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Functional Characterizations of Countable Tychonoff Spaces

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Several characterizations of countable Tychonoff spaces by means of function spaces equipped with various locally convex topologies are provided.

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1. Introduction

For a Tychonoff space X, we denote by $C_p(X)$ and $C_k(X)$ the space $C(X)$ of all continuous real-valued functions on X endowed with the pointwise topology τ_p and the compact-open topology τ_k, respectively. The subspace of $C(X)$ consisting of all bounded functions is denoted by $C^b(X)$. Unless otherwise stated $C^b(X)$ means always a Banach space with the sup-norm topology. Moreover, $C^b(X)$ considered as a subspace of $C_p(X)$ or $C_k(X)$ is denoted by $C_p^b(X)$ and $C_k^b(X)$, respectively. A locally convex space E endowed with the weak topology is denoted by E_w.

We start with the classic Velichko theorem; parts (iii) and (iv) supplement this result.

Theorem 1.1. *For a Tychonoff space X the following assertions are equivalent:*

(i) *X is finite;*

(ii) *$C_p(X)$ is σ-compact* ([1, Theorem 1.2.2])*;*

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(iii) $C_p(X)$ *is* σ*-pointwise bounded relatively sequentially complete* (*see* [10, Corollary 3.2])*;*

(iv) $C_p(X)$ *has a fundamental sequence of bounded sets* ([15, Proposition 2.13, Corollary 2.8]).

Thus X is finite if the function space $C_p(X)$ can be covered by an *increasing sequence* of compact-like subsets. It is natural to consider other types of ordered families of compact-type subsets. The following notion is widely studied in functional analysis, see for example [15] and references therein for papers dealing with this concept. A locally convex space E admits a *compact (bounded) resolution* if there is a family $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ of compact (bounded) sets in E covering E and such that $A_\alpha \subseteq A_\beta$ for all $\alpha \leq \beta$ in $\mathbb{N}^{\mathbb{N}}$. If additionally each compact (bounded) set in E is contained in some A_α, the above family is called a *fundamental compact (bounded) resolution.* Surely Theorem 1.1 fails when the condition (ii) replaced by $C_p(X)$ has a *compact resolution* as the case for metrizable compact X shows.

The most evident (and well known) functional characterization of countable Tychonoff spaces X states that X is countable if and only if $C_p(X)$ is a metrizable lcs. On the other hand, Tkachuk [20] characterized countable discrete spaces as follows.

Theorem 1.2. ([20]) *A Tychonoff space X is countable and discrete if and only if $C_p(X)$ has a fundamental compact resolution.*

Recently Guerrero Sánchez and Tkachuk (answering a question from [4] motivated by the above theorem) have proved in [14] that if $C_p(X)$ is *strongly dominated by a second countable space*, then X is countable.

Let us recall the next interesting result of this type due to Mercourakis and Stamati, see [17, Corollary 1.10].

Theorem 1.3. ([17]) *A compact Hausdorff space X is finite if and only if $C(X)_w$ has a fundamental compact resolution.*

Note that the above result originally was formulated for $SWKA$ Banach spaces $C(X)$, but Lemma 1.7 from [17] shows that a Banach space E is $SWKA$ if and only if E_w has a fundamental compact resolution.

In [6] we obtained another functional characterization of countable spaces X.

Theorem 1.4. ([6]) *A Tychonoff space X is countable if and only if $C_p(X)$ has a fundamental bounded resolution.*

Other functional characterizations of countable spaces involving so called generalized metric conditions were obtained by Corson, Michael and Sakai. The following result was proved by Corson, see [18, Proposition 10.8].

Theorem 1.5. ([18]) *For a compact Hausdorff space X, the space $C(X)_w$ is an $\aleph_0$-space if and only if X is countable.*

This theorem was extended in [12] to $\aleph$-spaces $C_k(X)_w$, where $C_k(X)$ is a Fréchet space. Next theorem (dealing with generalized metric conditions for spaces $C_p(X)$) gathers together six equivalent conditions (much less evident) for X to be countable, the first fifth were proved by Sakai [19]. The equivalence (i)$\Leftrightarrow$(vi) is due to Gartside and Reznichenko [13].

Theorem 1.6. ([13, 19]) *For a Tychonoff space X the following conditions are equivalent:*

(i) *X is countable.*

(ii) *$C_p(X)$ is an $\aleph_0$-space.*

(iii) *$C_p(X)$ is an $\aleph$-space.*

(iv) *$C_p(X)$ has a strong Pytkeev property.*

(v) *$C_p(X)$ has a countable cs*-network at zero.*

(vi) *$C_p(X)$ is stratifiable.*

Being motivated by the above theorems, we provide several functional characterizations of countable (in particular, finite) spaces by mean of another important classes of function spaces endowed with different (but natural) locally convex topologies. Our first main result below extends Theorem 1.1. The equivalence between (i) and (iii) extends Theorem 1.3 of Mercourakis and Stamati.

Theorem 1.7. *For a Tychonoff space X the following assertions are equivalent:*

(i) *X is finite;*

(ii) *$C_p^b(X)$ has a fundamental compact resolution;*

(iii) *$C_k^b(X)_w$ has a fundamental compact resolution;*

(iv) *$C^b(X)_w$ has a fundamental compact resolution.*

Our second main result below is a natural counterpart to Theorem 1.2 and extends Theorem 1.3.

Theorem 1.8. *A Tychonoff space X is countable and discrete if and only if the space $C_k(X)_w$ has a fundamental compact resolution.*

It is well known that for infinite compact K the space $C(K)$ contains a copy of c_0. Consequently $C(K)_w$ does not contain a fundamental compact resolution just by applying Christensen's theorem, see [15, Theorem 6.1]. By the same reason any space $C_k(X)$ containing a copy of c_0 does not have a fundamental compact resolution in the weak topology. The converse implication fails. Indeed, if X is an uncountable discrete space, then $C_p(X) = C_k(X) = \mathbb{R}^X = C_k(X)_w$ does not contain a copy of c_0 neither admits a fundamental compact resolution.

The next theorem supplements Theorem 1.7.

Theorem 1.9. *A Tychonoff space X is countable if and only if $C_p^b(X)$ has a fundamental bounded resolution.*

We prove Theorems 1.7–1.9 in Section 2. We also characterize those Tychonoff spaces X for which the Banach space $C^b(X)$ has a weakly compact resolution, see Proposition 2.2.

2. Proofs

Let X be a Tychonoff space. Denote by $\mathfrak{F}(X)$ and $\mathcal{K}(X)$ the families of all finite subsets and all compact subsets of X, respectively. For a subset A of X and $\varepsilon > 0$, set

$$[A;\varepsilon] := \big\{f \in C(X) : |f(x)| < \varepsilon \text{ for every } x \in A\big\}.$$

Then the families $\{[F;\varepsilon] : F \in \mathfrak{F}(X), \varepsilon > 0\}$ and $\{[K;\varepsilon] : K \in \mathcal{K}(X), \varepsilon > 0\}$ form a basis for the pointwise topology τ_p and the compact-open topology τ_k on $C(X)$, respectively.

We need the following assertion.

Proposition 2.1. *Let X be a Tychonoff space and let $H = C_k(X)$ or $H = C_k^b(X)$. If H_w has a fundamental compact resolution, then every compact subset of X is finite.*

Proof. If $H = C_k(X)$ or $H = C_k^b(X)$, we set $E := C_p(X)$ or $E := C_p^b(X)$, respectively, and note that in both cases E has a compact resolution. Let $\big\{A_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\big\}$ be a fundamental compact resolution for H_w.

Now suppose for a contradiction that there exists an infinite compact subset K of X. Since $C_p^b(X)$ is a dense linear subspace of $C_p(X)$, according to [15, Example 4.1], in both cases the space $C_p(X)$ is web-compact. Hence, Orihuela's angelicity theorem [15, Theorem 4.5] yields that each compact set of $X \hookrightarrow C_p(C_p(X))$ is Fréchet–Urysohn. Consequently, since K is infinite, it contains an infinite convergent sequence Q. Since Q is countable, it is a metrizable compact set. Note that Q is (homeomorphic to) a metrizable and compact subspace of βX. Therefore, by [3], there is a continuous linear extender map $\varphi : C_p(Q) \to C_p(\beta X)$. Consequently, if $S : C_p(\beta X) \to E$ is the restriction map $Sg = g|_X$, the mapping $\psi = S \circ \varphi : C_p(Q) \to E$ is a continuous linear extender, i. e. $\psi(f)|_Q = f$ for every $f \in C(Q)$. This ensures that the map $\psi : C(Q) \to H_w$ has closed graph. Since $C(Q)$ is a Banach space and H_w has a compact resolution, Theorem 1 of [8] implies that the map $\psi : C(Q) \to H_w$ is continuous, so that ψ is weakly continuous.

Claim: The inverse family $\big\{\psi^{-1}(A_\alpha) : \alpha \in \mathbb{N}^\mathbb{N}\big\}$ is a resolution for $C(Q)$ consisting of weakly compact sets.

Indeed, if $\{f_n\}_{n\in\mathbb{N}}$ is a sequence in $\psi^{-1}(A_\delta) \subseteq C(Q)$ for $\delta \in \mathbb{N}^\mathbb{N}$, then $\{\psi(f_n)\}_{n\in\mathbb{N}}$ is a sequence in the compact set A_δ. So, there is a cluster point $h \in A_\delta$ of the sequence $\{\psi(f_n)\}_{n\in\mathbb{N}}$. If $T : H \to C(Q)$ stands for the linear restriction map defined by $T(g) = g|_Q$, clearly T is a weakly continuous map, which implies that $T(h)$ is a cluster point of $\{T(\psi(f_n)\}_{n\in\mathbb{N}}$ in $C(Q)_w$. Now since

$$T(\psi(f_n)) = \psi(f_n)|_Q = f_n, \quad \forall n \in \mathbb{N},$$

one has that $h|_Q$ is a cluster point of $\{f_n\}_{n\in\mathbb{N}}$ in $C(Q)_w$. So, we have shown that $\psi^{-1}(A_\delta)$ is a weakly countably compact subset of $C(Q)$, thus a compact set keeping in mind that $C(Q)_w$ is angelic (since Q is compact). Consequently the family $\big\{\psi^{-1}(A_\alpha) : \alpha \in \mathbb{N}^\mathbb{N}\big\}$ is a weakly compact resolution for $C(Q)$, as claimed.

Let P be a compact subset of $C(Q)_w$. Then $\psi(P)$ is a compact set in H_w. Therefore there is a $\gamma \in \mathbb{N}^{\mathbb{N}}$ such that $\psi(P) \subseteq A_\gamma$. Therefore $P \subseteq \psi^{-1}(A_\gamma)$, which means that $\left\{\psi^{-1}(A_\alpha) : \alpha \in \mathbb{N}^{\mathbb{N}}\right\}$ swallows the compact sets of $C(Q)_w$. Hence, Theorem 1.3 implies that Q is a finite set, a contradiction. $\square$

Now we prove Theorem 1.7.

Proof of Theorem 1.7. The implications (i) $\Rightarrow$ (ii), (iii), (iv) are trivial.

(ii) $\Rightarrow$ (i) Assume that the space $C_p^b(X)$ has a fundamental compact resolution.

Claim 1: For the discrete space $\mathbb{N}$, the space $C_p^b(\mathbb{N})$ does not have a fundamental compact resolution.

Indeed, otherwise the Christensen theorem implies that $C_p^b(\mathbb{N})$ is completely metrizable. So $C_p^b(\mathbb{N})$ is a Baire space. However this is impossible since $C^b(\mathbb{N}) = \bigcup_{n\in\mathbb{N}}[-n,n]^{\mathbb{N}}$, where the compact sets $[-n,n]^{\mathbb{N}}$ are nowhere dense in $C_p^b(\mathbb{N})$.

Claim 2: The space X is pseudocompact.

Indeed, suppose for a contradiction that X is not pseudocompact. Then we can choose a sequence $\{x_n : n \in \mathbb{N}\}$ of points in X and a discrete sequence $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$ of closed subsets of X such that U_n is a neighborhood of x_n for every $n \in \mathbb{N}$. For every $n \in \mathbb{N}$, choose a continuous function $f_n : X \to [0,1]$ such that $f_n(X \setminus U_n) = \{0\}$ and $f_n(x_n) = 1$. Define

$$H := \left\{ \sum_{n\in\mathbb{N}} a_n f_n : \ (a_n) \in \ell_\infty \right\}.$$

Clearly, H is a linear subspace of $C_p^b(X)$. We show that H is closed (even in $C_p(X)$). Let $h_i = \sum_{n\in\mathbb{N}} a_n^i f_n$ be a net in H converging pointwise to $h \in C_p^b(X)$. Then for every $n \in \mathbb{N}$, $h_i(x_n) = a_n^i$ converges to $h(x_n) := a_n$. Since the family $\mathcal{U}$ is discrete, it is clear that $h = \sum_{n\in\mathbb{N}} a_n f_n$ and hence $h \in H$. So H, with the relative pointwise topology, also has a fundamental compact resolution. Moreover, the map

$$T : H \to C_p^b(\mathbb{N}), \quad T\left(\sum_{n\in\mathbb{N}} a_n f_n\right) := (a_n),$$

is a linear topological isomorphism. Indeed, clearly T is linear and bijective. Set

$$Y := \{x \in X : \ f_n(x) \neq 0 \text{ for some } n \in \mathbb{N}\}.$$

To show that T is open let F be a finite subset of X and $\epsilon > 0$. Set $F' := F \cap Y$. For every $y \in F'$, let $n(y)$ be the unique $n \in \mathbb{N}$ such that $f_{n(y)}(y) \neq 0$. Then

$$T([F;\epsilon]) = \bigcap_{y\in F'} \left\{ (a_n) \in C_p^b(\mathbb{N}) : |a_{n(y)}| < \epsilon / |f_{n(y)}(y)| \right\}$$

and hence T is open. The map T is continuous since for every finite $L \subseteq \mathbb{N}$ and each $\epsilon > 0$, we have $T([F;\epsilon]) \subseteq [L;\epsilon]$, where $F := \{x_n : n \in L\}$. Therefore, the space $C_p^b(\mathbb{N})$ also has a fundamental compact resolution. But this contradicts Claim 1. Thus X must be pseudocompact.

Claim 3. The space X is finite.

Indeed, by Claim 2, the space X is pseudocompact. Therefore the space $C_p(X) = C_p^b(X)$ also has a fundamental compact resolution. Now Theorem 1.2 implies that X is countable and discrete. Thus X being pseudocompact must be finite.

(iii) ⇒ (i) Let $C_k^b(X)_w$ have a fundamental compact resolution. Then, by Proposition 2.1, all compact sets of X are finite. Therefore $C_p(X) = C_k(X)$, which implies that $C_k^b(X)_w = C_p^b(X)$ has a fundamental compact resolution. Thus X must be finite by (ii).

(iv) ⇒ (i) If $C^b(X)_w$ has a fundamental compact resolution, so does $C(\beta X)_w$. So Theorem 1.3 ensures that βX must be finite. □

Below we prove Theorem 1.8.

Proof of Theorem 1.8. If X is countable and discrete, then the Polish space $C_k(X)_w = \mathbb{R}^X$ has a fundamental compact resolution.

Conversely, let $C_k(X)_w$ have a fundamental compact resolution. Then, by Proposition 2.1, all compact sets of X are finite. Therefore $C_k(X)_w = C_p(X)$. Thus X is countable and discrete by Theorem 1.2. □

Denote by $\mathfrak{F}(X)$ the family of all finite subsets of a space X.

Proof of Theorem 1.9. Let $\{A_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ be a fundamental bounded resolution for $C_p^b(X)$ consisting of pointwise closed subsets. For every $\alpha \in \mathbb{N}^\mathbb{N}$, let B_α be the closure of A_α in $C_p(X)$. We claim that the family $\mathcal{B} := \{B_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ is a fundamental bounded resolution for $C_p(X)$. Indeed, fix arbitrarily a bounded subset B of $C_p(X)$. For every $F \in \mathfrak{F}(X)$ and each $f \in B$, set $d(f;F) := \max\{|f(x)| : x \in F\}$ and define

$$f_F(x) := \begin{cases} f(x) & \text{if } |f(x)| \le d(f;F), \\ \text{sign } (f(x)) \cdot d(f;F) & \text{if } |f(x)| > d(f;F). \end{cases}$$

Then $f_F \in C^b(X)$ with $f_F(x) = f(x)$ for each $x \in F$ and $|f_F(x)| \le |f(x)|$ for every $x \in X$. Set $T_B := \{f_F : f \in B \text{ and } F \in \mathfrak{F}(X)\}$. Clearly, T_B is a pointwise bounded subset of $C^b(X)$ whose closure in $C_p(X)$ contains B. Therefore, there is $\alpha \in \mathbb{N}^\mathbb{N}$ such that $T_B \subseteq A_\alpha$ and hence $B \subseteq B_\alpha$. This proves the claim. Now Theorem 1.4 implies that X is countable. Conversely, if X is countable then $C_p^b(X)$ is metrizable, and hence it has a fundamental bounded resolution, see [6, Corollary 2.2]. □

Recall that a compact space K is called *Talagrand compact* if $C(K)_w$ is K-analytic. Below we characterize the existence of compact resolutions in $C^b(X)_w$.

Proposition 2.2. *Let X be a Tychonoff space. Then $C^b(X)$ has a weakly compact resolution if and only if βX is a Talagrand compact. Consequently, if X is realcompact, then $C^b(X)_w$ has a compact resolution if and only if X is a Talagrand compact.*

Proof. Note that the Stone extension map $f \mapsto f^\beta$ is a linear isometry from $C^b(X)$ onto $C(\beta X)$ when both Banach spaces equipped with the supremum norm, which becomes a weakly continuous isomorphism between both spaces. Now the assertion follows from the angelicity of $C^b(X)_w$ and Corollary 3.6 of [15].

Now we prove the last assertion. The 'if part' is clear. Conversely, if $C^b(X)_w$ has a compact resolution, then it is weakly K-analytic. Hence X is pseudocompact by [9, Lemma 3.1]. Since X is assumed to be realcompact, it must be compact. So $\beta X = X$ and the 'only if' statement follows. □

Example 2.3. For concrete examples consider the ordinal space $X = [0, \omega_1)$, where ω_1 is the first uncountable ordinal, equipped with the order topology. Observe that X is not realcompact, but since $\beta X = [0, \omega_1]$ is not Talagrand compact, the previous proposition prevents $C^b(X)_w$ to have a compact resolution. On the other hand, if we consider the one-point Lindelöfication Y of space $[0, \omega_1)$, this time provided with the discrete topology, then Y is a Lindelöf P-space (see [1, Example 4.2.15]), hence realcompact. By Proposition 2.2, neither $C^b(Y)_w$ has a compact resolution. According to [2, Example 7.14] there exists a Talagrand compact space K with a point $x \in K$ such that K coincides with the Stone-Čech compactification of $Z = K \setminus \{x\}$. By the previous proposition the space $C^b(Z)_w$ has a compact resolution.

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