^p = \begin{cases} 1 & \text{if } m = 1, 2, \\ 0 & \text{if } m \geq 3, \end{cases}$$

we have
$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{E\|F_{mn}\|^p}{(mn)^p}\le 2\sum_{n=1}^{\infty}\frac{1}{n^p}<\infty.$$

Next, we put $S_{mn}=\frac{1}{mn}\mathrm{cl}\sum_{i=1}^{m}\sum_{j=1}^{n}F_{ij}$. For every $\omega\in\bigcap_{n=1}^{\infty}A_n$,

$$S_{1n}(\omega)=\frac{1}{n}\sum_{j=1}^{n}F_{1j}(\omega)=\frac{1}{n}\sum_{j=1}^{n}[0,1]=[0,1]$$

and
$$S_{2n}(\omega)=\frac{1}{n}\sum_{j=1}^{n}F_{2j}(\omega)=\frac{1}{n}\sum_{j=1}^{n}[-1,0]=[-1,0].$$

Therefore, the limit of the double array $\{S_{mn}:m\ge 1,n\ge 1\}$ as $m\vee n\to\infty$ with respect to the gap topology does not exist almost surely. Thus, the conclusion of Theorem 3.8 is not true without the condition (ii).

4. SLLN for a double array of independent closed valued random variables in a separable Banach space

Now we introduce the SLLN for a double array of random sets in a Rademacher type p Banach space.

Theorem 4.1. *Suppose that $\mathfrak{X}$ is a Rademacher type p Banach space, where $1\le p\le 2$. Let $\{F_{mn}:m\ge 1,\ n\ge 1\}$ be a double array of independent random sets satisfying the following conditions:*

(i) $\displaystyle\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{E\|F_{mn}\|^p}{(mn)^p}<\infty,$

(ii) *there exists $X\in k(\mathfrak{X})$ such that*

$$X\subset s\text{-}\liminf_{m\vee n\to\infty}(\mathrm{cl}E(F_{mn},\mathcal{A}_{F_{mn}})),\tag{18}$$

$$\limsup_{m\vee n\to\infty}s(x^*,\mathrm{cl}EF_{mn})\le s(x^*,X),\ \ x^*\in\mathfrak{X}^*.\tag{19}$$

Then $\displaystyle\frac{1}{mn}\,\mathrm{cl}\sum_{i=1}^{m}\sum_{j=1}^{n}F_{ij}(\omega)\to\overline{\mathrm{co}}X$ *a.s. in the gap topology as $m\vee n\to\infty$.*

Proof. By applying [6, Theorem 3.8], we get

$$\overline{\mathrm{co}}X\subset s\text{-}\liminf_{m\vee n\to\infty}\frac{1}{mn}\,\mathrm{cl}\sum_{i=1}^{m}\sum_{j=1}^{n}F_{ij}(\omega)\ \text{a.s.}$$

Similarly as in the proof of Theorem 3.8, we obtain

$$\overline{\mathrm{co}}X+r.B\subset s\text{-}\liminf_{m\vee n\to\infty}\Big(\frac{1}{mn}\,\mathrm{cl}\sum_{i=1}^{m}\sum_{j=1}^{n}F_{ij}(\omega)+r.B\Big).$$

By (18), (19), and Lemma 3.2, we get

$$E(s(x^*, F_{mn})) = s(x^*, \mathrm{cl}EF_{mn}) \to s(x^*, X) = s(x^*, \overline{\mathrm{co}}X) \text{ as } m \vee n \to \infty.$$

Therefore, $\displaystyle\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n} E(s(x^*, F_{ij})) \to s(x^*, \overline{\mathrm{co}}X)$ as $m \vee n \to \infty$.

Moreover, since $\{F_{mn} : m \geq 1, n \geq 1\}$ is a double array of independent random sets, $\{s(x^*, F_{mn}) : m \geq 1, \ n \geq 1\}$ is a double array of independent random variables with

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{E|s(x^*, F_{mn})|^p}{(mn)^p} \leq \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{E\|F_{mn}\|^p}{(mn)^p} < \infty.$$

Hence, by applying the SLLN for the double array of independent random variables in $\mathbf{L}^p(\mathbb{R})$ (see [21]), we obtain

$$\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}\Big(s(x^*, F_{ij}(\omega)) - Es(x^*, F_{ij})\Big) \to 0 \ \text{ a.s. as } m \vee n \to \infty.$$

Thus, there is a null subset N_2 of Ω such that for every $\omega \in \Omega \setminus N_2$, as $m \vee n \to \infty$

$$s\Big(x^*, \frac{1}{mn}\mathrm{cl}\sum_{i=1}^{m}\sum_{j=1}^{n} F_{ij}(\omega)\Big) = \frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n} s(x^*, F_{ij}(\omega)) \to s(x^*, \overline{\mathrm{co}}X).$$

For the rest of proof, we proceed as in the proof of Theorem 3.8. □

The following theorem will establish the SLLN for a double array of pairwise independent identically distributed random sets with respect to the gap topology.

Theorem 4.2. *Suppose that $\{F_{mn} : m \geq 1, n \geq 1\}$ is a double array of pairwise independent random sets, having the same distribution as X which is a $k(\mathfrak{X})$-valued random set satisfying $E(\|X\| \log^+ \|X\|) < \infty$. Then,*

$$\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n} F_{ij}(\omega) \to \overline{\mathrm{co}}EX$$

almost surely, in the gap topology, as $m \vee n \to \infty$.

Proof. By applying [6, Theorem 3.7], we get

$$\overline{\mathrm{co}}EX \subset s\text{-}\liminf_{m\vee n\to\infty}\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n} F_{ij}(\omega) \text{ a.s.}$$

Therefore, there is a null subset N_1 of Ω so that, for all $\omega \in \Omega \setminus N_1$ and $x \in \overline{\mathrm{co}}EX$, we can choose

$$x_{mn} \in \frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n} F_{ij}(\omega) \ \text{ for all } m \geq 1, \ n \geq 1$$

such that $\lim_{m \vee n \to \infty} x_{mn} = x$, which yields

$$\lim_{m \vee n \to \infty} (x_{mn} + y) = x + y \text{ for any } y \in r.B,$$

where B is the closed unit ball with respect to the fixed arbitrary equivalent norm of $\mathfrak{X}$ and $r > 0$. Hence, we obtain

$$\overline{\mathrm{co}} EX + r.B \subset s\text{-}\liminf_{m \vee n \to \infty} \Big(\frac{1}{mn} \sum_{i=1}^{m} \sum_{j=1}^{n} F_{ij}(\omega) + r.B \Big).$$

Since $X(\omega) \in k(\mathfrak{X}), \omega \in \Omega$, we have that $EX \in k(\mathfrak{X})$. By Lemma 3.4, $\overline{\mathrm{co}} EX + r.B = \overline{\mathrm{co}}(EX + r.B)$.

Similarly as in the proof of Theorem 3.8, there exists a countable Mackey dense subset D^* of S^* satisfying

$$\overline{\mathrm{co}} EX + r.B = \bigcap_{x^* \in D^*} \{x : \langle x^*, x \rangle \leq s(x^*, \overline{\mathrm{co}} EX) + s(x^*, r.B)\}.$$

Next, since $\{F_{mn} : m \geq 1, n \geq 1\}$ is a double array of pairwise independent random sets, having the same distribution as X, $\{s(x^*, F_{mn}) : m \geq 1, n \geq 1\}$ is a double array of pairwise independent identically distributed random variables and

$$E(|s(x^*, X)| \log^+ |s(x^*, X)|) \leq E(\|X\| \log^+ \|X\|) < \infty,$$

and so, by virtue of Etemadi's SLLN [8], there exists a null subset N_2 of Ω such that for every $\omega \in \Omega \setminus N_2$, as $m \vee n \to \infty$,

$$\begin{aligned} s\Big(x^*, \frac{1}{mn} \sum_{i=1}^{m} \sum_{j=1}^{n} F_{ij}(\omega)\Big) &= \frac{1}{mn} \sum_{i=1}^{m} \sum_{j=1}^{n} s(x^*, F_{ij}(\omega)) \\ &\to \quad E(s(x^*, X)) = s(x^*, \overline{\mathrm{co}} EX) < \infty\,. \end{aligned}$$

For the rest of the proof, we proceed as in the proof of Theorem 3.8 noting that $E(\overline{\mathrm{co}} X) = \overline{\mathrm{co}} EX$ (see [18, Theorem 1.17, page 154]). □

Remark 4.3. *The compactness of values of the random set X ensures that we can use Lemma* 3.4 *to prove Theorem* 4.2. *Moreover, if X is a convex compact valued random set, then using the embedding method, we get the same result with respect to the Hausdorff distance convergence.*

Open problem. Can an example be given to show that if X is only a closed valued random set, then the conclusion in Theorem 4.2 is not true?

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