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Near Equality and Almost Convexity of Functions with Applications to Optimization and Error Bounds*

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We investigate near equality and almost convexity of extended real valued functions defined on finite-dimensional Euclidean spaces. The main result states that an almost convex function (respectively, its domain, lower level set) is nearly equal to (respectively, the domain, lower level set of) its closure, convex hull and closed convex hull. It is proved that almost convexity of an extended real valued function is equivalent to near equality of itself and another almost convex function. Moreover, it is shown that the operations given by sum, scalar multiple, pointwise supremum, epi-sum and epi-multiple of almost convex functions preserve almost convexity, the formulation of the subdifferential of sum and scalar multiple of almost convex functions on the relative interior of their domain is analogous to that of convex functions under suitable additional assumptions and the proximal average of almost convex functions enjoys analogous properties of lower semi-continuous and convex functions. The epi-sum of almost convex functions is proved to be convex under additional assumptions and the Moreau envelope of an almost convex function is shown to be convex and continuously differentiable with the gradient given by the one related to its closure.

Applications to almost convex optimization problems are provided, in particular, under suitable assumptions, the solution set is proved to be nearly convex and the solution sets of two almost convex optimization problems are shown to be nearly equal if the related objective functions are nearly equal. Another application shows that the classical Hoffman's error bound holds for almost convex inequalities under a generalized Slater condition. Several examples are given to illustrate these results.

Keywords: Near equality, almost convexity, near convexity, proximal average, optimization, error bounds.

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1. Introduction

It is well known that convexity has a great importance in the study of optimization problems in many areas of applied mathematics, in which convex sets and functions play a central role. For this reason, convexity has been extended in many aspects, such as almost convexity [28], abstract convexity [38]. Among these, near convexity is a very important concept, which can find nice applications in near equality of sets which is an important generalization of equality of sets (see, for example, [6, 13]) and duality of optimization problems (see, for example, [8, 10, 11, 17]).

The concept of nearly convex sets was first investigated by Green and Gustin [19]. The nearly convex functions were introduced by Aleman [1] as p-convex (see also, for example, [16]), which generalize the older midconvex functions [32]. For related works, please see [12, 23]. In 2013, Bauschke, Moffat and Wang [6] introduced near equality for sets which extended that given by Brezis and Haraux [13] by using relative interior instead of interior, and proved many interesting properties of this notion and also showed it is useful in the study of nearly convex sets in the sense of Rockafellar and Wets [37, Theorem 12.41, p.554].

In this paper, we shall employ near convexity of sets in the sense of Rockafellar and Wets [37] (see also, for example, [6]), recalling that this concept is equivalent to almost convexity introduced in [11]. We first introduce near equality of two extended real valued functions by using near equality of their epigraphs in the sense of Bauschke, Moffat and Wang [6]. Similar to near equality of sets, near equality of two extended real valued functions is also an equivalence relation and satisfies a squeeze theorem. Then we recall almost convexity of an extended real valued function introduced in [11, 18]. We say that an extended real valued function is almost convex if its epigraph is nearly convex, or equivalently, almost convex. We prove that almost convex functions enjoy many useful properties: in particular, an almost convex function (respectively, its domain, lower level set) is nearly equal to (respectively, the domain, lower level set of) its closure, convex hull and closed convex hull, and almost convexity of an extended real valued function is equivalent to near equality of itself and another almost convex function. We show that the operations given by sum, scalar multiple, pointwise supremum, epi-sum and epi-multiple of almost convex functions preserve almost convexity.

The formulation of the subdifferential of sum and scalar multiple of almost convex functions on the relative interior of their domain is similar to that of convex functions under suitable assumptions and the proximal average of almost convex functions enjoys many nice properties analogous to those obtained by Bauschke, Lucet and Trienis for lower semi-continuous and convex functions [4]. Moreover, we prove that the epi-sum of almost convex functions is convex under additional assumptions and the Moreau envelope function of an almost convex function is convex and continuously differentiable with the gradient given by the one related to its closure. As applications, we prove that under some assumptions the solution set of an almost convex optimization problem is nearly convex, the solution sets of two almost convex optimization problems are nearly equal if the two almost convex objective functions involved do. A further application shows that the classical Hoffman's error bound also holds for almost convex inequalities under a generalized Slater condition.

The paper is organized as follows. In Section 2, we recall some preliminaries on convex sets and functions in finite-dimensional Euclidean spaces which will be useful in the sequel. In Section 3, we first give the definitions of near equality and near convexity of sets and recall some of their properties. Then we define near equality and recall almost convexity of extended real valued functions. In Section 4, we discuss many properties of almost convexity of extended real valued functions and the relations between near equality and almost convexity of extended real valued functions. We investigate several properties of proximal average of almost convex functions in Section 5. In Section 6, we consider two applications of the obtained results. In the first application, we investigate near equality and near convexity of solution sets and optimality conditions for almost convex optimization problems. The second application is concerned with global error bounds for almost convex inequalities. Several examples are provided to illustrate these results.

2. Preliminaries on convex analysis

Let $\mathbb{N}$ be the positive integers and $\mathbb{R}^m$ be the m-dimensional Euclidean space, where $m \in \mathbb{N}$. All vectors in $\mathbb{R}^m$ are to be regarded as column vectors. Let $\mathbb{R}^m_+ := \{x := (x_1, \ldots, x_m)^\top \in \mathbb{R}^m : \text{each } x_i \geq 0\}$ and $\mathbb{R}^m_{++} := \{x := (x_1, \ldots, x_m)^\top \in \mathbb{R}^m : \text{each } x_i > 0\}$, where the superscript $\top$ denotes transpose. Denote by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ the inner product and Euclidean 2-norm in $\mathbb{R}^m$, respectively. The *distance* $d(x, K)$ of a point $x \in \mathbb{R}^m$ from $K \subseteq \mathbb{R}^m$ is $d(x, K) := \inf_{y \in K} \|x - y\|$. The open and closed *balls* with radius $r > 0$ and center $x_0 \in \mathbb{R}^m$ are given by $B(x_0, r) := \{x \in \mathbb{R}^m : \|x - x_0\| < r\}$ and $\overline{B(x_0, r)} := \{x \in \mathbb{R}^m : \|x - x_0\| \leq r\}$, respectively.

We denote by $\widetilde{\mathbb{R}} := \mathbb{R} \cup \{+\infty\}$ the extended real numbers. The operations on $\overline{\mathbb{R}} := \widetilde{\mathbb{R}} \cup \{-\infty\}$ are the usual ones on $\mathbb{R}$ to which we add the following natural ones: $a + (+\infty) = (+\infty) + a := +\infty, \forall a \in \widetilde{\mathbb{R}}$; $a \cdot (+\infty) := +\infty, \forall a \in (0, +\infty]$; $0 \cdot (+\infty) := 0$; $\inf \emptyset := +\infty$; $+\infty \geq a \geq -\infty, \forall a \in \overline{\mathbb{R}}$. Denote by $\widetilde{\mathbb{R}}^m := \{x := (x_1, \ldots, x_m)^\top : \text{each } x_i \in \widetilde{\mathbb{R}}\}$. Let K be a nonempty subset of $\mathbb{R}^m$. Denote by $\operatorname{cl} K, \operatorname{aff} K, \operatorname{conv} K$ and $\overline{\operatorname{conv}}\, K := \operatorname{cl} \operatorname{conv} K$ the closure, affine hull, convex hull and closed convex hull of K, respectively.

We denote by $\operatorname{int} K$ and $\operatorname{ri} K$ the *interior* and *relative interior* (i.e., the interior with respect to the affine hull $\operatorname{aff} K$) of K, respectively. Denote by $\operatorname{rbd} K := \operatorname{cl} K \backslash \operatorname{ri} K$ the relative boundary of K. General Minkowski scalar multiples and sums of sets in $\mathbb{R}^m$ are defined by $tK := \{ta : a \in K\}$ and $K + M := \{a + b : a \in K, b \in M\}$, respectively, where $t \in \mathbb{R}$ and $K, M \subseteq \mathbb{R}^m$.

Let $f \colon \mathbb{R}^m \to \widetilde{\mathbb{R}}$ be an extended real valued function. Denote by $\operatorname{dom} f := \{x \in \mathbb{R}^m : f(x) < +\infty\}$ the effective domain of f. The epigraph of f is denoted by $\operatorname{epi} f := \{(x, r) \in \mathbb{R}^m \times \mathbb{R} : f(x) \leq r\}$. The lower level sets $\operatorname{lev}_{\leq \alpha} f$ and $\operatorname{lev}_{< \alpha} f$ are given by $\operatorname{lev}_{\leq \alpha} f := \{x \in \mathbb{R}^m : f(x) \leq \alpha\}$ and $\operatorname{lev}_{< \alpha} f := \{x \in \mathbb{R}^m : f(x) < \alpha\}$, respectively, where $\alpha \in \mathbb{R}$. We say that f is *proper*, if $\operatorname{dom} f \neq \emptyset$; otherwise it is improper. The function f is *lower semi-continuous* (for short, lsc) at $\bar{x} \in \mathbb{R}^m$ if

$$\liminf_{x \to \bar{x}} f(x) \geq f(\bar{x}),$$

and lsc if this holds for each $\bar{x} \in \mathbb{R}^m$. The *subdifferential* $\partial f(x)$ of a proper function $f \colon \mathbb{R}^m \to \widetilde{\mathbb{R}}$ at $x \in \mathbb{R}^m$ is defined by

$$\partial f(x) := \{s \in \partial f(x) : f(y) - f(x) \geq \langle s, y - x \rangle \text{ for each } y \in \mathbb{R}^m\}.$$

It is clear that if f is proper convex, then $\partial f(x) = \emptyset$ when $x \notin \operatorname{dom} f$, and $\partial f(x) \neq \emptyset$ when $x \in \operatorname{ri}(\operatorname{dom} f)$ [35, Theorem 23.4]. Let K be a nonempty subset of $\mathbb{R}^m$. The *indicator function* of K is defined by

$$\delta_K(x) := \begin{cases} 0, & x \in K, \\ +\infty, & x \notin K. \end{cases}$$

Clearly, $\operatorname{epi} \delta_K = K \times \mathbb{R}_+$. The *normal cone* to K at $x \in \mathbb{R}^m$ is defined by $N_K(x) := \{x^* \in \mathbb{R}^m : \langle x^*, y - x \rangle \leq 0 \text{ for each } y \in K\}$ when $x \in K$, and $N_K(x) := \emptyset$ when $x \notin K$. It is clear that $\partial \delta_K(x) = N_K(x)$ for each $x \in \mathbb{R}^m$. The *conjugate function* of a proper function $f \colon \mathbb{R}^m \to \widetilde{\mathbb{R}}$ is defined by

$$f^*(s) := \sup_{x \in \operatorname{dom} f} \{\langle x, s \rangle - f(x)\}.$$

The *support function* of K is defined by $\sigma_K(s) := \sup_{x \in K} \langle s, x \rangle$.

It is well known that $\sigma_K(s) = \sigma_{\operatorname{cl} K}(s) = \sigma_{\operatorname{conv} K}(s) = \sigma_{\overline{\operatorname{conv}} K}(s) = i_K^*(s)$. Let $K \subseteq \mathbb{R}^m$ be convex. We say that f is convex on K, if $f(tx + (1-t)y) \leq tf(x) + (1-t)f(y), \forall x, y \in K, \forall t \in [0,1]$ and convex, if this holds for $K := \mathbb{R}^m$. The closure (or, lsc hull), convex hull and closed convex hull (or, lsc convex hull) of f, denoted by $\operatorname{cl} f$, $\operatorname{conv} f$ and $\overline{\operatorname{conv}} f := \operatorname{cl}(\operatorname{conv} f)$, resp., are defined by

$$\operatorname{epi}(\operatorname{cl} f) := \operatorname{cl}(\operatorname{epi} f),$$

or equivalently $(\operatorname{cl} f)(x) := \liminf_{y \to x} f(y)$ for each $x \in \mathbb{R}^m$ [37, 1(6) and 1(7)],

$$(\operatorname{conv} f)(x) := \inf\{\sum_{i=0}^{m} \lambda_i f(x^i) : \sum_{i=0}^{m} \lambda_i x^i = x, x^i \in \mathbb{R}^m, \lambda_i \geq 0, \sum_{i=0}^{m} \lambda_i = 1\},$$

or equivalently $(\operatorname{conv} f)(x) := \inf\{r \in \mathbb{R} : (x, r) \in \operatorname{conv}(\operatorname{epi} f)\}$ [35, Theorem 5.3] (that is, $\operatorname{conv} f$ is the function obtained by taking $\operatorname{epi}(\operatorname{conv} f)$ to be the epigraphical closure of $\operatorname{conv}(\operatorname{epi} f)$) (see, [37], [21, IV1.3(g)]), and

$$\operatorname{epi}(\overline{\operatorname{conv}} f) := \overline{\operatorname{conv}}(\operatorname{epi} f).$$

It is well known that $\operatorname{cl} f \leq f, \operatorname{conv} f \leq f$ (i.e., $\operatorname{cl} f$ and $\operatorname{conv} f$ are majorized by f) and $\overline{\operatorname{conv}} f \leq \operatorname{cl} f$ (respectively, $\operatorname{conv} f$) $\leq f$ (see, for example, [9, 40]). Also,

f is lsc if and only if $\mathrm{lev}_{\leq\alpha} f$ is closed for each $\alpha \in \mathbb{R}$, and f is lsc (resp., convex) if and only if $\mathrm{epi}\, f$ is closed (resp., convex), which implies $\mathrm{dom}\, f$ is closed (resp., convex). Moreover, the convexity of f implies $\mathrm{lev}_{\leq\alpha} f$ and $\mathrm{lev}_{<\alpha} f$ are convex for each $\alpha \in \mathbb{R}$. For functions $f, g \colon \mathbb{R}^m \to \widetilde{\mathbb{R}}$, the epi-sum, also called the epi-addition or the inf-convolution, is the function $f \,\square\, g \colon \mathbb{R}^m \to \widetilde{\mathbb{R}}$ defined by

$$(f \,\square\, g)(x) := \inf_{y+z=x} \{f(y) + g(z)\}$$

and the epi-multiple $\lambda \star f \colon \mathbb{R}^m \to \widetilde{\mathbb{R}}$ is defined for $\lambda > 0$ by

$$(\lambda \star f)(x) := \lambda f(\lambda^{-1} x).$$

The Moreau envelope function $e_\lambda f := f \,\square\, \frac{1}{2\lambda}\|\cdot\|^2$ and proximal mapping $P_\lambda f$ are defined for $\lambda > 0$ by, respectively,

$$e_\lambda f(x) := \inf_{w\in\mathbb{R}^m} \{f(w) + \frac{1}{2\lambda}\|x - w\|^2\}, \text{ and}$$
$$P_\lambda f(x) := \mathrm{argmin}_{w\in\mathbb{R}^m} \{f(w) + \frac{1}{2\lambda}\|x - w\|^2\},$$

Notice that the distance function $d(\cdot, K) = \delta_K \,\square\, \|\cdot\|$, where $K \subseteq \mathbb{R}^m$.

The following well-known result is useful in the sequel.

Proposition 2.1. *For $f, g, h \colon \mathbb{R}^m \to \widetilde{\mathbb{R}}$, we have*

$$h \leq f \leq g \iff \mathrm{epi}\, g \subseteq \mathrm{epi}\, f \subseteq \mathrm{epi}\, h,$$

which implies $\mathrm{dom}\, g \subseteq \mathrm{dom}\, f \subseteq \mathrm{dom}\, h$, $\mathrm{lev}_{\leq\alpha}\, g \subseteq \mathrm{lev}_{\leq\alpha}\, f \subseteq \mathrm{lev}_{\leq\alpha}\, h$ *and* $\mathrm{lev}_{<\alpha}\, g \subseteq \mathrm{lev}_{<\alpha}\, f \subseteq \mathrm{lev}_{<\alpha}\, h$, *where $\alpha \in \mathbb{R}$. In particular, we have the equivalence* $f = g \iff \mathrm{epi}\, f = \mathrm{epi}\, g$, *which implies* $\mathrm{dom}\, f = \mathrm{dom}\, g$, $\mathrm{lev}_{\leq\alpha}\, f = \mathrm{lev}_{\leq\alpha}\, g$ *and* $\mathrm{lev}_{<\alpha}\, f = \mathrm{lev}_{<\alpha}\, g$, *where $\alpha \in \mathbb{R}$.*

We also need the following lemma. We recall that the sets A and B in $\mathbb{R}^m$ are said to be properly linearly separable if they are linearly separable and, moreover, they are not both contained in the separating hyperplane [35].

Lemma 2.2. *Let A, B be nonempty subsets of $\mathbb{R}^m$ and A_i $(i \in I)$ be nonempty convex subsets of $\mathbb{R}^m$, where I is an index set. Let $T \colon \mathbb{R}^m \to \mathbb{R}^p$ be a linear operator. Then the following statements are true:*

(i) *A and B are properly linearly separable if and only if there exists a vector $a \in \mathbb{R}^m$ such that $\inf_{x\in A}\langle a, x\rangle \geq \sup_{x\in B}\langle a, x\rangle$ and $\sup_{x\in A}\langle a, x\rangle > \inf_{x\in B}\langle a, x\rangle$. Furthermore, if A, B are convex, then $\mathrm{ri}\,(\mathrm{cl}\, A) = \mathrm{ri}\, A$, $\mathrm{cl}\,(\mathrm{ri}\, A) = \mathrm{cl}\, A$ and moreover, A and B are properly linearly separable if and only if $\mathrm{ri}\, A \cap \mathrm{ri}\, B = \emptyset$.*

(ii) *If A is convex, then $\mathrm{ri}\, A$ is a nonempty convex set.*

(iii) $\mathrm{cl}\, A + \mathrm{cl}\, B \subseteq \mathrm{cl}\,(A + B)$ *and* $\mathrm{ri}\,(A + B) = \mathrm{ri}\, A + \mathrm{ri}\, B$ *when A, B are convex.*

(iv) $\mathrm{aff}\,(A \times B) = \mathrm{aff}\, A \times \mathrm{aff}\, B$ *and* $\mathrm{cl}\,(A \times B) = \mathrm{cl}\, A \times \mathrm{cl}\, B$.

(v) *If* $\mathrm{ri}\, A \neq \emptyset$ *and* $\mathrm{ri}\, B \neq \emptyset$*, then* $\mathrm{ri}\,(A \times B) = \mathrm{ri}\, A \times \mathrm{ri}\, B$.

(vi) *If* $\cap_{i\in I} (\mathrm{ri}\, A_i) \neq \emptyset$*, then* $\mathrm{cl}\,(\cap_{i\in I} A_i) = \cap_{i\in I} (\mathrm{cl}\, A_i)$. *Furthermore, if* I *is finite, then also* $\mathrm{ri}\,(\cap_{i\in I} A_i) = \cap_{i\in I} (\mathrm{ri}\, A_i)$.

(vii) *If* A *is convex, then* $\mathrm{ri}\,(TA) = T(\mathrm{ri}\, A)$ *and* $\mathrm{cl}\,(TA) \supseteq T(\mathrm{cl}\, A)$. *In particular,* $\mathrm{ri}\,(\alpha A) = \alpha(\mathrm{ri}\, A)$ *and* $\mathrm{cl}\,(\alpha A) = \alpha(\mathrm{cl}\, A)$*, where* $\alpha \in \mathbb{R}$.

Proof. Statements (i)–(iii), (vi) and (vii) follow from [35, Theorems 6.2, 6.3 and 6.6, Corollary 6.6.2, Theorems 6.5, 11.1 and 11.3] and (iv) is clear (see, for example, [40]). It is sufficient to prove (v). If $\mathrm{ri}\,(A \times B) = \emptyset$, then we have $\mathrm{ri}\,(A \times B) \subseteq \mathrm{ri}\, A \times \mathrm{ri}\, B$ since $\mathrm{ri}\, A \neq \emptyset$ and $\mathrm{ri}\, B \neq \emptyset$. Suppose $\mathrm{ri}\,(A \times B) \neq \emptyset$. Let $(a, b) \in \mathrm{ri}\,(A \times B)$. Then there is $r > 0$ such that $B((a, b), r) \cap \mathrm{aff}\,(A \times B) \subseteq A \times B$, where $B((a, b), r) := \{(x, y) \in \mathbb{R}^m \times \mathbb{R}^m : \|x - a\| + \|y - b\| < r\}$. Set $B_1(a, \frac{r}{2}) := \{x \in \mathbb{R}^m : \|x - a\| < \frac{r}{2}\}$ and $B_2(b, \frac{r}{2}) := \{y \in \mathbb{R}^m : \|y - b\| < \frac{r}{2}\}$. Then $B_1(a, \frac{r}{2}) \times B_2(b, \frac{r}{2}) \subseteq B((a, b), r)$. From the first argument of (iv) it follows that $B_1(a, \frac{r}{2}) \cap \mathrm{aff}\, A \subseteq A$ and $B_2(b, \frac{r}{2}) \cap \mathrm{aff}\, B \subseteq B$, which yields $a \in \mathrm{ri}\, A$ and $b \in \mathrm{ri}\, B$. This proves $\mathrm{ri}\,(A \times B) \subseteq \mathrm{ri}\, A \times \mathrm{ri}\, B$.

Conversely, let $a \in \mathrm{ri}\, A$ and $b \in \mathrm{ri}\, B$. Then there are $r_1, r_2 > 0$ such that we have $B_1(a, r_1) \cap \mathrm{aff}\, A \subseteq A$ and $B_2(b, r_2) \cap \mathrm{aff}\, B \subseteq B$. Set $r := \min\{r_1, r_2\}$. Then $B((a, b), r) \subseteq B_1(a, r_1) \times B_2(b, r_2)$. Again from the first argument of (iv) we have $B((a, b), r) \cap \mathrm{aff}\,(A \times B) \subseteq A \times B$, that is $(a, b) \in \mathrm{ri}\,(A \times B)$. This yields $\mathrm{ri}\, A \times \mathrm{ri}\, B \subseteq \mathrm{ri}\,(A \times B)$. □

The following lemmas are well known.

Lemma 2.3. [35, Theorem 23.8] [37, Propositions 1.38 and 1.39, Exercises 1.28, 2.24 and 2.46, Theorem 2.26] *Let* $\alpha \geq 0$, $\beta \geq 0$, $\gamma, \lambda > 0$ *and* $f, g \colon \mathbb{R}^m \to \widetilde{\mathbb{R}}$. *Then the following statements are true:*

(i) $\mathrm{cl}\,(\alpha f) = \alpha \mathrm{cl}\, f$ *and* $\mathrm{cl}\,(\mathrm{cl}\, f + \mathrm{cl}\, g) = \mathrm{cl}\, f + \mathrm{cl}\, g$.

(ii) $\partial(\gamma f)(x) = \gamma \partial f(x)$ *and, moreover, if* f, g *are convex and* $\mathrm{ri}\,(\mathrm{dom}\, f) \cap \mathrm{ri}\,(\mathrm{dom}\, g) \neq \emptyset$*, then* $\mathrm{cl}\,(f + g) = \mathrm{cl}\, f + \mathrm{cl}\, g$ *and* $\partial(f + g)(x) = \partial f(x) + \partial g(x)$.

(iii) *If* f, g *are convex, then* $\alpha f + \beta g$, $f \,\square\, g$ *and* $\gamma \star f$ *are convex and, moreover,* $(\alpha + \beta) \star f = (\alpha \star f) \,\square\, (\beta \star f)$.

(iv) $\mathrm{epi}\,(f \,\square\, g) = \mathrm{epi}\, f + \mathrm{epi}\, g$ *as long as the infimum defining* $(f \,\square\, g)(x)$ *is attained when finite, and* $\mathrm{epi}\,(\gamma \star f) = \gamma(\mathrm{epi}\, f)$.

(v) $\alpha \star (f \,\square\, g) = (\alpha \star f) \,\square\, (\alpha \star g)$ *and* $\alpha \star (\beta \star f) = (\alpha\beta) \star f$.

(vi) *If* f *is proper lsc and convex, then the Moreau envelope function* $e_\lambda f$ *is convex and continuously differentiable with the gradient* $\nabla e_\lambda f(x) = \frac{1}{\lambda}(x - P_\lambda f(x))$.

Lemma 2.4. [37, Theorem 11.1] *For any function* $f \colon \mathbb{R}^m \to \widetilde{\mathbb{R}}$ *with* $\mathrm{conv}\, f$ *proper, both* f^* *and* f^{**} *are proper lsc and convex, and* $f^{**} = \overline{\mathrm{conv}} f$. *Thus* $f^{**} \leq f$*, and when* f *is itself proper lsc and convex, one has* $f^{**} = f$. *Anyway, regardless of such assumptions, one always has*

$$f^* = (\mathrm{cl}\, f)^* = (\mathrm{conv}\, f)^* = (\overline{\mathrm{conv}}\, f)^*.$$

Lemma 2.5. [35, Theorems 7.4 and 23.4, Corollary 7.4.1] *For a proper convex function* $f\colon \mathbb{R}^m \to \widetilde{\mathbb{R}}$, $\mathrm{cl}\, f$ *agrees with* f *except perhaps at points of* $\mathrm{rbd}\,(\mathrm{dom}\, f)$ *and* $\partial f(x)$ *is nonempty for each* $x \in \mathrm{ri}\,(\mathrm{dom}\, f)$. *In particular we have the identities* $\mathrm{cl}\,(\mathrm{dom}\, f) = \mathrm{cl}\,(\mathrm{dom}\,(\mathrm{cl}\, f))$ *and* $\mathrm{ri}\,(\mathrm{dom}\, f) = \mathrm{ri}\,(\mathrm{dom}\,(\mathrm{cl}\, f))$.

Lemma 2.6. [35, Theorem 7.6] *For a proper convex function* $f\colon \mathbb{R}^m \to \widetilde{\mathbb{R}}$ *and* $\alpha > \inf_{x\in\mathbb{R}^m} f(x)$, *we have the identities* $\mathrm{cl}\,(\mathrm{lev}_{\leq\alpha} f) = \mathrm{cl}\,(\mathrm{lev}_{<\alpha} f) = \mathrm{lev}_{\leq\alpha}\,(\mathrm{cl}\, f)$ *and* $\mathrm{ri}\,(\mathrm{lev}_{\leq\alpha} f) = \mathrm{ri}\,(\mathrm{lev}_{<\alpha} f) = \{x \in \mathrm{ri}\,(\mathrm{dom}\, f) : f(x) < \alpha\}$.

3. Basic results on near equality, near and almost convexity

In this section, we recall near equality, near convexity and almost convexity of sets and of extended real valued functions [37, 11], analysing their mutual relations and the connections with related concepts existing in the literature as the further definition of near convexity introduced in [8]. Then, we introduce near equality of extended real valued functions.

We first recall the notion of near equality of sets introduced by Bauschke, Moffat and Wang [6].

Definition 3.1. [6, Definition 2.3] Let A, B be subsets of $\mathbb{R}^m$. We say that A and B are *nearly equal*, denoted by $A \approx B$, if $\mathrm{cl}\, A = \mathrm{cl}\, B$ and $\mathrm{ri}\, A = \mathrm{ri}\, B$.

The following definition of near convexity of sets is due to Rockafellar and Wets [37], see also, Bauschke, Moffat and Wang [6]. Based on Lemma 4.1 in [18], almost convexity of sets was first introduced by Boţ, Kassay and Wanka [11].

Definition 3.2. Let A be a subset of $\mathbb{R}^m$.

(a) [37, Theorem 12.41] We say that A is *nearly convex*, if there is a convex subset C of $\mathbb{R}^m$ such that
$$C \subseteq A \subseteq \mathrm{cl}\, C. \tag{1}$$

(b) [11, Definition 2.1] We say that A is *almost convex*, if $\mathrm{cl}\, A$ is convex and $\mathrm{ri}(\mathrm{cl}\, A) \subseteq A$.

As pointed out by Boţ (see the acknowledgements in [6]), the concept of near convexity in Definition 3.2 (a) is equivalent to the almost convexity in Definition 3.2 (b). For completeness we report here the proof.

Proposition 3.3. *Let* $A \subseteq \mathbb{R}^m$. *Then* A *is an almost convex set if and only if it is a nearly convex set.*

Proof. Assume that A is nearly convex. Then, there exists a convex set $C \subseteq \mathbb{R}^m$ such that (1) is fulfilled. Since (1) implies $\mathrm{cl}\, A = \mathrm{cl}\, C$, it follows that $\mathrm{cl}\, A$ is convex. Moreover, (1) implies that $\mathrm{aff}\, A = \mathrm{aff}\, C$ and so $\mathrm{ri}\,(\mathrm{cl}\, A) = \mathrm{ri}\,(\mathrm{cl}\, C) = \mathrm{ri}\, C \subseteq \mathrm{ri}\, A \subseteq A$.

Assume that A is almost convex. Then, $\mathrm{cl}\, A$ is convex and $\mathrm{ri}\,(\mathrm{cl}\, A) \subseteq A$. Setting $C := \mathrm{ri}\,(\mathrm{cl}\, A)$ we have $\mathrm{cl}\, C = \mathrm{cl}\,[\mathrm{ri}\,(\mathrm{cl}\, A)] = \mathrm{cl}\, A$, so that (1) is fulfilled. □

Remark 3.4. (a) $A \subseteq \mathbb{R}^m$ is nearly convex if and only if there is a convex subset C of $\mathbb{R}^m$ such that $\operatorname{ri} C \subseteq A \subseteq \operatorname{cl} C$. The necessity is clear. The sufficiency follows immediately from the fact that $\operatorname{ri} C$ is convex and $\operatorname{cl} C = \operatorname{cl}(\operatorname{ri} C)$ (see, Lemma 2.2 (i) and (ii)). As a consequence, $\operatorname{cl} A = \operatorname{cl} C$ and $\operatorname{ri} A = \operatorname{ri} C$ which yields $\operatorname{ri} A \neq \emptyset$ (see, [6]). The connections with the concept of near equality of sets are pointed out in Lemma 3.5.

(b) In [1, 8], a subset A of $\mathbb{R}^m$ is said to be nearly convex, if there exists $\alpha \in (0,1)$ such that $\alpha x + (1-\alpha)y \in A$ for each $x, y \in A$. We observe that if a set A is nearly convex in the sense of [1] and $\operatorname{ri} A \neq \emptyset$, then A is nearly convex in the sense of Definition 3.2. To prove this claim, we recall that if a set A is nearly convex in the sense of [1] with $\operatorname{ri} A \neq \emptyset$, then $\operatorname{ri} A$ and $\operatorname{cl} A$ are convex sets [1] and $\operatorname{ri} A = \operatorname{ri}(\operatorname{cl} A)$ [10]. Therefore, setting $C := \operatorname{ri}(\operatorname{cl} A)$ we have

$$C = \operatorname{ri}(\operatorname{cl} A) = \operatorname{ri} A \subseteq A \subseteq \operatorname{cl} A = \operatorname{cl}(\operatorname{cl} A) = \operatorname{cl}(\operatorname{ri}(\operatorname{cl} A)) = \operatorname{cl} C.$$

Moreover, it is not difficut to show that there exist nearly convex sets in the sense of Definition 3.2 that are not nearly convex in the sense of [1]. For example, set

$$A := \{(x_1, x_2)^\top \in \mathbb{R}^2 : 0 < x_1 < 1,\ 0 < x_2 < 1\} \cup \{(0,0)^\top, (0,1)^\top, (1,0)^\top, (1,1)^\top\}.$$

The following lemmas will be useful in the sequel.

Lemma 3.5. [6, Propositions 2.4 and 2.5, Lemmas 2.7 and 2.9] *Let A, B, C be subsets of $\mathbb{R}^m$. Then the following statements are true:*

(i) $A \approx A$. (ii) $A \approx B \iff B \approx A$. (iii) $A \approx B$ *and* $B \approx C \Longrightarrow A \approx C$.

(iv) $A \subseteq B \subseteq C$ *and* $A \approx C \Longrightarrow A \approx B \approx C$.

(v) *A is nearly convex, say $C \subseteq A \subseteq \operatorname{cl} C$, where $C \subseteq \mathbb{R}^m$ is a convex set*
$\iff$ *A is nearly equal to a convex set*
$\iff$ *A is nearly equal to a nearly convex set* $\Longrightarrow A \approx C$.

With the same arguments used in the proof of Proposition 3.3 it is possible to obtain the following result.

Lemma 3.6. [11, Lemma 2.1] *Let A be a nearly convex subset of $\mathbb{R}^m$. Then* $\operatorname{ri}(\operatorname{cl} A) = \operatorname{ri} A$.

A direct consequence of the previous lemma is the following separation theorem for almost convex sets.

Proposition 3.7. *Let A and B be nonempty nearly convex subsets of $\mathbb{R}^m$. Then A and B are properly linearly separable if and only if* $\operatorname{ri} A \cap \operatorname{ri} B = \emptyset$.

Proof. Since $\inf_{x \in K}\langle a, x\rangle = \inf_{x \in \operatorname{cl} K}\langle a, x\rangle$ and $\sup_{x \in K}\langle a, x\rangle = \sup_{x \in \operatorname{cl} K}\langle a, x\rangle$ for each $a \in \mathbb{R}^m$ and each nonempty subset K of $\mathbb{R}^m$, from Lemma 2.2 (i) we have A and B are properly linearly separable if and only if $\operatorname{cl} A$ and $\operatorname{cl} B$ are properly linearly separable. Since A and B are nearly convex, $\operatorname{cl} A$ and $\operatorname{cl} B$ are convex (see, Remark 3.4 (a)). Again from Lemma 2.2 (i) it follows that $\operatorname{cl} A$ and $\operatorname{cl} B$ are properly linearly separable if and only if $\operatorname{ri}(\operatorname{cl} A) \cap \operatorname{ri}(\operatorname{cl} B) = \emptyset$ (see, [35, Theorem 11.3]), or equivalently, $\operatorname{ri} A \cap \operatorname{ri} B = \emptyset$, in view of Lemma 3.6. $\square$

Lemma 3.8. [11, Lemma 2.3, Theorem 2.1] *Let A_i be a nearly convex subset of $\mathbb{R}^m$ for each $i \in I$, say $\mathbf{A}_i \subseteq A_i \subseteq \operatorname{cl} \mathbf{A}_i$, where I is an index set and $\mathbf{A}_i \subseteq \mathbb{R}^m$ is convex for each $i \in I$ and assume that $\cap_{i\in I} (\operatorname{ri} A_i) \neq \emptyset$. Then*

(i) $\cap_{i\in I} A_i$ *is nearly convex and* $\cap_{i\in I} A_i \approx \cap_{i\in I} \mathbf{A}_i$.

(ii) $\operatorname{cl}(\cap_{i\in I} A_i) = \cap_{i\in I} (\operatorname{cl} A_i)$. *Furthermore, if I is finite, then also* $\operatorname{ri}(\cap_{i\in I} A_i) = \cap_{i\in I} (\operatorname{ri} A_i)$.

(iii) *If I is finite, then $\prod_{i\in I} A_i$ is nearly convex and $\prod_{i\in I} A_i \approx \prod_{i\in I} \mathbf{A}_i$.*

Lemma 3.9. [11, Lemma 2.4, Theorem 2.2, Corollary 2.1] *Let A be a nearly convex subset of $\mathbb{R}^m$ and let $T\colon \mathbb{R}^m \to \mathbb{R}^p$ be a linear operator. Then*

(i) $T(A)$ *is nearly convex.*

(ii) $\operatorname{ri}(T(A)) = T(\operatorname{ri} A))$ *and* $\operatorname{cl}(TA) \supseteq T(\operatorname{cl} A)$.

(iii) *If $T^{-1}(\operatorname{ri} A) \neq \emptyset$, then $\operatorname{ri}(T^{-1}(A)) = T^{-1}(\operatorname{ri} A)$ and so $T^{-1}(A)$ is nearly convex.*

Now, we introduce near equality of extended real valued functions and recall the concept of almost convexity [11] of extended real valued functions as follows:

Definition 3.10. Let $f, g\colon \mathbb{R}^m \to \widetilde{\mathbb{R}}$.

(a) f and g are said to be *nearly equal*, denoted by $f \approx g$, if $\operatorname{epi} f \approx \operatorname{epi} g$, i.e.,

$$\operatorname{cl}(\operatorname{epi} f) = \operatorname{cl}(\operatorname{epi} g) \quad \text{and} \quad \operatorname{ri}(\operatorname{epi} f) = \operatorname{ri}(\operatorname{epi} g).$$

(b) f is said to be *almost convex*, if $\operatorname{epi} f$ is an almost convex (or equivalently, a nearly convex) (see, Proposition 3.3) set (in $\mathbb{R}^m \times \mathbb{R}$).

Remark 3.11. (a) Since $\operatorname{epi}(\operatorname{cl} f) = \operatorname{cl}(\operatorname{epi} f)$ and $\operatorname{epi}(\operatorname{cl} g) = \operatorname{cl}(\operatorname{epi} g)$, from Proposition 2.1 we have the condition $\operatorname{cl}(\operatorname{epi} f) = \operatorname{cl}(\operatorname{epi} g)$ in Definition 3.10 (a) is equivalent to $\operatorname{cl} f = \operatorname{cl} g$.

(b) In [8], the function $f\colon \mathbb{R}^m \to \widetilde{\mathbb{R}}$ is said to be nearly convex, if there is an $\alpha \in (0, 1)$ such that $f(\alpha x+(1-\alpha)y) \leq \alpha f(x)+(1-\alpha)f(y)$ for each $x, y \in \operatorname{dom} f$. In this case, f is nearly convex if and only if $\operatorname{epi} f$ is nearly convex in the sense of Remark 3.4 (b). Consequently, if f is nearly convex and $\operatorname{ri}(\operatorname{epi} f) \neq \emptyset$ (or $\operatorname{ri}(\operatorname{dom} f) \neq \emptyset$), then f is almost convex.

4. Properties of near equality and almost convexity of functions

In this section, we shall investigate properties of near equality and almost convexity of extended real valued functions. We first show near equality of two extended real valued functions is an equivalence relation and satisfies squeeze theorem.

Proposition 4.1. *Let $f, g, h\colon \mathbb{R}^m \to \widetilde{\mathbb{R}}$. Then the following statements are true:*

(i) $f \approx f$. (ii) $f \approx g \iff g \approx f$. (iii) $f \approx g$ *and* $g \approx h \Longrightarrow f \approx h$.

(iv) $h \leq f \leq g$ *and* $g \approx h \Longrightarrow h \approx f \approx g$.

Proof. Conclusions (i)–(iii) follow immediately from Lemma 3.5. We prove (iv). Since $h \leq f \leq g$, from Proposition 2.1 we have $\operatorname{epi} g \subseteq \operatorname{epi} f \subseteq \operatorname{epi} h$. Since $g \approx h$, that is, $\operatorname{epi} g \approx \operatorname{epi} h$, it follows from Lemma 3.5 (iv) that $\operatorname{epi} g \approx \operatorname{epi} f \approx \operatorname{epi} h$, which yields $h \approx f \approx g$. □

We give the following characterization of almost convexity of extended real valued functions which has been partially proved in [18].

Theorem 4.2. *Let $f\colon \mathbb{R}^m \to \widetilde{\mathbb{R}}$. The following statements are equivalent:*

(i) *f is almost convex (i.e., $\operatorname{epi} f$ is nearly convex).*

(ii) $\operatorname{cl} f$ *is convex and* $\operatorname{ri}(\operatorname{epi}(\operatorname{cl} f)) \subseteq \operatorname{epi} f$.

(iii) *There exists a convex function $\mathbf{f}\colon \mathbb{R}^m \to \widetilde{\mathbb{R}}$ such that* $\operatorname{ri}(\operatorname{epi} \mathbf{f}) \subseteq \operatorname{epi} f \subseteq \operatorname{cl}(\operatorname{epi} \mathbf{f})$*, which yields* $\operatorname{cl} f = \operatorname{cl} \mathbf{f}$.

Proof. (i) $\Longrightarrow$ (ii): If $\operatorname{epi} f$ is nearly convex, then there exists a convex subset $C \subseteq \mathbb{R}^m \times \mathbb{R}$ such that $C \subseteq \operatorname{epi} f \subseteq \operatorname{cl} C$. This yields $\operatorname{cl} C = \operatorname{cl}(\operatorname{epi} f) = \operatorname{epi}(\operatorname{cl} f)$, which implies that $\operatorname{cl} f$ is lsc and convex and $\operatorname{ri} C = \operatorname{ri}(\operatorname{cl} C) = \operatorname{ri}(\operatorname{epi}(\operatorname{cl} f))$ and $\operatorname{cl} C = \operatorname{cl}(\operatorname{epi}(\operatorname{cl} f))$ (see, Lemma 2.2 (i) and (ii)). It thus follows that $\operatorname{ri}(\operatorname{epi}(\operatorname{cl} f)) \subseteq \operatorname{epi} f$, which proves the assertion.

(ii) $\Longrightarrow$ (iii): By (ii) we have $\operatorname{ri}(\operatorname{epi}(\operatorname{cl} f)) \subseteq \operatorname{epi} f \subseteq \operatorname{cl}(\operatorname{epi} f) = \operatorname{cl}(\operatorname{epi}(\operatorname{cl} f))$. Then setting $\mathbf{f} := \operatorname{cl} f$ yields (iii).

(iii) $\Longrightarrow$ (i): It follows immediately from the convexity of $\operatorname{epi} \mathbf{f}$, taking into account Lemma 2.2 (i) and Proposition 2.1. □

The equivalence between the first two statements in Theorem 4.2 can be found in [18]. We note that in statement (iii) of Theorem 4.2 the convex function $\mathbf{f}$ can be chosen as $\operatorname{cl} f$, provided that (ii) holds, but this is not necessary as the following example shows.

Example 4.3. Consider $f\colon \mathbb{R}^2 \to \widetilde{\mathbb{R}}$ and $\mathbf{f}\colon \mathbb{R}^2 \to \widetilde{\mathbb{R}}$ defined by

$$f(x) := \begin{cases} x_1, & \text{if } x_1 > 0, \\ x_2(1-x_2), & \text{if } x_1 = 0,\ 0 \leq x_2 \leq 1, \\ 0, & \text{if } x_1 = 0,\ x_2 < 0, \\ 0, & \text{if } x_1 = 0,\ x_2 > 1, \\ +\infty, & \text{if } x_1 < 0, \end{cases} \qquad \mathbf{f}(x) := \begin{cases} x_1, & \text{if } x_1 > 0, \\ 1-x_2, & \text{if } x_1 = 0,\ x_2 \leq 1, \\ 0, & \text{if } x_1 = 0,\ x_2 > 1, \\ +\infty, & \text{if } x_1 < 0. \end{cases}$$

Similar to Boţ, Kassay and Wanka [11, Lemma 2.5] we can provide an exact formulation of the relative interior of the epigraph of an almost convex function, which coincides with the one stated in the convex case (see, for example, [35, Lemma 7.3]).

Proposition 4.4. *Let $f\colon \mathbb{R}^m \to \widetilde{\mathbb{R}}$ be almost convex with f proper. Then*

$$\operatorname{ri}(\operatorname{epi} f) = \{(x, y) \in \operatorname{ri}(\operatorname{dom} f) \times \mathbb{R} : y > f(x)\}.$$

Proof. We follow the proof of [11, Lemma 2.5]. Since f is a proper almost convex function, from Lemma 3.6 we have $\mathrm{ri}\,(\mathrm{epi}\, f) = \mathrm{ri}\,(\mathrm{epi}\,(\mathrm{cl}\, f)) \neq \emptyset$. We first note that $\mathbf{P}_{\mathbb{R}^m}(\mathrm{epi}\, f) = \mathrm{dom} f$ and that

$$(x, y) \in \mathrm{ri}\,(\mathrm{epi}\, f) \iff x \in \mathbf{P}_{\mathbb{R}^m}\mathrm{ri}\,(\mathrm{epi}\, f) \text{ and } y \in \mathbf{P}_{\mathbb{R}}[\mathrm{ri}\,(\mathrm{epi}\, f) \cap (\{x\} \times \mathbb{R})],$$

where $\mathbf{P}_{\mathbb{R}^m} : \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}^m$ denotes the projection operator on $\mathbb{R}^m$. By Lemma 3.8(ii) and Lemma 3.9 (ii), for all $x \in \mathbf{P}_{\mathbb{R}^m}\mathrm{ri}\,(\mathrm{epi}\, f) = \mathrm{ri}\,(\mathbf{P}_{\mathbb{R}^m}(\mathrm{epi}\, f)) = \mathrm{ri}\,(\mathrm{dom} f)$, we have

$$\emptyset \neq \mathrm{ri}\,(\mathrm{epi}\, f) \cap (\{x\} \times \mathbb{R}) = \mathrm{ri}\,(\mathrm{epi}\, f) \cap \mathrm{ri}(\{x\} \times \mathbb{R}) = \mathrm{ri}\,[(\mathrm{epi}\, f) \cap (\{x\} \times \mathbb{R})].$$

Then, again by Lemma 2.2 (iii) and Lemma 3.9 (ii), $(x, y) \in \mathrm{ri}\,(\mathrm{epi}\, f)$, if and only if, $x \in \mathrm{ri}\,(\mathrm{dom} f)$ and

$$\begin{aligned} y \ &\in \mathbf{P}_{\mathbb{R}}[\mathrm{ri}\,(\mathrm{epi}\, f) \cap (\{x\} \times \mathbb{R})] = \mathrm{ri}\,[\mathbf{P}_{\mathbb{R}}\,((\mathrm{epi}\, f) \cap (\{x\} \times \mathbb{R}))] \\ &= \mathrm{ri}\,[\mathbf{P}_{\mathbb{R}}\{(x, y) : y \geq f(x)\}] = \mathrm{ri}\,[f(x) + \mathbb{R}_+] = f(x) + \mathbb{R}_{++}. \end{aligned}$$

This completes the proof. □

For a proper almost convex function $f : \mathbb{R}^m \to \widetilde{\mathbb{R}}$, from Proposition 4.4, it follows that, $\mathrm{ri}\,(\mathrm{dom}\, f) \neq \emptyset \Leftrightarrow \mathrm{ri}\,(\mathrm{epi}\, f) = \mathrm{ri}\,(\mathrm{epi}\,(\mathrm{cl}\, f)) \neq \emptyset \Leftrightarrow \mathrm{ri}\,(\mathrm{dom}\,(\mathrm{cl}\, f)) \neq \emptyset$. Moreover, f is proper if and only if $\mathrm{cl}\, f$ is proper, in view of Theorem 4.2 (iii) and Lemma 2.2 (i). We next prove that an almost convex function (respectively, its domain, lower level set) is nearly equal to (respectively, the domain, lower level set of) its closure, convex hull and closed convex hull. We first give the following

Lemma 4.5. *Let $h : \mathbb{R}^m \to \widetilde{\mathbb{R}}$. Then $\inf_{x\in\mathbb{R}^m} h(x) = \inf_{x\in\mathbb{R}^m}(\mathrm{cl}\, h)(x)$.*

Proof. Let $\rho := \inf_{x\in\mathbb{R}^m} h(x)$ and $\tau := \inf_{x\in\mathbb{R}^m}(\mathrm{cl}\, h)(x)$. We declare that $\rho = \tau$. In fact, $(\mathrm{cl}\, h)(x) := \liminf_{y\to x} h(y) = \sup_{\alpha>0} \inf_{y\in B(x,\alpha)} h(y) \geq \rho$ for each $x \in \mathbb{R}^m$, and therefore $\tau = \inf_{x\in\mathbb{R}^m}(\mathrm{cl}\, h)(x) \geq \rho$. Since $h(x) \geq (\mathrm{cl}\, h)(x)$ for each $x \in \mathbb{R}^m$, we have $\tau = \inf_{x\in\mathbb{R}^m}(\mathrm{cl}\, h)(x) \leq \rho$, which implies $\rho = \tau$. This completes the proof. □

Proposition 4.6. *Let $f : \mathbb{R}^m \to \widetilde{\mathbb{R}}$ be almost convex with f proper, say $\mathrm{ri}\,(\mathrm{epi}\, \mathbf{f}) \subseteq \mathrm{epi}\, f \subseteq \mathrm{cl}\,(\mathrm{epi}\, \mathbf{f})$, where $\mathbf{f} : \mathbb{R}^m \to \widetilde{\mathbb{R}}$ is a convex function. Then the following statements are true:*

(i) $f \approx \mathrm{cl}\, f \approx \mathrm{conv}\, f \approx \overline{\mathrm{conv}}\, f \approx \mathbf{f}$.

(ii) $\mathrm{cl}\, f = \overline{\mathrm{conv}}\, f = \mathrm{cl}\, \mathbf{f}$ *and* $\mathrm{cl}\, f$ *is convex.*

(iii) $\mathrm{dom}\, f \approx \mathrm{dom}\,(\mathrm{cl}\, f) \approx \mathrm{dom}\,(\mathrm{conv}\, f) \approx \mathrm{dom}\,(\overline{\mathrm{conv}}\, f) \approx \mathrm{dom}\, \mathbf{f}$, *and* $\mathrm{dom}\, f$ *is nearly convex.*

(iv) $\mathrm{dom}\,(\mathrm{cl}\, f) = \mathrm{dom}\,(\overline{\mathrm{conv}}\, f) = \mathrm{dom}\,(\mathrm{cl}\, \mathbf{f})$.

(v) $\mathrm{lev}_{\leq\alpha} f \approx \mathrm{lev}_{\leq\alpha}\,(\mathrm{cl}\, f) \approx \mathrm{lev}_{\leq\alpha}\,(\mathrm{conv}\, f) \approx \mathrm{lev}_{\leq\alpha}\,(\overline{\mathrm{conv}}\, f) \approx \mathrm{lev}_{\leq\alpha}\, \mathbf{f} \approx \mathrm{lev}_{<\alpha} f \approx \mathrm{lev}_{<\alpha}\,(\mathrm{cl}\, f) \approx \mathrm{lev}_{<\alpha}\,(\mathrm{conv}\, f) \approx \mathrm{lev}_{<\alpha}\,(\overline{\mathrm{conv}}\, f) \approx \mathrm{lev}_{<\alpha}\, \mathbf{f}$, *and* $\mathrm{lev}_{\leq\alpha} f$ *and* $\mathrm{lev}_{<\alpha} f$ *are nearly convex, as long as* $\alpha > \inf_{x\in\mathbb{R}^m} f(x)$.

(vi) $\operatorname{lev}_{\le\alpha}(\operatorname{cl} f) = \operatorname{lev}_{\le\alpha}(\overline{\operatorname{conv}} f) = \operatorname{lev}_{\le\alpha}(\operatorname{cl}\mathbf{f})$ *and* $\operatorname{lev}_{<\alpha}(\operatorname{cl} f) = \operatorname{lev}_{<\alpha}(\overline{\operatorname{conv}} f) = \operatorname{lev}_{<\alpha}(\operatorname{cl}\mathbf{f})$, *where* $\alpha \in \mathbb{R}$.

Proof. Since $\mathbf{f}$ is convex, so is $\operatorname{epi}\mathbf{f}$. Since $\operatorname{ri}(\operatorname{epi}\mathbf{f}) \subseteq \operatorname{epi} f \subseteq \operatorname{cl}(\operatorname{epi}\mathbf{f})$, from Lemma 3.5 (v) we have $\operatorname{epi} f \approx \operatorname{epi}\mathbf{f}$ and so $f \approx \mathbf{f}$. We also have $\operatorname{ri}(\operatorname{epi}\mathbf{f}) \subseteq \operatorname{epi} f \subseteq \operatorname{cl}(\operatorname{epi} f) = \operatorname{epi}(\operatorname{cl} f)$ (resp., $\operatorname{conv}(\operatorname{epi} f) \subseteq \operatorname{epi}(\operatorname{conv} f)) \subseteq \operatorname{epi}(\overline{\operatorname{conv}} f) = \overline{\operatorname{conv}}(\operatorname{epi} f) \subseteq \operatorname{cl}(\operatorname{epi}\mathbf{f}) = \operatorname{epi}(\operatorname{cl}\mathbf{f})$, which yields $f \ge \operatorname{cl} f$ (reps., $\operatorname{conv} f) \ge \overline{\operatorname{conv}} f \ge \operatorname{cl}\mathbf{f}$ in view of Proposition 2.1.

(i) It is clear that $\operatorname{cl}(\operatorname{epi}\mathbf{f}) = \operatorname{epi}(\operatorname{cl}\mathbf{f}) = \operatorname{cl}(\operatorname{epi}(\operatorname{cl}\mathbf{f}))$. Since $\operatorname{epi}\mathbf{f}$ is convex, $\operatorname{ri}(\operatorname{epi}\mathbf{f}) = \operatorname{ri}(\operatorname{cl}(\operatorname{epi}\mathbf{f})) = \operatorname{ri}(\operatorname{epi}(\operatorname{cl}\mathbf{f}))$ (by Lemma 2.2 (i)) and as a consequence, $\mathbf{f} \approx \operatorname{cl}\mathbf{f}$. It thus follows from Proposition 4.1 (iii) and (iv) that $f \approx \operatorname{cl} f \approx (\operatorname{conv} f) \approx (\overline{\operatorname{conv}} f) \approx \mathbf{f}$.

(ii) From $\operatorname{cl} f = \operatorname{cl}\mathbf{f}$ (by Remark 3.11 (b)) and $f \ge \operatorname{cl} f$ (resp., $\operatorname{conv} f) \ge \overline{\operatorname{conv}} f \ge \operatorname{cl}\mathbf{f}$ we have $\operatorname{cl} f = \overline{\operatorname{conv}} f = \operatorname{cl}\mathbf{f}$ and so $\operatorname{cl} f$ is convex.

(iii) Note that $\operatorname{dom}\mathbf{f}$ and $\operatorname{dom}(\operatorname{cl}\mathbf{f})$ are convex. Denote by $\mathbf{P}_{\mathbb{R}^m} : \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}^m$ the projection operator on $\mathbb{R}^m$. Then from Proposition 2.1 and Lemma 2.2 (vii) one has

$$\begin{aligned}\operatorname{ri}(\operatorname{dom}\mathbf{f}) &= \operatorname{ri}(\mathbf{P}_{\mathbb{R}^m}(\operatorname{epi}\mathbf{f})) = \mathbf{P}_{\mathbb{R}^m}\operatorname{ri}(\operatorname{epi}\mathbf{f}) \subseteq \mathbf{P}_{\mathbb{R}^m}(\operatorname{epi} f) = \operatorname{dom} f\\ &\subseteq \operatorname{dom}(\operatorname{cl} f)\ (\text{resp.}, \operatorname{dom}(\operatorname{conv} f)) \subseteq \operatorname{dom}(\overline{\operatorname{conv}} f) \subseteq \operatorname{dom}(\operatorname{cl}\mathbf{f}).\end{aligned}$$

Now Lemma 2.5 allows $\operatorname{cl}(\operatorname{dom}\mathbf{f}) = \operatorname{cl}(\operatorname{dom}(\operatorname{cl}\mathbf{f}))$. By Lemma 2.2 (i) we obtain $\operatorname{ri}(\operatorname{dom}\mathbf{f}) = \operatorname{ri}(\operatorname{cl}(\operatorname{dom}\mathbf{f})) = \operatorname{ri}(\operatorname{cl}(\operatorname{dom}(\operatorname{cl}\mathbf{f}))) = \operatorname{ri}(\operatorname{dom}(\operatorname{cl}\mathbf{f}))$, which implies $\operatorname{dom}\mathbf{f} \approx \operatorname{dom}(\operatorname{cl}\mathbf{f})$. Therefore, from Lemma 3.5 (iii)–(v) it follows that $\operatorname{dom} f \approx \operatorname{dom}(\operatorname{cl} f) \approx \operatorname{dom}(\operatorname{conv} f) \approx \operatorname{dom}(\overline{\operatorname{conv}} f) \approx \operatorname{dom}\mathbf{f}$. The near convexity of $\operatorname{dom} f$ follows immediately from Lemma 3.5 (v).

(iv) This follows immediately from (ii).

(v) If $\mathbf{f}$ is improper, then this is clear. Suppose now that $\mathbf{f}$ is proper. From $f \ge \operatorname{cl} f$ (resp., $\operatorname{conv} f) \ge \overline{\operatorname{conv}} f \ge \operatorname{cl}\mathbf{f}$ we have $\operatorname{lev}_{\le\alpha} f \subseteq \operatorname{lev}_{\le\alpha}(\operatorname{cl} f)$ (resp., $\operatorname{lev}_{\le\alpha}(\operatorname{conv} f)$) $\subseteq \operatorname{lev}_{\le\alpha}(\overline{\operatorname{conv}} f) \subseteq \operatorname{lev}_{\le\alpha}(\operatorname{cl}\mathbf{f}) = \operatorname{cl}(\operatorname{lev}_{\le\alpha}\mathbf{f})$, where the last equality follows from Lemma 2.6, noticing that $\alpha > \inf_{x\in\mathbb{R}^m} f(x) = \inf_{x\in\mathbb{R}^m}(\operatorname{cl} f)(x)$ (see, Lemma 4.5) and recalling that $\operatorname{cl}\mathbf{f} = \operatorname{cl} f$ (see, (ii)). Observe that

$$\operatorname{lev}_{\le\alpha} f = \mathbf{P}_{\mathbb{R}^m}[(\operatorname{epi} f) \cap \{(u,v) \in \mathbb{R}^m \times \mathbb{R} : v \le \alpha\}],$$

where $\mathbf{P}_{\mathbb{R}^m} : \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}^m$ denotes the projection operator on $\mathbb{R}^m$. Since $\operatorname{ri}(\operatorname{epi}\mathbf{f}) \subseteq \operatorname{epi} f$ and $\operatorname{ri}\{(u,v) \in \mathbb{R}^m \times \mathbb{R} : v \le \alpha\} = \{(u,v) \in \mathbb{R}^m \times \mathbb{R} : v < \alpha\}$, it follows that

$$\begin{aligned}\operatorname{ri}(\operatorname{lev}_{\le\alpha}\mathbf{f}) &= \operatorname{ri}(\mathbf{P}_{\mathbb{R}^m}[(\operatorname{epi}\mathbf{f}) \cap \{(u,v) \in \mathbb{R}^m \times \mathbb{R} : v \le \alpha\}])\\ &= \mathbf{P}_{\mathbb{R}^m}[\operatorname{ri}(\operatorname{epi}\mathbf{f}) \cap \operatorname{ri}\{(u,v) \in \mathbb{R}^m \times \mathbb{R} : v \le \alpha\}]\\ &\subseteq \mathbf{P}_{\mathbb{R}^m}[\operatorname{ri}(\operatorname{epi}\mathbf{f}) \cap \{(u,v) \in \mathbb{R}^m \times \mathbb{R} : v \le \alpha\}] \ \subseteq\ \operatorname{lev}_{\le\alpha} f,\end{aligned}$$

where the second equality follows from Lemma 2.2 (vi) and (vii). Hence,

$$\begin{aligned}\mathrm{ri}\,(\mathrm{lev}_{\leq\alpha}\,\mathbf{f}) &\subseteq \mathrm{lev}_{\leq\alpha}\,f \subseteq \mathrm{lev}_{\leq\alpha}\,(\mathrm{cl}\,f)\ \ (\mathrm{resp.}, \mathrm{lev}_{\leq\alpha}\,(\mathrm{conv}\,f))\\ &\subseteq \mathrm{lev}_{\leq\alpha}\,(\overline{\mathrm{conv}}\,f) \subseteq \mathrm{cl}\,(\mathrm{lev}_{\leq\alpha}\,\mathbf{f}).\end{aligned}$$

Since $\mathrm{lev}_{\leq\alpha}\,\mathbf{f}$ is convex, from Lemma 3.5 (iii)–(v) it follows that $\mathrm{lev}_{\leq\alpha}f$ is nearly convex and $\mathrm{lev}_{\leq\alpha}f \approx \mathrm{lev}_{\leq\alpha}\,(\mathrm{cl}\,f) \approx \mathrm{lev}_{\leq\alpha}\,(\mathrm{conv}\,f) \approx \mathrm{lev}_{\leq\alpha}\,(\overline{\mathrm{conv}}\,f) \approx \mathrm{lev}_{\leq\alpha}\,\mathbf{f}$. Similarly, we can prove that $\mathrm{lev}_{<\alpha}f$ is nearly convex and moreover $\mathrm{lev}_{<\alpha}f \approx \mathrm{lev}_{<\alpha}\,(\mathrm{cl}\,f) \approx \mathrm{lev}_{<\alpha}\,(\mathrm{conv}\,f) \approx \mathrm{lev}_{<\alpha}\,(\overline{\mathrm{conv}}\,f) \approx \mathrm{lev}_{<\alpha}\,\mathbf{f}$. Again from Lemma 2.6, one has $\mathrm{lev}_{\leq\alpha}\mathbf{f} \approx \mathrm{lev}_{<\alpha}\,\mathbf{f}$ and therefore it follows from Lemma 3.5 (iii) that the conclusion holds.

(vi) This follows immediately from (ii). □

Remark 4.7. From Proposition 4.6 (iii) and Proposition 4.4 it follows that an almost convex function f with f proper agrees with $\mathrm{cl}\,f$ on $\mathrm{ri}\,(\mathrm{dom}\,f)$, which extends Lemma 2.5 to the case of almost convex functions along with Proposition 4.8. This also yields that an almost convex function $f\colon \mathbb{R}^m \to \mathbb{R}$ is convex and so continuous.

The following proposition characterizes the subdifferential of a proper almost function on the relative interior of its domain.

Proposition 4.8. *Let $f\colon \mathbb{R}^m \to \widetilde{\mathbb{R}}$ be almost convex with f proper. Then we have $\partial f(x) = \partial(\mathrm{cl}\,f)(x) \neq \emptyset$ for each $x \in \mathrm{ri}\,(\mathrm{dom}\,f)$.*

Proof. Notice that $\mathrm{cl}\,f$ is convex and $\mathrm{ri}\,(\mathrm{dom}\,f) = \mathrm{ri}\,(\mathrm{dom}\,(\mathrm{cl}\,f))$ (see, Proposition 4.6 (ii) and (iii)). Let $x \in \mathrm{ri}\,(\mathrm{dom}\,f)$. Then $f(x) = (\mathrm{cl}\,f)(x)$ (see, Remark 4.7), and $\partial(\mathrm{cl}\,f)(x) \neq \emptyset$ (see, Lemma 2.5). Therefore, it follows from Lemma 2.4 that

$$\begin{aligned}s \in \partial f(x) &\Longleftrightarrow f(x) + f^*(s) = \langle x, s\rangle \Longleftrightarrow (\mathrm{cl}\,f)\,(x) + (\mathrm{cl}\,f)^*(s) = \langle x, s\rangle\\ &\Longleftrightarrow s \in \partial(\mathrm{cl}\,f)(x),\end{aligned}$$

where the first and third equivalence relation follows immediately from the definition of subdifferential. This proves $\partial(\mathrm{cl}\,f)(x) = \partial\,f(x)$ for each $x \in \mathrm{ri}\,(\mathrm{dom}\,f)$. □

By means of Proposition 4.6 we can extend Lemma 2.6 to almost convex functions.

Proposition 4.9. *Let $f\colon \mathbb{R}^m \to \widetilde{\mathbb{R}}$ be almost convex with f proper and assume $\alpha > \inf_{x\in\mathbb{R}^m} f(x)$. Then, we have $\mathrm{cl}\,(\mathrm{lev}_{\leq\alpha}f) = \mathrm{cl}\,(\mathrm{lev}_{<\alpha}f) = \mathrm{lev}_{\leq\alpha}\,(\mathrm{cl}\,f)$ and $\mathrm{ri}\,(\mathrm{lev}_{\leq\alpha}f) = \mathrm{ri}\,(\mathrm{lev}_{<\alpha}f) = \{x \in \mathrm{ri}\,(\mathrm{dom}\,f) : f(x) < \alpha\}$.*

Proof. By Proposition 4.6 (ii), $\mathrm{cl}\,f$ is convex and by Proposition 4.6 (v),

$$\mathrm{lev}_{\leq\alpha}f \approx \mathrm{lev}_{\leq\alpha}\,(\mathrm{cl}\,f) \approx \mathrm{lev}_{<\alpha}f, \tag{2}$$

which yields that $\mathrm{lev}_{\leq\alpha}f$ is nearly convex (by Lemma 3.5 (v)) and

$$\mathrm{cl}\,(\mathrm{lev}_{\leq\alpha}f) = \mathrm{cl}\,(\mathrm{lev}_{<\alpha}f) = \mathrm{cl}\,(\mathrm{lev}_{\leq\alpha}\,(\mathrm{cl}\,f)) = \mathrm{lev}_{\leq\alpha}\,(\mathrm{cl}\,f),$$

where the last equality follows from Lemma 2.6 applied to the convex function $\mathrm{cl}\,f$, noticing that $\alpha > \inf_{x\in\mathbb{R}^m} f(x) = \inf_{x\in\mathbb{R}^m}(\mathrm{cl}\,f)(x)$ (see, Lemma 4.5). Furthermore, (2) yields

$$\begin{aligned}\mathrm{ri}\,(\mathrm{lev}_{\le\alpha} f) &= \mathrm{ri}\,(\mathrm{lev}_{<\alpha} f) = \mathrm{ri}\,(\mathrm{lev}_{\le\alpha}\,(\mathrm{cl}\, f)) = \mathrm{lev}_{<\alpha}\,(\mathrm{cl}\, f)\\ &= \{x \in \mathrm{ri}\,(\mathrm{dom}\,(\mathrm{cl}\, f)) : (\mathrm{cl}\, f)(x) < \alpha\},\end{aligned}$$

where the last two equalities follow from Lemma 2.6 applied to the convex function $\mathrm{cl}\, f$. Finally, by Proposition 4.6 (iii), we have $\mathrm{ri}\,(\mathrm{dom}\, f) = \mathrm{ri}\,(\mathrm{dom}\,(\mathrm{cl}\, f))$ and so

$$\begin{aligned}\{x \in \mathrm{ri}\,(\mathrm{dom}\,(\mathrm{cl}\, f)) : (\mathrm{cl}\, f)(x) < \alpha\} &= \{x \in \mathrm{ri}\,(\mathrm{dom}\, f) : (\mathrm{cl}\, f)(x) < \alpha\}\\ &= \{x \in \mathrm{ri}\,(\mathrm{dom}\, f) : f(x) < \alpha\},\end{aligned}$$

where the last equality follows from the fact that we have $f(x) = (\mathrm{cl}\, f)(x)$ for each $x \in \mathrm{ri}\,(\mathrm{dom}\, f)$ (see Remark 4.7). This completes the proof. □

In the sequel we characterize the conjugate function of an almost convex function.

Proposition 4.10. *Let $f \colon \mathbb{R}^m \to \widetilde{\mathbb{R}}$ be almost convex with f proper, say* $\mathrm{ri}\,(\mathrm{epi}\,\mathbf{f}) \subseteq \mathrm{epi}\, f \subseteq \mathrm{cl}\,(\mathrm{epi}\,\mathbf{f})$*, where $\mathbf{f} \colon \mathbb{R}^m \to \widetilde{\mathbb{R}}$ is a convex function. Then, the following statements are true:*

(i) $f^* = (\mathrm{cl}\, f)^* = (\mathrm{conv}\, f)^* = (\overline{\mathrm{conv}}\, f)^* = \mathbf{f}^*$.

(ii) $\mathrm{dom}\, f^* = \mathrm{dom}\,(\mathrm{cl}\, f)^* = \mathrm{dom}\,(\mathrm{conv}\, f)^* = \mathrm{dom}\,(\overline{\mathrm{conv}}\, f)^* = \mathrm{dom}\,\mathbf{f}^*$.

(iii) $\mathrm{lev}_{\le\alpha}\, f^* = \mathrm{lev}_{\le\alpha}\,(\mathrm{cl}\, f)^* = \mathrm{lev}_{\le\alpha}\,(\mathrm{conv}\, f)^* = \mathrm{lev}_{\le\alpha}\,(\overline{\mathrm{conv}}\, f)^* = \mathrm{lev}_{\le\alpha}\,\mathbf{f}^*$ *and* $\mathrm{lev}_{<\alpha}\, f^* = \mathrm{lev}_{<\alpha}\,(\mathrm{cl}\, f)^* = \mathrm{lev}_{<\alpha}\,(\mathrm{conv}\, f)^* = \mathrm{lev}_{<\alpha}\,(\overline{\mathrm{conv}}\, f)^* = \mathrm{lev}_{<\alpha}\,\mathbf{f}^*$, *where $\alpha \in \mathbb{R}$.*

Proof. From Proposition 4.6 (ii) we have $\mathrm{cl}\, f = \mathrm{cl}\,\mathbf{f}$. Therefore the equalities $f^* = (\mathrm{cl}\, f)^* = (\mathrm{conv}\, f)^* = (\overline{\mathrm{conv}}\, f)^* = (\mathrm{cl}\,\mathbf{f})^* = \mathbf{f}^*$ follow immediately from Lemma 2.4.

(ii) and (iii) follow immediately from (i). □

The following proposition shows that the indicator functions of two sets are nearly equal if and only if the two sets involved do under the assumption that their relatively interior is nonempty, where the two sets do not need to be convex.

Proposition 4.11. *Let K and M be nonempty subsets of $\mathbb{R}^m$. Then, the following statements are true:*

(i) *If* $\mathrm{ri}\, K \neq \emptyset$ *and* $\mathrm{ri}\, M \neq \emptyset$, *then* $\delta_K \approx \delta_M \Longleftrightarrow K \approx M$ $\Longrightarrow \mathrm{cl}\, K = \mathrm{cl}\, M$ and $\sigma_K = \sigma_M$.

(ii) δ_K *is almost convex* $\Longleftrightarrow \mathrm{epi}\,\delta_K = K \times \mathbb{R}_+$ *is nearly convex* $\Longleftrightarrow K$ *is nearly convex* $\Longrightarrow \mathrm{cl}\, K$ *is convex.*

(iii) *If K is nearly convex, then we have* $\mathrm{cl}\,\delta_K = \overline{\mathrm{conv}}\,\delta_K = \delta_{\mathrm{cl}\, K} \le \mathrm{conv}\,\delta_K$ *and* $\partial\,\delta_K(x) = N_K(x) = N_{\mathrm{cl}\, K}(x) \neq \emptyset$ *for each* $x \in \mathrm{ri}\, K$.

Proof. (i) Note that $\delta_K \approx \delta_M \Longleftrightarrow \mathrm{epi}\,\delta_K \approx \mathrm{epi}\,\delta_M \Longleftrightarrow K \times \mathbb{R}_+ \approx M \times \mathbb{R}_+ \Longleftrightarrow \mathrm{cl}\,(K \times \mathbb{R}_+) = \mathrm{cl}\,(M \times \mathbb{R}_+)$ and $\mathrm{ri}\,(K \times \mathbb{R}_+) = \mathrm{ri}\,(M \times \mathbb{R}_+) \Longleftrightarrow \mathrm{cl}\, K = \mathrm{cl}\, M$ and $\mathrm{ri}\, K = \mathrm{ri}\, M \Longleftrightarrow K \approx M$, where the fourth equivalence follows from Lemma 2.2 (iv) and (v). From $K \approx M$, we have $\mathrm{cl}\, K = \mathrm{cl}\, M$ and so $\sigma_K = \sigma_M$.

(ii) The equivalence that δ_K is almost convex $\Longleftrightarrow$ $\operatorname{epi}\delta_K = K \times \mathbb{R}_+$ is nearly convex follows immediately from the definition.

Since δ_K is almost convex, i.e., there is a convex function $\mathbf{f}\colon \mathbb{R}^m \to \widetilde{\mathbb{R}}$ such that $\operatorname{ri}(\operatorname{epi}\mathbf{f}) \subseteq \operatorname{epi}\delta_K \subseteq \operatorname{cl}(\operatorname{epi}\mathbf{f})$. Now Proposition 4.6 (iii) implies that $\operatorname{dom}\delta_K \approx \operatorname{dom}\mathbf{f}$. Since $\operatorname{dom}\delta_K = K$ and $\operatorname{dom}\mathbf{f}$ is convex, it follows that $K \approx \operatorname{dom}\mathbf{f}$ and so from Lemma 3.5 (v), K is nearly convex. The equality $\operatorname{cl}K = \operatorname{cl}(\operatorname{dom}\mathbf{f})$ follows immediately from $K \approx \operatorname{dom}\mathbf{f}$.

Since K is nearly convex, there is a convex set $C \subseteq \mathbb{R}^m$ such that $C \subseteq K \subseteq \operatorname{cl}C$. This allows $\delta_C \geq \delta_K \geq \delta_{\operatorname{cl}C}$ and then by Proposition 2.1, $\operatorname{epi}\delta_C \subseteq \operatorname{epi}\delta_K \subseteq \operatorname{epi}\delta_{\operatorname{cl}C} = \operatorname{cl}C \times \mathbb{R}_+ = \operatorname{cl}(C \times \mathbb{R}_+) = \operatorname{cl}(\operatorname{epi}\delta_C)$, where the second equality follows from Lemma 2.2 (iv). Since C is convex, so are δ_C and $\operatorname{epi}\delta_C$. This implies that δ_K is almost convex.

(iii) Since K is nearly convex, from (ii) we have δ_K is almost convex and $\operatorname{cl}K$ is convex. Then, there is a convex set $C \subseteq \mathbb{R}^m \times \mathbb{R}$ such that $C \subseteq \operatorname{epi}\delta_K \subseteq \operatorname{cl}C$. Since C is convex, so is $\operatorname{cl}C$. It follows that $\operatorname{epi}(\operatorname{cl}\delta_K) := \operatorname{cl}(\operatorname{epi}\delta_K) = \operatorname{cl}C$ and $C \subseteq \operatorname{conv}(\operatorname{epi}\delta_K) \subseteq \operatorname{cl}C$ and so $\operatorname{epi}(\overline{\operatorname{conv}}\,\delta_K) := \operatorname{cl}(\operatorname{epi}(\operatorname{conv}\delta_K)) := \overline{\operatorname{conv}}(\operatorname{epi}\delta_K) = \operatorname{cl}C$, which yields $\operatorname{epi}(\operatorname{cl}\delta_K) = \operatorname{epi}(\overline{\operatorname{conv}}\,\delta_K) \supseteq \operatorname{epi}(\operatorname{conv}\delta_K)$ and therefore, $\operatorname{cl}\delta_K = \overline{\operatorname{conv}}\,\delta_K \leq \operatorname{conv}\delta_K$, in view of Proposition 2.1. Since $\operatorname{epi}\delta_K = K \times \mathbb{R}_+$, one has $\operatorname{epi}(\operatorname{cl}\delta_K) := \operatorname{cl}(\operatorname{epi}\delta_K) = \operatorname{cl}K \times \mathbb{R}_+$ (see, Lemma 2.2 (iv)) and thus $\operatorname{cl}\delta_K = \delta_{\operatorname{cl}K}$. Notice that $\operatorname{ri}(\operatorname{dom}\delta_K) = \operatorname{ri}K$, $\partial\delta_K(x) = N_K(x)$ and $\partial(\operatorname{cl}\delta_K)(x) = \partial\delta_{\operatorname{cl}K}(x) = N_{\operatorname{cl}K}(x)$ for each $x \in \mathbb{R}^m$. The conclusion $\partial\,\delta_K(x) = N_K(x) = N_{\operatorname{cl}K}(x) \neq \emptyset$ for each $x \in \operatorname{ri}K$ follows immediately from Proposition 4.8. $\square$

Under certain assumptions, almost convexity of an extended real valued function is proved to be equivalent to near equality of itself and another almost convex function.

Theorem 4.12. *Let $f\colon \mathbb{R}^m \to \widetilde{\mathbb{R}}$ be a proper function. The following statements are equivalent:*

(i) *f is almost convex.* (ii) *$f \approx \overline{\operatorname{conv}}\,f$.* (iii) *$f \approx \operatorname{conv} f$.*

(iv) *$f \approx \mathbf{f}$, where $\mathbf{f}\colon \mathbb{R}^m \to \widetilde{\mathbb{R}}$ is a convex function.*

(v) *$f \approx g$, where $g\colon \mathbb{R}^m \to \widetilde{\mathbb{R}}$ is an almost convex function, say $\operatorname{ri}(\operatorname{epi}\mathbf{g}) \subseteq \operatorname{epi}g \subseteq \operatorname{cl}(\operatorname{epi}\mathbf{g})$, where $\mathbf{g}\colon \mathbb{R}^m \to \widetilde{\mathbb{R}}$ is a convex function.*

Proof. (i)$\Longrightarrow$(ii): This follows from Proposition 4.6 (i).

(ii)$\Longrightarrow$(iii): Observe that $f \approx \overline{\operatorname{conv}}\,f \Longleftrightarrow \operatorname{epi}f \approx \operatorname{epi}(\overline{\operatorname{conv}}\,f)$.
This leads to $\operatorname{cl}(\operatorname{epi}f) = \operatorname{cl}(\operatorname{epi}(\overline{\operatorname{conv}}\,f))$ and $\operatorname{ri}(\operatorname{epi}f) = \operatorname{ri}(\operatorname{epi}(\overline{\operatorname{conv}}\,f))$.
Since $\operatorname{cl}(\operatorname{epi}(\overline{\operatorname{conv}}\,f)) = \operatorname{epi}(\overline{\operatorname{conv}}\,f) = \operatorname{cl}(\operatorname{epi}(\operatorname{conv}f))$ and $\operatorname{ri}(\operatorname{epi}(\overline{\operatorname{conv}}\,f)) = \operatorname{ri}(\operatorname{cl}(\operatorname{epi}(\operatorname{conv}f))) = \operatorname{ri}(\operatorname{epi}(\operatorname{conv}f))$ (by Lemma 2.2 (i)), we obtain $\operatorname{cl}(\operatorname{epi}f) = \operatorname{cl}(\operatorname{epi}(\operatorname{conv}f))$ and $\operatorname{ri}(\operatorname{epi}f) = \operatorname{ri}(\operatorname{epi}(\operatorname{conv}f))$, that is, $f \approx \operatorname{conv}f$.

(iii)$\Longrightarrow$(iv): This is clear by setting $\mathbf{f} := \operatorname{conv}f$.

(iv)$\Longrightarrow$(v): This is clear, since a convex function is almost convex.

(v)$\Longrightarrow$(i): Suppose (v) holds. Then $f \approx g$ and there exists a convex function $\mathbf{g}\colon \mathbb{R}^m \to \widetilde{\mathbb{R}}$ such that $\mathrm{ri}\,(\mathrm{epi}\,\mathbf{g}) \subseteq \mathrm{epi}\,g \subseteq \mathrm{cl}\,(\mathrm{epi}\,\mathbf{g})$. From Proposition 4.6 (i) we have $g \approx \mathbf{g}$ and so $f \approx \mathbf{g}$ (by Proposition 4.1 (iii)), that is $\mathrm{cl}\,(\mathrm{epi}\,f) = \mathrm{cl}\,(\mathrm{epi}\,\mathbf{g})$ and $\mathrm{ri}\,(\mathrm{epi}\,f) = \mathrm{ri}\,(\mathrm{epi}\,\mathbf{g})$. As a consequence, $\mathrm{ri}\,(\mathrm{epi}\,\mathbf{g}) = \mathrm{ri}\,(\mathrm{epi}\,f) \subseteq \mathrm{epi}\,f \subseteq \mathrm{cl}\,(\mathrm{epi}\,f) = \mathrm{cl}\,(\mathrm{epi}\,\mathbf{g})$, which implies that f is almost convex. $\square$

We also give the following characterization of near equality of (respectively, the domains, the lower level sets of) almost convex functions.

Theorem 4.13. *Let $f, g\colon \mathbb{R}^m \to \widetilde{\mathbb{R}}$ be almost convex with f and g proper, say $\mathrm{ri}\,(\mathrm{epi}\,\mathbf{f}) \subseteq \mathrm{epi}\,f \subseteq \mathrm{cl}\,(\mathrm{epi}\,\mathbf{f})$ and $\mathrm{ri}\,(\mathrm{epi}\,\mathbf{g}) \subseteq \mathrm{epi}\,g \subseteq \mathrm{cl}\,(\mathrm{epi}\,\mathbf{g})$, where $\mathbf{f}, \mathbf{g}\colon \mathbb{R}^m \to \widetilde{\mathbb{R}}$ are convex functions. For any $s, t \in \{f, \mathbf{f}, \mathrm{cl}\,f, \mathrm{conv}\,f, \overline{\mathrm{conv}}\,f\}$ and $u, v \in \{g, \mathbf{g}, \mathrm{cl}\,g, \mathrm{conv}\,g, \overline{\mathrm{conv}}\,g\}$, we have $s \approx u \Longleftrightarrow t \approx v$, which implies $\mathrm{cl}\,f = \mathrm{cl}\,g$ and so $\mathrm{dom}\,(\mathrm{cl}\,f) = \mathrm{dom}\,(\mathrm{cl}\,g)$, $\mathrm{lev}_{\le\alpha}(\mathrm{cl}\,f) = \mathrm{lev}_{\le\alpha}(\mathrm{cl}\,g)$ and $\mathrm{lev}_{<\alpha}(\mathrm{cl}\,f) = \mathrm{lev}_{<\alpha}(\mathrm{cl}\,g)$, where $\alpha \in \mathbb{R}$.*

Proof. Without loss of generality, let $s := f$, $t := \mathbf{f}$, $u := g$, $v := \mathbf{g}$. Since $f, g\colon \mathbb{R}^m \to \widetilde{\mathbb{R}}$ are almost convex, from Proposition 4.6 (i) one has $f \approx \mathbf{f}$ and $g \approx \mathbf{g}$. Then $f \approx g \Longleftrightarrow \mathbf{f} \approx \mathbf{g}$ follows from Proposition 4.1 (ii) and (iii).

From $f \approx g$ one has $\mathrm{epi}\,(\mathrm{cl}\,f) = \mathrm{cl}\,(\mathrm{epi}\,f) = \mathrm{cl}\,(\mathrm{epi}\,g) = \mathrm{epi}\,(\mathrm{cl}\,g)$. Then by Proposition 2.1 $\mathrm{cl}\,f = \mathrm{cl}\,g$ and as a consequence, $\mathrm{dom}\,(\mathrm{cl}\,f) = \mathrm{dom}\,(\mathrm{cl}\,g)$, $\mathrm{lev}_{\le\alpha}(\mathrm{cl}\,f) = \mathrm{lev}_{\le\alpha}(\mathrm{cl}\,g)$ and $\mathrm{lev}_{<\alpha}(\mathrm{cl}\,f) = \mathrm{lev}_{<\alpha}(\mathrm{cl}\,g)$. $\square$

Similarly, from Proposition 4.6 (iii) and (v), we can state the following results.

Theorem 4.14. *Let $f, g\colon \mathbb{R}^m \to \widetilde{\mathbb{R}}$ be almost convex with f and g proper, say $\mathrm{ri}\,(\mathrm{epi}\,\mathbf{f}) \subseteq \mathrm{epi}\,f \subseteq \mathrm{cl}\,(\mathrm{epi}\,\mathbf{f})$ and $\mathrm{ri}\,(\mathrm{epi}\,\mathbf{g}) \subseteq \mathrm{epi}\,g \subseteq \mathrm{cl}\,(\mathrm{epi}\,\mathbf{g})$, where $\mathbf{f}, \mathbf{g}\colon \mathbb{R}^m \to \widetilde{\mathbb{R}}$ are convex functions. For any*

$$E, F \in \{\mathrm{dom}\,f,\ \mathrm{dom}\,\mathbf{f},\ \mathrm{dom}\,(\mathrm{cl}\,f),\ \mathrm{dom}\,(\mathrm{conv}\,f),\ \mathrm{dom}\,(\overline{\mathrm{conv}}\,f)\} \quad \textit{and}$$
$$G, H \in \{\mathrm{dom}\,g,\ \mathrm{dom}\,\mathbf{g},\ \mathrm{dom}\,(\mathrm{cl}\,g),\ \mathrm{dom}\,(\mathrm{conv}\,g),\ \mathrm{dom}\,(\overline{\mathrm{conv}}\,g)\},$$

we have $E \approx G \Longleftrightarrow F \approx H$.

Theorem 4.15. *Let $f, g\colon \mathbb{R}^m \to \widetilde{\mathbb{R}}$ be almost convex with f and g proper, say $\mathrm{ri}\,(\mathrm{epi}\,\mathbf{f}) \subseteq \mathrm{epi}\,f \subseteq \mathrm{cl}\,(\mathrm{epi}\,\mathbf{f})$ and $\mathrm{ri}\,(\mathrm{epi}\,\mathbf{g}) \subseteq \mathrm{epi}\,g \subseteq \mathrm{cl}\,(\mathrm{epi}\,\mathbf{g})$, where $\mathbf{f}, \mathbf{g}\colon \mathbb{R}^m \to \widetilde{\mathbb{R}}$ are convex functions. Let $\alpha > \max\{\inf_{x\in\mathbb{R}^m} f(x), \inf_{x\in\mathbb{R}^m} g(x)\}$. For any*

$$\begin{aligned} E, F \in \{&\mathrm{lev}_{\le\alpha}f,\ \mathrm{lev}_{\le\alpha}(\mathrm{cl}\,f),\ \mathrm{lev}_{\le\alpha}(\mathrm{conv}\,f),\ \mathrm{lev}_{\le\alpha}(\overline{\mathrm{conv}}\,f),\ \mathrm{lev}_{\le\alpha}\mathbf{f},\\ &\mathrm{lev}_{<\alpha}f,\ \mathrm{lev}_{<\alpha}(\mathrm{cl}\,f),\ \mathrm{lev}_{<\alpha}(\mathrm{conv}\,f),\ \mathrm{lev}_{<\alpha}(\overline{\mathrm{conv}}\,f),\ \mathrm{lev}_{<\alpha}\mathbf{f}\} \quad \textit{and} \end{aligned}$$

$$\begin{aligned} G, H \in \{&\mathrm{lev}_{\le\alpha}g,\ \mathrm{lev}_{\le\alpha}(\mathrm{cl}\,g),\ \mathrm{lev}_{\le\alpha}(\mathrm{conv}\,g),\ \mathrm{lev}_{\le\alpha}(\overline{\mathrm{conv}}\,g),\ \mathrm{lev}_{\le\alpha}\mathbf{g},\\ &\mathrm{lev}_{<\alpha}g,\ \mathrm{lev}_{<\alpha}(\mathrm{cl}\,g),\ \mathrm{lev}_{<\alpha}(\mathrm{conv}\,g),\ \mathrm{lev}_{<\alpha}(\overline{\mathrm{conv}}\,g),\ \mathrm{lev}_{<\alpha}\mathbf{g}\}, \end{aligned}$$

we have $E \approx G \Longleftrightarrow F \approx H$.

From Theorems 4.13, 4.14 and 4.15 we have:

Corollary 4.16. *Let $f, g\colon \mathbb{R}^m \to \widetilde{\mathbb{R}}$ be almost convex with f and g proper, say $\mathrm{ri}\,(\mathrm{epi}\,\mathbf{f}) \subseteq \mathrm{epi}\,f \subseteq \mathrm{cl}\,(\mathrm{epi}\,\mathbf{f})$ and $\mathrm{ri}\,(\mathrm{epi}\,\mathbf{g}) \subseteq \mathrm{epi}\,g \subseteq \mathrm{cl}\,(\mathrm{epi}\,\mathbf{g})$, where $\mathbf{f}, \mathbf{g}\colon \mathbb{R}^m \to \widetilde{\mathbb{R}}$ are convex functions. If $f \approx g$, then $E \approx G$ in Theorems 4.14 and 4.15 holds.*

The next result characterizes the set of almost convex functions by using near equality.

Theorem 4.17. *Define* $\mathcal{F}_{\mathrm{lc}} := \{f\colon \mathbb{R}^m \to \widetilde{\mathbb{R}} : f \text{ is proper lsc and convex}\}$ *and* $\mathcal{F}_{\mathrm{a}} := \{f\colon \mathbb{R}^m \to \widetilde{\mathbb{R}} : f \text{ is proper almost convex}\}$. *For each* $u \in \mathcal{F}_{\mathrm{lc}}$*, define* $\mathcal{F}^u := \{f\colon \mathbb{R}^m \to \widetilde{\mathbb{R}} : f \approx u\}$. *Then the following statements are true:*

(i) $\mathcal{F}_{\mathrm{a}} = \cup_{u\in\mathcal{F}_{\mathrm{lc}}}\mathcal{F}^u$ *and* $\mathcal{F}^u \cap \mathcal{F}^v = \emptyset$ *for any* $u, v \in \mathcal{F}_{\mathrm{lc}}$ *with* $u \neq v$.

(iii) *For any* $u, v \in \mathcal{F}_{\mathrm{lc}}$*,* $f \in \mathcal{F}^u$ *and* $g \in \mathcal{F}^v$*, we have* $u = v \Longleftrightarrow f \approx g$.

(iii) *For any* $u \in \mathcal{F}_{\mathrm{lc}}$ *and* $f, g \in \mathcal{F}^u$*, we have* $f \approx g$*,* $\mathrm{ri}\,(\mathrm{dom}\, f) = \mathrm{ri}\,(\mathrm{dom}\, g) = \mathrm{ri}\,(\mathrm{dom}\, u)$ *and* $f = g = u$ *on* $\mathrm{ri}\,(\mathrm{dom}\, u)$.

Proof. (i) Let $h \in \mathcal{F}_{\mathrm{a}}$. From Theorem 4.12 one has $h \approx \overline{\mathrm{conv}}\, h$. Since $\overline{\mathrm{conv}}\, h \in \mathcal{F}_{\mathrm{lc}}$, it follows that $h \in \mathcal{F}^{\overline{\mathrm{conv}}\, h}$. This yields $\mathcal{F}_{\mathrm{a}} \subseteq \cup_{u\in\mathcal{F}_{\mathrm{lc}}}\mathcal{F}^u$.
Let $h \in \cup_{u\in\mathcal{F}_{\mathrm{lc}}}\mathcal{F}^u$. Then, there exists $u \in \mathcal{F}_{\mathrm{lc}}$ such that $h \in \mathcal{F}^u$ and so $h \approx u$. Again from Theorem 4.12 we have $h \in \mathcal{F}_{\mathrm{a}}$, that is, $\cup_{u\in\mathcal{F}_{\mathrm{lc}}}\mathcal{F}^u \subseteq \mathcal{F}_{\mathrm{a}}$. This proves $\mathcal{F}_{\mathrm{a}} = \cup_{u\in\mathcal{F}_{\mathrm{lc}}}\mathcal{F}^u$.
Let $u, v \in \mathcal{F}_{\mathrm{lc}}$ with $u \neq v$. Suppose to the contrary that $\mathcal{F}^u \cap \mathcal{F}^v \neq \emptyset$. Then there exists $h \in \mathcal{F}^u \cap \mathcal{F}^v$, i.e., $h \approx u$ and $h \approx v$. Now, Proposition 4.1 (ii) and (iii) imply $u \approx v$ and as a consequence, $u = \mathrm{cl}\, u = \mathrm{cl}\, v = v$ (by Remark 3.11 (a)), a contradiction. This allows $\mathcal{F}^u \cap \mathcal{F}^v = \emptyset$.

(ii) Let $u, v \in \mathcal{F}_{\mathrm{lc}}$, $f \in \mathcal{F}^u$ and $g \in \mathcal{F}^v$. Then $f \approx u$ and $g \approx u$. If $u = v$, then, from Proposition 4.1 (ii) and (iii), one has $f \approx g$.
If $f \approx g$, then again from Proposition 4.1 (ii) and (iii), $u \approx v$. Now, Remark 3.11 (a) implies $u = \mathrm{cl}\, u = \mathrm{cl}\, v = v$.

(iii) Let $u \in \mathcal{F}_{\mathrm{lc}}$ and any $f, g \in \mathcal{F}^u$. Then $f \approx u$ and $g \approx u$ and (ii) yields $f \approx g$. Now, Theorem 4.12 implies f and g are almost convex, and Remark 3.11 (a) leads to $\mathrm{cl}\, f = \mathrm{cl}\, u = u = \mathrm{cl}\, g$. Then, from Proposition 4.6 (iii), we have $\mathrm{dom}\, f \approx \mathrm{dom}\,(\mathrm{cl}\, f)$ and $\mathrm{dom}\, g \approx \mathrm{dom}\,(\mathrm{cl}\, g)$. It follows that $\mathrm{ri}\,(\mathrm{dom}\, f) = \mathrm{ri}\,(\mathrm{dom}\, g) = \mathrm{ri}\,(\mathrm{dom}\, u)$, which yields $f = g = u$ on $\mathrm{ri}\,(\mathrm{dom}\, u)$, along with Proposition 4.4. □

We now present the operations on sum, scalar multiple and pointwise supremum of almost convex functions.

Theorem 4.18. *Let* $\lambda_i \in \mathbb{R}_+$ *and* $f_i\colon \mathbb{R}^m \to \widetilde{\mathbb{R}}$ *be almost convex with* f_i *proper for each* $i \in I$*, where* $I := \{1, ..., n\}$ *and* $n \in \mathbb{N}$. *Then* $\sum_{i\in I} \lambda_i f_i$ *is almost convex.*

Proof. It suffices to prove that $f_1 + f_2$ and $\lambda_1 f_1$ are almost convex. We first prove that $f_1 + f_2$ is almost convex. If $f_1 + f_2 \equiv +\infty$, then this is clear. Let $\mathrm{dom}\,(f_1 + f_2) \neq \emptyset$. The proof is similar to that in [11, Theorem 4.5]. For completeness, we include it here.

Define linear operators $V\colon \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}$ and $W\colon \mathbb{R}^m \times \mathbb{R} \times \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}$ by $V(x, r) := (x, x, r)$ and $W(x, r, y, s) := (x, y, r + s)$, respectively.

We declare $\operatorname{epi}(f_1+f_2) = V^{-1}(W(\operatorname{epi} f_1 \times \operatorname{epi} f_2))$. In fact,

$$\begin{aligned}(x,r) \in \operatorname{epi}(f_1+f_2) &\Longleftrightarrow f_1(x)+f_2(x) \leq r \Longleftrightarrow f_1(x) \leq r - f_2(x)\\ &\Longleftrightarrow ((x,f_1(x)),(x,r-f_1(x))) \in \operatorname{epi} f_1 \times \operatorname{epi} f_2\\ &\Longleftrightarrow (x,x,r) \in W(\operatorname{epi} f_1 \times \operatorname{epi} f_2), \Longleftrightarrow (x,r) \in V^{-1}(W(\operatorname{epi} f_1 \times \operatorname{epi} f_2)).\end{aligned}$$

Since f_1 and f_2 are almost convex, i.e., $\operatorname{epi} f_1$ and $\operatorname{epi} f_2$ are nearly convex, from Lemma 3.8 (iii) and Lemma 3.9 (i) we have $W(\operatorname{epi} f_1 \times \operatorname{epi} f_2)$ is almost convex. From Lemma 3.9 (iii), we only need to show $V^{-1}(\operatorname{ri}(W(\operatorname{epi} f_1 \times \operatorname{epi} f_2))) \neq \emptyset$. Again since f_1 and f_2 are almost convex, from Lemma 2.2 (v) we have $\operatorname{ri}(\operatorname{epi} f_1 \times \operatorname{epi} f_2) = \operatorname{ri}(\operatorname{epi} f_1) \times \operatorname{ri}(\operatorname{epi} f_2) \neq \emptyset$. From Lemma 3.9 (ii) one has $\operatorname{ri}(W(\operatorname{epi} f_1 \times \operatorname{epi} f_2)) = W(\operatorname{ri}(\operatorname{epi} f_1 \times \operatorname{epi} f_2)) \neq \emptyset$. Letting $(x_0,x_0,r_0) \in \operatorname{ri}(W(\operatorname{epi} f_1 \times \operatorname{epi} f_2))$ yields $(x_0,r_0) \in V^{-1}(\operatorname{ri}(W(\operatorname{epi} f_1 \times \operatorname{epi} f_2)))$.

We now prove $\lambda_1 f_1$ is almost convex. If $\lambda_1 = 0$, then $\lambda_1 f_1 = 0$ is convex and so almost convex. Then

$$\begin{aligned}(x,r) \in \operatorname{epi}(\lambda_1 f_1) &\Longleftrightarrow \lambda_1 f_1(x) \leq r \Longleftrightarrow f_1(x) \leq \frac{r}{\lambda_1}\\ &\Longleftrightarrow (x, \frac{r}{\lambda_1}) \in \operatorname{epi} f_1 \Longleftrightarrow (x,r) \in U(\operatorname{epi} f_1),\end{aligned}$$

where $U\colon \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}^m \times \mathbb{R}$ is the linear operator given by $U(x,r) := (x, \lambda_1 r)$. This yields $\operatorname{epi}(\lambda_1 f_1) = U(\operatorname{epi} f_1)$. Since f_1 is almost convex, that is, $\operatorname{epi} f_1$ is nearly convex, from Lemma 3.9 (i) we have that $U(\operatorname{epi} f_1)$ is nearly convex and as a consequence, $\lambda_1 f_1$ is almost convex. □

Theorem 4.19. *Let $\lambda_i \in \mathbb{R}_+$ and $f_i, u_i\colon \mathbb{R}^m \to \widetilde{\mathbb{R}}$ be proper for each $i \in I$, where $I := \{1,\dots,n\}$ and $n \in \mathbb{N}$. If f_i is almost convex and $u_i \approx f_i$ for each $i \in I$, then $\sum_{i\in I} \lambda_i u_i$ is almost convex.*

Proof. From Theorem 4.12, u_i is almost convex for each $i \in I$ and the conclusion follows immediately from Theorem 4.18. □

Theorem 4.20. *Let $f_i\colon \mathbb{R}^m \to \widetilde{\mathbb{R}}$ be almost convex for each $i \in I$, say* $\operatorname{ri}(\operatorname{epi} \mathbf{f}_i) \subseteq \operatorname{epi} f_i \subseteq \operatorname{cl}(\operatorname{epi} \mathbf{f}_i)$ $(i \in I)$*, where $\mathbf{f}_i\colon \mathbb{R}^m \to \widetilde{\mathbb{R}} (i \in I)$ are convex functions and I is an index set. If $\cap_{i\in I} \operatorname{ri}(\operatorname{epi} f_i) \neq \emptyset$, then the function F is almost convex and $F \approx \mathbf{F}$, where the functions F and $\mathbf{F}$ are defined by $F(x) := \sup_{i\in I} f_i(x)$ and $\mathbf{F}(x) := \sup_{i\in I} \mathbf{f}_i(x)$, respectively.*

Proof. Since f_i is almost convex for each $i \in I$, $\operatorname{epi} f_i$ is nearly convex. Then by Lemma 3.8 (i) it follows that $\operatorname{epi} F = \cap_{i\in I}(\operatorname{epi} f_i)$ is nearly convex and $\operatorname{epi} F := \cap_{i\in I}(\operatorname{epi} f_i) \approx \cap_{i\in I}(\operatorname{epi} \mathbf{f}_i) =: \operatorname{epi} \mathbf{F}$, which yields that $F \approx \mathbf{F}$. □

Finally, we investigate the epi-sum and the epi-multiple operations of almost convex functions.

Theorem 4.21. *Let $\alpha, \beta > 0$ and $f, g\colon \mathbb{R}^m \to \widetilde{\mathbb{R}}$ be almost convex with f and g proper, say $\operatorname{ri}(\operatorname{epi} \mathbf{f}) \subseteq \operatorname{epi} f \subseteq \operatorname{cl}(\operatorname{epi} \mathbf{f})$ and $\operatorname{ri}(\operatorname{epi} \mathbf{g}) \subseteq \operatorname{epi} g \subseteq \operatorname{cl}(\operatorname{epi} \mathbf{g})$, where $\mathbf{f}, \mathbf{g}\colon \mathbb{R}^m \to \widetilde{\mathbb{R}}$ are convex functions. Then the following statements are true:*

(i) $\alpha \star (\beta \star f)$ *is almost convex and* $\alpha \star (\beta \star f) \approx \alpha \star (\beta \star \mathbf{f})$.

(ii) *If the infimum defining* $(f \,\Box\, g)(x)$ *is attained when finite, then* $f \,\Box\, g$ *is almost convex and* $f \,\Box\, g \approx \mathbf{f} \,\Box\, \mathbf{g}$.

(iii) *If the infimum defining* $((\alpha \star f) \,\Box\, (\alpha \star g))(x)$ *is attained when finite, then* $\alpha \star (f \,\Box\, g)$ *is almost convex and* $\alpha \star (f \,\Box\, g) \approx \alpha \star (\mathbf{f} \,\Box\, \mathbf{g})$.

(iv) *If the infimum defining* $((\alpha \star f) \,\Box\, (\beta \star g))(x)$ *is attained when finite, then* $(\alpha \star f) \,\Box\, (\beta \star g)$ *is almost convex and* $(\alpha \star f) \,\Box\, (\beta \star g)) \approx (\alpha \star \mathbf{f}) \,\Box\, (\beta \star \mathbf{g})$.

(v) *If the infimum defining* $(((\alpha\beta) \star f) \,\Box\, ((\alpha\beta) \star g))(x)$ *is attained when finite, then* $\alpha \star (\beta \star (f \,\Box\, g))$ *is almost convex and* $\alpha \star (\beta \star (f \,\Box\, g)) \approx \alpha \star (\beta \star (\mathbf{f} \,\Box\, \mathbf{g}))$.

(vi) $(\alpha + \beta) \star f$ *is almost convex and* $(\alpha + \beta) \star f \approx (\alpha \star \mathbf{f}) \,\Box\, (\beta \star \mathbf{f})$.

Proof. From $\mathrm{ri}\,(\mathrm{epi}\,\mathbf{f}) \subseteq \mathrm{epi}\, f \subseteq \mathrm{cl}\,(\mathrm{epi}\,\mathbf{f})$ and $\mathrm{ri}\,(\mathrm{epi}\,\mathbf{g}) \subseteq \mathrm{epi}\, g \subseteq \mathrm{cl}\,(\mathrm{epi}\,\mathbf{g})$ we get

$$\begin{aligned} \mathrm{ri}\,(\mathrm{epi}\,\mathbf{f} + \mathrm{epi}\,\mathbf{g}) &= \mathrm{ri}\,(\mathrm{epi}\,\mathbf{f}) + \mathrm{ri}\,(\mathrm{epi}\,\mathbf{g}) \subseteq \mathrm{epi}\, f + \mathrm{epi}\, g \\ &\subseteq \mathrm{cl}\,(\mathrm{epi}\,\mathbf{f}) + \mathrm{cl}\,(\mathrm{epi}\,\mathbf{g}) \subseteq \mathrm{cl}\,(\mathrm{epi}\,\mathbf{f} + \mathrm{epi}\,\mathbf{g}), \end{aligned}$$

where the first and the last inclusion follow from Lemma 2.2 (iii).

(i) From Lemma 2.3 (v) we have $\alpha \star (\beta \star f) = (\alpha\beta) \star f$ and so from Lemma 2.2 (vii), Lemma 2.3 (iii) and (iv),

$$\begin{aligned} \mathrm{ri}\,(\mathrm{epi}\,(\alpha \star (\beta \star \mathbf{f}))) &= \mathrm{ri}\,(\mathrm{epi}\,((\alpha\beta) \star \mathbf{f})) = \mathrm{ri}\,(\alpha\beta\, \mathrm{epi}\,\mathbf{f}) = \alpha\beta\, \mathrm{ri}\,(\mathrm{epi}\,\mathbf{f}) \\ &\subseteq \mathrm{epi}\,((\alpha\beta) \star f) = \alpha\beta\, \mathrm{epi}\, f \subseteq \alpha\beta\, (\mathrm{cl}\,(\mathrm{epi}\,\mathbf{f})) = \mathrm{cl}\,((\alpha\beta)\, \mathrm{epi}\,\mathbf{f}) \\ &= \mathrm{cl}\,(\mathrm{epi}\,((\alpha\beta) \star \mathbf{f})) = \mathrm{cl}\,(\mathrm{epi}\,(\alpha \star (\beta \star \mathbf{f}))) \end{aligned}$$

and $\alpha \star (\beta \star \mathbf{f})$ is convex, which yields $\mathrm{epi}\,(\alpha \star (\beta \star \mathbf{f}))$ is convex and so $\alpha \star (\beta \star f)$ is almost convex and $\alpha \star (\beta \star f) \approx \alpha \star (\beta \star \mathbf{f})$ in view of Proposition 4.6 (i).

(ii) Since $\mathbf{f}$ and $\mathbf{g}$ are convex, it follows from Lemma 2.3 (iii) and (iv) and $\mathrm{ri}\,(\mathrm{epi}\,\mathbf{f} + \mathrm{epi}\,\mathbf{g}) \subseteq \mathrm{epi}\, f + \mathrm{epi}\, g \subseteq \mathrm{cl}\,(\mathrm{epi}\,\mathbf{f} + \mathrm{epi}\,\mathbf{g})$ that

$$\mathrm{ri}\,(\mathrm{epi}\,(\mathbf{f} \,\Box\, \mathbf{g})) \subseteq \mathrm{epi}\,(f \,\Box\, g) \subseteq \mathrm{cl}\,(\mathrm{epi}\,(\mathbf{f} \,\Box\, \mathbf{g}))$$

and $\mathrm{epi}\,(\mathbf{f} \,\Box\, \mathbf{g})$ is convex. This yields $f \,\Box\, g$ is almost convex and $f \,\Box\, g \approx \mathbf{f} \,\Box\, \mathbf{g}$ (see, Proposition 4.6 (i)).

(iii) Observe from Lemma 2.3 (v), $\alpha \star (f \,\Box\, g) = (\alpha \star f) \,\Box\, (\alpha \star g)$ and so the conclusion follows immediately from (i) and (ii).

(iv) The conclusion follows immediately from (i) and (ii).

(v) Observe from Lemma 2.3 (v), $\alpha \star (\beta \star (f \,\Box\, g)) = \alpha\beta \star (f \,\Box\, g)$ and so the conclusion follows immediately from (iii).

(vi) Since $\mathrm{ri}\,(\mathrm{epi}\,\mathbf{f}) \subseteq \mathrm{epi}\, f \subseteq \mathrm{cl}\,(\mathrm{epi}\,\mathbf{f})$, from Lemma 2.2 (vii) and Lemma 2.3 (iii) and (iv), one has

$$\begin{aligned} \mathrm{ri}\,(\mathrm{epi}\,((\alpha \star \mathbf{f}) \,\Box\, (\beta \star \mathbf{f}))) &= \mathrm{ri}\,(\mathrm{epi}\,((\alpha + \beta) \star \mathbf{f})) = \mathrm{ri}\,((\alpha + \beta)\, \mathrm{epi}\,\mathbf{f}) \\ &= (\alpha + \beta)\, \mathrm{ri}\,(\mathrm{epi}\,\mathbf{f}) \subseteq (\alpha + \beta)\, \mathrm{epi}\, f = \mathrm{epi}\,((\alpha + \beta) \star f) \subseteq (\alpha + \beta)\, \mathrm{cl}\,(\mathrm{epi}\,\mathbf{f}) \\ &= \mathrm{cl}\,((\alpha + \beta)\, \mathrm{epi}\,\mathbf{f}) = \mathrm{cl}\,(\mathrm{epi}\,((\alpha + \beta) \star \mathbf{f})) = \mathrm{cl}\,(\mathrm{epi}\,((\alpha \star \mathbf{f}) \,\Box\, (\beta \star \mathbf{f}))). \end{aligned}$$

Since $(\alpha \star \mathbf{f}) \,\square\, (\beta \star \mathbf{f})$ is convex (see, Lemma 2.3 (iii)), it follows that $(\alpha + \beta) \star f$ is almost convex and $(\alpha + \beta) \star f \approx (\alpha \star \mathbf{f}) \,\square\, (\beta \star \mathbf{f})$. □

It is well known that for a convex set $K \subseteq \mathbb{R}^m$, the distance function $d(\cdot, K)$ is convex (see, for example, [37, Example 2.25]) and that for a convex function $f\colon \mathbb{R}^m \to \widetilde{\mathbb{R}}$, the Moreau envelope function $e_\lambda f$ is convex in view of Lemma 2.3 (iii), and moreover if f is lsc and proper, then the Moreau envelope function $e_\lambda f$ is convex and continuously differentiable (see, Lemma 2.3 (vi)). We next extend the previous results to the case where the involved functions are almost convex. Note that $d(\cdot, K) = \delta_K \,\square\, \|\cdot\|$ and the Moreau envelope function $e_\lambda f = f \,\square\, \frac{1}{2\lambda}\|\cdot\|^2$, where $K \subseteq \mathbb{R}^m$, $\lambda > 0$.

We first give the following lemma.

Lemma 4.22. *Let $P \subseteq \mathbb{R}^m$ be a nonempty set. Then $d(x, P) = d(x, \operatorname{cl} P)$ for each $x \in \mathbb{R}^m$. Furthermore, if P is bounded, then $\Delta_P = \Delta_{\operatorname{cl} P}$, where Δ_P ad $\Delta_{\operatorname{cl} P}$ denote the diameters of P and $\operatorname{cl} P$, respectively.*

Proof. Let $x \in \mathbb{R}^m$. Since $P \subseteq \operatorname{cl} P$, $d(x, P) \geq d(x, \operatorname{cl} P)$. We prove $d(x, P) \leq d(x, \operatorname{cl} P)$. For any $\epsilon > 0$, there exists $\bar{x} \in \operatorname{cl} P$ such that $\|x - \bar{x}\| < d(x, \operatorname{cl} P) + \epsilon$. Since $\bar{x} \in \operatorname{cl} P$, then there exists $\{x^k\} \subseteq P$ such that $\lim_{k\to+\infty} x^k = \bar{x}$ and so $\lim_{k\to+\infty} \|x - x^k\| = \|x - \bar{x}\|$. It follows that for any k sufficiently large,

$$d(x, P) \leq \|x - x^k\| < d(x, \operatorname{cl} P) + \epsilon,$$

which yields $d(x, P) \leq d(x, \operatorname{cl} P)$ since ϵ is arbitrary.

The inequality $\Delta_P \leq \Delta_{\operatorname{cl} P}$ is clear. Furthermore, if P is bounded, then $\operatorname{cl} P$ is bounded and closed and so there exist $\bar{x}, \bar{y} \in \operatorname{cl} P$ such that $\Delta_{\operatorname{cl} P} = \|\bar{x} - \bar{y}\|$. Therefore, there exist $\{x^k\}, \{y^k\} \subseteq P$ such that $\lim_{k\to+\infty} x^k = \bar{x}$ and $\lim_{k\to+\infty} y^k = \bar{y}$ and so $\|x^k - y^k\| \to \|\bar{x} - \bar{y}\|$, which yields

$$\Delta_P \geq \lim_{k\to+\infty} \|x^k - y^k\| = \|\bar{x} - \bar{y}\| = \Delta_{\operatorname{cl} P}. \qquad \square$$

Proposition 4.23. *Let $K, D \subseteq \mathbb{R}^m$ be nonempty. Then the following statements are true:*

(i) *If K is nearly convex, then the distance function $d(\cdot, K)$ is convex.*

(ii) *If $D \approx K$, then $d(\cdot, D) = d(\cdot, K)$. Furthermore, if K is nearly convex, then $d(\cdot, D)$ is convex.*

Proof. (i) Since K is nearly convex, there is a convex subset C of $\mathbb{R}^m$ such that $C \subseteq K \subseteq \operatorname{cl} C$. Then from Lemma 4.22 it follows that $d(x, \operatorname{cl} C) \leq d(x, K) \leq d(x, C) = d(x, \operatorname{cl} C)$ for each $x \in \mathbb{R}^m$, which yields $d(x, K) = d(x, \operatorname{cl} C)$. As a consequence, $d(\cdot, K)$ is convex.

(ii) Since $D \approx K$, one has $\operatorname{cl} D = \operatorname{cl} K$ and from Lemma 4.22 it follows that $d(x, D) = d(x, \operatorname{cl} D) = d(x, \operatorname{cl} K) = d(x, K)$ for each $x \in \mathbb{R}^m$. Since K is nearly convex, the conclusion follows directly from (i). □

Under suitable additional conditions we can prove $f \,\square\, g$ is convex. To this aim, we first provide some assumptions which ensure that the epi-sum of two extended real valued functions is equivalent to that of their closure.

Lemma 4.24. *Let* $f, g\colon \mathbb{R}^m \to \widetilde{\mathbb{R}}$. *If* $\mathrm{cl}\,(f(x-\cdot)+g) = \mathrm{cl}\, f(x-\cdot) + \mathrm{cl}\, g$ *for each* $x \in \mathbb{R}^m$, *then* $f \,\square\, g = (\mathrm{cl}\, f) \,\square\, (\mathrm{cl}\, g)$.

Proof. For each $x \in \mathbb{R}^m$, one has

$$\begin{aligned}(f \,\square\, g)(x) &:= \inf_{z\in\mathbb{R}^m}[f(x-z)+g(z)] = \inf_{z\in\mathbb{R}^m}(f(x-\cdot)+g)(z)\\ &= \inf_{z\in\mathbb{R}^m}[\mathrm{cl}\,(f(x-\cdot)+g)](z) = \inf_{z\in\mathbb{R}^m}[\mathrm{cl}\, f(x-\cdot)+\mathrm{cl}\, g](z)\\ &= \inf_{z\in\mathbb{R}^m}[(\mathrm{cl}\, f)(x-z)+(\mathrm{cl}\, g)(z)] =: ((\mathrm{cl}\, f) \,\square\, (\mathrm{cl}\, g))(x),\end{aligned}$$

where the third equality follows from Lemma 4.5, the fourth equality follows from the assumption $\mathrm{cl}\,(f(x-\cdot)+g) = \mathrm{cl}\, f(x-\cdot)+\mathrm{cl}\, g$ and the fifth equality follows from the fact that

$$\begin{aligned}[\mathrm{cl}\, f(x-\cdot)](z) &:= \liminf_{y\to z} f(x-y) = \sup_{\alpha>0}\inf_{y\in B(z,\alpha)} f(x-y)\\ &= \sup_{\alpha>0}\inf_{u\in B(x-z,\alpha)} f(u) = \liminf_{u\to x-z} f(u) =: (\mathrm{cl}\, f)(x-z).\end{aligned}$$

This yields $f \,\square\, g = (\mathrm{cl}\, f) \,\square\, (\mathrm{cl}\, g)$. $\square$

The following lemma (with $h := f(x-\cdot)$ in Lemma 4.25 (i) and (ii) for each $x \in \mathbb{R}^m$) provides sufficient conditions on f and g that ensure that

$$\mathrm{cl}\,(f(x-\cdot)+g) = \mathrm{cl}\, f(x-\cdot)+\mathrm{cl}\, g \quad \text{for each} \quad x \in \mathbb{R}^m.$$

Lemma 4.25. *Let* $h, g\colon \mathbb{R}^m \to \widetilde{\mathbb{R}}$. *Then the following statements hold:*

(i) *If* g *is convex with* $\mathrm{dom}\, g = \mathbb{R}^m$ *or* g *is continuous, then* $\mathrm{cl}\,(h+g) = \mathrm{cl}\, h + g$.

(ii) *If* $\mathrm{epi}\,\mathbf{h} \subseteq \mathrm{epi}\, h \subseteq \mathrm{cl}\,(\mathrm{epi}\,\mathbf{h})$ *and* $\mathrm{epi}\,\mathbf{g} \subseteq \mathrm{epi}\, g \subseteq \mathrm{cl}\,(\mathrm{epi}\,\mathbf{g})$, *where* $\mathbf{h}, \mathbf{g}\colon \mathbb{R}^m \to \widetilde{\mathbb{R}}$ *are convex functions, and* $\mathrm{ri}(\mathrm{dom}\,\mathbf{h}) \cap \mathrm{ri}(\mathrm{dom}\,\mathbf{g}) \neq \emptyset$, *then* $\mathrm{cl}(h+g) = \mathrm{cl}\, h + \mathrm{cl}\, g$.

Proof. (i) From Lemma 2.3 (i), it is clear that $\mathrm{cl}\, h + g = \mathrm{cl}\, h + \mathrm{cl}\, g = \mathrm{cl}\,(\mathrm{cl}\, h + \mathrm{cl}\, g) \leq \mathrm{cl}\,(h+g)$. If g is convex with $\mathrm{dom}\, g = \mathbb{R}^m$, then g is continuous (see, for example, [35, Corollary 10.1.1]). Then for each $x \in \mathbb{R}^m$,

$$\begin{aligned}\mathrm{cl}\,(h+g)(x) &:= \liminf_{y\to x}(h+g)(y) = \liminf_{y\to x}(h(y)+g(y))\\ &\leq \liminf_{y\to x} h(y) + \limsup_{y\to x} g(y) = \liminf_{y\to x} h(y) + \liminf_{y\to x} g(y)\\ &=: (\mathrm{cl}\, h)(x) + (\mathrm{cl}\, g)(x) = (\mathrm{cl}\, h + \mathrm{cl}\, g)(x) = (\mathrm{cl}\, h + g)(x),\end{aligned}$$

which implies $\mathrm{cl}\,(h+g) \leq \mathrm{cl}\, h + g$ and it follows thatcl $(h+g) = \mathrm{cl}\, h + g$.

(ii) Since $\mathrm{epi}\,\mathbf{h} \subseteq \mathrm{epi}\, h \subseteq \mathrm{cl}\,(\mathrm{epi}\,\mathbf{h})$ and $\mathrm{epi}\,\mathbf{g} \subseteq \mathrm{epi}\, g \subseteq \mathrm{cl}\,(\mathrm{epi}\,\mathbf{g})$, $\mathrm{cl}\, h = \mathrm{cl}\,\mathbf{h}$ and $\mathrm{cl}\, g = \mathrm{cl}\,\mathbf{g}$ and from Proposition 2.1 one has $\mathrm{cl}\, h \leq h \leq \mathbf{h}$ and $\mathrm{cl}\, g \leq g \leq \mathbf{g}$. From

Lemma 2.3 (ii) we have $\mathrm{cl}\,(\mathbf{h}+\mathbf{g}) = \mathrm{cl}\,\mathbf{h} + \mathrm{cl}\,\mathbf{g}$. As a consequence, from Lemma 2.3 (i) it follows that

$$\mathrm{cl}\,h + \mathrm{cl}\,g = \mathrm{cl}\,(\mathrm{cl}\,h + \mathrm{cl}\,g) \le \mathrm{cl}\,(h+g) \le \mathrm{cl}\,(\mathbf{h}+\mathbf{g}) = \mathrm{cl}\,\mathbf{h} + \mathrm{cl}\,\mathbf{g} = \mathrm{cl}\,h + \mathrm{cl}\,g,$$

which yields $\mathrm{cl}\,(h+g) = \mathrm{cl}\,h + \mathrm{cl}\,g$. □

The next result shows that $f \,\square\, g$ is convex under suitable additional conditions.

Theorem 4.26. *Let $f, g\colon \mathbb{R}^m \to \widetilde{\mathbb{R}}$ be almost convex with f and g proper. If $\mathrm{cl}\,(f(x-\cdot)+g) = \mathrm{cl}\,f(x-\cdot) + \mathrm{cl}\,g$ for each $x \in \mathbb{R}^m$, then $f \,\square\, g$ is convex.*

Proof. From Lemma 4.24, we have $f \,\square\, g = (\mathrm{cl}\,f) \,\square\, (\mathrm{cl}\,g)$. Since f and g are almost convex, from Proposition 4.3 (ii) one has $\mathrm{cl}\,f$ and $\mathrm{cl}\,g$ are convex and so is $(\mathrm{cl}\,f)\square(\mathrm{cl}\,g)$ in view of Lemma 2.2 (iii). This completes the proof. □

The following theorem proves that the Moreau envelope function of a proper almost convex function is convex and continuously differentiable and the gradient coincides with the one related to its closure. This plays an important role in the definition of the proximal point algorithm (see, [36]).

Theorem 4.27. *Let $f\colon \mathbb{R}^m \to \widetilde{\mathbb{R}}$ be almost convex with f proper. Then the Moreau envelope function $e_\lambda f$ is convex and continuously differentiable with the gradient $\nabla e_\lambda f(x) = \frac{1}{\lambda}(x - P_\lambda(\mathrm{cl}\,f)(x))$.*

Proof. Since f is almost convex, from Proposition 4.3 (ii) one has $\mathrm{cl}\,f$ is convex. From Lemmas 4.24 and 4.25 (i), one has $e_\lambda f = f \,\square\, \frac{1}{2\lambda}\|\cdot\|^2 = (\mathrm{cl}\,f) \,\square\, (\frac{1}{2\lambda}\|\cdot\|^2)$. Now the conclusion follows immediately from Lemma 2.3 (vi). □

We give the following example to illustrate Theorem 4.27.

Example 4.28. Consider $\lambda := \frac{1}{2}$ and $f\colon \mathbb{R}^2 \to \widetilde{\mathbb{R}}$ and $g\colon \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x) := \begin{cases} 1-x_2, & \text{if } x_1 = 0,\ 0 \le x_2 \le 1, \\ 0, & \text{if } x_1 = 0,\ -1 \le x_2 < 0, \\ 0, & \text{if } 0 < x_1 \le 1,\ -1 \le x_2 \le 1, \\ +\infty, & \text{else }, \end{cases} \quad \text{and} \quad g(x) := \frac{1}{2\lambda}\|x\|^2 = x_1^2 + x_2^2.$$

It is easy to see that f is almost convex and that

$$e_{\frac{1}{2}} f(x) = (f \,\square\, g)(x) = \begin{cases} 0, & \text{if } 0 \le x_1 \le 1,\ -1 \le x_2 \le 1, \\ x_1^2, & \text{if } x_1 < 0,\ -1 \le x_2 \le 1, \\ x_1^2 + (x_2+1)^2, & \text{if } x_1 < 0,\ x_2 < -1, \\ x_1^2 + (x_2-1)^2, & \text{if } x_1 < 0,\ x_2 > 1, \\ (x_2+1)^2, & \text{if } 0 \le x_1 \le 1,\ x_2 < -1, \\ (x_1-1)^2 + (x_2+1)^2, & \text{if } x_1 > 1,\ x_2 < -1, \\ (x_1-1)^2, & \text{if } x_1 > 1,\ -1 \le x_2 \le 1, \\ (x_1-1)^2 + (x_2-1)^2, & \text{if } x_1 > 1,\ x_2 > 1, \\ (x_2-1)^2, & \text{if } 0 \le x_1 \le 1,\ x_2 > 1, \end{cases}$$

and hence $e_{\frac{1}{2}}f(x) = (f \,\square\, g)(x) = d^2(x,K)$, where $K := \{(x_1,x_2)^\top : 0 \le x_1 \le 1, -1 \le x_2 \le 1\}$ is closed, convex and bounded (i.e., compact). Moreover, we have $(\mathrm{cl}\, f)(x) = \delta_K(x)$, $(\mathrm{cl}\, f)\square g(x) = d^2(x,K)$ and $P_{\frac{1}{2}}(\mathrm{cl}\, f)(x) = \mathbf{P}_K(x)$, where $\mathbf{P}_K\colon \mathbb{R}^2 \to K$ denotes the projection operator on K. Clearly, $e_{\frac{1}{2}}f$ is convex and continuously differentiable with the gradient

$$\nabla e_{\frac{1}{2}}f(x) = 2(x - \mathbf{P}_K(x)) = 2(x - P_{\frac{1}{2}}(\mathrm{cl}\, f)(x)). \qquad \square$$

We now provide the formulation of the subdifferential of the sum and the scalar multiple of almost convex functions.

Theorem 4.29. *Let $\lambda_i > 0$ and $f_i\colon \mathbb{R}^m \to \widetilde{\mathbb{R}}$ be almost convex with f_i proper for each $i \in I$, where $I := \{1,\dots,n\}$ and $n \in \mathbb{N}$. If $\cap_{i\in I}\mathrm{ri}\,(\mathrm{dom}\, f_i) \neq \emptyset$ and $\mathrm{cl}\,(\sum_{i\in I}\lambda_i f_i) = \sum_{i\in I}\mathrm{cl}\,(\lambda_i f_i)$, then $\partial(\sum_{i\in I}\lambda_i f_i)(x) = \sum_{i\in I}\lambda_i\partial f_i(x)$ for each $x \in \mathrm{ri}\,(\mathrm{dom}\,(\sum_{i\in I}\lambda_i f_i)) = \cap_{i\in I}\mathrm{ri}\,(\mathrm{dom}\, f_i)$.*

Proof. From Theorem 4.18, $\sum_{i\in I}\lambda_i f_i$ is almost convex. Since for each $\lambda_i > 0$, $\mathrm{dom}\,(\sum_{i\in I}\lambda_i f_i) = \cap_{i\in I}\mathrm{dom}\,(\lambda_i f_i) = \cap_{i\in I}\mathrm{dom}\, f_i$. Notice that for each $i \in I$, $\mathrm{cl}\, f_i$ is convex (see, Proposition 4.6 (ii)), $\mathrm{cl}\,(\lambda_i f_i) = \lambda_i \mathrm{cl}\, f_i$ (see, Lemma 2.3 (i)), $\mathrm{ri}\,(\mathrm{dom}\, f_i) = \mathrm{ri}\,(\mathrm{dom}\,(\mathrm{cl}\, f_i))$, $\mathrm{dom}\,(\sum_{i\in I}\lambda_i f_i)$ and $\mathrm{dom}\, f_i$ are nearly convex (see, Proposition 4.6 (iii)). Since $\cap_{i\in I}\mathrm{ri}\,(\mathrm{dom}\, f_i) \neq \emptyset$, we directly obtain the equation $\mathrm{ri}\,(\mathrm{dom}\,(\sum_{i\in I}\lambda_i f_i)) = \cap_{i\in I}\mathrm{ri}\,(\mathrm{dom}\, f_i)$ in view of Lemma 3.8 (ii). From Proposition 4.8 one has $\partial(\sum_{i\in I}\lambda_i f_i)(x) = \partial(\mathrm{cl}\,(\sum_{i\in I}\lambda_i f_i))(x)$ for each $x \in \mathrm{ri}\,(\mathrm{dom}\,(\sum_{i\in I}\lambda_i f_i))$. Since $\mathrm{cl}\,(\sum_{i\in I}\lambda_i f_i) = \sum_{i\in I}\mathrm{cl}\,(\lambda_i f_i)$ and

$$\cap_{i\in I}\mathrm{ri}\,(\mathrm{dom}\,(\lambda_i \mathrm{cl}\, f_i)) = \cap_{i\in I}\mathrm{ri}\,(\mathrm{dom}\,(\mathrm{cl}\, f_i)) = \cap_{i\in I}\mathrm{ri}\,(\mathrm{dom}\, f_i) \neq \emptyset,$$

it follows that for each $x \in \mathrm{ri}\,(\mathrm{dom}\,(\sum_{i\in I}\lambda_i f_i)) = \cap_{i\in I}\mathrm{ri}\,(\mathrm{dom}\, f_i)$,

$$\begin{aligned}\partial(\sum_{i\in I}\lambda_i f_i)(x) &= \partial(\sum_{i\in I}\mathrm{cl}\,(\lambda_i f_i))(x) = \partial(\sum_{i\in I}\lambda_i \mathrm{cl}\, f_i)(x)\\ &= \sum_{i\in I}\lambda_i\partial(\mathrm{cl}\, f_i)(x) = \sum_{i\in I}\lambda_i\partial f_i(x),\end{aligned}$$

where the second to fourth equalities follow from the facts that for each $i \in I$, $\mathrm{cl}\,(\lambda_i f_i) = \lambda_i\mathrm{cl}\, f_i$, Lemma 2.3 (ii) and Proposition 4.8, respectively. $\square$

We note that sufficient conditions ensuring that $\mathrm{cl}\,(\sum_{i\in I}\lambda_i f_i) = \sum_{i\in I}\mathrm{cl}\,(\lambda_i f_i)$ are stated in Lemma 4.25.

5. Proximal average of almost convex functions

In this section, we shall investigate the proximal average of almost convex functions. The formula for the proximal average was first given by Bauschke, Matoušková and Reich [5] to provide an explicit constructive proof of Moreau's observation [29] that the set of all so-called proximal mappings is convex. For works related to proximal average, please see [3, 4].

Denote by $\mathcal{F}$ the set of all proper functions $f\colon \mathbb{R}^m \to \widetilde{\mathbb{R}}$, i.e.,

$$\mathcal{F} := \{f\colon \mathbb{R}^m \to \widetilde{\mathbb{R}} : f \text{ is proper}\},$$

and let $\mathcal{F}_{\mathrm{lc}}$ be as in Theorem 4.17, i.e.,

$$\mathcal{F}_{\mathrm{lc}} := \{f\colon \mathbb{R}^m \to \widetilde{\mathbb{R}} : f \text{ is proper lsc and convex}\}.$$

Definition 5.1. The *proximal average operator* is

$$\mathcal{P}\colon \mathcal{F} \times [0,1] \times \mathcal{F} \to \mathcal{F}$$
$$(f,t,g) \mapsto ((1-t)(f+\frac{1}{2}\|\cdot\|^2)^* + t(g+\frac{1}{2}\|\cdot\|^2)^*)^* - \frac{1}{2}\|\cdot\|^2.$$

If $\mathcal{F} := \mathcal{F}_{\mathrm{lc}}$, then Definition 5.1 collapses to that given in [3, 4, 5].

The following lemmas will be useful in the sequel.

Lemma 5.2. [4, Propositions 4.2, 4.8, 4.14, Theorems 4.3, 4.11, Remark 4.4] *Let $u, v \in \mathcal{F}_{\mathrm{lc}}$ and $t \in [0,1]$. Then the following statements are true:*

(i) $\mathcal{P}(u,0,v) = u, \mathcal{P}(u,1,v) = v$ *and* $\mathcal{P}(u,t,v) = \mathcal{P}(v,1-t,u)$.

(ii) $\mathcal{P}(u,t,v) \in \mathcal{F}_{\mathrm{lc}}$ *and* $(w_t)^* := (\mathcal{P}(u,t,v))^* = \mathcal{P}(u^*,t,v^*) =: (w^*)_t$.

(iii) $\mathcal{P}(u,\frac{1}{2},u^*) = \frac{1}{2}\|\cdot\|^2$.

(iv) *For* $w_t := \mathcal{P}(u,t,v)$ *and* $t \in (0,1)$ *we have* $\mathrm{cl}\,(\mathrm{dom}\, w_t) = \mathrm{cl}\,((1-t)\mathrm{dom}\, u + t\mathrm{dom}\, v)$ *and* $\mathrm{int}\,(\mathrm{dom}\, w_t) = \mathrm{int}\,((1-t)\mathrm{dom}\, u + t\mathrm{dom}\, v)$.

(v) *If* $0 \le u$ *and* $0 \le v$, *then* $0 \le w_t \le (1-t)u + tv$, *where* $w_t := \mathcal{P}(u,t,v)$ *and* $t \in (0,1)$.

The following theorem shows that proximal average of almost convex functions enjoys many nice properties of that of lsc and convex functions obtained by Bauschke, Lucet and Trienis [4].

Theorem 5.3. *Let $t \in [0,1]$ and $f, g \in \mathcal{F}$ such that f and g are almost convex, say* $\mathrm{ri}\,(\mathrm{epi}\,\mathbf{f}) \subseteq \mathrm{epi}\, f \subseteq \mathrm{cl}\,(\mathrm{epi}\,\mathbf{f})$ *and* $\mathrm{ri}\,(\mathrm{epi}\,\mathbf{g}) \subseteq \mathrm{epi}\, g \subseteq \mathrm{cl}\,(\mathrm{epi}\,\mathbf{g})$, *where* $\mathbf{f}, \mathbf{g}\colon \mathbb{R}^m \to \widetilde{\mathbb{R}}$ *are convex functions. Then the following statements are true:*

(i) $\mathcal{P}(f,t,g) = \mathcal{P}(\mathrm{cl}\, f,t,\mathrm{cl}\, g) = \mathcal{P}(\mathrm{cl}\,\mathbf{f},t,\mathrm{cl}\,\mathbf{g}) = \mathcal{P}(\mathbf{f},t,\mathbf{g})$.

(ii) $\mathcal{P}(f,0,g) = \mathrm{cl}\, f, \mathcal{P}(f,1,g) = \mathrm{cl}\, g$ *and* $\mathcal{P}(f,t,g) = \mathcal{P}(g,1-t,f)$.

(iii) $\mathcal{P}(f,t,g) \in \mathcal{F}_{\mathrm{lc}}$ *and* $(h_t)^* := (\mathcal{P}(f,t,g))^* = \mathcal{P}(f^*,t,g^*) =: (h^*)_t$.

(iv) $\mathcal{P}(f,\frac{1}{2},f^*) = \frac{1}{2}\|\cdot\|^2$.

(v) $\mathrm{cl}\,(\mathrm{dom}\, h_t) = \mathrm{cl}\,((1-t)\mathrm{dom}\, f + t\mathrm{dom}\, g)$ *and* $\mathrm{ri}\,(\mathrm{dom}\, h_t) = \mathrm{ri}\,((1-t)\mathrm{dom}\, f + t\mathrm{dom}\, g)$, *where* $h_t := \mathcal{P}(f,t,g)$ *and* $t \in (0,1)$.

(vi) *If* $0 \le f$ *and* $0 \le g$, *then* $0 \le h_t \le (1-t)f + tg$, *where* $h_t := \mathcal{P}(f,t,g)$ *and* $t \in (0,1)$.

Proof. (i) Since $f, g \in \mathcal{F}$ are almost convex and $\frac{1}{2}\|\cdot\|^2$ is convex, from Theorem 4.18 it follows that $f+\frac{1}{2}\|\cdot\|^2$ and $g+\frac{1}{2}\|\cdot\|^2$ are almost convex. Since $\mathrm{dom}(\frac{1}{2}\|\cdot\|^2) = \mathbb{R}^m$, Proposition 4.10 (i), along with Lemma 4.25 (i), yields

$$\begin{aligned}\mathcal{P}(f,t,g) &= ((1-t)(\mathrm{cl}(f+\frac{1}{2}\|\cdot\|^2))^* + t(\mathrm{cl}(g+\frac{1}{2}\|\cdot\|^2))^*)^* - \frac{1}{2}\|\cdot\|^2\\ &= ((1-t)(\mathrm{cl}\, f+\frac{1}{2}\|\cdot\|^2)^* + t(\mathrm{cl}\, g+\frac{1}{2}\|\cdot\|^2)^*)^* - \frac{1}{2}\|\cdot\|^2\\ &= \mathcal{P}(\mathrm{cl}\, f,t,\mathrm{cl}\, g) = \mathcal{P}(\mathrm{cl}\,\mathbf{f},t,\mathrm{cl}\,\mathbf{g}),\end{aligned}$$

where $\mathrm{cl}\, f = \mathrm{cl}\,\mathbf{f}$ and $\mathrm{cl}\, g = \mathrm{cl}\,\mathbf{g}$ follow from Proposition 4.6 (ii). The equality $\mathcal{P}(\mathrm{cl}\,\mathbf{f},t,\mathrm{cl}\,\mathbf{g}) = \mathcal{P}(\mathbf{f},t,\mathbf{g})$ follows immediately by using similar arguments.

(ii) From (i) and Lemma 5.2 (i) it follows that $\mathcal{P}(f,0,g) = \mathcal{P}(\mathrm{cl}\, f,0,\mathrm{cl}\, g) = \mathrm{cl}\, f$, $\mathcal{P}(f,1,g) = \mathcal{P}(\mathrm{cl}\, 1,t,\mathrm{cl}\, g) = \mathrm{cl}\, g$ and $\mathcal{P}(f,t,g) = \mathcal{P}(\mathrm{cl}\, f,t,\mathrm{cl}\, g) = \mathcal{P}(\mathrm{cl}\, g,1-t,\mathrm{cl}\, f) = \mathcal{P}(g,1-t,f)$.

(iii) From (i) one has $\mathcal{P}(f,t,g)\in\mathcal{F}_{\mathrm{lc}}$ and Lemma 5.2 (ii) allows $(h_t)^* := (\mathcal{P}(f,t,g))^* = (\mathcal{P}(\mathrm{cl}\, f,t,\mathrm{cl}\, g))^* = \mathcal{P}((\mathrm{cl}\, f)^*,t,(\mathrm{cl}\, g)^*) = \mathcal{P}(f^*,t,g^*) =: (h^*)_t$, where the fourth equality follows from Lemma 2.4 or Proposition 4.10 (i).

(iv) From (i), Proposition 4.10 (i) and Lemma 5.2 (iii) we have $\mathcal{P}(f,\frac{1}{2},f^*) = \mathcal{P}(\mathrm{cl}\, f,\frac{1}{2},\mathrm{cl}\, f^*) = \mathcal{P}(\mathrm{cl}\, f,\frac{1}{2},(\mathrm{cl}\, f)^*) = \frac{1}{2}\|\cdot\|^2$.

(v) From Proposition 4.6 (ii), we have that $\mathrm{cl}\, f$ and $\mathrm{cl}\, g$ are convex and so are $\mathrm{dom}\,(\mathrm{cl}\, f)$ and $\mathrm{dom}\,(\mathrm{cl}\, g)$. From Proposition 4.6 (iii), we have $\mathrm{dom}\, f \approx \mathrm{dom}\,(\mathrm{cl}\, f)$, $\mathrm{dom}\, g \approx \mathrm{dom}\,(\mathrm{cl}\, g)$, $\mathrm{dom}\, f$ and $\mathrm{dom}\, g$ are nearly convex, which implies $\mathrm{cl}\,(\mathrm{dom}\, f) = \mathrm{cl}\,(\mathrm{dom}\,(\mathrm{cl}\, f))$, $\mathrm{cl}\,(\mathrm{dom}\, g) = \mathrm{cl}\,(\mathrm{dom}\,(\mathrm{cl}\, g))$, $\mathrm{ri}\,(\mathrm{dom}\, f) = \mathrm{ri}\,(\mathrm{dom}\,(\mathrm{cl}\, f))$ and finally $\mathrm{ri}\,(\mathrm{dom}\, g) = \mathrm{ri}\,(\mathrm{dom}\,(\mathrm{cl}\, g))$. From (i) and Lemma 5.2 (iv) it follows that

$$\begin{aligned}\mathrm{cl}\,(\mathrm{dom}\, h_t) &= \mathrm{cl}\,(\mathrm{dom}\,\mathcal{P}(\mathrm{cl}\, f,t,\mathrm{cl}\, g)) = \mathrm{cl}\,((1-t)\mathrm{dom}\,(\mathrm{cl}\, f) + t\mathrm{dom}\,(\mathrm{cl}\, g))\\ &= \mathrm{cl}\,((1-t)\mathrm{cl}\,(\mathrm{dom}\,(\mathrm{cl}\, f)) + t\mathrm{cl}\,(\mathrm{dom}\,(\mathrm{cl}\, g)))\\ &= \mathrm{cl}\,((1-t)\mathrm{cl}\,(\mathrm{dom}\, f) + t\mathrm{cl}\,(\mathrm{dom}\, g)) = \mathrm{cl}\,((1-t)\mathrm{dom}\, f + t\mathrm{dom}\, g),\end{aligned}$$

where the third equality follows from Lemma 2.2 (vii) and the fifth equality follows from Lemma 3.8 (iii) and Lemma 3.9 (ii). Since f and g are almost convex, from Proposition 4.6 (iii) we have $\mathrm{dom}\, f$ and $\mathrm{dom}\, g$ are nearly convex and so from Lemma 3.8 (iii) and Lemma 3.9 (i), $(1-t)\mathrm{dom}\, f + t\mathrm{dom}\, g$ is nearly convex. As a consequence,

$$\begin{aligned}\mathrm{ri}\,(\mathrm{dom}\, h_t) &= \mathrm{ri}\,(\mathrm{cl}\,(\mathrm{dom}\, h_t)) = \mathrm{ri}\,(\mathrm{cl}\,((1-t)\mathrm{dom}\, f + t\mathrm{dom}\, g))\\ &= \mathrm{ri}\,((1-t)\mathrm{dom}\, f + t\mathrm{dom}\, g),\end{aligned}$$

where the first equality follows from (iii) and Lemma 2.2 (i) and the third equality follows from Lemma 3.6.

(vi) If $0 \le f$ and $0 \le g$, then from (i), Lemma 5.2 (v) and Proposition 4.6 (ii) we obtain

$$\begin{aligned}0 \le h_t &= \mathcal{P}(\mathrm{cl}\, f,t,\mathrm{cl}\, g) = \mathcal{P}(\mathrm{cl}\,\mathbf{f},t,\mathrm{cl}\,\mathbf{g})\\ &\le (1-t)\mathrm{cl}\,\mathbf{f} + t\mathrm{cl}\,\mathbf{g} = (1-t)\mathrm{cl}\, f + t\mathrm{cl}\, g \le (1-t)f + tg.\end{aligned}$$

6. Some applications to optimization problems and error bounds

In this section, we consider two applications of the results obtained in Section 4. The first application is related to near equality and near convexity of solution sets for almost convex optimization problems. The second one is concerned with the classical Hoffman' error bound for almost convex inequalities. Some examples are given to illustrate our results.

6.1. Applications to almost convex optimization problems

In this subsection, we shall investigate near equality and near convexity of solution sets and give some refinements on Fenchel duality for almost convex optimization problems.

Consider the following optimization problem with a set constraint (for short, OP):

$$\inf f(x) \quad \text{subject to} \quad x \in S,$$

where $S \subseteq \mathbb{R}^m$ and $f\colon \mathbb{R}^m \to \widetilde{\mathbb{R}}$ is a function. Denote by S_f the solution set of OP.

In the case $S := \mathbb{R}^m$, [11, Theorem 2.3] states that any local solution $\bar{x} \in \operatorname{dom} f$ of OP is a global solution of OP provided that f is almost convex. We first prove the following proposition which extends the classical results of convex optimization problems to the case of OP.

Proposition 6.1. *The following assertions hold:*

(i) *If $\bar{x} \in S$ and $0 \in \partial f(\bar{x}) + N_S(\bar{x})$, then $\bar{x} \in S_f$.*

(ii) *Suppose that $f\colon \mathbb{R}^m \to \widetilde{\mathbb{R}}$ is almost convex and S is nearly convex. If, additionally,* $\operatorname{ri}(\operatorname{dom} f) \cap \operatorname{ri} S \neq \emptyset$ *and $\bar{x} \in S_f$, then $0 \in \partial f(\bar{x}) + N_S(\bar{x})$.*

Proof. (i) By assumption there exists $x^* \in \partial f(\bar{x}) \cap (-N_S(\bar{x}))$. Then,

$$f(x) - f(\bar{x}) \geq \langle x^*, x - \bar{x}\rangle \geq 0 \quad \text{for each} \quad x \in S,$$

which proves that $\bar{x} \in S_f$.

(ii) We preliminarily observe that $\bar{x} \in S_f$ if and only if $\operatorname{epi} f \cap (S \times (-\infty, f(\bar{x})) = \emptyset$. Notice that $\operatorname{epi} f$ and $S \times (-\infty, f(\bar{x}))$ are nearly convex (see, Definition 3.10 and Lemma 3.8 (iii)). It follows from Proposition 3.7 that $\operatorname{epi} f$ and $S \times (-\infty, f(\bar{x}))$ are properly linearly separable, i.e., there exist $(\lambda^*, \theta^*) \in \mathbb{R}^m \times \mathbb{R}$ with $(\lambda^*, \theta^*) \neq 0$, and $\alpha \in \mathbb{R}$ such that

$$\langle \lambda^*, x\rangle + \theta^* r \geq \alpha \quad \text{for each } (x, r) \text{ with } x \in \operatorname{dom} f \text{ and } r \geq f(x) \tag{3}$$

and $$\langle \lambda^*, x\rangle + \theta^* r \leq \alpha \quad \text{for each } (x, r) \text{ with } x \in S \text{ and } r < f(\bar{x}), \tag{4}$$

with strict inequality for at least one point in any of the previous inequalities. Next we prove that $\theta^* \neq 0$. Indeed, assume that $\theta^* = 0$. Then, the previous inequalities become

$$\langle \lambda^*, x\rangle \geq \alpha \;\text{ for each } x \in \operatorname{dom} f \qquad \text{and} \qquad \langle \lambda^*, x\rangle \leq \alpha \;\text{ for each } x \in S,$$

with strict inequality for at least one point in any of the previous inequalities. This yields that the sets $\operatorname{dom} f$ and S are properly linearly separable. By Proposition 3.7, it follows that $\operatorname{ri}(\operatorname{dom} f) \cap \operatorname{ri} S = \emptyset$, a contradiction. Therefore, $\theta^* \neq 0$.

Setting $x := \bar{x}$ and $r := f(\bar{x})$ in (3) and setting $x := \bar{x}$ and letting $r \nearrow f(\bar{x})$ in (4), we obtain that $\alpha = \langle \lambda^*, \bar{x} \rangle + \theta^* f(\bar{x})$. By (4) we have

$$\langle \lambda^*, x - \bar{x} \rangle \leq \theta^*(f(\bar{x}) - r) \quad \text{for each } (x, r) \text{ with } x \in S \text{ and } r < f(\bar{x}), \tag{5}$$

which implies $\theta^* > 0$. Moreover, taking the limit $r \nearrow f(\bar{x})$ in (5) yields that $\lambda^* \in N_S(\bar{x})$. Setting $r := f(x)$ in (3), where $x \in \operatorname{dom} f$, and recalling that $\alpha = \langle \lambda^*, \bar{x} \rangle + \theta^* f(\bar{x})$, it follows that

$$f(x) - f(\bar{x}) \geq \langle -\frac{\lambda^*}{\theta^*}, x - \bar{x} \rangle \quad \text{for each } x \in \operatorname{dom} f$$

and so

$$f(x) - f(\bar{x}) \geq \langle -\frac{\lambda^*}{\theta^*}, x - \bar{x} \rangle \quad \text{for each } x \in \mathbb{R}^m,$$

which yields $-\frac{\lambda^*}{\theta^*} \in \partial f(\bar{x})$. Since $\frac{\lambda^*}{\theta^*} \in N_S(\bar{x})$, it follows that $0 \in \partial f(\bar{x}) + N_S(\bar{x})$. □

The following theorem states that, under suitable assumptions, the solution set of an almost convex optimization problem with $S \neq \mathbb{R}^m$ is nearly convex.

Theorem 6.2. *Suppose $f : \mathbb{R}^m \to \widetilde{\mathbb{R}}$ is almost convex, S is nearly convex and $S_f \neq \emptyset$. Assume that there exists $\hat{x} \in \mathbb{R}^m$ such that $f(\hat{x}) < f(x_0)$, where $x_0 \in S_f$, and $\operatorname{ri} S \cap \operatorname{ri}(\operatorname{lev}_{\leq f(x_0)} f) \neq \emptyset$. Then, S_f is nearly convex.*

Proof. Since $f(x_0) > f(\hat{x}) \geq \inf_{x \in \mathbb{R}^m} f(x) = \inf_{x \in \mathbb{R}^m} (\operatorname{cl} f)(x)$ (see, Lemma 4.5), from Proposition 4.6 (v) it follows that $\operatorname{lev}_{\leq f(x_0)} f$ is nearly convex. Since $\operatorname{ri} S \cap \operatorname{ri}(\operatorname{lev}_{\leq f(x_0)} f) \neq \emptyset$ and S is a nearly convex set, by Lemma 3.8 (i), $S_f = S \cap (\operatorname{lev}_{\leq f(x_0)} f)$ is nearly convex. □

We consider the case where $S := \mathbb{R}^m$. We first prove the following preliminary result.

Lemma 6.3. *Let $S := \mathbb{R}^m$. Suppose that $f : \mathbb{R}^m \to \widetilde{\mathbb{R}}$ and $S_f \neq \emptyset$. Then, $S_f \subseteq S_{\operatorname{cl} f}$.*

Proof. Let $\bar{x} \in S_f$. Then, $(\operatorname{cl} f)(x) := \liminf_{y \to x} f(y) \geq f(\bar{x})$ for each $x \in \mathbb{R}^m$. Since $f(\bar{x}) \geq (\operatorname{cl} f)(\bar{x})$ is always true, it follows that $f(\bar{x}) = (\operatorname{cl} f)(\bar{x})$ and so $\bar{x} \in S_{\operatorname{cl} f}$. □

The next theorem shows that, under some assumptions, the solution set of an almost convex optimization problem with $S := \mathbb{R}^m$ is nearly convex.

Theorem 6.4. *Let $S := \mathbb{R}^m$. Suppose $f : \mathbb{R}^m \to \widetilde{\mathbb{R}}$ is almost convex. If $S_f \neq \emptyset$ and the following condition holds: $\cap_{\epsilon_0 > \epsilon > 0} \operatorname{ri}(\operatorname{lev}_{\leq f(x_0)+\epsilon} f) \neq \emptyset$, where $x_0 \in S_f$ and $\epsilon_0 > 0$, then $\operatorname{cl} S_f = S_{\operatorname{cl} f}$ and $S_f \approx S_{\operatorname{cl} f}$. Consequently, S_f is nearly convex.*

Proof. Since $S_f \neq \emptyset$, f is proper. We first note that $S_f = \cap_{\epsilon_0 > \epsilon > 0} (\operatorname{lev}_{\leq f(x_0)+\epsilon} f)$ is the intersection of nearly convex sets (by Proposition 4.6 (v)). Since we have

$\bigcap_{\epsilon_0>\epsilon>0} \operatorname{ri}(\operatorname{lev}_{\leq f(x_0)+\epsilon} f) \neq \emptyset$, from Lemma 3.8 (i) we get that S_f is nearly convex and from Lemma 3.8 (ii) and Proposition 4.9 it follows that

$$\operatorname{cl} S_f = \bigcap_{\epsilon_0>\epsilon>0} \operatorname{cl}(\operatorname{lev}_{\leq f(x_0)+\epsilon} f) = \bigcap_{\epsilon_0>\epsilon>0} (\operatorname{lev}_{\leq (\operatorname{cl} f)(x_0)+\epsilon} \operatorname{cl} f) = S_{\operatorname{cl} f},$$

recalling that in the proof of Lemma 6.3, we have shown that $x_0 \in S_f$ implies $f(x_0) = (\operatorname{cl} f)(x_0)$ and $x_0 \in S_{\operatorname{cl} f}$. Since S_f is nearly convex and $\operatorname{cl} S_f = S_{\operatorname{cl} f}$, by Lemma 3.6 one has

$$\operatorname{ri} S_f = \operatorname{ri}(\operatorname{cl} S_f) = \operatorname{ri} S_{\operatorname{cl} f} \quad \text{and} \quad S_f \approx S_{\operatorname{cl} f}. \qquad \square$$

Example 6.5. Consider the almost convex function f defined in Example 4.3. It is easy to see that $S_f = \{(x_1, x_2)^\top \in \mathbb{R}^2 : x_1 = 0, x_2 \notin (0,1)\}$ and so it is a closed nonconvex set. In consequence, S_f is not almost convex. Note that $\bigcap_{\epsilon_0>\epsilon>0} \operatorname{ri}(\operatorname{lev}_{\leq f(x_0)+\epsilon} f) = \emptyset$, where $x_0 \in S_f$ and $\epsilon_0 > 0$, so that the assumption of Theorem 6.4 is not fulfilled. $\square$

The following theorem shows that under certain assumptions the solution sets of two almost convex optimization problems are nearly equal if the two objective functions involved are nearly equal.

Theorem 6.6. *Let $S := \mathbb{R}^m$ and $f\colon \mathbb{R}^m \to \widetilde{\mathbb{R}}$. If $g\colon \mathbb{R}^m \to \widetilde{\mathbb{R}}$ is almost convex, $f \approx g$, $S_f \neq \emptyset$, $S_g \neq \emptyset$, $\bigcap_{\epsilon_0>\epsilon>0} \operatorname{ri}(\operatorname{lev}_{f(x_0)+\epsilon} f) \neq \emptyset$, $\bigcap_{\epsilon_0>\varepsilon>0} \operatorname{ri}(\operatorname{lev}_{g(y_0)+\varepsilon} g) \neq \emptyset$, where $x_0 \in S_f$, $\epsilon_0 > 0$, $y_0 \in S_g$, $\varepsilon_0 > 0$, then $S_f \approx S_g$.*

Proof. Since g is almost convex and $f \approx g$, from Theorem 4.12 one has f is almost convex. Then the conclusion is a direct consequence of Theorem 6.4 taking into account that f and g are nearly equal. Indeed, from Theorem 4.13 one has $\operatorname{cl} f = \operatorname{cl} g$ and from Theorem 6.4 we have S_f and S_g are nearly convex, $S_f \approx \operatorname{cl} S_f$, $S_g \approx \operatorname{cl} S_g$ and the following equalities hold:

$$\operatorname{cl} S_f = S_{\operatorname{cl} f} = S_{\operatorname{cl} g} = \operatorname{cl} S_g.$$

Now Lemma 3.5 (i) and (ii) yield $S_f \approx S_g$. $\square$

Consider the following optimization problem with inequality constraints:

(OPIC) $\quad \inf f(x)$ subject to $x \in K := \{x \in S : g_i(x) \leq 0,\ i = 1, \ldots, l\}$,

where $S \subseteq \mathbb{R}^m$, $f, g_i\colon \mathbb{R}^m \to \widetilde{\mathbb{R}}$ $(i = 1, \ldots, l)$ and $l \in \mathbb{N}$. We always suppose that K is nonempty and the optimal value is finite. The Lagrange function of OPIC, say $L_{fg}\colon \mathbb{R}^m \times \mathbb{R}^l_+ \to \widetilde{\mathbb{R}}$, is given by

$$L_{fg}(x, \lambda) := f(x) + \sum_{i=1}^{l} \lambda_i g_i(x).$$

Let $E_{fg}(x) := \sup_{\lambda \in \mathbb{R}^l_+} L_{fg}(x, \lambda)$. Then OPIC is equivalent to

$$\inf E_{fg}(x) \quad \text{subject to} \quad x \in S.$$

Denote by $S_{E_{fg}}$ the solution set of OPIC.

Conclusions (i) and (ii) of the following lemma follow immediately from Theorems 4.18 and 4.20, respectively.

Lemma 6.7. *Let $f, g_i\colon \mathbb{R}^m \to \widetilde{\mathbb{R}}$ be almost convex for each $i = 1, \ldots, l$. Then the following statements are true:*

(i) *For each $\lambda \in \mathbb{R}^l_+$, $x \mapsto L_{fg}(x, \lambda)$ is almost convex.*

(ii) *If $\cap_{\lambda\in\mathbb{R}^l_+} \mathrm{ri}\,(\mathrm{epi}\, L_{fg}(\cdot, \lambda)) \neq \emptyset$, then E_{fg} is almost convex.*

Remark 6.8. The assumption $\cap_{\lambda\in\mathbb{R}^l_+} \mathrm{ri}\,(\mathrm{epi}\, L_{fg}(\cdot, \lambda)) \neq \emptyset$ in Lemma 6.7 (ii) can be characterized by using similar arguments as in the proof of Theorem 4.18.

Theorem 6.9. *Suppose that $f, g_i\colon \mathbb{R}^m \to \widetilde{\mathbb{R}}$ $(i = 1, \ldots, l)$ are almost convex, S is nearly convex and $S_{E_{fg}} \neq \emptyset$. Assume that there exists $\hat{x} \in \mathbb{R}^m$ such that $f(\hat{x}) < f(x_0)$, where $x_0 \in S_{E_{fg}}$, and*

$$\mathrm{ri}\, S \cap (\cap_{i=1}^l \mathrm{ri}\,(\mathrm{lev}_{\leq 0}\, g_i)) \cap \mathrm{ri}\,(\mathrm{lev}_{\leq f(x_0)}\, f) \neq \emptyset. \tag{6}$$

Then, $S_{E_{fg}}$ is nearly convex.

Proof. Since $K = S \cap (\cap_{i=1}^l (\mathrm{lev}_{\leq 0}\, g_i))$, by (6) and applying Lemma 3.8 (i) and Proposition 4.6 (v), it follows that K is nearly convex. Moreover, by Lemma 3.8 (ii) we have that $\mathrm{ri}\, K = \mathrm{ri}\, S \cap (\cap_{i=1}^l \mathrm{ri}\,(\mathrm{lev}_{\leq 0}\, g_i)) \neq \emptyset$.

Again from (6), $\mathrm{ri}\, K \cap \mathrm{ri}\,(\mathrm{lev}_{\leq f(x_0)}\, f) \neq \emptyset$ and applying Theorem 6.2 with $S := K$ we obtain that $S_{E_{fg}}$ is nearly convex. □

The following remark is given to illustrate the assumption (6).

Remark 6.10. We show that the assumption (6) is equivalent to

$$\mathrm{ri}\, S \cap \mathrm{ri}\,(\mathrm{lev}_{\leq E_{fg}(x_0)}\, E_{fg}) \neq \emptyset$$

under the assumption that $\cap_{i=1}^l \mathrm{ri}\,(\mathrm{lev}_{\leq 0}\, g_i) \neq \emptyset$ and $\mathrm{ri}\,(\mathrm{lev}_{\leq f(x_0)}\, f) \neq \emptyset$. In fact, according to Lemma 3.8 (ii) it suffices to prove

$$\mathrm{lev}_{\leq E_{fg}(x_0)}\, E_{fg} = \cap_{i=1}^l (\mathrm{lev}_{\leq 0}\, g_i) \cap (\mathrm{lev}_{\leq f(x_0)}\, f),$$

where $x_0 \in S_{E_{fg}}$. The inclusion $\mathrm{lev}_{\leq E_{fg}(x_0)}\, E_{fg} \supseteq \cap_{i=1}^l (\mathrm{lev}_{\leq 0}\, g_i) \cap (\mathrm{lev}_{\leq f(x_0)}\, f)$ is clear. For the converse inclusion, let $x \in \mathrm{lev}_{\leq E_{fg}(x_0)}\, E_{fg}$. Then

$$E_{fg}(x) := \sup_{\lambda\in\mathbb{R}^l_+} (f(x) + \sum_{i=1}^l \lambda_i g_i(x)) \leq E_{fg}(x_0) := \sup_{\lambda\in\mathbb{R}^l_+} (f(x_0) + \sum_{i=1}^l \lambda_i g_i(x_0)).$$

Since $x_0 \in S_{E_{fg}}$, $g_i(x_0) \leq 0$ for each $i = 1, \ldots, l$, it follows that

$$E_{fg}(x) := \sup_{\lambda\in\mathbb{R}^l_+} (f(x) + \sum_{i=1}^l \lambda_i g_i(x)) \leq E_{fg}(x_0) := f(x_0),$$

which implies that $g_i(x) \leq 0$ for each $i = 1, \ldots, l$ and so $f(x) \leq f(x_0)$. This proves $\mathrm{lev}_{\leq E_{fg}(x_0)}\, E_{fg} \subseteq \cap_{i=1}^l (\mathrm{lev}_{\leq 0}\, g_i) \cap (\mathrm{lev}_{\leq f(x_0)}\, f)$. □

The following theorem follows immediately from Lemma 6.7 (ii) and Theorem 6.4.

Theorem 6.11. *Let $S := \mathbb{R}^m$ and $f, g_i : \mathbb{R}^m \to \widetilde{\mathbb{R}}$ be almost convex for $i = 1, ..., l$. If $S_{E_{fg}} \neq \emptyset$, $\cap_{\lambda \in \mathbb{R}^l_+} \mathrm{ri}\,(\mathrm{epi}\, L_{fg}(\cdot, \lambda)) \neq \emptyset$ and $\cap_{\epsilon_0 > \epsilon > 0} \mathrm{ri}\,(\mathrm{lev}_{E_{fg}(x_0)+\epsilon}\, E_{fg}) \neq \emptyset$, where $x_0 \in S_{E_{fg}}$ and $\epsilon_0 > 0$, then $S_{E_{fg}}$ is nearly convex.*

Consider the following optimization problem concerning the sum of two functions:

$$\text{(OPS)} \qquad \inf f(x) + g(Mx) \quad \text{subject to} \quad x \in \mathbb{R}^m,$$

where M is a $m \times m$ real matrix and $f, g \colon \mathbb{R}^m \to \widetilde{\mathbb{R}}$. Denote by S_{fg} the solution set of OPS. Set $h(x) := f(x) + (g \circ M)(x) := f(x) + g(Mx)$.

We prove that the solution set of OPS is nearly convex under suitable assumptions.

Theorem 6.12. *Let M be a $m \times m$ real matrix and let $f, g \colon \mathbb{R}^m \to \widetilde{\mathbb{R}}$ be almost convex with f and g proper. If $S_{fg} \neq \emptyset$ and $\cap_{\epsilon_0 > \varepsilon > 0} \mathrm{ri}\,(\mathrm{lev}_{\leq h(x_0)+\varepsilon}\, h) \neq \emptyset$, where $x_0 \in S_{fg}$ and $\epsilon_0 > 0$, then S_{fg} is nearly convex.*

Proof. From [11, Theorem 4.5], h is almost convex and by the assumptions, Theorem 6.4 implies S_{fg} is nearly convex. □

6.2. Applications to error bounds for almost convex inequalities

In this subsection, we shall investigate error bounds for almost convex inequalities. It is shown that the classical Hoffman' error bound also holds for almost convex inequalities under a generalized Slater condition.

Consider the following system of inequalities:

$$x \in X_0 \quad \text{and} \quad g_i(x) \leq 0 \quad \text{for each} \quad i = 1, \ldots, l, \tag{7}$$

where $X_0 \subseteq \mathbb{R}^m$ is nonempty, $g_i \colon \mathbb{R}^m \to \widetilde{\mathbb{R}}$ $(i = 1, \ldots, l)$ are functions and $l \in \mathbb{N}$. Let K be the set of solutions of (7), i.e.,

$$K := \{x \in X_0 : g_i(x) \leq 0 \text{ for } i = 1, \ldots, l\} = X_0 \cap (\cap_{i=1}^l \mathrm{lev}_{\leq 0}\, g_i).$$

It is clear that $K \subseteq \cap_{i=1}^l \mathrm{dom}\, g_i$ and $K = \{x \in X_0 \cap (\cap_{i=1}^l \mathrm{dom}\, g_i) : 0 \in g(x) + \mathbb{R}^l_+\}$, where $g(x) := (g_1(x), \ldots, g_l(x))^\top$. In the following, we shall assume that K is nonempty. We are interested in the following global error bound for approximate solutions of (7): there is $\lambda > 0$ such that

$$d(x, K) \leq \lambda \|(g(x))_+\| \quad \text{for each} \quad x \in X_0, \tag{8}$$

where $(g(x))_+$ denotes the vector obtained by replacing each negative component of $g(x)$ by zero, i.e., $(g(x))_+ := (\max\{g_1(x), 0\}, \ldots, \max\{g_l(x), 0\})^\top$, and $d(x, K) := \inf_{y \in K} \|x - y\|$. Let $P \subseteq \mathbb{R}^l$ be nonempty convex. Recall that $h := (h_1, \ldots, h_l)^\top \colon \mathbb{R}^m \to \mathbb{R}^l$ is $\mathbb{R}^l_+$-convex on P, if for each $x, y \in P$ and $t \in [0, 1]$, we have $th(x) + (1-t)h(y) \in h(tx + (1-t)y) + \mathbb{R}^l_+$. It is clear that h is $\mathbb{R}^l_+$-convex on P if and only if h_i is convex on P for each $i = 1, \ldots, l$. For convenience, we set $\frac{+\infty}{+\infty} := +\infty$ and set $\|x\| := +\infty$ if $x \in \widetilde{\mathbb{R}}^m \backslash \mathbb{R}^m$.

In 1952, Hoffman [22] proved that (8) holds if $X_0 := \mathbb{R}^m$ and g is an affine mapping (which is often referred to as Hoffman's error bound). Robinson [33] proved that Hoffman's error bound holds under the assumptions that g is cone-convex and that a generalized Slater condition holds in a normed linear space.

We state here Robinson's error bound in finite-dimensional Euclidean spaces.

Lemma 6.13. [33] *Let X_0 be convex and $g_i \colon \mathbb{R}^m \to \widetilde{\mathbb{R}}$ $(i = 1, \ldots, l)$ be convex. If* (7) *satisfies a generalized Slater condition* (*that is, there exist a real number $\delta > 0$ and a point $x^0 \in X_0 \cap (\cap_{i=1}^l (\operatorname{dom} g_i))$ such that*

$$\delta\overline{B(0,1)} \subseteq g(x^0) + \mathbb{R}^l_+\,, \tag{9}$$

then the following inequality holds:

$$d(x, K) \leq \frac{\rho}{\rho + \delta}\|x - x^0\| \quad \text{for each} \quad x \in X_0 \cap (\cap_{i=1}^l (\operatorname{dom} g_i)),$$

where $\rho := d(0, g(x) + \mathbb{R}^l_+) = \|(g(x))_+\|$. Moreover, if the set K is bounded, then

$$d(x, K) \leq \frac{\Delta_K}{\delta} d(0, g(x) + \mathbb{R}^l_+) = \frac{\Delta_K}{\delta}\|(g(x))_+\|$$

for each $x \in X_0 \cap (\cap_{i=1}^l (\operatorname{dom} g_i))$, where Δ_K is the diameter of K.

As is well known, error bounds have important applications in sensitivity analysis and in convergence analysis of some algorithms (see, for example, [2, 21, 27, 31]), and are closely related to that of metric regularity, weak sharp minima and calmness for multifunctions (see, for example, [14, 15, 20, 26]). Recently, the study of error bounds for convex systems has received a growing interest in the mathematical programming literature (see, for example, [24, 25, 26, 30, 39]).

We need the following lemma.

Lemma 6.14. *If $0 \leq \alpha \leq \beta$ and $\gamma > 0$, then $\frac{\alpha}{\alpha+\gamma} \leq \frac{\beta}{\beta+\gamma}$.*

Proof. Observe that if $0 \leq \alpha \leq \beta$ and $\gamma > 0$, then $\alpha\beta + \alpha\gamma \leq \alpha\beta + \beta\gamma$, which is equivalent to $\frac{\alpha}{\alpha+\gamma} \leq \frac{\beta}{\beta+\gamma}$. □

Consider the following system of inequalities:

$$x \in \operatorname{cl} X_0 \quad \text{and} \quad (\operatorname{cl} g_i)(x) \leq 0 \quad \text{for each} \quad i = 1, \ldots, l. \tag{10}$$

Let $\bar{K}$ be the set of solutions of (10), i.e., $\bar{K} := \{x \in \operatorname{cl} X_0 : (\operatorname{cl} g_i)(x) \leq 0$ for $i = 1, \ldots, l\} = \operatorname{cl} X_0 \cap (\cap_{i=1}^l \operatorname{lev}_{\leq 0} (\operatorname{cl} g_i))$. It is clear that $\bar{K} \subseteq \cap_{i=1}^l \operatorname{dom} (\operatorname{cl} g_i)$ and $\bar{K} = \{x \in \operatorname{cl} X_0 \cap (\cap_{i=1}^l \operatorname{dom} (\operatorname{cl} g_i)) : 0 \in (\operatorname{cl} g)(x) + \mathbb{R}^l_+\}$, where $(\operatorname{cl} g)(x) := ((\operatorname{cl} g_1)(x), \ldots, (\operatorname{cl} g_l)(x))^\top$.

We have the following results.

Proposition 6.15. *Then the following statements are true:*

(i) *If the generalized Slater condition* (9) *holds* (*that is, there exist a real number* $\delta > 0$ *and a point* $x^0 \in \bar{X}_0 \cap (\cap_{i=1}^l (\mathrm{dom}\, g_i))$ *such that the inclusion* (9) *holds*), *then* $x^0 \in \mathrm{cl}\, X_0 \cap (\cap_{i=1}^l (\mathrm{dom}\,(\mathrm{cl}\, g_i)))$ *such that*

$$\delta \overline{B(0,1)} \subseteq (\mathrm{cl}\, g)(x^0) + \mathbb{R}_+^l . \tag{11}$$

(ii) *If* X_0 *is nearly convex and* $g_i \colon \mathbb{R}^m \to \widetilde{\mathbb{R}}$ $(i = 1, \ldots, l)$ *are almost convex with* $\mathrm{ri}\, X_0 \cap (\cap_{i=1}^l \mathrm{ri}\,(\mathrm{lev}_{\leq 0}\, g_i)) \neq \emptyset$, *then the set* $\bar{K}$ *is closed and convex and* $\mathrm{cl}\, \bar{K} \supseteq K \supseteq \mathrm{ri}\, \bar{K}$, *which implies* K *is nearly convex and* $K \approx \bar{K}$.

Proof. (i) Since $x^0 \in X_0 \cap (\cap_{i=1}^l (\mathrm{dom}\, g_i))$ and since $\mathrm{cl}\, g_i \leq g_i$ for each $i = 1, ..., l$, we conclude $\cap_{i=1}^l \mathrm{lev}_{\leq 0}\,(\mathrm{cl}\, g_i) \supseteq \cap_{i=1}^l \mathrm{lev}_{\leq 0}\, g_i$ and $g(x^0) - (\mathrm{cl}\, g)(x^0) \in \mathbb{R}_+^m$, i.e., $0 \in (\mathrm{cl}\, g)(x^0) - g(x^0) + \mathbb{R}_+^l$. Then $x^0 \in \mathrm{cl}\, X_0 \cap (\cap_{i=1}^l (\mathrm{dom}\,(\mathrm{cl}\, g_i)))$ and from (9) it follows that

$$\delta \overline{B(0,1)} \subseteq g(x^0) + (\mathrm{cl}\, g)(x^0) - g(x^0) + \mathbb{R}_+^l + \mathbb{R}_+^l \subseteq (\mathrm{cl}\, g)(x^0) + \mathbb{R}_+^l,$$

which proves (11).

(ii) Suppose X_0 is nearly convex and $g_i \colon \mathbb{R}^m \to \widetilde{\mathbb{R}}$ $(i = 1, \ldots, l)$ are almost convex with $\mathrm{ri}\, X_0 \cap (\cap_{i=1}^l \mathrm{ri}\,(\mathrm{lev}_{\leq 0}\, g_i)) \neq \emptyset$. Then $\mathrm{cl}\, X_0$ is closed and convex. From Proposition 4.6 (v) we have $\mathrm{lev}_{\leq 0}\, g_i \approx \mathrm{lev}_{\leq 0}\,(\mathrm{cl}\, g_i)$ for each $i = 1, \ldots, l$, which yields $\mathrm{cl}\,(\mathrm{lev}_{\leq 0}\, g_i) = \mathrm{cl}\,(\mathrm{lev}_{\leq 0}\,(\mathrm{cl}\, g_i))$ and $\mathrm{ri}\,(\mathrm{lev}_{\leq 0}\, g_i) = \mathrm{ri}\,(\mathrm{lev}_{\leq 0}\,(\mathrm{cl}\, g_i))$. From Proposition 4.6 (ii) we have $\mathrm{cl}\, g_i$ is closed and convex and so $\bar{K}$ is closed and convex. Since $\mathrm{cl}\, g_i \leq g_i$ for each $i = 1, \ldots, l$, one has $\cap_{i=1}^l \mathrm{lev}_{\leq 0}\,(\mathrm{cl}\, g_i) \supseteq \cap_{i=1}^l \mathrm{lev}_{\leq 0}\, g_i$. Since $\mathrm{ri}\, X_0 \cap (\cap_{i=1}^l \mathrm{ri}\,(\mathrm{lev}_{\leq 0}\, g_i)) \neq \emptyset$, it follows that

$$\begin{aligned}
\mathrm{cl}\, \bar{K} = \bar{K} &= \mathrm{cl}\, X_0 \cap (\cap_{i=1}^l \mathrm{lev}_{\leq 0}\,(\mathrm{cl}\, g_i)) \supseteq X_0 \cap (\cap_{i=1}^l \mathrm{lev}_{\leq 0}\, g_i) \\
&= K \supseteq \mathrm{ri}\,(X_0 \cap (\cap_{i=1}^l \mathrm{lev}_{\leq 0}\, g_i)) = \mathrm{ri}\, X_0 \cap (\cap_{i=1}^l \mathrm{ri}\,(\mathrm{lev}_{\leq 0}\, g_i)) \\
&= \mathrm{ri}\,(\mathrm{cl}\, X_0) \cap (\cap_{i=1}^l \mathrm{ri}\,(\mathrm{lev}_{\leq 0}\,(\mathrm{cl}\, g_i))) = \mathrm{ri}\,(\mathrm{cl}\, X_0 \cap (\cap_{i=1}^l \mathrm{lev}_{\leq 0}\,(\mathrm{cl}\, g_i))) = \mathrm{ri}\, \bar{K},
\end{aligned}$$

where the fourth and sixth qualities follow from Lemma 3.8 (ii) and the fifth equality follows from Lemma 3.6. This yields $\mathrm{cl}\, \bar{K} \supseteq K \supseteq \mathrm{ri}\, \bar{K}$ and so K is nearly convex and $K \approx \bar{K}$ in view of Lemma 3.5 (v). □

Proposition 6.16. *The inequality* $\|((\mathrm{cl}\, g)(x))_+\| \leq \|(g(x))_+\|$ *holds for each point* $x \in \mathbb{R}^m$.

Proof. Observe that $\mathrm{cl}\, g_i \leq g_i$ for each $i = 1, ..., l$. Let $x \in \mathbb{R}^m$ and $I := \{1, ..., l\}$. The proof consists of the following three cases:

If $(\mathrm{cl}\, g_i)(x) \geq 0$ for each $i \in I$, then $0 \leq (\mathrm{cl}\, g_i)(x) \leq g_i(x)$, and for each $i \in I$ $\max\{(\mathrm{cl}\, g_i)(x), 0\} \leq \max\{g_i(x), 0\}$. This yields $\|((\mathrm{cl}\, g)(x))_+\| \leq \|(g(x))_+\|$.

If $(\mathrm{cl}\, g_i)(x) < 0$ for each $i \in I$, then $\max\{(\mathrm{cl}\, g_i)(x), 0\} = 0 \leq \max\{g_i(x), 0\}$ for each $i \in I$, which implies $\|((\mathrm{cl}\, g)(x))_+\| \leq \|(g(x))_+\|$;

If there is a nonempty subset $I_0 \subseteq I$ such that $(\mathrm{cl}\, g_i)(x) \geq 0$ for each $i \in I_0$ and $(\mathrm{cl}\, g_i)(x) < 0$ for each $i \in I \backslash I_0$, then $0 \leq (\mathrm{cl}\, g_i)(x) \leq g_i(x)$ for each $i \in I_0$.

This leads to $\max\{(\operatorname{cl} g_i)(x), 0\} \leq \max\{g_i(x), 0\}$ for each $i \in I_0$ and also to $\max\{(\operatorname{cl} g_i)(x), 0\} = 0 \leq \max\{g_i(x), 0\}$ for each $i \in I \backslash I_0$. This proves the inequality $\|((\operatorname{cl} g)(x))_+\| \leq \|(g(x))_+\|$. □

By Lemma 4.22 and Proposition 6.16, any error bound result existing in the literature related to system (10) can be directly applied to system (7). We next prove system (7) has a global error bound under suitable assumptions.

Theorem 6.17. *Let X_0 be nearly convex and $g_i\colon \mathbb{R}^m \to \widetilde{\mathbb{R}}$ $(i = 1, \ldots, l)$ be almost convex with $\operatorname{ri} X_0 \cap (\cap_{i=1}^l \operatorname{ri}(\operatorname{lev}_{\leq 0} g_i)) \neq \emptyset$ and the generalized Slater condition* (9) *holds. Then the following inequality holds:*

$$d(x, K) \leq \frac{\rho}{\rho + \delta}\|x - x^0\| \quad \text{for each } x \in \operatorname{cl} X_0, \tag{12}$$

where $\rho := d(0, g(x) + \mathbb{R}^l_+) = \|(g(x))_+\|$ and x^0 is the same as in (9). *Furthermore, if the set K is bounded, then*

$$d(x, K) \leq \frac{\Delta_K}{\delta} d(0, g(x) + \mathbb{R}^l_+) = \frac{\Delta_K}{\delta}\|(g(x))_+\| \quad \text{for each } x \in \operatorname{cl} X_0, \tag{13}$$

where Δ_K is the diameter of K, which implies (8) *is true.*

Proof. From Proposition 6.15(ii), we have that $\bar{K}$ is closed and convex and $K \approx \bar{K}$. Then Proposition 4.23 (ii) yields $d(x, K) = d(x, \bar{K})$. Recall that $\operatorname{cl} X_0$ is closed and convex, and $\operatorname{cl} g_i$ $(i = 1, \ldots, l)$ are convex and real valued on the set $\operatorname{cl} X_0 \cap (\cap_{i=1}^l (\operatorname{dom}(\operatorname{cl} g_i)))$ (i.e., $\operatorname{cl} g$ is $\mathbb{R}^l_+$-convex on $\operatorname{cl} X_0 \cap (\cap_{i=1}^l (\operatorname{dom}(\operatorname{cl} g_i)))$ by Proposition 4.6 (ii). Now from Proposition 6.15 (i), Lemma 6.13 yields

$$d(x, K) = d(x, \bar{K}) \leq \frac{\|((\operatorname{cl} g)(x))_+\|}{\|((\operatorname{cl} g)(x))_+\| + \delta}\|x - x^0\|$$

for each $x \in \operatorname{cl} X_0 \cap (\cap_{i=1}^l \operatorname{dom}(\operatorname{cl} g_i))$, where x^0 is the same as in (9). Again since $\operatorname{cl} g_i \leq g_i$ for each $i = 1, \ldots, l$, $\cap_{i=1}^l \operatorname{dom}(\operatorname{cl} g_i) \supseteq \cap_{i=1}^l \operatorname{dom} g_i$ and from Proposition 6.16, one has $\|((\operatorname{cl} g)(x))_+\| \leq \|(g(x))_+\|$ for each $x \in \mathbb{R}^m$. Consequently,

$$d(x, K) \leq \frac{\|((\operatorname{cl} g)(x))_+\|}{\|((\operatorname{cl} g)(x))_+\| + \delta}\|x - x^0\| \leq \frac{\|(g(x))_+\|}{\|(g(x))_+\| + \delta}\|x - x^0\| \tag{14}$$

for each $x \in \operatorname{cl} X_0 \cap (\cap_{i=1}^l \operatorname{dom} g_i)$, where the second equality follows from Lemma 6.14. If $x \in \operatorname{cl} X_0 \backslash (\cap_{i=1}^l \operatorname{dom} g_i)$, then $\|(g(x))_+\| = +\infty$ and so (14) is true. This proves (12).

Furthermore, if the set K is bounded, then from Lemma 4.22 and Proposition 6.15 (i), Lemma 6.13 yields

$$d(x, K) = d(x, \bar{K}) \leq \frac{\Delta_{\bar{K}}}{\delta}\|((\operatorname{cl} g)(x))_+\| \leq \frac{\Delta_K}{\delta}\|(g(x))_+\|$$

for each $x \in \mathrm{cl}\, X_0 \cap (\cap_{i=1}^l \mathrm{dom}\,(\mathrm{cl}\, g_i))$. Again since $\cap_{i=1}^l \mathrm{dom}\,(\mathrm{cl}\, g_i) \supseteq \cap_{i=1}^l \mathrm{dom}\, g_i$, it follows that

$$d(x,K) \le \frac{\Delta_K}{\delta}\|(g(x))_+\|$$

for each $x \in \mathrm{cl}\, X_0 \cap (\cap_{i=1}^l \mathrm{dom}\, g_i)$. If $x \in \mathrm{cl}\, X_0 \backslash (\cap_{i=1}^l \mathrm{dom}\, g_i)$, then $\|(g(x))_+\| = +\infty$ and so the above inequality holds trivially. This proves (13), which implies (8). □

The following result follows from Theorem 6.17 immediately.

Corollary 6.18. *Let $X_0 := \mathbb{R}^m$ and $g_i \colon \mathbb{R}^m \to \widetilde{\mathbb{R}}$ be almost convex satisfying $\cap_{i=1}^l \mathrm{ri}\,(\mathrm{lev}_{\le 0}\, g_i) \ne \emptyset$, and assume that the generalized Slater condition (9) holds. Then the following inequality holds:*

$$d(x,K) \le \frac{\rho}{\rho+\delta}\|x - x^0\| \quad \textit{for each } x \in \mathbb{R}^m,$$

where $\rho := d(0, g(x)+\mathbb{R}^l_+) = \|(g(x))_+\|$ and x^0 is the same as in (9). Furthermore, if the set K is bounded, then (8) is true, i.e.,

$$d(x,K) \le \frac{\Delta_K}{\delta} d(0, g(x)+\mathbb{R}^l_+) = \frac{\Delta_K}{\delta}\|(g(x))_+\| \quad \textit{for each } x \in \mathbb{R}^m,$$

where Δ_K is the diameter of K.

The following two examples are given to illustrate Corollary 6.18.

Example 6.19. Consider functions $g_1, g_2 \colon \mathbb{R}^2 \to \widetilde{\mathbb{R}}$ defined by

$$g_1(x) := \begin{cases} x_1 - 1, & \text{if } x_1 > -1, \\ x_2(1-x_2) - 1, & \text{if } x_1 = -1,\ 0 \le x_2 \le 1, \\ -1, & \text{if } x_1 = -1,\ x_2 < 0, \\ -1, & \text{if } x_1 = -1,\ x_2 > 1, \\ +\infty, & \text{if } x_1 < -1, \end{cases}$$

and

$$g_2(x) := \begin{cases} x_1 - 1, & \text{if } x_1 > -1, \\ -x_2, & \text{if } x_1 = -1,\ 0 \le x_2 \le 1, \\ -1, & \text{if } x_1 = -1,\ x_2 < 0, \\ -1, & \text{if } x_1 = -1,\ x_2 > 1, \\ +\infty, & \text{if } x_1 < -1. \end{cases}$$

It is easy to check that g_1 and g_2 are almost convex, $\mathrm{lev}_{\le 0}\, g_1 = \{(x_1,x_2)^\top : -1 \le x_1 \le 1, x_2 \in \mathbb{R}\} = \mathrm{lev}_{\le 0}\, g_2$ and so $K := \mathrm{lev}_{\le 0}\, g_1 \cap \mathrm{lev}_{\le 0}\, g_2 = \{(x_1,x_2)^\top : -1 \le x_1 \le 1, x_2 \in \mathbb{R}\}$ and $\mathrm{ri}\,(\mathrm{lev}_{\le 0}\, g_1 \cap \mathrm{lev}_{\le 0}\, g_2) = \{(x_1,x_2)^\top : -1 < x_1 < 1, x_2 \in \mathbb{R}\}$.

Setting $x^0 := (0,0)^\top$ leads to $g(x^0) := (g_1(x^0), g_2(x^0))^\top = (-1,-1)^\top$ and so

$$\delta\overline{B(0,1)} \subseteq g(x^0) + \mathbb{R}^2_+,$$

which implies that the generalized Slater condition (9) holds, where $\delta := 1$. Observe that

$$(g(x))_+ = (\max\{g_1(x),0\}, \max\{g_1(x),0\})^\top = \begin{cases} (x_1-1, x_1-1)^\top, & \text{if } x_1 > 1, \\ (0,0)^\top, & \text{if } -1 \le x_1 \le 1, \\ (+\infty,+\infty)^\top, & \text{if } x_1 < -1, \end{cases}$$

and
$$d(x,K) = \begin{cases} x_1 - 1, & \text{if } x_1 > 1, \\ 0, & \text{if } -1 \le x_1 \le 1, \\ -1 - x_1, & \text{if } x_1 < -1. \end{cases}$$

Then
$$\rho := \|(g(x))_+\| = \begin{cases} \sqrt{2}(x_1 - 1), & \text{if } x_1 > 1, \\ 0, & \text{if } -1 \le x_1 \le 1, \\ +\infty, & \text{if } x_1 < -1, \end{cases}$$

which implies that $\quad d(x,K) \le \dfrac{\rho}{\rho+\delta}\|x - x^0\| \quad$ for each $x \in \mathbb{R}^m$.

Example 6.20. Consider the functions $g_1, g_2 \colon \mathbb{R}^2 \to \widetilde{\mathbb{R}}$ defined by

$$g_1(x) := \begin{cases} x_1 - 1, & \text{if } x_1 > -1, \\ x_2(1-x_2) - 1, & \text{if } x_1 = -1,\ 0 \le x_2 \le 1, \\ -1, & \text{if } x_1 = -1,\ x_2 < 0, \\ -1, & \text{if } x_1 = -1,\ x_2 > 1, \\ +\infty, & \text{if } x_1 < -1, \end{cases}$$

and
$$g_2(x) := \begin{cases} x_1 - 1, & \text{if } x_1 > -1,\ -2 \le x_2 \le 2, \\ -x_2, & \text{if } x_1 = -1,\ 0 \le x_2 \le 1, \\ -1, & \text{if } x_1 = -1,\ -2 \le x_2 < 0, \\ -1, & \text{if } x_1 = -1,\ 2 \ge x_2 > 1, \\ +\infty, & \text{else}. \end{cases}$$

It is easy to check that g_1 and g_2 are almost convex, $\operatorname{lev}_{\le 0} g_1 = \{(x_1,x_2)^\top : -1 \le x_1 \le 1, x_2 \in \mathbb{R}\}$, $\operatorname{lev}_{\le 0} g_2 = \{(x_1,x_2)^\top : -1 \le x_1 \le 1, -2 \le x_2 \le 2\}$ and so $K := \operatorname{lev}_{\le 0} g_1 \cap \operatorname{lev}_{\le 0} g_2 = \{(x_1,x_2)^\top : -1 \le x_1 \le 1, -2 \le x_2 \le 2\}$ and $\operatorname{ri}(\operatorname{lev}_{\le 0} g_1 \cap \operatorname{lev}_{\le 0} g_2) = \{(x_1,x_2)^\top : -1 < x_1 < 1, -2 < x_2 < 2\}$.

Setting $x^0 := (0,0)^\top$ leads to $g(x^0) := (g_1(x^0), g_2(x^0))^\top = (-1,-1)^\top$ and so

$$\delta\overline{B(0,1)} \subseteq g(x^0) + \mathbb{R}^2_+,$$

which implies that the generalized Slater condition (9) holds, where $\delta := 1$. Observe that $\Delta_K = 2\sqrt{5}$,

$$(g(x))_+ = (\max\{g_1(x), 0\}, \max\{g_1(x), 0\})^\top$$
$$= \begin{cases} (x_1 - 1, x_1 - 1)^\top, & \text{if } x_1 > 1,\ -2 \le x_2 \le 2, \\ (0,0)^\top, & \text{if } -1 \le x_1 \le 1,\ -2 \le x_2 \le 2, \\ (x_1 - 1, +\infty)^\top, & \text{if } x_1 > 1,\ -2 > x_2 \text{ or } x_2 > 2, \\ (+\infty, +\infty)^\top, & \text{else,} \end{cases}$$

and
$$d(x, K) = \begin{cases} x_1 - 1, & \text{if } x_1 > 1,\ -2 \le x_2 \le 2, \\ -1 - x_1, & \text{if } x_1 < -1,\ -2 \le x_2 \le 2, \\ \ \ 0, & \text{if } -1 \le x_1 \le 1,\ -2 \le x_2 \le 2, \\ x_2 - 2, & \text{if } x_2 > 2,\ -1 \le x_1 \le 1, \\ -2 - x_2, & \text{if } x_2 < -2,\ -1 \le x_1 \le 1. \end{cases}$$

Then
$$\rho := \|(g(x))_+\| = \begin{cases} \sqrt{2}(x_1 - 1), & \text{if } x_1 > 1,\ -2 \le x_2 \le 2, \\ 0, & \text{if } -1 \le x_1 \le 1,\ -2 \le x_2 \le 2, \\ +\infty, & \text{else,} \end{cases}$$

which implies that $d(x, K) \le \dfrac{\Delta_K}{\delta} \|(g(x))_+\|$ for each $x \in \mathbb{R}^m$.

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