

The Plasticity of Non-Overlapping Convex Sets in $\mathbb{R}^2$

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We study a generalization of the weighted Fermat-Torricelli problem in the plane, which is derived by replacing vertices of a convex polygon by 'small' closed convex curves with weights being positive real numbers on the curves, we also study its generalized inverse problem. Our solution of the problems is based on the first variation formula of the length of line segments that connect the weighted Fermat-Torricelli point with its projections onto given closed convex curves. We find the 'plasticity' solutions for non-overlapping circles with variable radius.

Keywords: Fermat-Torricelli problem, convex, curve, variation, inverse problem, plasticity of non-overlapping closed convex sets.

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1. Introduction

The extremum problem formulated by Fermat and after a few years solved by Torricelli, is as follows: Given three points A, B and C in the Euclidean plane $\mathbb{R}^2$, the task is to find a point P such that the sum of distances $PA + PB + PC$ is minimal. The *weighted Fermat-Torricelli* (F-T) problem is to find the (unique) point that minimizes the sum of the weighted distances (i.e., multiplied by positive numbers – weights) from three given points in $\mathbb{R}^2$. The study of the weighted F-T problem in the plane and its inverse is given in [5, 20]. For historical remarks and generalizations (e.g. in Banach spaces) of the weighted F-T problem, the reader can consult [1, 9, 10, 11, 12].

In this paper, we study the generalized F-T problem for $n \geq 3$ convex sets in $\mathbb{R}^2$ and provide a method of its study. We are based on a technique of differentiation of the length of geodesics on a C^2 -surface with respect to arc length, see e.g. [14], and applying the parametrization method of [3, 4].

Let $A_1 A_2 \dots A_n$ ($n \geq 3$) be a convex polygon in $\mathbb{R}^2$ and γ_i a convex curve surrounding A_i that meets orthogonally at points D_{ij} ($j \neq i$) two sides containing A_i , and the curves do not intersect inside the polygon, see Fig. 2.1 for $n = 3$. Segments on the edges of the polygon and n arcs of γ_i bound a curvilinear $2n$ -gon Ω . For $P \in \Omega$, let $A'_i = \pi_i(P)$ be projections of P onto γ_i . Note that the segments

PA'_i intersect γ_i orthogonally. For $n = 3$, let φ_Q be the angle between the line segments RP and SP for $Q, R, S \in \{A'_1, A'_2, A'_3\}$ and $Q \neq R \neq S$.

Problem 1.1. (The generalized F-T problem in $\mathbb{R}^2$) Find a point $P \in \mathbb{R}^2$ such that

$$\sum_{i=1}^n w_i d(P, \gamma_i) \rightarrow \min, \quad (1)$$

where $w_1, \dots, w_n$ are given positive numbers (weights) and d the distance.

Problem 1.2. (The inverse generalized F-T problem in $\mathbb{R}^2$) Given a generalized F-T point P inside Ω with the vertices defined by orthogonal projections of P onto γ_i ($i = 1, \dots, n$) find positive weights $w_i \in \mathbb{R}$ such that

$$\sum_{i=1}^n w_i = 1.$$

The paper is organized as follows. Section 2 is devoted to the solution of Problem 1.1. In Sections 3 and 4 we characterize solutions of Problem 1.2.

2. The generalized F-T problem for closed convex sets in $\mathbb{R}^2$

In this section we assume that the weights w_i are positive real numbers which correspond on convex curves γ_i , and generalize results of [20], where $\gamma_i = A_i$.

Proposition 2.1. The function $f = \sum_i w_i d(P, \pi_i P)$ is convex in Ω .

Proof. It is sufficient to show that the function $f_i = w_i d(P, \pi_i(P))$ is strictly convex in Ω . We shall compute the second differential of f_i at a point P in any direction X . Let P_s be a smooth curve with the properties $P_0 = P$ and $\frac{d}{ds} P_s = \alpha X$ (for some $\alpha \neq 0$) such that s is the natural parameter of $\pi_i(P_s)$ (the part of γ_i). Denote by $f_i(s) = w_i(s) l_i(s)$, where $l_i(s) = d(P_s, \pi_i(P_s)) \geq 0$. It is known that $l'_i(0) = 0$ and $l''_i(0) > 0$ (see in [8], [7, Proposition 6.1, Corollary 6.1]). By conditions, we get $f''_i = w_i l''_i > 0$ at $s = 0$. Hence $d^2 f_i(X, X) > 0$. The sum $f = \sum_i f_i$ of strictly convex functions is also strictly convex. $\square$

Theorem 2.2. The solution (P_F) of Problem 1.1 exists and is unique.

Proof. By Proposition 2.1, the objective function in (1) is strictly convex; hence, it has one minimum point in Ω , see [2], [13, p. 263]. $\square$

The next theorem and corollary can be easily extended for any $n > 3$.

Theorem 2.3. If the generalized F-T point P is an interior point of Ω (Fig. 2.1) then each angle φ_i can be expressed as a function of w_i , ($i = 1, 2, 3$), as

$$\cos \varphi_1 = \frac{w_1^2 - w_2^2 - w_3^2}{2w_2w_3}, \quad \cos \varphi_2 = \frac{w_2^2 - w_1^2 - w_3^2}{2w_1w_3}, \quad \cos \varphi_3 = \frac{w_3^2 - w_1^2 - w_2^2}{2w_1w_2}. \quad (2)$$

Proof. Let $\gamma_i : \vec{r}_i(s) \equiv (x_i(s), y_i(s), z_i(s))$ be three convex curves in $\mathbb{R}^2$, $\gamma_{Pi} : \vec{r}_{Pi}(s) \equiv (x_{Pi}(s), y_{Pi}(s), z_{Pi}(s))$ be three smooth curves in $\mathbb{R}^2$ starting from P ,

$$l_i(s) \equiv \sqrt{(x_i(s) - x_{Pi}(s))^2 + (y_i(s) - y_{Pi}(s))^2 + (z_i(s) - z_{Pi}(s))^2}$$

the length of the line segment $\gamma_i(s)\gamma_{Pi}(s)$, and $d(P, \gamma_i) \equiv l_Q$ for $Q \in \{A'_1, A'_2, A'_3\}$ the distance. Let us join Q and P by the line segment defined by the line $c_Q(s, t) = c_{QP}(t)$, where $t \in [0, 1]$ is a canonical parameter such that $c_{QP}(0) = Q$ and $c_{QP}(1) = P$. The function

$$l_Q(P) = l_Q(c_{QP}(t)) = l_Q(c_Q(s, t)) = l_Q(s)$$

is differentiable with respect to s . Denote by $\varphi_c^i(s)$ the angle between the vectors $-\frac{dc}{ds} \equiv \overrightarrow{\gamma_i(s)\gamma_{Pi}(s)}$ and $\frac{d}{ds} \vec{r}_{Pi}$ at P and by $\alpha_i(s)$ the angle between $\frac{dc}{ds}$ and $\frac{d}{ds} \vec{r}_i$ at Q . We will prove that $\alpha_i(s) = \frac{\pi}{2}$ for $i = 1, 2, 3$.

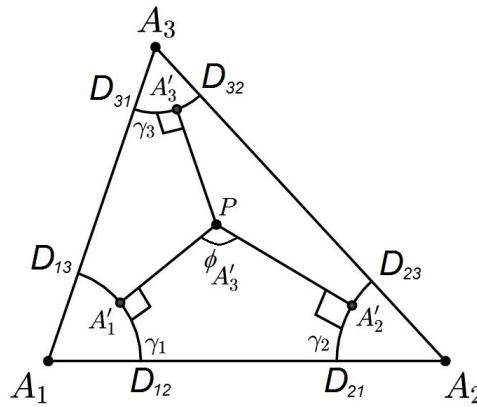


Figure 2.1: The generalized F-T problem for three closed convex curves in $\mathbb{R}^2$.

If any of these angles is not $\frac{\pi}{2}$, say, for A'_1 , then we move $A'_1 \rightarrow A''_1$ along γ_1 and the length $l_P(A''_1) < l_P(A'_1)$ by the first variation formula, see [14, Lemma 3.5.1],

$$\frac{d}{ds} l_Q(s) = \cos \varphi_c^i(s) + \cos \alpha_i(s). \quad (3)$$

Therefore, $\cos \alpha_i(s) = 0$ and $\frac{d}{ds} l_Q(s) = \cos \varphi_c^i(s)$.

If $Q = A'_1$ then we have the line segment $c_{A'_1P}(s)$ that connects the points A'_1 and P , and $c_{A'_2}(s, t)$ is a line that passes through A'_2 and $c_{A'_3}(s, t)$ is a line that passes through A'_3 . Therefore,

$$\frac{d}{ds} l_{A'_1}(s) = \cos \varphi_{A'_3}(s) \quad \text{and} \quad \frac{d}{ds} l_{A'_1}(s) = \cos \varphi_{A'_2}(s),$$

where $c(s)$ counts from the point A'_2 to P , and from A'_3 to P , respectively.

Similarly, for $Q = A'_2$ and $Q = A'_3$, we obtain

$$\frac{d}{ds} l_{A'_2}(s) = \cos \varphi_{A'_3}(s), \quad \frac{d}{ds} l_{A'_2}(s) = \cos \varphi_{A'_1}(s), \quad (4)$$

$$\frac{d}{ds} l_{A'_3}(s) = \cos \varphi_{A'_2}(s), \quad \frac{d}{ds} l_{A'_3}(s) = \cos \varphi_{A'_1}(s). \quad (5)$$

Since t is a canonical parameter, we choose the parametrization

$$l_{A_1}(s) = \int_0^s \left\| \frac{d}{dt} \gamma_{A_1 P}(t) \right\| dt = \int_0^s \left\| \frac{d}{dt} \gamma_{A_1}(s, t) \right\| dt = s,$$

that is

$$l_{A'_1}(s) = l_{A_1} = s. \quad (6)$$

We assume that the distances $l_{A'_2}$, $l_{A'_3}$ can be expressed as functions of $l_{A'_1}$,

$$l_{A'_2} = l_{A'_2}(l_{A'_1}), \quad l_{A'_3} = l_{A'_3}(l_{A'_1}).$$

From this and (1) the following equation is obtained:

$$w_1 l_{A'_1} + w_2 l_{A'_2}(l_{A'_1}) + w_3 l_{A'_3}(l_{A'_1}) \rightarrow \min.$$

Differentiating this with respect to the variable $l_{A'_1}$ and using (6), we get

$$w_1 + w_2 \frac{dl_{A'_2}}{dl_{A'_1}} + w_3 \frac{dl_{A'_3}}{dl_{A'_1}} = 0. \quad (7)$$

From (4) and (5) we get

$$\frac{dl_{A'_2}}{dl_{A'_1}} = \cos \varphi_{A'_3}(s) \quad \text{and} \quad \frac{dl_{A'_3}}{dl_{A'_1}} = \cos \varphi_{A'_2}(s). \quad (8)$$

Replacing (8) and (6) in (7), we obtain

$$w_1 + w_2 \cos \varphi_{A'_3}(l_{A'_1}) + w_3 \cos \varphi_{A'_2}(l_{A'_1}) = 0. \quad (9)$$

Similarly, working cyclically, we choose the parametrization

$$l_{A'_2}(s') = \int_0^{s'} \left\| \frac{d}{dt} \gamma_{A'_2}(s', t) \right\| dt = s', \quad l_{A'_3}(s'') = \int_0^{s''} \left\| \frac{d}{dt} \gamma_{A'_3}(s'', t) \right\| dt = s''.$$

Differentiating (1) with respect to l_{A_2} for $s' = l_{A'_2}$ and $l_{A'_3}$ for $s'' = l_{A'_3}$, we get

$$\begin{aligned} w_1 \cos \varphi_{A'_3}(l_{A'_2}) + w_2 + w_3 \cos \varphi_{A'_1}(l_{A'_2}) &= 0 \quad \text{and} \\ w_1 \cos \varphi_{A'_2}(l_{A'_3}) + w_2 \cos \varphi_{A'_1}(l_{A'_3}) + w_3 &= 0. \end{aligned} \quad (10)$$

From the uniqueness of the generalized F-T point P , we obtain

$$\varphi_{A'_Q}(l_Q) = \varphi_{A'_Q}(l_R) = \varphi_{A'_Q}(l_S) = \varphi_{A'_Q},$$

and that the solution of the linear system (9)–(10) is (2). $\square$

Remark 2.4. (a) Setting $w_1 = w_2 = w_3$ in (2), we get $\varphi_{A'_Q} = 120^\circ$. Hence, if the generalized equally weighted F-T point P is an interior point of Ω for $n = 3$ then $\varphi_{A'_Q} = 120^\circ$ (Isogonal property of the generalized F-T point for equal weights).

(b) If each closed convex curve γ_i ($i = 1, 2, 3$) approaches to the circle $C(A_i, r_i)$ with the same perimeter then the limiting case of Problem 1.1 for $n = 3$ is the F-T problem in $\mathbb{R}^2$ and $P \rightarrow P_F$, where P_F is the F-T point of $\triangle A_1 A_2 A_3$.

3. The generalized inverse weighted F-T problem for closed convex sets in $\mathbb{R}^2$

In this section we generalize results of [17, 18], where $\gamma_i = A_i$. We shall give the definition of dynamic plasticity for n non-overlapping convex sets in $\mathbb{R}^2$.

Definition 3.1. We call *dynamic plasticity of n non-overlapping convex sets C_i in $\mathbb{R}^2$* the set of solutions $\{(w_1)_{1\dots n}, \dots, (w_n)_{1\dots n}\}$ of Problem 1.2 for n sets C_i with corresponding variable weights $(w_i)_{1\dots n}$.

Proposition 3.2. (see [20, 3]) *Given the generalized F-T point P to be an interior point of Ω with the vertices defined by the three orthogonal projections A'_i of P onto γ_i ($i = 1, 2, 3$) lie on three line segments PA'_i and form the given angles φ_i , the positive weights w_i are the solution of Problem 1.2:*

$$w_Q = \left(1 + \frac{\sin \varphi_R}{\sin \varphi_Q} + \frac{\sin \varphi_S}{\sin \varphi_Q}\right)^{-1} \quad \text{for } Q, R, S \in \{1, 2, 3\} \text{ and } Q \neq R \neq S.$$

Let $(w_i)_{1234}$ be the weight corresponding to the point A'_i of the closed convex curve γ_i , which is a vertex of the convex quadrilateral $A'_1 A'_2 A'_3 A'_4$, and let $(w_j)_{jkl}$ be the weight corresponding to the point A_j of the closed convex curve γ_j , which is a vertex of $\triangle A_j A_k A_l$ ($j, k, l = 1, \dots, 4$). Furthermore, assume that P lies at the interior of $\triangle A'_1 A'_2 A'_3 \cap \triangle A'_1 A'_2 A'_4$ and at the exterior of $\triangle A'_1 A'_3 A'_4$.

The following theorem deals with equations of dynamic plasticity with respect to four closed convex sets in $\mathbb{R}^2$ and their corresponding weights.

Theorem 3.3. (The dynamic plasticity of four non-overlapping closed convex sets) *Consider Problem 1.2 for $n = 4$. The dynamic plasticity of four non-overlapping closed convex sets (e.g. circles) C_i ($i = 1, \dots, 4$) and their corresponding weights $(w_i)_{1\dots 4}$ is given by the following three equations:*

$$\left(\frac{w_2}{w_1}\right)_{1\dots 4} = \left(\frac{w_2}{w_1}\right)_{123} \left[1 - \left(\frac{w_4}{w_1}\right)_{1\dots 4} \left(\frac{w_1}{w_4}\right)_{134}\right], \quad (11)$$

$$\left(\frac{w_3}{w_1}\right)_{1\dots 4} = \left(\frac{w_3}{w_1}\right)_{123} \left[1 - \left(\frac{w_4}{w_1}\right)_{1\dots 4} \left(\frac{w_1}{w_4}\right)_{124}\right], \quad (12)$$

$$\sum_{i=1}^4 (w_i)_{1\dots 4} = \text{const}. \quad (13)$$

Proof. By applying the variational method that was used in the proof of Theorem 2.3, we obtain the weighted ‘cosine’ equations:

$$w_1 + w_2 \cos \angle A'_1 P A'_2 + w_3 \cos \angle A'_1 P A'_3 + w_4 \cos \angle A'_1 P A'_4 = 0, \quad (14)$$

$$w_1 \cos \angle A'_2 P A'_1 + w_2 + w_3 \cos \angle A'_2 P A'_3 + w_4 \cos \angle A'_2 P A'_4 = 0, \quad (15)$$

$$w_1 \cos \angle A'_3 P A'_1 + w_2 \cos \angle A'_3 P A'_2 + w_3 + w_4 \cos \angle A'_3 P A'_4 = 0. \quad (16)$$

Solving (14) and (15) with respect to w_1 and w_2 , we derive the weighted ‘sine’ equations:

$$\begin{aligned} -w_1 \sin \angle A'_2 P A'_1 + w_3 \sin \angle A'_2 P A'_3 + w_4 \sin \angle A'_2 P A'_4 &= 0, \\ -w_2 \sin \angle A'_1 P A'_2 + w_3 \sin \angle A'_1 P A'_3 + w_4 \sin \angle A'_1 P A'_4 &= 0. \end{aligned} \quad (17)$$

Solving (14) and (16) with respect to w_1 and w_3 , we derive the weighted ‘sine’ equation:

$$-w_1 \sin \angle A'_3 P A'_1 + w_2 \sin \angle A'_3 P A'_2 - w_4 \sin \angle A'_3 P A'_4 = 0. \quad (18)$$

Applying Proposition 3.2 with respect to $\triangle A'_1 A'_2 A'_3$, $\triangle A'_1 A'_2 A'_4$ and $\triangle A'_1 A'_3 A'_4^*$, where A'_4^* is the symmetric point of A'_4 with respect to P , we get

$$\left(\frac{w_2}{w_1}\right)_{123} = \frac{\sin \angle A'_1 P A'_3}{\sin \angle A'_2 P A'_3}, \quad \left(\frac{w_1}{w_4}\right)_{124} = \frac{\sin \angle A'_2 P A'_4}{\sin \angle A'_1 P A'_2}, \quad -\left(\frac{w_1}{w_4}\right)_{134} = \frac{\sin \angle A'_3 P A'_4}{\sin \angle A'_1 P A'_3}.$$

By replacing these solutions of Proposition 3.2 in (18) and (3)₁, we get (11) and (12), respectively. $\square$

Remark 3.4. A similar system of weighted ‘sine’ equations has been obtained in [18, Theorem 1], where four closed convex sets degenerate to four fixed points.

Theorem 3.5. *Given four rays which meet at the generalized F-T point P and their orthogonal projections with respect to four closed convex curves γ_i ($i = 1, \dots, 4$) forming a convex quadrilateral in $\mathbb{R}^2$, an increase of the weight that corresponds to one ray causes a decrease to the two weights that correspond to the two neighboring rays and an increase to the weight that corresponds to the opposite ray.*

Proof. Taking into account that the four rays which meet at the generalized F-T point P intersects each γ_i at a right angle, we derive that the first variation formula of the length of a ray with respect to arc length (see [14, Lemma 3.5.1]) coincides with the first variation formula of the length of the ray with respect to arc length from four fixed points, respectively, and thus obtain the same angular relation. By following the process that was used in [18], we derive the same plasticity equations of Theorem 3.3. Assuming

$$\sum_{1234} w = \sum_{123} w = \sum_{124} w = \sum_{134} w = \sum_{234} w,$$

by Theorem 3.3 we obtain $(w_i)_{1234} = a_i(w_4)_{1234} + (w_i)_{123}$ for $i = 1, 2, 3$, where

$$\begin{aligned} a_1 &= \frac{\left(\frac{w_1}{w_4}\right)_{134} \left(\frac{w_2}{w_1}\right)_{123} + \left(\frac{w_1}{w_4}\right)_{124} \left(\frac{w_3}{w_1}\right)_{123} - 1}{1 + \left(\frac{w_2}{w_1}\right)_{123} + \left(\frac{w_3}{w_1}\right)_{123}}, \\ a_2 &= a_1 \left(\frac{w_2}{w_1}\right)_{123} - \left(\frac{w_1}{w_4}\right)_{134} \left(\frac{w_2}{w_1}\right)_{123}, \\ a_3 &= a_1 \left(\frac{w_3}{w_1}\right)_{123} - \left(\frac{w_1}{w_4}\right)_{124} \left(\frac{w_3}{w_1}\right)_{123}. \end{aligned}$$

From this the claim follows. $\square$

We will discuss a connection of the generalized F-T problem for n circles $C(A_i, r_i)$ in $\mathbb{R}^2$ and a degenerate Steiner problem for n circles $C(A_i, r_i)$ and one mobile vertex P (or circle $C(P, r)$) at the convex hull of $\{A_1, \dots, A_n\}$. The Steiner problem reads:

Problem 3.6. Find all networks of minimal length spanning points $\{A_1, \dots, A_n\}$ in $\mathbb{R}^2$.

A related problem is the degenerate Steiner problem:

Definition 3.7. A generalized F-T tree for n given circles C_i in $\mathbb{R}^2$ is the solution of the following problem: “Let $\{C_1, C_2, \dots, C_n\}$ be n given non-overlapping circles and a positive real number (weight) corresponds to each circle C_i in $\mathbb{R}^2$ and one mobile circle $C(P, r)$ located at the convex hull of $\{A_1, A_2, \dots, A_n\}$. Describe all the minimal weighted networks (networks of minimal length) spanning $\{C_1, C_2, \dots, C_n\}$.” $\square$

The unique generalized F-T tree with respect to circles $\{C_i\}$ consists of n (weighted) line segments $A'_i P$ (branches) which intersect at the generalized F-T point P , where A'_i is the intersection point of C_i with the segment $A_i P$. The geometric plasticity of weighted F-T tree networks for quadrilaterals on surfaces was defined in [18]. We shall extend this definition regarding the generalized F-T tree problem for n circles C_i . The generalized F-T tree solution may also be viewed as a branching solution and the generalized F-T point P – as the branching point.

Definition 3.8. We call *geometric plasticity of a weighted generalized F-T tree* a network through n non-overlapping circles C_i for given weights $(w_i)_{1 \dots n}$, which correspond to each circle the set of branching solutions of n variable branches $A'_i P$ that preserve the generalized F-T point P at the same location of the n -gons formed by the endpoints of $A'_i P$. $\square$

The field of branching solutions to various one-dimensional variational problems has been introduced in [8]. Note that the geometric plasticity of a weighted generalized F-T tree for n non-overlapping circles $C_i(A_i, r_i)$ permits a parallel translation of the circles in direction of rays defined by $A_i P$.

We generalize the geometric plasticity principle derived in [18, Theorem 3, Quadrilaterals] for n given circles C_i in $\mathbb{R}^2$.

Theorem 3.9. Let $C_i(A_i, r_i)$ be n non-overlapping circles such that $A_1 \dots A_n$ is an n -gon in $\mathbb{R}^2$ and each vertex A_i possesses a non-negative weight w_i for $i=1, \dots, n$. Assume that the floating case of the generalized F-T problem is valid:

$$\left\| \sum_{j \neq i} w_j \vec{u}(A_j, A_i) \right\| > w_i, \quad i \in \{1, \dots, n\},$$

and select r_i ($i = 1, \dots, n$) such that P does not belong to the disk bounded by $C_i(A_i, r_i)$. Assume that P is connected with every vertex A_i for $i = 1, \dots, n$, a circle C'_i with center A'_i has a non-negative weight w_i at the line that is defined by the line segment PA_i , an n -gon $A'_1 \dots A'_n$ has the property

$$\left\| \sum_{j \neq i} w_j \vec{u}(A'_j, A'_i) \right\| > w_i, \quad i = 1, \dots, n,$$

and P does not belong to the disk $C_i(A'_i, r'_i)$ for $i = 1, \dots, n$. Then the generalized F-T point P' equals to P (the geometric plasticity principle).

Proof. P is the generalized F-T floating point of circles $C_1 \dots C_n$ whose centers form a convex polygon $A_1 \dots A_n$ in $\mathbb{R}^2$. Thus, the weighted floating equilibrium condition holds:

$$\sum_{i=1}^n w_i \vec{u}(P, A_i) = 0.$$

If P' is the generalized F-T floating point of the circles $C'_1 \dots C'_n$ then

$$\sum_{i=1}^n w_i \vec{u}(P', A'_i) = 0;$$

hence, $P \equiv P'$ because A'_i is located at the ray defined by PA_i starting from P . $\square$

Therefore, the plasticity of a generalized F-T tree of n circles deals with the simultaneous occurrence of both the dynamic plasticity and the geometric plasticity of the corresponding variable weighted tree network.

Proposition 3.10. *Consider Problem 1.2 with respect to n circles $C(A_i, r_i)$. The following equations point out the plasticity of the system:*

$$\begin{aligned} \left(\frac{w_2}{w_1}\right)_{1\dots n} &= \left(\frac{w_2}{w_1}\right)_{123} \left[1 - \left(\frac{w_4}{w_1}\right)_{1\dots n} \left(\frac{w_1}{w_4}\right)_{134} - \dots - \left(\frac{w_n}{w_1}\right)_{1\dots n} \left(\frac{w_1}{w_n}\right)_{13n}\right], \\ \left(\frac{w_3}{w_1}\right)_{1\dots n} &= \left(\frac{w_3}{w_1}\right)_{123} \left[1 - \left(\frac{w_4}{w_1}\right)_{1\dots n} \left(\frac{w_1}{w_4}\right)_{124} - \dots - \left(\frac{w_n}{w_1}\right)_{1\dots n} \left(\frac{w_1}{w_n}\right)_{12n}\right]. \end{aligned} \quad (19)$$

The weight $(w_i)_{1\dots n}$ corresponds to the vertex A_i of $A_1 \dots A_n$, and the weight $(w_j)_{jkl}$ corresponds to the vertex A_j of $\triangle A_j A_k A_l$ ($j, k, l = 1, \dots, n$).

Proof. We follow the process used in [21, Proposition 4.4] for a convex n -gon, and assume that $n-3$ branches grow simultaneously from the point P and belong to $\angle A_1 P A_4$ such that the vector $\overrightarrow{PA_i}$ belongs to $\angle A_1 P A_{i-1}$ for $i = 5, \dots, n$. Applying the cosine law to $\triangle PA_i A_3$ for $i \in \{1, \dots, n\}$ and $i \neq 3$, allows us to consider the distance $d(P, A_i)$ as a function of two variables, $d(P, A_3)$ and $\angle PA_3 A_2$:

$$d(P, A_i)^2 = d(P, A_3)^2 + d(A_3, A_i)^2 - 2d(P, A_3)d(A_3, A_i) \cos(\angle A_2 A_3 A_i - \angle P A_3 A_2).$$

Differentiating the objective function $f(P) = \sum_i w_i d(P, A_i)$ with respect to the variable $\angle PA_3 A_2$ (see also the differentiation in Corollary 2.4), we obtain

$$w_1 \sin \angle A_1 P A_3 - w_2 \sin \angle A_2 P A_3 + w_4 \sin \angle A_3 P A_4 + \dots + w_n \sin \angle A_3 P A_n = 0.$$

From this we have

$$\left(\frac{w_2}{w_1}\right)_{1\dots n} = \frac{\sin \angle A_3 P A_1}{\sin \angle A_2 P A_3} \left[1 + \left(\frac{w_4}{w_1}\right)_{1\dots n} \frac{\sin \angle A_3 P A_4}{\sin \angle A_3 P A_1} + \dots + \left(\frac{w_n}{w_1}\right)_{1\dots n} \frac{\sin \angle A_3 P A_n}{\sin \angle A_3 P A_1}\right].$$

Taking into account Proposition 3.2 with respect to $\triangle A_1 A_2 A_3$ and $\triangle A_1 A_3 A_i^*$, where A_i^* is the symmetric point of A_i with respect to P for $i = 4, \dots, n$, we get

$$\left(\frac{w_2}{w_1}\right)_{123} = \frac{\sin \angle A_1 P A_3}{\sin \angle A_2 P A_3}, \quad -\left(\frac{w_1}{w_n}\right)_{13n} = \frac{\sin \angle A_3 P A_n}{\sin \angle A_1 P A_3}.$$

Similarly, differentiating the objective function $f(P) = \sum_i w_i d(P, A_i)$ with respect to the variable $\angle PA_2A_3$, we obtain

$$-w_1 \sin \angle A_1PA_2 + w_3 \sin \angle A_2PA_3 + \dots + w_n \sin \angle A_2PA_n = 0.$$

From Proposition 3.2 with respect to $\triangle A_1A_2A_i$ for $i = 3, \dots, n$, we get

$$\left(\frac{w_3}{w_1}\right)_{123} = \frac{\sin \angle A_1PA_2}{\sin \angle A_2PA_3}, \quad \left(\frac{w_1}{w_i}\right)_{12i} = \frac{\sin \angle A_2PA_i}{\sin \angle A_1PA_2}. \quad \square$$

Remark 3.11. (1) Choosing a proper orientation of angles that was used in [18], we also derive the plasticity equations (19) applying the first variation formula of line segments with respect to arc length and the parametrization that was used in [3].

(2) The dynamic plasticity of the degenerate case of quadrilaterals and convex pentagons as a limiting case of closed hexahedra is studied in [18, Theorem 1 and Corollary 3] and [17, Theorem 6], respectively.

Define $\sum_{1\dots n} w := \sum_{i=1}^n \left(\frac{w_i}{w_1}\right)_{1\dots n}$.

Corollary 3.12. *Let*

$$\sum_{1\dots n} w = \sum_{123} w = \sum_{124} w = \sum_{134} w = \dots = \sum_{1(n-1)n} w.$$

Then $(w_i)_{1\dots n} = a_{i,4}(w_4)_{1\dots n} + \dots + a_{i,n}(w_n)_{1\dots n} + a_{i,n+1}$ for $i = 1, 2, 3$, where

$$\begin{aligned} (a_{1,4}, \dots, a_{1,n}, a_{1,n+1}) &= \left[\frac{\left(\frac{w_1}{w_4}\right)_{134} \left(\frac{w_2}{w_1}\right)_{123} + \left(\frac{w_1}{w_4}\right)_{124} \left(\frac{w_3}{w_1}\right)_{123} - 1}{1 + \left(\frac{w_2}{w_1}\right)_{123} + \left(\frac{w_3}{w_1}\right)_{123}}, \right. \\ &\quad \left. \dots, \frac{\left(\frac{w_1}{w_n}\right)_{13n} \left(\frac{w_2}{w_1}\right)_{123} + \left(\frac{w_1}{w_n}\right)_{12n} \left(\frac{w_3}{w_1}\right)_{123} - 1}{1 + \left(\frac{w_2}{w_1}\right)_{123} + \left(\frac{w_3}{w_1}\right)_{123}}, (w_1)_{123} \right] \\ (a_{2,4}, \dots, a_{2,n}, a_{2,n+1}) &= \left[a_{1,4} \left(\frac{w_2}{w_1}\right)_{123} - \left(\frac{w_1}{w_4}\right)_{134} \left(\frac{w_2}{w_1}\right)_{123}, \right. \\ &\quad \left. \dots, a_{1,n} \left(\frac{w_2}{w_1}\right)_{123} - \left(\frac{w_1}{w_n}\right)_{13n} \left(\frac{w_2}{w_1}\right)_{123}, (w_2)_{123} \right] \\ (a_{3,4}, \dots, a_{3,n}, a_{3,n+1}) &= \left[a_{1,4} \left(\frac{w_3}{w_1}\right)_{123} - \left(\frac{w_1}{w_4}\right)_{124} \left(\frac{w_3}{w_1}\right)_{123}, \right. \\ &\quad \left. \dots, a_{1,n} \left(\frac{w_3}{w_1}\right)_{123} - \left(\frac{w_1}{w_n}\right)_{12n} \left(\frac{w_3}{w_1}\right)_{123}, (w_3)_{123} \right]. \end{aligned}$$

Proof. This is a direct consequence of (19): by taking into account that

$$\sum_{1\dots n} w := (w_1)_{1\dots n} \left(1 + \sum_{j=2}^n \frac{w_j}{w_1}\right)_{1\dots n} = \sum_{ijk} w$$

for $i, j, k = 1, \dots, n$ and by solving with respect to $(w_i)_{1\dots n}$ we derive the plasticity equations for $i = 1, 2, 3$ which depend on $n-3$ variables $(w_i)_{1\dots n}$ for $i = 4, \dots, n$. $\square$

Remark 3.13. If $n - 4$ weights are zero, the plasticity principle holds by Theorem 3.3. Thus, Corollary 3.12 for $n = 4$ is a direct consequence of Theorem 3.3.

Remark 3.14. The condition

$$\sum_{1\dots n} w = \sum_{123} w = \sum_{124} w = \sum_{134} w = \dots = \sum_{1(n-1)n} w$$

can be interpreted as equality between isoperimetric conditions of the weights $(w_i)_{ijk}$, $(w_j)_{ijk}$, $(w_k)_{ijk}$ with respect to the triangles $\triangle A_i A_j A_k$. These weights corresponds to the side lengths of the dual restricted weighted F-T problem for these specific triangles by letting the rest $n - 3$ weights zero with the isoperimetric condition of initial generalized F-T problem for n given weights $w_{1\dots n}$. The dual F-T problem is connected with the F-T problem by exchanging the lengths of the F-T problem with weights and the new weights become edge lengths of the dual F-T problem for triangles.

The next proposition provides a control to Problem 1.2 and extends the result [1, Proposition 19.6].

Proposition 3.15. (Absorbed case of Problem 1.1) *Let $W_P \subset \mathbb{R}^2$ consists of even number of circles with $m = 2k$ ($k \geq 2$), and their centers $A_1, A_2, \dots, A_{2k-1}$ be distributed around A_m in such a way that the interior of $\angle A_{i_1} A_m A_{i_2}$ contains at least one of the vectors $-w_j \vec{u}(A_m, A_j)$, $j = 1, \dots, m - 1$. Then $P = P_m$.*

Proof. We can set $\sum_{i=1}^{2k-1} w_i = 1$, where w_i is the weight corresponding to the vertex A_i ($i = 1, 2, \dots, 2k - 1$) and $w_{2k} = 1$ corresponds to A_{2k} . By the weighted absorbed case [1, p. 250], we need to get $\|\sum_{i=1}^{2k-1} w_i \vec{u}(A_m, A_i)\| \leq 1$, where $\vec{u}(A_m, A_i)$ is a unit vector in the direction of $A_m A_i$. Set $w_i \vec{u}(A_m, A_i) = \vec{u}'(A_m, A_i)$. Then the proof goes line by line in the same way as given in [1, Proposition 19.6]. $\square$

4. Some new types of evolution of five non-overlapping circles in $\mathbb{R}^2$

We obtain two new types of evolutionary structures of five non-overlapping circles with variable radius such that their (fixed) centers form a convex pentagon by deriving the iteration plasticity equations and their radius may be taken to be a scaling of weights w_i . The evolution of five such circles is a generalization of the evolution of weighted pentagons in $\mathbb{R}^2$ which has been given in [17, Theorem 7].

We may select a scaling constant with respect to weights of evolutionary circles with variable radius, corresponding to Problem 1.2 for $n = 3$ not overlapping circles.

Theorem 4.1. *The evolution of five non-overlapping circles, whose centers form a convex pentagon, might be of the following two types:*

Type A: For Problem 1.2 with $n = 3$, two branches grow simultaneously from P inside $\angle A_1 P A_3$.

Type B: For Problem 1.2 with $n = 3$, one branch grows from P inside $\angle A_1 P A_3$, the other branch grows inside $\angle A_1 P A_2$, but both branches do not grow simultaneously.

Proof. The characterization of type A is given by Corollary 3.12 for $n = 5$. These iteration plasticity equations give an increase of weights $(w_4)_{1\dots 5}$, $(w_5)_{1\dots 5}$ and $(w_2)_{1\dots 5}$, and a decrease of weights $(w_1)_{1\dots 5}$ and $(w_3)_{1\dots 5}$.

One may characterize type B by extracting Problem 1.2 for $n = 3$ from Problem 1.2 for four evolutionary non-overlapping circles and derive the plasticity of new problem. Thus, from the plasticity principle of four non-overlapping circles the weights (radii) $(w_4)_{1\dots 4}$ and $(w_2)_{1\dots 4}$ increase, and the weights (radii) $(w_1)_{1\dots 4}$ and $(w_3)_{1\dots 4}$ decrease. Therefore, the fourth and second circles are decreased and the first and third circles are increased, keeping their centers at vertices of a fixed convex quadrilateral, because their radii are taken to be scalings of weights w_i ($i = 1, \dots, 4$).

Composing the vectors $w_4\vec{u}(P, A_4)$ and $w_3\vec{u}(P, A_3)$ we get $w_{3=4}\vec{u}(P, A_{3=4})$. Therefore, modified Problem 1.2 for three new evolutionary non-overlapping circles depends on ratios of weights w_1 , w_2 and $w_{3=4}$. Taking into account the plasticity principle of four non-overlapping circles whose centers form a convex quadrilateral $A_1A_5A_2A_{3=4}$, we find that the weights $(w_5)_{152(3=4)}$ and $(w_{34})_{152(3=4)}$ increase, while $(w_1)_{152(3=4)}$ and $(w_2)_{152(3=4)}$ decrease. Indices in parentheses mean possible reduction of pentagon to quadrilateral (composing vectors of third and fourth rays) starting from $\triangle A_1A_2A_5$. So, the weight $w_{125(3=4)}$ corresponds to the reduced weight of the evolutionary quadrilateral $A_1A_2A_5A_{3=4}$. $\square$

Remark 4.2. Following Theorem 4.1, we can derive various types of evolution of n non-overlapping circles with respect to growing of branches from P . The termination of evolution may be studied when the closed convex sets are circles, whose radii may be taken to be a scaling of weights w_i from Corollary 3.12, such that at every circle might appear a leaf of tree-like type (see [15] for formation of leaves and [16, 19] for creation of tree leaf types).

Note that the algorithm and equations used for modeling the formation of leaves in [15] did not take into account the main branch of the leaf and used a selection principle to control solar energy. We may overpass the obstacle of creating a monster leaf by controlling the solar energy using conditions for the weights referring as isoperimetric conditions of the weights. Concerning our model n leaves may be derived by the generalized F-T problem for n circles (or closed convex sets) and $2k$ branches which could be moved via parallel translation inside these circles. We call a leaf a circle (or closed convex set) enriched with the structure of a finite number of parallel translated V shaped branches inside the circle along its main branch. $\square$

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