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On some Class of Polytopes in an Idempotent, Symmetrical and Non-Associative Convex Structure

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$\mathbb{B}$-convexity was defined by the author and D. Horvath [*$\mathbb{B}$-convexity*, Optimization 53(2) (2004) 103–127] as a suitable Painlevé-Kuratowski limit of linear convexities. Recently, an alternative algebraic formulation over the whole Euclidean vector space was proposed by the author in further articles [*Some remarks on an idempotent and non-associative convex structure*, J. Convex Analysis 22 (2015) 259–289; *Separation properties in some idempotent and symmetrical convex structure*, J. Convex Analysis 24(4) (2017) 1143–1168].

In this paper the structure of the polytopes arising in such a context is analyzed. It is first established that this notion of convexity has the interval and decompositions properties but fails the recursive property. Among the key results of the paper, it is proven that a polytope defined over $\mathbb{R}^n$ is the Painlevé-Kuratowski limit of a suitable sequence of generalized convex polytopes. Some analogues to the Helly, Radon and Caratheodory theorems are derived in limit. A separation theorem for $\mathbb{B}$-polytopes is stated extending an earlier result recently established in the last paper mentioned above for fixed $\mathbb{B}$-convex sets. Finally, considering the Kuratowski-Painlevé limit of a suitable sequence of linear halfspaces the external representation of a $\mathbb{B}$-polytope is established over $\mathbb{R}^n$. It follows that the interior of a $\mathbb{B}$-polytope defined over $\mathbb{R}^n$ satisfies the convexity criterion proposed in the second paper mentioned above.

Keywords: Generalized mean, convexity, convex hull, duality, semilattice, $\mathbb{B}$-convexity.

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1. Introduction

$\mathbb{B}$-convexity was defined in [9], as a suitable Painlevé-Kuratowski limit of linear convexities. An algebraic formulation was established over $\mathbb{R}^n_+$ making the formal transformation $+ \to \max$ over the nonnegative Euclidean orthant. In such a case $\mathbb{B}$-convex sets are homeomorphic to the convex sets derived from a large class of idempotent algebraic structures introduced in [15]. Along this line, $\mathbb{B}$-convex functions were analyzed in [1]. Hahn-Banach like separation properties [11] as well as fixed point results have been established in [10] and [16]. Some alternative idempotent convex structure were also proposed in [2] and [18]. In [6] a special class of idempotent magma was considered in which associativity was relaxed to

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preserve both symmetry and idempotence. It has been shown that such a notion of convexity is equivalently characterized from the Kuratowski-Painlevé limit of the generalized convex hull of two points defined in [9]. More recently, some separation results over $\mathbb{R}^n$ were established in [7].

This paper continues and extends some investigation started in [6]. It is shown that this definition of $\mathbb{B}$-convexity satisfies the interval property. This we do by using some limit properties of $\mathbb{B}$-polytopes for two points. However, it is also shown that the new formalism proposed in [6] fails the recursive property.

Considering the general case of a $\mathbb{B}$-polytope, involving several points, an extension of the definition of intermediate points introduced in [6] is proposed. To do that, a special notion of idempotent and nonassociative determinants is defined. Hence, it is possible to give an algebraic characterization of the intersection between a $\mathbb{B}$-polytope and an n-dimensional orthant. Along this line, it is proven that a $\mathbb{B}$-polytope is the Painlevé-Kuratowski limit of the sequence of generalized convex sets involved in $\mathbb{B}$-convexity. As an immediate consequence, a convergence result in the sense of the Hausdorff metric is obtained. Among other things, it is shown that some classical results like Caratheodory, Helly and Radon's properties hold in limit.

The existence of this limit has some important implications to analyze the properties of $\mathbb{B}$-polytopes over $\mathbb{R}^n$. In [7] a convex separation framework was proposed as well as an extension of some known results established over posets. In this paper, a general separation theorem is derived for two $\mathbb{B}$-polytopes. Along this line, it is shown that they have an external representation over $\mathbb{R}^n$ with respect to the class of halfspaces introduced in [7].

The paper unfolds as follows. Section 1 presents the framework of $\mathbb{B}$-convexity. Section 2 focusses on the convexity of a $\mathbb{B}$-polytope for two points, the interval, decomposition and recursive properties are analyzed. Section 3 presents some geometrical aspects of the notion of $\mathbb{B}$-polytopes and provides some analogues of the Helly, Radon, and Caratheodory properties in limit. Section 4 considers dual properties of $\mathbb{B}$-polytopes. A separation result is established and it is shown that they have an external representation.

2. Idempotent and non-associative convex structure

2.1. Generalized means and convexity

The notion of generalized means was introduced by Hardy, Littlewood and Polya [13] and extended to the field of optimization theory by Avriel [3, 4] and Ben Tal [5]. In particular Avriel introduced the r-convex functions that generalize the log-convexity concept. This notion was extended from an algebraic point of view by Ben Tal [5] who showed that transposing an algebraic structure via an isomorphism, a particular form of convexity can be derived over a vector space.
In [9], $\mathbb{B}$-convexity is introduced as a limit of linear convexities. More precisely, for all $p \in \mathbb{N}$, a bijection $\varphi_p\colon \mathbb{R} \longrightarrow \mathbb{R}$ defined by

$$\varphi_p\colon \lambda \longrightarrow \lambda^{2p+1}, \tag{1}$$

and $\Phi_p(x_1,...,x_n) = (\varphi_p(x_1),...,\varphi_p(x_n))$ was considered; this is basically the approach of Ben-Tal [5] and Avriel [3]. A field structure on $\mathbb{R}$, for which φ_p becomes a field isomorphism, can then be defined. Given this change of notation via φ_p and Φ_p we can define a $\mathbb{R}$-vector space structure on $\mathbb{R}^n$ by

$$\lambda \overset{\varphi_p}{\cdot} x = \Phi_p^{-1}(\varphi_p(\lambda).\Phi_p(x)) = \lambda.x \quad \text{for all } \lambda \in \mathbb{R}, \text{ and}$$

$$x \overset{\varphi_p}{+} y = \Phi_p^{-1}(\Phi_p(x) + \Phi_p(y)) \quad \text{for all } x, y \in \mathbb{R}^n.$$

We call these two operations the *indexed scalar product* and the *indexed sum* (indexed by φ). For all positive natural numbers m, let us denote $[m] = \{1,...,m\}$. The φ_p-*sum*, denoted $\overset{\varphi_p}{\sum}$, of $(x_1,...,x_m) \in \mathbb{R}^n$ is defined by

$$\sum_{i\in[m]}^{\varphi_p} x_i = \Phi_p^{-1}\Big(\sum_{i\in[m]} \Phi_p(x_i)\Big) \tag{2}$$

and the Φ_p-*convex hull* of a finite set $A \subset \mathbb{R}^n$ is defined by

$$Co^{\Phi_p}(A) = \left\{\sum_{x\in A}^{\varphi} t_x \overset{\varphi_p}{\cdot} x : \sum_{x\in A}^{\varphi_p} t_x = \varphi_p^{-1}(1), x \in A\, \varphi_p(t_x) \geq 0\right\}, \tag{3}$$

which can be rewritten as

$$Co^{\Phi_p}(A) = \Big\{\Phi_p^{-1}\big(\sum_{x\in A} t_x^{2p+1}.\Phi_p(x)\big) : \big(\sum_{x\in A} t_x^{2p+1}\big)^{\frac{1}{2p+1}} = 1, x \in A, t_x \geq 0\Big\}. \tag{4}$$

This definition can be made more explicit by showing that:

$$Co^{\Phi_p}(A) = \Phi_p^{-1}\big(Co\,(\Phi_p(A))\big). \tag{5}$$

For simplicity, throughout the paper we denote for all $x, y \in \mathbb{R}^n$:

$$x \overset{p}{+} y = x \overset{\varphi_p}{+} y. \tag{6}$$

Moreover, for all $L \subset \mathbb{R}$, we simplify the notations denoting $Co^p(L) = Co^{\Phi_p}(L)$. The *Kuratowski-Painlevé*[1] *lower limit* of the sequence of sets $\{A^{(p)}\}_{p\in\mathbb{N}}$, denoted $Li_{p\to\infty}A^{(p)}$, is the set of points x for which there exists a sequence $\{x^{(p)}\}$ of points such that $x^{(p)} \in A^{(p)}$ for all p and $x = \lim_{p\to\infty} x^{(p)}$; the *Kuratowski-Painlevé upper limit* of the sequence of sets $\{A^{(p)}\}_{p\in\mathbb{N}}$, denoted $Ls_{p\to\infty A^{(p)}}$, is the set of points x for which there exists a subsequence $\{x^{(p_k)}\}_{k\in\mathbb{N}}$ of points such that $x^{(p_k)} \in A^{(p_k)}$ for all k and $x = \lim_{k\to\infty} x^{(p_k)}$; a sequence $\{A^{(p_k)}\}_{k\in\mathbb{N}}$ of subsets of $\mathbb{R}^n$ is said to *converge*, in the Kuratowski-Painlevé sense, to a set A if $Ls_{p\to\infty}A^{(p)} = A = Li_{p\to\infty}A^{(p)}$, in which case we write $A = Lim_{p\to\infty} A^{(p)}$.

We now define $\mathbb{B}$-convexity and its characterization in terms of Kuratowski-Painlevé limit of Φ_p-convex sets. Let $A \subset \mathbb{R}^n_+$ be a finite part. The definition

[1] Though this terminology is usual, it seems that Peano also was at the origin of this notion.

of $\mathbb{B}$-convex sets is based on the definition of the Kuratowski-Painlevé limit of a sequence of Φ_p-convex sets. For a non-empty finite subset $A \subset \mathbb{R}^n$ we let $Co^\infty(A) = Ls_{p\to\infty} Co^p(A)$. When $A \subset \mathbb{R}^n_+$, it was shown in [9] that we have:

$$Co^\infty(A) = \left\{ \bigvee_{x\in A} t_x x : t_x \in [0,1], \max_{x\in A} t_x = 1 \right\} = \underset{p\to\infty}{Lim}\, Co^p(A).$$

Moreover, it is also proved that the sequence $\{Co^p(A)\}_{p\in\mathbb{N}}$ is Hausdorff convergent.

$\mathbb{B}$-convex sets have a number of remarkable properties. In particular they are linked to the linear convex hull operator Co^p. If $L \subset \mathbb{R}^n_+$ is $\mathbb{B}$-convex, then $L = Lim_{p\to\infty} Co^p(L) = \bigcap_{p=0}^{\infty} Co^p(L)$. For all $S \subset \mathbb{R}^n_+$, we denote $Co^\infty(S) = Lim_{p\to\infty} Co^p(S)$. Properties of this operation are developed in [9]. In particular, for all $S \subset \mathbb{R}^n_+$, $Co^\infty(S)$ is $\mathbb{B}$-convex

2.2. An algebraic structure extended to the whole Euclidean vector space

For all $x, y \in \mathbb{R}$ we have
$$\lim_{p\to+\infty} x \overset{p}{+} y = \begin{cases} x & \text{if} \quad |x| > |y| \\ \frac{1}{2}(x+y) & \text{if} \quad |x| = |y| \\ y & \text{if} \quad |x| < |y|. \end{cases} \tag{7}$$

In the following it is established that, though the operation $\boxplus$ does not satisfy associativity, it can be extended by constructing a non-associative algebraic structure which returns to a given n-tuple a real value. For all $x \in \mathbb{R}^n$ and all subset I of $[n]$[2], let us consider the map $\xi_I[x]\colon \mathbb{R} \longrightarrow \mathbb{Z}$ defined for all $\alpha \in \mathbb{R}$ by

$$\xi_I[x](\alpha) = \mathrm{Card}\{i \in I : x_i = \alpha\} - \mathrm{Card}\{i \in I : x_i = -\alpha\}. \tag{8}$$

This map measures the symmetry of the occurrences of a given value α in the components of a vector x.

For all $x \in \mathbb{R}^n$, let $\mathcal{J}_I(x)$ be a subset of I defined by

$$\mathcal{J}_I(x) = \Big\{ j \in I : \xi_I[x](x_j) \neq 0 \Big\} = I \backslash \big(\psi_I[x]^{-1}(0)\big), \tag{9}$$

where $\psi[x](j) = \xi_I[x](x_j)$ for all $j \in I$. $\mathcal{J}_I(x)$ is called *the residual index set* of x. It is obtained by dropping from I all the i's such that $\mathrm{Card}\{j \in I : x_j = x_i\} = \mathrm{Card}\{j \in I : x_j = -x_i\}$.

For all positive natural numbers n and for all subsets I of $[n]$, let $\digamma_I\colon \mathbb{R}^n \longrightarrow \mathbb{R}$ be the map defined for all $x \in \mathbb{R}^n$ by

$$\digamma_I(x) = \begin{cases} \max_{i\in\mathcal{J}_I(x)} x_i & \text{if} \quad \xi_I[x]\big(\max_{i\in\mathcal{J}_I(x)} |x_i|\big) > 0 \\ \min_{i\in\mathcal{J}_I(x)} x_i & \text{if} \quad \xi_I[x]\big(\max_{i\in\mathcal{J}_I(x)} |x_i|\big) < 0 \\ 0 & \text{if} \quad \xi_I[x]\big(\max_{i\in\mathcal{J}_I(x)} |x_i|\big) = 0. \end{cases} \tag{10}$$

where $\xi_I[x]$ is the map defined in (8) and $\mathcal{J}_I(x)$ is the residual index set of x. The operation that takes an n-tuple $(x_1, ..., x_n)$ of $\mathbb{R}^n$ and returns a single real

[2]For all positive natural numbers n, $[n] = \{1, ..., n\}$.

element $\digamma_I(x_1, ..., x_n)$ is called a n-ary extension of the binary operation $\boxplus$. For all natural numbers $n \geq 1$, if I is a nonempty subset of $[n]$, then, for all n-tuples $x = (x_1, ..., x_n)$, one can define the operation

$$\boxplus_{i \in I} x_i = \lim_{p \to \infty} \sum_{i \in I}^{\varphi_p} x_i = \digamma_I(x). \tag{11}$$

Clearly, this operation encompasses as a special case the binary operation defined in equation (7). Clearly, for all $(x_1, x_2) \in \mathbb{R}^2$, $\boxplus_{i \in \{1,2\}} x_i = x_1 \boxplus x_2$.

There are some basic properties inheritable from the above algebraic structure. If all the elements of the family $\{x_i\}_{i \in I}$ are mutually non symmetrical, then:

$$\boxplus_{i \in I} x_i = \arg\max_{\lambda} \big\{ |\lambda| : \lambda \in \{x_i\}_{i \in I} \big\}.$$

The algebraic structure $(\mathbb{R}, \boxplus, \cdot)$ can be extended to $\mathbb{R}^n$. Let us consider m vectors $x^{(1)}, ..., x^{(m)} \in \mathbb{R}^n$, and define

$$\boxplus_{j \in [m]} x^{(j)} = \Big(\boxplus_{j \in [m]} x_1^{(j)}, ..., \boxplus_{j \in [m]} x_n^{(j)} \Big). \tag{12}$$

Along this line an extended definition of $\mathbb{B}$-convexity for $\mathbb{R}^n$ was suggested in [6]: A subset C of $\mathbb{R}^n$ is called $\mathbb{B}^\sharp$-convex if for all $x, y \in C$ and all $t \in [0, 1]$, $x \boxplus ty \in C$. It has been equivalently established that a set $C \subset \mathbb{R}^n$ is $\mathbb{B}^\sharp$-convex if $Co^\infty(\{x, y\}) \subset C$ for all $x, y \in C$.

3. Interval property

3.1. Intermediate points

For the sake of simplicity we write $Co^{(p)}(x, y)$ in place of $Co^{(p)}(\{x, y\})$. A notion of intermediate points was considered in [6] to show that the sequence of sets $\{Co^{(p)}(x, y)\}_{p \in \mathbb{N}}$ has a Painlevé-Kuratowski limit denoted $Co^\infty(x, y)$ that can be expressed trough an explicit algebraic formula in terms of the idempotent and non-associative algebraic structure defined in the paper. To do that, a notion of intermediate points was introduced in [6]. In the following the key results are briefly summarized. For all $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$, let us consider the map $\gamma \colon \mathbb{R}^n \times \mathbb{R}^n \times [0, +\infty] \longrightarrow \mathbb{R}^n$ defined by:

$$\gamma(x, y, t) = (\max\{1, t\})^{-1} \big(x \boxplus ty \big), \quad \text{for all } t \geq 0 \tag{13}$$

and by $\gamma(x, y, +\infty) = y$. For all $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$, let $\mathcal{I}(x, y)$ be the subset defined by $\mathcal{I}(x, y) = \{i \in [n] : x_i y_i < 0\}$ and let $n(x, y)$ be its cardinal. Remark that $\gamma(x, y, 0) = x$. For all $i \in \mathcal{I}(x, y)$ and all $t_i^\star \in \mathbb{R}_{++}$ a point $\gamma \in \mathbb{R}^n$ is called *i-intermediate point* between x and y if there is some $t_i^\star \in]0, +\infty[$ such that

$$\gamma_i(x, y, t_i^\star) := \big(\gamma(x, y, t_i^\star) \big)_i = 0. \tag{14}$$

For all $i \in \mathcal{I}(x,y)$, the map $t \mapsto \gamma_i(x,y,t)$ has a unique zero $t_i^\star = -\frac{x_i}{y_i} = |\frac{x_i}{y_i}| > 0$ and there is a unique i-intermediate point

$$\gamma(x,y,t_i^\star) = \Big(\frac{|y_i|}{\max\{|x_i|,|y_i|\}}\,x\Big) \boxplus \Big(\frac{|x_i|}{\max\{|x_i|,|y_i|\}}\,y\Big). \tag{15}$$

Moreover, $\lim_{t\to 0}\gamma(x,y,t) = x$ and $\lim_{t\to+\infty}\gamma(x,y,t) = y$. Define $\Theta(x,y) = \{0,+\infty,-\frac{x_i}{y_i} : i \in \mathcal{I}(x,y)\}$. If $\mathcal{I}(x,y) = \emptyset$, then $\Theta(x,y) = \{x,y\}$. For all $x,y \in \mathbb{R}^n$ such that $\mathcal{I}(x,y) \neq \emptyset$, let $\{i_m\}_{m\in[n(x,y)]}$ be an index sequence of $\mathcal{I}(x,y)$ such that

$$i_m \in \arg\max_{i\in\mathcal{I}(x,y)} \Big\{-\frac{x_i}{y_i} : i \geq m\Big\} \quad \text{for all} \quad m \in [n(x,y)].$$

Let $\boxdot$ denote the Hadamard product. We have for all $m \in [n(x,y)-1]$:

$$\gamma\big(x,y,-\frac{x_{i_m}}{y_{i_m}}\big) \boxdot \gamma\big(x,y,-\frac{x_{i_{m+1}}}{y_{i_{m+1}}}\big) \geq 0. \tag{16}$$

Moreover $x \boxdot \gamma\big(x,y,-\frac{x_{i_1}}{y_{i_1}}\big) \geq 0$ and $\gamma\big(x,y,-\frac{x_{i_{n(x,y)}}}{y_{i_{n(x,y)}}}\big) \boxdot y \geq 0$. Define $t_{i_0}^\star = 0$, $t_{i_{n(x,y)+1}}^\star = +\infty$ and $t_{i_m}^\star = -\frac{x_{i_m}}{y_{i_m}}$ for all $m \in [n(x,y)]$. A sequence $\{t_{i_m}^\star\}_{m=0}^{n(x,y)+1}$ of $\Theta(x,y)$ satisfying the conditions of equation (16) is called an *intermediate sequence*. Moreover, let $\Gamma(x,y) = \{\gamma(x,y,t_i^\star) : t_i^\star \in \Theta(x,y)\}$ be the set of the i-intermediate points together with x and y. It was established in [6] that:

$$Co^\infty(x,y) = \bigcup_{m=0}^{n(x,y)} \mathbb{B}\Big[\gamma(x,y,t_{i_m}^\star), \gamma(x,y,t_{i_{m+1}}^\star)\Big]. \tag{17}$$

In order to show that $Co^\infty(x,y)$ is the Painlevé-Kuratowski limit of the sequence $\{Co^p(x,y)\}_{p\in\mathbb{N}}$ it is useful to define a notion of i-intermediate points of order p. For all natural numbers p, let us consider the map $\gamma^{(p)}: \mathbb{R}^n \times \mathbb{R}^n \times [0,+\infty] \longrightarrow \mathbb{R}$ defined by

$$\gamma^{(p)}(x,y,t) = \Big(\frac{1}{1 \overset{p}{+} t}\Big)x \overset{p}{+} \Big(\frac{t}{1 \overset{p}{+} t}\Big)y, \quad \text{for all } t \geq 0 \tag{18}$$

and by $\gamma^{(p)}(x,y,+\infty) = y$. For all $i \in \mathcal{I}(x,y)$, we say that $\gamma^{(p)}(x,y,t_i^\star)$ is an *i-intermediate point of order* p between x and y if $\gamma_i^{(p)}(x,y,t_i^\star) = 0$. For all $p \in \mathbb{N}$, there is a uniqueness i-intermediate point of order p

$$\gamma^{(p)}(x,y,t_i^\star) = \Big(\frac{|y_i|}{|x_i| \overset{p}{+} |y_i|}\Big)x \overset{p}{+} \Big(\frac{|x_i|}{|x_i| \overset{p}{+} |y_i|}\Big)y, \tag{19}$$

with $t_i^\star = -\frac{x_i}{y_i} = |\frac{x_i}{y_i}|$. It was shown in [6] that for all $i \in \mathcal{I}(x,y)$

$$\lim_{p\to\infty} \gamma^{(p)}(x,y,t_i^\star) = \gamma(x,y,t_i^\star), \tag{20}$$

where $t^\star_{i_0} = 0$, $t^\star_{i_{n(x,y)+1}} = +\infty$ and $t^\star_{i_m} = -\frac{x_{i_m}}{y_{i_m}}$ for all $m \in [n(x,y)]$. Moreover:

$$Co^p(x,y) = \bigcup_{m=0}^{n(x,y)} Co^p\Big(\gamma^{(p)}\big(x,y,t^\star_{i_m}\big), \gamma^{(p)}\big(x,y,t^\star_{i_{m+1}}\big)\Big). \tag{21}$$

Along this line, the following result was established in [6]:

Proposition 3.1. *For all $x, y \in \mathbb{R}^n$, let $\{t^\star_{i_m}\}_{m=0}^{n(x,y)+1}$ be an intermediate sequence of $\Theta(x,y)$. Then*

$$Co^\infty(x,y) = \underset{p\to+\infty}{Lim}\, Co^p(x,y) = \bigcup_{m=0}^{n(x,y)} \mathbb{B}\Big[\gamma(x,y,t^\star_{i_m}), \gamma(x,y,t^\star_{i_{m+1}})\Big].$$

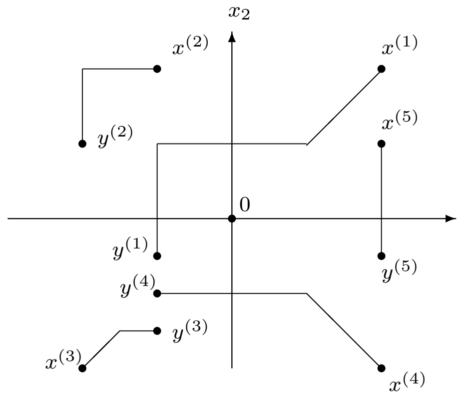

Figure 3.1: Examples of sets $Co^\infty(x,y)$

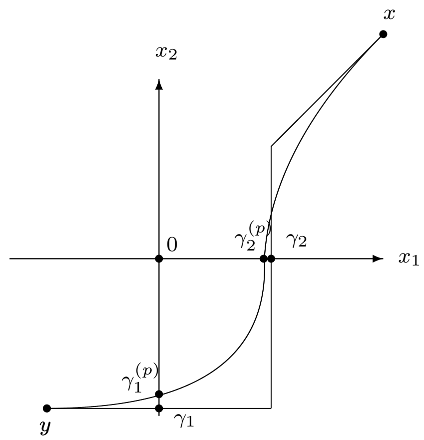

Figure 3.2: Limit of $Co^p(x,y)$

As an immediate consequence of Proposition 3.2, one can give the following formulation of a $\mathbb{B}$-polytope in the case of two points [3]:

Proposition 3.2. [6] *For all $x, y \in \mathbb{R}^n$,*

$$Co^\infty(x,y) = \Big\{tx \boxplus rx \boxplus sy \boxplus wy : \max\{t,r,s,w\} = 1, t,r,s,w \geq 0\Big\}.$$

Moreover, it was established that this limit entirely characterizes a $\mathbb{B}^\sharp$-convex set.

Proposition 3.3. [6] *A subset C of $\mathbb{R}^n$ is $\mathbb{B}^\sharp$-convex if and only if for all $x, y \in \mathbb{R}^n$ $Co^\infty(x,y) \subset C$.*

In the following we establish that $Co^\infty(x,y)$ is $\mathbb{B}^\sharp$-convex.

3.2. On the convexity of $Co^\infty(x,y)$

To prove the interval property, we need to establish that for all $x, y \in \mathbb{R}^n$, $Co^\infty(x,y)$ is $\mathbb{B}^\sharp$-convex. Given two sequences $\{x^{(p)}\}_{p\in\mathbb{N}}$ and $\{y^{(p)}\}_{p\in\mathbb{N}}$ of $\mathbb{R}^n$, the next result establishes a sufficient criterion for the convergence of the sequence $\{x^{(p)} \overset{p}{+} y^{(p)}\}_{p\in\mathbb{N}}$ to $x \boxplus y$. This criterion will be useful to prove the convexity of $Co^\infty(x,y)$.

[3] We use the convention that for all $(\alpha,\beta,\gamma,\delta) \in \mathbb{R}^4$, $\alpha \boxplus \beta \boxplus \gamma \boxplus \delta = \digamma_{[4]}(\alpha,\beta,\gamma,\delta)$.

Lemma 3.4. *Let $\{x^{(p)}\}_{p\in\mathbb{N}}$ and $\{y^{(p)}\}_{p\in\mathbb{N}}$ be two sequences of $\mathbb{R}^n$, which respectively converge to x and y. Suppose that there exists a subset I of $[n]$ such that: (i) $x_i^{(p)} = -y_i^{(p)}$ for all $i \in I$; (ii) $x_i \neq -y_i$ for all $i \in [n]\backslash I$. Then:*

$$\lim_{p\to\infty} x^{(p)} \overset{p}{+} y^{(p)} = x \boxplus y.$$

Proof. Suppose that $i \in I$. In such a case $\big(x_i^{(p)2p+1} + y_i^{(p)2p+1}\big)^{\frac{1}{2p+1}} = 0$. Moreover, one has $x_i = -y_i$ and $x_i \boxplus y_i = 0$. Consequently,

$$\big(x_i^{(p)2p+1} + y_i^{(p)2p+1}\big)^{\frac{1}{2p+1}} = x_i \boxplus y_i = 0$$

for all $p \in \mathbb{N}$ and all $i \in I$. This implies $\lim_{p\to\infty} x_i^{(p)} \overset{p}{+} y_i^{(p)} = x_i \boxplus y_i$ for all $i \in I$.

Suppose now that $i \in [n]\backslash I$, $x_i \neq -y_i$. If $x_i = y_i$, then, since $x_i \neq -y_i$, we deduce that $x_i \neq 0$ and $y_i \neq 0$. Consequently, there is some $p_0 \in \mathbb{N}$, such that for all $p > p_0$, $x_i^{(p)} y_i^{(p)} \neq 0$. Suppose for instance that $x_i, y_i \in \epsilon\mathbb{R}_+$ with ϵ is $+1$ or -1. From [9] we have

$$\lim_{p\to\infty} \big(x_i^{(p)2p+1} + y_i^{(p)2p+1}\big)^{\frac{1}{2p+1}} = \epsilon \max\{x_i, y_i\} = x_i \boxplus y_i.$$

Assume now that $|x_i| \neq |y_i|$. Suppose, for example, that $|x_i| > |y_i|$. We have

$$\lim_{p\to\infty} (x_i^{(p)2p+1} + y_i^{(p)2p+1})^{\frac{1}{2p+1}} = \lim_{p\to\infty} x_i^{(p)} \Big(1 + (\frac{y_i^{(p)}}{x_i^{(p)}})^{2p+1}\Big)^{\frac{1}{2p+1}}.$$

Since $\left|\frac{y_i}{x_i}\right| < 1$, there is some $c \in]0,1[$ and some natural number p_c such that $\left|\frac{y_i^{(p)}}{x_i^{(p)}}\right| < c$ for all $p > p_c$. This implies that for all $p > p_c$

$$\Big(1 - c^{2p+1}\Big)^{\frac{1}{2p+1}} < \Big(1 + (\frac{y_i^{(p)}}{x_i^{(p)}})^{2p+1}\Big)^{\frac{1}{2p+1}} < \Big(1 + c^{2p+1}\Big)^{\frac{1}{2p+1}}.$$

Taking the limit on both side yields $\lim_{p\to\infty} \Big(1 + (\frac{y_i^{(p)}}{x_i^{(p)}})^{2p+1}\Big)^{\frac{1}{2p+1}} = 1$, which completes the proof. □

Remark 3.5. Let $\{\alpha_p\}_{p\in\mathbb{N}}$ and $\{\beta_p\}_{p\in\mathbb{N}}$ be two sequences of real numbers. In general one cannot deduce that $\lim_{p\to\infty} \alpha_p \overset{p}{+} \beta_p = \alpha \boxplus \beta$ from the fact that that $\lim_{p\to\infty} \alpha_p = \alpha$ and $\lim_{p\to\infty} \beta_p = \beta$. Suppose for example that for all natural numbers p, $\alpha_p = 2^{\frac{1}{2p+1}}$ and $\beta_p = -1$. Therefore, we have $\lim_{p\to\infty} \alpha_p = 1$ and $\lim_{p\to\infty} \beta_p = -1$. Hence $\alpha \boxplus \beta = 1 \boxplus (-1) = 0$. However,

$$\alpha_p \overset{p}{+} \beta_p = \big((2^{\frac{1}{2p+1}})^{2p+1} - 1\big)^{\frac{1}{2p+1}} = 1^{\frac{1}{2p+1}} = 1.$$

Therefore, in such a case, $\lim_{p\to\infty} \alpha_p \overset{p}{+} \beta_p \neq \alpha \boxplus \beta$. □

The next statement uses the fact that an intermediate point between two intermediate points of order p in $Co^p(x,y)$ is also an intermediate point between x an y. Such a property is not evident a priori for two intermediate points of $Co^\infty(x,y)$. It was shown in [6] that if $\{u^{(p)}\}_{p\in\mathbb{N}}$ and $\{v^{(p)}\}_{p\in\mathbb{N}}$ are two copositive sequences of an n-dimensional orthant K (which implies that $u^{(p)} \boxdot v^{(p)} \geq 0$ for all natural numbers p) respectively converging to u and v, then $\lim_{p\to\infty} u^{(p)} \overset{p}{+} v^{(p)} = u \boxplus v$.

Proposition 3.6. *Let $x, y \in \mathbb{R}^n$, and assume that $\mathcal{I}(x,y) \neq \emptyset$. For all $u, v \in Co^\infty(x,y)$ and all $i \in \mathcal{I}(u,v)$, any i-intermediate point between u and v, is a i-intermediate point between x and y. Equivalently, we have:*

$$\Gamma(u,v) = \{u,v\} \cup \big\{\gamma(x,y,t_i^\star) : t_i^\star \in \Theta(x,y), i \in \mathcal{I}(u,v)\big\}.$$

Proof. Suppose that $u, v \in Co^\infty(x,y)$. Since from (17)

$$\bigcup_{m=0}^{n(x,y)} \mathbb{B}\Big[\gamma(x,y,t_{i_m}^\star), \gamma(x,y,t_{i_{m+1}}^\star)\Big] = Co^\infty(x,y)$$

there exists $i_m, i_l \in I(x,y)$ such that

$$u \in \mathbb{B}\Big[\gamma(x,y;-\frac{x_{i_m}}{y_{i_m}}), \gamma(x,y;-\frac{x_{i_{m+1}}}{y_{i_{m+1}}})\Big] \quad \text{and} \quad v \in \mathbb{B}\Big[\gamma(x,y;-\frac{x_{i_l}}{y_{i_l}}), \gamma(x,y;-\frac{x_{i_{l+1}}}{y_{i_{l+1}}})\Big],$$

where we assume that $i_l \geq i_m$. If u and v are copositive then $i_m = i_l$. If $i_l > i_m$, then $\mathcal{I}(u,v) = \bigcup_{k=m}^{l}\{i_k\} \subset \mathcal{I}(x,y)$. Suppose by hypothesis that $i_m < i_l$, the case $i_m = i_l$ being trivial.

By hypothesis we know that there are $\alpha_u, \beta_u, \alpha_v, \beta_v \in [0,1]$ with $\max\{\alpha_u, \beta_u\} = 1$ and $\max\{\alpha_v, \beta_v\} = 1$ such that

$$u = \alpha_u \gamma(x,y;-\frac{x_{i_m}}{y_{i_m}}) \boxplus \alpha_u \gamma(x,y;-\frac{x_{i_{m+1}}}{y_{i_{m+1}}}),$$

and

$$v = \alpha_v \gamma(x,y;-\frac{x_{i_l}}{y_{i_l}}) \boxplus \beta_v \gamma(x,y;-\frac{x_{i_{l+1}}}{y_{i_{l+1}}}).$$

Let us denote

$$u^{(p)} := \Big[\frac{\alpha_u}{\alpha_u \overset{p}{+} \beta_u}\gamma^{(p)}(x,y;-\frac{x_{i_m}}{y_{i_m}}) \overset{p}{+} \frac{\beta_u}{\alpha_u \overset{p}{+} \beta_u}\gamma^{(p)}(x,y;-\frac{x_{i_{m+1}}}{y_{i_{m+1}}})\Big],$$

and

$$v^{(p)} := \Big[\frac{\alpha_v}{\alpha_v \overset{p}{+} \beta_v}\gamma^{(p)}(x,y;-\frac{x_{i_l}}{y_{i_l}}) \overset{p}{+} \frac{\beta_v}{\alpha_v \overset{p}{+} \beta_v}\gamma^{(p)}(x,y;-\frac{x_{i_{l+1}}}{y_{i_{l+1}}})\Big].$$

Since $\gamma^{(p)}(x,y;-\frac{x_{i_m}}{y_{i_m}})$ and $\gamma^{(p)}(x,y;-\frac{x_{i_{m+1}}}{y_{i_{m+1}}})$ are copositive:

$$\lim_{p\to\infty} u^{(p)} = \alpha_u \gamma(x,y;-\frac{x_{i_m}}{y_{i_m}}) \boxplus \beta_u \gamma(x,y;-\frac{x_{i_{m+1}}}{y_{i_{m+1}}}).$$

Moreover, $\gamma^{(p)}(x,y;-\frac{x_{i_l}}{y_{i_l}})$ and $\gamma^{(p)}(x,y;-\frac{x_{i_{l+1}}}{y_{i_{l+1}}})$ are also copositive and we have:

$$\lim_{p\to\infty} v^{(p)} = \alpha_v \gamma(x,y;-\frac{x_{i_l}}{y_{i_l}}) \boxplus \beta_v \gamma(x,y;-\frac{x_{i_{l+1}}}{y_{i_{l+1}}}).$$

By construction, note that from equation (21):

$$u^{(p)} \in Co^p\Big(\gamma^{(p)}(x,y;-\frac{x_{i_m}}{y_{i_m}}),\gamma^{(p)}(x,y;-\frac{x_{i_{m+1}}}{y_{i_{m+1}}})\Big)$$

and
$$v^{(p)} \in Co^p\Big(\gamma^{(p)}(x,y;-\frac{x_{i_l}}{y_{i_l}}),\gamma^{(p)}(x,y;-\frac{x_{i_{l+1}}}{y_{i_{l+1}}})\Big).$$

Furthermore, since $Co^p(x,y)$ is ϕ_p-convex, using from equation (19), there are an i-intermediate point between u and v and an i-intermediate point of order p between $u^{(p)}$ and $v^{(p)}$ and an intermediate value $t_i = -\frac{x_i}{y_i}$ with

$$-\frac{x_{i_{m+1}}}{y_{i_{m+1}}} \leq -\frac{x_i}{y_i} \leq -\frac{x_{i_l}}{y_{i_l}}.$$

Hence, since $u^{(p)}, v^{(p)} \in Co^p(x,y),\ t_i = -\frac{x_i}{y_i}$ is such that

$$\frac{1}{1 \overset{p}{+} t_i}u^p \overset{p}{+} \frac{t_i}{1 \overset{p}{+} t_i}v^{(p)} = \gamma^{(p)}(x,y;-\frac{x_i}{y_i}).$$

However, from Lemma 3.4 and equation (20):

$$\begin{aligned}\lim_{p\to\infty} \frac{1}{1 \overset{p}{+} t_i}u^p \overset{p}{+} \frac{t_i}{1 \overset{p}{+} t_i}v^{(p)} &= \frac{1}{\max\{1,t_i\}}u \boxplus \frac{t_i}{\max\{1,t_i\}}v \\ &= \lim_{p\to\infty} \gamma^{(p)}(x,y;-\frac{x_i}{y_i}) = \gamma(x,y;-\frac{x_i}{y_i}).\end{aligned}$$

Consequently, $\gamma(x,y;-\frac{x_i}{y_i})$ is an i-intermediate point between u and v. □

The next statement is an immediate consequence of Proposition 3.6.

Proposition 3.7. *For all $x, y \in \mathbb{R}^n$ and for all $u, v \in Co^\infty(x,y)$, we have $Co^\infty(u,v) \subset Co^\infty(x,y)$.*

Proof. Suppose that $u, v \in \mathbb{B}^\sharp[x,y]$. There exist $i_m, i_l \in \mathcal{I}(x,y)$ such that

$$\begin{aligned}u &\in \mathbb{B}\Big[\gamma(x,y;-\frac{x_{i_m}}{y_{i_m}}),\gamma(x,y;-\frac{x_{i_{m+1}}}{y_{i_{m+1}}})\Big] \\ v &\in \mathbb{B}\Big[\gamma(x,y;-\frac{x_{i_l}}{y_{i_l}}),\gamma(x,y;-\frac{x_{i_{l+1}}}{y_{i_{l+1}}})\Big].\end{aligned}$$

Therefore:
$$\begin{aligned}Co^\infty(u,v) &= \mathbb{B}\Big[u,\gamma(x,y;-\frac{x_{i_{m+1}}}{y_{i_{m+1}}})\Big]\cup \\ &\cup \bigcup_{k=m+1}^{l-1}\mathbb{B}\Big[\gamma(x,y;-\frac{x_{i_k}}{y_{i_k}}),\gamma(x,y;-\frac{x_{i_{k+1}}}{y_{i_{k+1}}})\Big] \cup \mathbb{B}\Big[\gamma(x,y;-\frac{x_{i_l}}{y_{i_l}}),v\Big].\end{aligned}$$

Since $\mathbb{B}\Big[u,\gamma(x,y;-\frac{x_{i_{m+1}}}{y_{i_{m+1}}})\Big] \cup \mathbb{B}\Big[\gamma(x,y;-\frac{x_{i_l}}{y_{i_l}}),v\Big] \subset Co^\infty(x,y)$ and

$$\bigcup_{k=m+1}^{l-1}\mathbb{B}\Big[\gamma(x,y;-\frac{x_{i_k}}{y_{i_k}}),\gamma(x,y;-\frac{x_{i_{k+1}}}{y_{i_{k+1}}})\Big] \subset Co^\infty(x,y),$$

we deduce that $Co^\infty(u,v) \subset Co^\infty(x,y)$. □

Proposition 3.8. *For all $x, y \in \mathbb{R}^n$, $Co^\infty(x,y)$ is $\mathbb{B}^\sharp$-convex. Moreover, we have $Co^\infty(x,y) = \mathbb{B}^\sharp[x,y]$.*

Proof. The first statement follows from Proposition 3.7. Moreover, this implies that $\mathbb{B}^\sharp[x,y] \subset Co^\infty(x,y)$. However, since $\mathbb{B}^\sharp[x,y]$ is $\mathbb{B}^\sharp$-convex, $Co^\infty(x,y) \subset \mathbb{B}^\sharp[x,y]$, which ends the proof. $\square$

Given a convexity, a convex-hull operator associates to each subset S of $\mathbb{R}^n$, the smallest convex set which contains it. Let $[\![\,\cdot\,]\!]$ this maps which associate to each set the convex hull $[\![\,\cdot\,]\!]$. For all $x, y \in \mathbb{R}^n$, the *interval property* means that if $u, v \in [\![\,x, y\,]\!]$, then $[\![\,u, v\,]\!] \subset [\![\,x, y\,]\!]$. Let us consider the interval defined for all $x, y \in \mathbb{R}^n$ by $[\![\,x, y\,]\!] = \mathbb{B}^\sharp[x,y]$.

Proposition 3.9. *$\mathbb{B}^\sharp$-convex sets have the interval property.*

Proof. Since $\mathbb{B}^\sharp[x,y] = Co^\infty(x,y)$ it is $\mathbb{B}^\sharp$-convex. Thus, from [6], for all $u, v \in \mathbb{B}^\sharp[x,y]$, $Co^\infty(u,v) \subset \mathbb{B}^\sharp[x,y]$ and consequently $\mathbb{B}^\sharp[u,v] \subset \mathbb{B}^\sharp[x,y]$, which ends the proof. $\square$

A convexity has the *decomposition property* if for all $z \in [\![x,y]\!]$, we have $[\![x,y]\!] = [\![x,z]\!] \cup [\![z,y]\!]$. In the following, we prove that $\mathbb{B}^\sharp$-convex sets have this property. To do that, one considers the intermediate points between two points x and y defined in [6].

Proposition 3.10. *$\mathbb{B}^\sharp$-convex sets have the decomposition property. Moreover, for all $x, y \in \mathbb{R}^n$ $\mathbb{B}^\sharp[x,y] = \mathbb{B}^\sharp\big[\Gamma(x,y)\big]$.*

Proof. Suppose that $z \in \mathbb{B}^\sharp[x,y]$. There exists $i_m \in \mathcal{I}(x,y)$ such that:

$$z \in \mathbb{B}\Big[\gamma(x,y;-\frac{x_{i_m}}{y_{i_m}}), \gamma(x,y;-\frac{x_{i_{m+1}}}{y_{i_{m+1}}})\Big].$$

Therefore:

$$\mathbb{B}^\sharp[x,z] = \bigcup_{k=0}^{m-1} \mathbb{B}\Big[\gamma(x,y;-\frac{x_{i_k}}{y_{i_k}}), \gamma(x,y;-\frac{x_{i_{k+1}}}{y_{i_{k+1}}})\Big] \cup \mathbb{B}\Big[\gamma(x,y;-\frac{x_{i_m}}{y_{i_m}}), z\Big]$$

and

$$\mathbb{B}^\sharp[z,y] = \bigcup_{k=m+1}^{n(x,y)+1} \mathbb{B}\Big[\gamma(x,y;-\frac{x_{i_k}}{y_{i_k}}), \gamma(x,y;-\frac{x_{i_{k+1}}}{y_{i_{m+1}}})\Big] \cup \mathbb{B}\Big[z, \gamma(x,y;-\frac{x_{i_{k+1}}}{y_{i_{m+1}}})\Big]$$

Since the decomposition property holds for oppositive vectors, we have:

$$\mathbb{B}^\sharp\Big[\gamma(x,y;-\frac{x_{i_m}}{y_{i_k}}), \gamma(x,y;-\frac{x_{i_{k+1}}}{y_{i_{m+1}}})\Big] = \mathbb{B}^\sharp\Big[\gamma(x,y;-\frac{x_{i_m}}{y_{i_k}}), z\Big] \cup \mathbb{B}^\sharp\Big[z, \gamma(x,y;-\frac{x_{i_{k+1}}}{y_{i_{m+1}}})\Big].$$

Hence:

$$\mathbb{B}^\sharp[x,z] \cup \mathbb{B}^\sharp[z,y] = \bigcup_{k=0}^{n(x,y)-1} \mathbb{B}\Big[\gamma(x,y;-\frac{x_{i_k}}{y_{i_k}}), \gamma(x,y;-\frac{x_{i_{k+1}}}{y_{i_{k+1}}})\Big] = \mathbb{B}^\sharp[x,z],$$

which ends the first part of the proof. Moreover, since we have $\Gamma[x,y] \subset \mathbb{B}^\sharp[x,y]$, the decomposition property implies that

$$\mathbb{B}^\sharp\big[\Gamma(x,y)\big] = \bigcup_{k=0}^{n(x,y)-1} \mathbb{B}\Big[\gamma(x,y;-\frac{x_{i_k}}{y_{i_k}}),\gamma(x,y;-\frac{x_{i_{k+1}}}{y_{i_{k+1}}})\Big] = \mathbb{B}^\sharp[x,z]. \qquad \square$$

3.3. On the recursive property

In this section, we focus on the case of a $\mathbb{B}$-polytope of the form $Co^\infty(A)$. It is shown that the recursive property is not satisfied for such an algebraic structure.

For all $x \in \mathbb{R}^n$, the recursive property means that

$$[\![S \cup \{x\}]\!] = \bigcup_{y\in S} [\![\{x\} \cup \{y\}]\!] \tag{22}$$

Let us consider the points $x = (-1,-1)$, $y = (-1,1)$, and $z = (1,1)$ in $\mathbb{R}^2$.

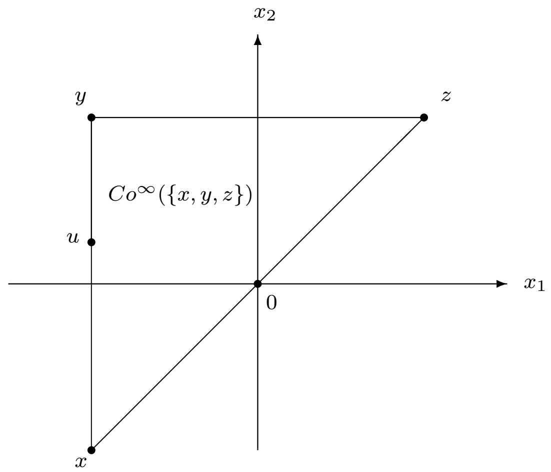

Figure 3.3: Non-Recursive Structure

It is easy to see that for all $u \in \mathbb{B}^\sharp[x,y]\backslash\{x\} = Co^\infty(x,y)\backslash\{x\}$ and that we have

$$\mathbb{B}^\sharp[u,z] = \mathbb{B}^\sharp[u,y] \cup \mathbb{B}^\sharp[y,z] = Co^\infty(u,y) \cup Co^\infty(y,z). \tag{23}$$

Moreover, for all p

$$\mathbb{B}^\sharp[x,z] = Co^\infty(x,z) = Co^p(x,z). \tag{24}$$

Thus:

$$Co^p(\{x,y,z\}) = Co^\infty(\{x,y,z\}). \tag{25}$$

Hence $Co^\infty(\{x,y,z\})$ is the triangle $[x,y,z]$. It follows that

$$\bigcup_{u\in Co^\infty(x,y)} Co^\infty(u,z) = Co^\infty(x,y) \cup Co^\infty(y,z) \cup Co^\infty(x,z). \tag{26}$$

However, one can see that, for example, the point $(-\frac{1}{2},\frac{1}{2}) \notin \bigcup_{u\in Co^\infty(x,y)} Co^\infty(u,z)$.

Hence
$$Co^{\infty}(\{x,y,z\}) \neq \bigcup_{u\in Co^{\infty}(x,y)} Co^{\infty}(u,z). \tag{27}$$

Consequently, the recursive property does not hold for $\mathbb{B}$-polytopes. Similarly, it is easy to see that $\mathbb{B}^{\sharp}[\{x,y,z\}]$ is the triangle $[x,y,z]$. Therefore, $\mathbb{B}^{\sharp}[\{x,y,z\}] \neq \bigcup_{u\in\mathbb{B}^{\sharp}[x,y]} \mathbb{B}^{\sharp}(u,z)$.

In general, given an arbitrary subset S of $\mathbb{R}^n$, $Co^{\infty}(S)$ is not identical to $\mathbb{B}^{\sharp}[S]$. Suppose that $S =]x,y] \cup [y,z]$. Clearly, S is not closed. However, for all natural numbers p, $Co^p(S) = [x,y,z]$, which implies that $Co^{\infty}(S) = [x,y,z]$. Moreover, $\mathrm{cl}\,(S) = [x,y] \cup [y,z] \neq [x,y,z]$. Since S is a $\mathbb{B}^{\sharp}$-convex set, it follows $\mathrm{cl}\,\mathbb{B}^{\sharp}[S] = S \neq Co^{\infty}(S)$. Notice also that

$$\mathbb{B}^{\sharp}[\mathrm{cl}\,(S)] = [x,y,z] \neq [x,y] \cup [y,z] = \mathrm{cl}\,(S) = \mathrm{cl}\,(\mathbb{B}^{\sharp}[S]).$$

It is not clear, however, that for all finite parts A of $\mathbb{R}^n$ $Co^{\infty}(A)$ and $\mathbb{B}^{\sharp}[A]$ are different. It was shown above that this is not the case when $\mathrm{Card}\,A = 2$.

4. $\mathbb{B}$-polytopes and Painlevé-Kuratowski Limit

The purpose of this section is to show that, for all finite parts A of $\mathbb{R}^n$, the $\mathbb{B}$-polytope $Co^{\infty}(A)$ is the Kuratowski-Painlevé limit of the sequence of sets $\{Co^p(A)\}_{p\in\mathbb{N}}$ (namely the upper and lower limits are identical). To do that, for all natural numbers p, an explicit algebraic form of the extreme points of the ϕ_p-polytope $Co^p(A) \cap K$ is given, where K is an n-dimensional orthant of $\mathbb{R}^n$. Along this line, it is shown that these extreme points have an explicit limit that allows to characterize $Co^{\infty}(A) \cap K$. To achieve this objective an useful notion of determinant in limit is introduced in the next subsection. A more complete framework is proposed in [8].

4.1. φ_p-linear Endomorphisms and Determinant in Limit

Let $\mathcal{L}(\mathbb{R}^n)$ denote the set of all the linear endomorphisms defined over $\mathbb{R}^n$. A map $f\colon \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is *φ_p-linear* if for all $\lambda \in \mathbb{R}$, $f(\lambda x \overset{p}{+} y) = \lambda f(x) \overset{p}{+} f(y)$. Parallel to the notation above, let $\mathcal{L}^{(p)}(\mathbb{R}^n)$ denote the set of all the φ_p-linear endomorphisms.

We denote by $\mathbb{M}_n(\mathbb{R})$ the set of all the matrices defined over the group $\mathbb{R}^n$ and define the map $\Phi_p\colon \mathbb{M}_n(\mathbb{R}) \longrightarrow \mathbb{M}_n(\mathbb{R})$ for all matrices $A = (a_{i,j})_{\substack{i=1...n\\ j=1..n}} \in \mathbb{M}_n(\mathbb{R})$ by:

$$\Phi_p(A) = (\varphi_p(a_{i,j}))_{\substack{i=1...n\\ j=1...n}} = \left(a_{i,j}{}^{2p+1}\right)_{\substack{i=1...n\\ j=1...n}}. \tag{28}$$

Its reciprocal is the map $\Phi_p^{-1}\colon \mathbb{M}_n(\mathbb{R}) \longrightarrow \mathbb{M}_n(\mathbb{R})$ defined by:

$$\Phi_p^{-1}(A) = \left(\varphi_p^{-1}(a_{i,j})\right)_{\substack{i=1...n\\ j=1...n}} = \left(a_{i,j}{}^{\frac{1}{2p+1}}\right)_{\substack{i=1...n\\ j=1...n}}. \tag{29}$$

Φ_p is a natural extension of the map ϕ_p from $\mathbb{R}^n$ to $\mathcal{M}_n(\mathbb{R})$. $\Phi_p(A)$ is the $2p+1$ Hadamard power of matrix A. In addition, we introduce the matrix product:

$$A \overset{p}{\cdot} x = \sum_{j\in[n]}^{\varphi_p} x_j . a^j, \tag{30}$$

where a^j is the j-th column of A. It is straightforward to show that this formulation is equivalent to the following:

$$A \overset{p}{\cdot} x = \phi_p^{-1}\big(\Phi_p(A).\phi_p(x)\big). \tag{31}$$

It is easy to see that the map $x \mapsto A \overset{p}{\cdot} x$ is φ_p-linear. Conversely, if g is a φ_p-linear map, then it can be represented by a matrix A such that $g(x) = A \overset{p}{\cdot} x$ for all $x \in \mathbb{R}^n$. Notice that the identity matrix I is invariant with respect of Φ_p.

A φ_p-linear endomorphism is invertible if and only if its matrix representation $\Phi_p(A)$ is invertible. Therefore, it is useful to introduce a suitable notion of determinant. For all $n \times n$ matrices A, let $|A|$ denotes its determinant. The φ_p-determinant of a matrix $A \in \mathcal{M}_n(\mathbb{R})$ is defined by:

$$|A|_p = \varphi_p^{-1}|\Phi_p(A)|. \tag{32}$$

Let S_n denotes the set of all the permutations defined on $[n]$. Using the Leibnitz formula yields

$$|A|_p = \Big(\sum_{\sigma\in S_n} \operatorname{sgn}(\sigma) \prod_{i\in[n]} a_{i,\sigma(i)}^{2p+1}\Big)^{\frac{1}{2p+1}}. \tag{33}$$

This implies that is f is a linear endomorphism having a matrix representation A, then $|f^{(p)}|_p = |A|_p$.

Proposition 4.1. *Let $f\colon \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a linear endomorphism having a matrix representation A. For all $p \in \mathbb{N}$, let $f^{(p)}$ be its Φ_p-linear transformation. Then:*

$$\lim_{p\to\infty} |A|_p := |A|_\infty = \boxplus_{\sigma\in S_n} \big(\operatorname{sgn}(\sigma) \prod_{i\in[n]} a_{i,\sigma(i)}\big).$$

Proof. From [6] we have:

$$\lim_{p\to\infty} \Big(\sum_{\sigma\in S_n} \operatorname{sgn}(\sigma) \prod_{i\in[n]} a_{i,\sigma(i)}^{2p+1}\Big)^{\frac{1}{2p+1}} = \boxplus_{\sigma\in S_n} \big(\operatorname{sgn}(\sigma) \prod_{i\in[n]} a_{i,\sigma(i)}\big). \qquad \square$$

Let $f\colon \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a linear endomorphism and let A be its matrix representation in the canonical basis. The map $T^{(p)}\colon \mathcal{L}(\mathbb{R}^n) \longrightarrow \mathcal{L}^p(\mathbb{R}^n)$ defined by:

$$T^{(p)}(f)(x) = f^{(p)}(x) := \phi_p^{-1}\big(\Phi_p(A)\phi_p(x)\big)$$

is called the Φ_p-*transformation of* f.

Proposition 4.2. *Let $f\colon \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a linear endomorphism having a matrix representation A. For all $p \in \mathbb{N}$, let $f^{(p)}$ be its φ_p-linear transformation. If $|A|_\infty \neq 0$, then there is some $p_0 \in \mathbb{N}$ such that for all $p \geq p_0$ $f^{(p)}$ is φ_p-invertible and for all $b \in \mathbb{R}^n$, there exists a solution $x^{(p)}$ to the system $A \overset{p}{\cdot} x = b$ expressed as:*

$$x_i^{(p)} = \frac{|A^{(i)}|_p}{|A|_p} = \frac{\left|\Phi_p(A^{(i)})\right|^{\frac{1}{2p+1}}}{|\Phi_p(A)|^{\frac{1}{2p+1}}},$$

where $A^{(i)}$ is obtained from A by dropping column i and replacing it with b.

Moreover, we have $\lim\limits_{p\to\infty} x^{(p)} = x^\star$ with $x_i^\star = \dfrac{|A^{(i)}|_\infty}{|A|_\infty}$ for all $i \in [n]$.

Proof. Since $|A|_\infty \neq 0$ and $\lim_{p\to\infty} |A|_p = |A|_\infty$, there is some p_0 such that for all $p \geq p_0$, $|A|_p \neq 0$, which implies that A is φ_p-invertible. In such a case, there exists an uniqueness solution to the system $A \overset{p}{\cdot} x = b$, which is $x^{(p)} = A^{-1,p}b$. Moreover:

$$A \overset{p}{\cdot} x = b \iff \phi_p^{-1}\left(\Phi_p(A)\phi_p(x)\right) = b \iff \Phi_p(A)\phi_p(x) = \phi_p(b).$$

Since $f^{(p)}$ is φ_p-invertible, it follows that $\Phi_p(A)$ is invertible. Set $u = \phi_p(x)$. The system $\Phi_p(A)u = \phi_p(b)$ has a solution for all $p \geq p_0$. Applying the Cramer rule the solution is the vector $u^{(p)}$ satisfying the relation:

$$u^{(p)} = \frac{\left|[\Phi_p(A)]^{(i)}\right|}{|\Phi_p(A)|} = \frac{\left|\Phi_p(A^{(i)})\right|}{|\Phi_p(A)|}.$$

Since $x^{(p)} = \phi_p^{-1}(u^{(p)})$, we obtained the solution. From Proposition 4.1 we finally get $\lim_{p\to\infty} |A^{(i)}|_p = |A^{(i)}|_\infty$ and $\lim_{p\to\infty} |A|_p = |A|_\infty$. $\square$

In the following, we say that $x^\star \in \mathbb{R}^n$ is a *limit point* of the sequence of sets $\{x \in \mathbb{R}^n : A \overset{p}{\cdot} x = b\}_{p\in\mathbb{N}}$, if there exists a sequence $\{x^{(p)}\}_{p\in\mathbb{N}}$ with $A \overset{p}{\cdot} x^{(p)} = b$ for all p, and $\lim_{p\to\infty} x^{(p)} = x$.

Example 4.3. Let us consider the matrix $A \in \mathcal{M}(\mathbb{R}^3)$ defined by:

$$A = \begin{pmatrix} -1 & 3 & 4 \\ 2 & 5 & -1 \\ 4 & -2 & 1 \end{pmatrix}$$

and let $b = (1, 1, 1)$. We have

$$A \overset{p}{\cdot} x = b \iff \begin{cases} \left((-1)^{2p+1} {x_1}^{2p+1} + 3^{2p+1} {x_2}^{2p+1} + 4^{2p+1} {x_3}^{2p+1}\right)^{\frac{1}{2p+1}} & = 1 \\ \left(2^{2p+1} {x_1}^{2p+1} + 5^{2p+1} {x_2}^{2p+1} + (-1)^{2p+1} {x_3}^{2p+1}\right)^{\frac{1}{2p+1}} & = 1 \\ \left(4^{2p+1} {x_1}^{2p+1} + (-2)^{2p+1} {x_2}^{2p+1} + {x_3}^{2p+1}\right)^{\frac{1}{2p+1}} & = 1. \end{cases}$$

Moreover, by definition:

$$A^{(1)} = \begin{pmatrix} 1 & 3 & 4 \\ 1 & 5 & -1 \\ 1 & -2 & 1 \end{pmatrix}, \quad A^{(2)} = \begin{pmatrix} -1 & 1 & 4 \\ 2 & 1 & -1 \\ 4 & 1 & 1 \end{pmatrix}, \quad A^{(3)} = \begin{pmatrix} -1 & 3 & 1 \\ 2 & 5 & 1 \\ 4 & -2 & 1 \end{pmatrix}.$$

Hence:

$$\begin{vmatrix} -1 & 3 & 4 \\ 2 & 5 & -1 \\ 4 & -2 & 1 \end{vmatrix}_\infty = (-1)\cdot 5\cdot 1 \boxplus 3\cdot(-1)\cdot 4 \boxplus (-2)\cdot 2\cdot 4 \boxplus (-4\cdot 5\cdot 4) \boxplus (-1)\cdot(-2)\cdot 1 \boxplus (-2\cdot 3\cdot 1) = -80,$$

$$\begin{vmatrix} 1 & 3 & 4 \\ 1 & 5 & -1 \\ 1 & -2 & 1 \end{vmatrix}_\infty = 1\cdot 5\cdot 1 \boxplus 3\cdot(-1)\cdot 1 \boxplus 1\cdot(-2)\cdot 4 \boxplus (-4\cdot 5\cdot 1) \boxplus (-3\cdot 1\cdot 1) \boxplus (-2\cdot 1\cdot 1) = -20,$$

$$\begin{vmatrix} -1 & 1 & 4 \\ 2 & 1 & -1 \\ 4 & 1 & 1 \end{vmatrix}_\infty = (-1)\cdot 1\cdot 1 \boxplus 2\cdot 1\cdot 4 \boxplus 1\cdot(-1)\cdot 4 \boxplus (-4\cdot 1\cdot 4) \boxplus (-1\cdot 2\cdot 1) \boxplus (-1\cdot 1\cdot 1) = -16,$$

$$\begin{vmatrix} -1 & 3 & 1 \\ 2 & 5 & 1 \\ 4 & -2 & 1 \end{vmatrix}_\infty = (-1)\cdot 5\cdot 1 \boxplus (-2)\cdot 2\cdot 1 \boxplus 3\cdot 1\cdot 4 \boxplus (-4\cdot 5\cdot 1) \boxplus (-1\cdot 2\cdot 1) \boxplus (-2\cdot 3\cdot 1) = -20.$$

The point $x^\star$ with $x_1^\star = \dfrac{-20}{-80} = \dfrac{1}{4}$, $x_2^\star = \dfrac{-16}{-80} = \dfrac{1}{5}$, $x_3^\star = \dfrac{-20}{-80} = \dfrac{1}{4}$, is a limit point of the sequence of systems $\{x \in \mathbb{R}^n : A \overset{p}{\cdot} x = b\}_{p\in\mathbb{N}}$.

4.2. Generalized intermediate points

For all natural numbers m, let $\{e_1, ..., e_m\}$ denotes the canonical basis of $\mathbb{R}^m$. Moreover, for all finite subsets J of $[m]$, we define

$$\mathbb{R}^J = \Big\{ \sum_{j\in J} x_j e_j : x_j \in \mathbb{R}, \forall j \in J \Big\} \text{ and } \mathbb{R}^J_+ = \Big\{ \sum_{j\in J} x_j e_j : x_j \in \mathbb{R}_+, \forall j \in J \Big\}.$$

For all nonempty subsets J of $[m]$ we denote by $\Delta_J^{(p)}$ the φ_p-simplex defined by $\Delta_J^{(p)} = \{t \in \mathbb{R}^J_+ : \overset{\varphi_p}{\sum}_{j\in J} t_j = 1\}$. Let $A = \{x^{(1)}, ..., x^{(m)}\}$ be a finite subset of $\mathbb{R}^n$. For all subsets J of $[m]$ let $\xi_J \colon \mathbb{R}^{n\times m}_+ \times \mathbb{R}^m_+ \longrightarrow \mathbb{R}^n$ be the map defined by:

$$\xi_J^{(p)}\big(x^{(1)}, ..., x^{(m)}, t\big) = \overset{\varphi_p}{\sum_{j\in J}} t_j x^{(j)}. \tag{34}$$

Let I be a subset of $[n]$ such that $\mathrm{Card}(I) = \mathrm{Card}(J) - 1$. For all $t_{I,J}^{(p)} \in \Delta_J^{(p)}$, we say that the point $\xi_J^{(p)}\big(x^{(1)}, ..., x^{(m)}, t_{I,J}^{(p)}\big)$ is an (I,J)*-intermediate point of order* p *for* A, if $t_{I,J}^{(p)}$ is the uniqueness element of $\Delta_J^{(p)}$ such that

$$\Big(\xi_J^{(p)}\big(x^{(1)}, ..., x^{(m)}, t_{I,J}^{(p)}\big)\Big)_i = 0 \quad \text{for all } i \in I. \tag{35}$$

$t_{I,J}^{(p)}$ is called an (I,J)*-intermediate value of order* p *for* A.

In the following, $\Xi^{(p)}(A)$ denotes the set of all (I,J)-intermediate points of order p for A. It is immediate to see that this notion generalizes the concept of i-intermediate point defined in the earlier subsection and in [6].

For all subsets H of $\mathbb{N}\backslash\{0\}$, let ν_H denotes the uniqueness increasing bijection defined from H to $\left[\text{ Card}\,H\,\right]$ (i.e. for all $l,m\in H$, if $m>l$, then $\nu_I(m)>\nu_I(l)$). For the sake of simplicity, let us denote $n_I=\text{Card}\,I$ and $n_J=\text{Card}\,J$.

Suppose that $I\subset[n]$ and $J\subset[m]$. Let

$$\Lambda_{I,J}\big[x^{(1)},...,x^{(m)}\big]=(\lambda_{i,j})_{\substack{i\in[n_I+1]\\ j\in[n_J]}}$$

be the matrix of $\mathcal{M}_{n_I+1,n_J}(\mathbb{R})$ defined by:

$$\lambda_{i,j}=\begin{cases} x^{(\nu_J(j))}_{\nu_I(i)} & \text{if} \quad (i,j)\in[n_I]\times[n_J] \\ 1 & \text{if} \quad (i,j)\in\{n_I+1\}\times[n_J] \end{cases} \tag{36}$$

In the remainder we say that $\Lambda_{I,J}[x^{(1)},...,x^{(m)}]$ *is an* (I,J)*-intermediate matrix for* A. Notice that since $\text{Card}\,I=\text{Card}\,J-1$, $\Lambda_{I,J}[x^{(1)},...,x^{(m)}]$ is a square matrix. For the sake of simplicity we will denote $\Lambda_{I,J}:=\Lambda_{I,J}[x^{(1)},...,x^{(m)}]$.

Let $\kappa_J\colon\mathbb{R}^{n_J}\longrightarrow\mathbb{R}^J$ be the map defined by $\kappa_J(s)=\sum_{j\in J}s_{\nu_J(j)}e_j$. If $t=\kappa_J(s)$, then this implies that for all $j\in J$, $t_j=s_{\nu_J(j)}$.

By construction, it follows that $t^{(p)}_{I,J}\in\Delta^{(p)}_J$ is an intermediate value of order p for A if and only il there is some solution $s^\star\in\Delta_{[n_J]}$ of the φ_p-matrix equation:

$$\Lambda_{I,J}\overset{p}{\cdot}s=e_{n_I+1}, \tag{37}$$

such that $t^{(p)}_{I,J}=\kappa_J(s^\star)$. We say that an (I,J)-intermediate point is *regular* if $\Lambda_{I,J}$ is φ_p-invertible. In the following, for all $k\in[n_J]$, let

$$\Lambda^{(k)}_{I,J}=(\lambda^{(k)}_{i,j})_{\substack{i\in[n_I+1]\\ j\in[n_J]}}$$

denote the matrix obtained by replacing the j-th column of $\Lambda_{I,J}$ with e_{n_I+1}. Along this line, one can retrieve as a special case the intermediate points in the case where $m=2$.

Remark 4.4. Let $x^{(1)},x^{(2)}\in\mathbb{R}^n$. Suppose that there is some $i\in[n]$ such that $x^{(1)}_i x^{(2)}_i<0$. In such a case, we have

$$\Lambda_{\{1,2\},\{i\}}=\begin{pmatrix} x^{(1)}_i & x^{(2)}_i \\ 1 & 1\end{pmatrix}.$$

Since $$\begin{vmatrix} x^{(1)}_i & x^{(2)}_i \\ 1 & 1\end{vmatrix}_p=\Big(\big(x^{(1)}_i\big)^{2p+1}-\big(x^{(2)}_i\big)^{2p+1}\Big)^{\frac{1}{2p+1}}\neq 0,$$

the Cramer solutions of the system $\begin{pmatrix} x^{(1)}_i & x^{(2)}_i \\ 1 & 1\end{pmatrix}\overset{p}{\cdot}\begin{pmatrix} t_1 \\ t_2\end{pmatrix}=\begin{pmatrix} 0 \\ 1\end{pmatrix}$ are

$$t^{(p)}_{\{1,2\},1} = \frac{\begin{vmatrix} 0 & x_i^{(2)} \\ 1 & 1 \end{vmatrix}_p}{\begin{vmatrix} x_i^{(1)} & x_i^{(2)} \\ 1 & 1 \end{vmatrix}_p} = \frac{|x_i^{(2)}|}{(|x_i^{(1)}|^{2p+1} + |x_i^{(2)}|^{2p+1})^{\frac{1}{2p+1}}}$$

and
$$t^{(p)}_{\{1,2\},2} = \frac{\begin{vmatrix} x_i^{(1)} & 0 \\ 1 & 1 \end{vmatrix}_p}{\begin{vmatrix} x_i^{(1)} & x_i^{(2)} \\ 1 & 1 \end{vmatrix}_p} = \frac{|x_i^{(1)}|}{(|x_i^{(1)}|^{2p+1} + |x_i^{(2)}|^{2p+1})^{\frac{1}{2p+1}}}.$$

We retrieve the solutions established in [6].

In the following, an example is provided.

Example 4.5. Suppose that $n = 3$ and $m = 4$. Let $A = \{x^{(1)}, x^{(2)}, x^{(3)}, x^{(4)}\}$ with $x^{(1)} = (-2, 2, 3)$, $x^{(2)} = (3, -1, -4)$, $x^{(3)} = (1, 1, 1)$ and $x^{(4)} = (-1, -3, -1)$. Let $J = \{1, 2, 4\}$. We have

$$\xi_{\{1,2,4\}}(x^{(1)}, x^{(2)}, x^{(3)}, x^{(4)}; t_1, t_2, t_4) = \sum_{j\in\{1,2,4\}}^{\varphi_p} t_i x_i = t_1 \begin{pmatrix} -2 \\ 2 \\ 3 \end{pmatrix} \overset{p}{+} t_2 \begin{pmatrix} 3 \\ -1 \\ -4 \end{pmatrix} \overset{p}{+} t_3 \begin{pmatrix} -1 \\ -3 \\ -1 \end{pmatrix}.$$

Suppose that $I = \{1, 2\}$. Then:

$$\Lambda_{\{1,2\},\{1,2,4\}} = \begin{pmatrix} -2 & 3 & -1 \\ 2 & -1 & -3 \\ 1 & 1 & 1 \end{pmatrix}, \quad \Lambda^{(1)}_{\{1,2\},\{1,2,4\}} = \begin{pmatrix} 0 & 3 & -1 \\ 0 & -1 & -3 \\ 1 & 1 & 1 \end{pmatrix},$$

$$\Lambda^{(2)}_{\{1,2\},\{1,2,4\}} = \begin{pmatrix} -2 & 0 & -1 \\ 2 & 0 & -3 \\ 1 & 1 & 1 \end{pmatrix}, \quad \Lambda^{(3)}_{\{1,2\},\{1,2,4\}} = \begin{pmatrix} -2 & 3 & 0 \\ 2 & -1 & 0 \\ 1 & 1 & 1 \end{pmatrix}.$$

The Cramer solution of the φ_p-linear system $\Lambda_{\{1,2\},\{1,2,4\}} \overset{p}{\cdot} \begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ yields

$$t^{(p)}_{\{1,2\},\{1,2,4\},1} = s_1^\star = \frac{\left|\Lambda^{(1)}_{\{1,2\},\{1,2,4\}}\right|_p}{\left|\Lambda_{\{1,2\},\{1,2,4\}}\right|_p} = \frac{\big(-9^{2p+1} - 1\big)^{\frac{1}{2p+1}}}{\big(-9^{2p+1} - 2^{2p+1} - 1\big)^{\frac{1}{2p+1}}},$$

$$t^{(p)}_{\{1,2\},\{1,2,4\},2} = s_2^\star = \frac{\left|\Lambda^{(2)}_{\{1,2\},\{1,2,4\}}\right|_p}{\left|\Lambda_{\{1,2\},\{1,2,4\}}\right|_p} = \frac{\big(-2^{2p+1} - 6^{2p+1}\big)^{\frac{1}{2p+1}}}{\big(-9^{2p+1} - 2^{2p+1} - 1\big)^{\frac{1}{2p+1}}},$$

$$t^{(p)}_{\{1,2\},\{1,2,4\},4} = s_3^\star = \frac{\left|\Lambda^{(3)}_{\{1,2\},\{1,2,4\}}\right|_p}{\left|\Lambda_{\{1,2\},\{1,2,4\}}\right|_p} = \frac{\big(2^{2p+1} - 6^{2p+1}\big)^{\frac{1}{2p+1}}}{\big(-9^{2p+1} - 2^{2p+1} - 1\big)^{\frac{1}{2p+1}}}.$$

One can check that for all $p \in \mathbb{N}$, $t^{(p)}_{\{1,2\},\{1,2,4\}} \geq 0$. Moreover

$$\lim_{p\to\infty} t^{(p)}_{\{1,2\},\{1,2,4\},1} = 1, \quad \text{and} \quad \lim_{p\to\infty} t^{(p)}_{\{1,2\},\{1,2,4\},2} = \lim_{p\to\infty} t^{(p)}_{\{1,2\},\{1,2,4\},4} = \frac{2}{3}.$$

Lemma 4.6. *Let $A = \{x^{(1)}, ..., x^{(m)}\}$ be a finite subset of $\mathbb{R}^n$. If there is some $t^{(p)}_{I,J} \in \Delta^{(p)}_J$ such that $\xi^{(p)}_J\big(x^{(1)}, ..., x^{(m)}, t^{(p)}_{I,J}\big) = \sum^{\varphi_p}_{j\in J} t^{(p)}_{I,J,j}\, x^{(j)}$ is a regular (I,J)-intermediate point of order p for A, then*

$$t^{(p)}_{I,J} = \sum^{\varphi_p}_{j\in J} \frac{\big|\Lambda^{(\nu_J(j))}_{I,J}\big|_p}{\big|\Lambda_{I,J}\big|_p} e_j \quad \text{and} \quad \xi^{(p)}_J(x^{(1)}, ..., x^{(m)}, t^{(p)}_{I,J}) = \sum^{\varphi_p}_{j\in J} \frac{\big|\Lambda^{(\nu_J(j))}_{I,J}\big|_p}{\big|\Lambda_{I,J}\big|_p} x^{(j)}.$$

Proof. By definition, if $\xi^{(p)}_J\big(x^{(1)}, ..., x^{(m)}, t^{(p)}_{I,J}\big)$ is a regular (I,J)-intermediate point of order p for A, then the (I,J)-intermediate value is $t^{(p)}_{I,J} = \kappa_J(s^\star)$ where $s^\star$ is the uniqueness solution of the matrix equation:

$$\Lambda_{I,J} \overset{p}{\cdot} s = e_{n_I+1}. \tag{38}$$

Since $\xi^{(p)}_J\big(x^{(1)}, ..., x^{(m)}, t^{(p)}_{I,J}\big)$ is regular, it follows that the matrix $\Lambda_{I,J}$ is φ_p-invertible. For all $j \in J$, let us denote $\Lambda^{(\nu_J(j))}_{I,J}$ the matrix obtained by replacing the $\nu_J(j)$-th column with e_{n_I+1}. Since ν_J is an increasing bijection from J to $[n_J]$, from the Cramer formula, we have for all $j \in J$:

$$s^\star_{\nu_J(j)} = \frac{\big|\Lambda^{(\nu_J(j))}_{I,J}\big|_p}{\big|\Lambda_{I,J}\big|_p}.$$

Since by definition, $t^{(p)}_{I,J,j} = s^\star_j$ for all $j \in J$, this yields the solution. The second statement is then an immediate consequence. □

Lemma 4.7. *Suppose there is some $x = (x_1, ..., x_n) \in \mathbb{R}^n$ such that $\boxplus_{i\in[n]} x_i = 0$. Then for all $p \in \mathbb{N}$ we have $\sum^{\varphi_p}_{i\in[n]} x_i = 0$.*

Proof. Let $A[x] = \{\alpha \in \mathbb{R}_+ : |x_i| = \alpha, i \in [n]\}$. Since $\boxplus_{i\in[n]} x_i = 0$, for all α, we have $\xi[x](\alpha) = \text{Card}\{i : x_i = \alpha\} - \text{Card}\{i : x_i = -\alpha\} = 0$. Hence, for all $\alpha \in A[x]$, $\sum^{\varphi_p}_{|x_i|=\alpha} x_i = 0$. Thus

$$\sum^{\varphi_p}_{i\in[n]} x_i = \sum^{\varphi_p}_{\alpha\in A[x]} \sum^{\varphi_p}_{|x_i|=\alpha} x_i = 0. \qquad \square$$

In the following, we say that $\zeta_{I,J} \in \mathbb{R}^n$ is an *(I,J)-intermediate point for A in limit*, if there exist a natural number $p_{I,J}$ and a sequence of regular (I,J)-intermediate points $\Big\{\xi^{(p)}_J\big(x^{(1)}, ..., x^{(m)}, t^{(p)}_{I,J}\big)\Big\}_{p\geq p_{I,J}}$ with $t^{(p)}_{I,J} \in \Delta^{(p)}_J$ for all $p \geq p_{I,J}$ such that:

$$\zeta_{I,J} = \lim_{p\to\infty} \xi^{(p)}_J\big(x^{(1)}, ..., x^{(m)}, t^{(p)}_{I,J}\big). \tag{39}$$

Let us denote by $\Xi^\infty(A)$ the set of all (I,J)-intermediate points in the limit.

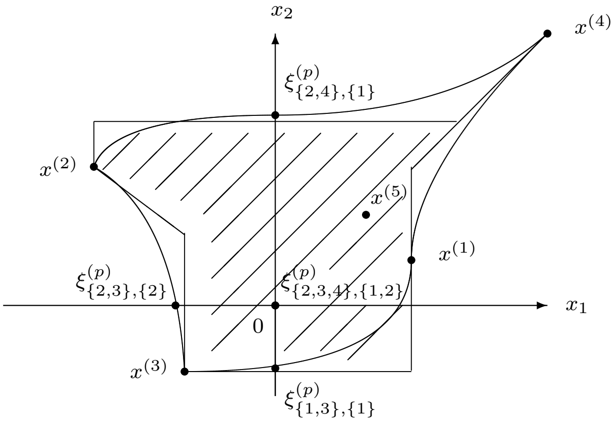

Figure 4.1: Φ_p-polytope and intermediate points

Lemma 4.8. *Let $A = \{x^{(1)}, ..., x^{(m)}\}$ be a finite subset of $\mathbb{R}^n$. Take some $(I,J) \in [n]\times[m]$ and suppose that there exists some natural number p and some $t^{(p)}_{I,J} \in \Delta^{(p)}_J$ such that $\xi_J\big(x^{(1)}, ..., x^{(m)}, t^{(p)}_{I,J}\big)$ is a regular (I,J)-intermediate point of order p. Then:*

(a) $|\Lambda_{I,J}|_\infty \neq 0$.

(b) *There exists a natural number $p_{I,J}$ such that $\Lambda_{I,J}$ is φ_p invertible for all $p \geq p_{I,J}$.*

(c) *There exists an (I,J)-intermediate point in limit $\zeta_{I,J}$ such that:*

$$\zeta_{I,J} = \lim_{p\to\infty} \sum_{j\in J}^{\varphi_p} \frac{\big|\Lambda^{(\nu_J(j))}_{I,J}\big|_p}{\big|\Lambda_{I,J}\big|_p} x^{(j)} = \boxplus_{\substack{j\in J\\ \sigma\in S_{n_I+1}}} \alpha_{\sigma,j} x^{(j)},$$

where $$\alpha_{\sigma,j} = \operatorname{sgn}(\sigma) \prod_{i\in[n_I+1]} \lambda^{(\nu_J(j))}_{i,\sigma(i)}, \qquad \frac{1}{\big|\Lambda_{I,J}\big|_\infty} \boxplus_{\substack{j\in J\\ \sigma\in S_{n_I+1}}} \alpha_{\sigma,j} = 1$$

and $$\boxplus_{\sigma\in S_{n_I+1}} \alpha_{\sigma,j} \geq 0 \quad \textit{for all } j.$$

Proof. (a) From Lemma 4.7, if $|\Lambda_{I,J}|_\infty = 0$, then, for all natural numbers p, we have $|\Lambda_{I,J}|_p = 0$. However, given some $p \geq p_{I,J}$, if $\xi_J\big(x^{(1)}, ..., x^{(m)}, t^{(p)}_{I,J}\big)$ is a regular (I,J)-intermediate point of order (p), by hypothesis we have $|\Lambda_{I,J}|_p \neq 0$, which is a contradiction. Hence, $|\Lambda_{I,J}|_\infty \neq 0$. (b) If there is some natural number p and some $t^{(p)}_{I,J} \in \Delta^{(p)}_J$ such that $\xi_J\big(x^{(1)}, ..., x^{(m)}, t^{(p)}_{I,J}\big)$ is a regular (I,J)-intermediate point of order p, then $\Lambda_{I,J}$ is a square matrix and $|\Lambda_{I,J}|_p \neq 0$. Then, from Lemma 4.7, $|\Lambda_{I,J}|_\infty \neq 0$. Since $\lim_{p\to\infty} |\Lambda_{I,J}|_p = |\Lambda_{I,J}|_\infty \neq 0$, there is some $p_{I,J} \in \mathbb{N}$ such that for all $p > p_{I,J}$, $|\Lambda_{I,J}|_p \neq 0$, which proves (b). (c) Moreover, from Lemma 4.6, we have

$$\lim_{p\to\infty}\sum_{j\in J}^{\varphi_p}\left|\Lambda_{I,J}^{(\nu_J(j))}\right|_p x^{(j)}=\lim_{p\to\infty}\sum_{j\in J}^{\varphi_p}\sum_{\sigma\in S_{n_I+1}}^{\varphi_p}\big(\operatorname{sgn}(\sigma)\prod_{i\in[n_I+1]}\lambda_{i,\sigma(i)}^{(\nu_J(j))}\big)$$

$$=\boxplus_{\substack{j\in J\\ \sigma\in S_{n_I+1}}}\big(\operatorname{sgn}(\sigma)\prod_{i\in[n_I+1]}\lambda_{i,\sigma(i)}^{(\nu_J(j))}\big)x^{(j)}.$$

Moreover, for all $p\in\mathbb{N}$, $\sum_{j\in J}^{\varphi_p}t_{I,J,j}^{(p)}=\sum_{j\in J}^{\varphi_p}\frac{\left|\Lambda_{I,J}^{(\nu_J(j))}\right|_p}{\left|\Lambda_{I,J}\right|_p}=1$. Thus:

$$\lim_{p\to\infty}\sum_{j\in J}^{\varphi_p}\frac{\left|\Lambda_{I,J}^{(\nu_J(j))}\right|_p}{\left|\Lambda_{I,J}\right|_p}=\boxplus_{\substack{j\in J\\ \sigma\in S_{n_I+1}}}\alpha_{\sigma,j}=1. \qquad \square$$

4.3. Extreme points and convergence in limit

In the following, given a ϕ_p-convex subset C of $\mathbb{R}^n$, we say that z is a ϕ_p-extreme point if there do not exist $u,v\in C$ and $(s,t)\in\Delta_2^{(p)}$, such that $z=su\overset{p}{+}tv$. Given a n-dimensional orthant K, the subset $Co^p(A)\cap K$ has a ϕ_p-internal representation. This means that there exists a finite subset $\mathcal{E}_K^{(p)}(A)$ of ϕ_p-extreme points such that $Co^p(A)\cap K=Co^p(\mathcal{E}_K^{(p)}(A))$.

Lemma 4.9. *Let $A=\{x^{(1)},...,x^{(m)}\}$ be a finite subset of $\mathbb{R}^n$. For all n-dimensional ortants K and for all natural numbers p, if $z_K^{(p)}$ is a ϕ_p-extreme point of $Co^p(A)\cap K$, then there exists $(I,J)\subset[n]\times[m]$ such that $z_K^{(p)}$ is a regular (I,J)-intermediate point of order p for A.*

Proof. Suppose that $z_K^{(p)}$ is a ϕ_p-extreme point of $Co^p(A)\cap K$. This implies that $z_K^{(p)}\in \mathrm{cl}Co^p(A)\cap K$. We consider three cases:

(i) $z_K^{(p)}\notin \mathrm{cl}\,K$ and $z_K^{(p)}\in\mathrm{cl}\,Co^p(A)$. In such a case $z_K^{(p)}\in\mathrm{int}\,K$. Hence, $z_K^{(p)}$ is an extreme point of $Co^p(A)$. Therefore, there is some $j\in[m]$, such that $z_K^{(p)}=x^{(j)}$. Hence, the proposition holds true with $I=\emptyset$ and $J=\{j\}$.

(ii) $z_K^{(p)}\in\mathrm{cl}\,K$ and $z_K^{(p)}\notin\mathrm{cl}\,Co^p(A)$. In such a case 0 is the uniqueness extreme point of K. If $0\in A$, the result is obvious. If $0\notin A$, then 0 is not an extreme point of $Co^p(A)$. Consequently, one can find a subset J of $[m]$, and some uniqueness $t\in\Delta_J^{(p)}$ such that $0=\sum_{j\in J}^{\varphi_p}t_jx^{(j)}$. From the Caratheodory theorem, $\mathrm{Card}(J)\le n+1$. Since t is uniqueness the φ_p-linear map $t\mapsto\sum_{j\in J}^{\varphi_p}t_jx^{(j)}$ has a full rank. Hence one can pick some I in $2^{[n]}$ such that the matrix $\Lambda_{I,J}$ is φ_p-invertible. Hence, if s is the uniqueness solution of the system $\Lambda_{I,J}\overset{p}{\cdot}s=e_{n_I+1}$, we have $t=\kappa_J(s)$. Therefore, $z_K^{(p)}=0$ is a regular (I,J)-intermediate point of order p, with $I=[n]$.

(iii) $z_K^{(p)}\in\mathrm{cl}\,K$ and $z_K^{(p)}\in\mathrm{cl}\,Co^p(A)$. There exists some $\epsilon=(\epsilon_1,...,\epsilon_n)\in K$, with ϵ_i is -1 or 1 for all i and such that $K=\{x\in\mathbb{R}^n:\epsilon_ix_i\ge 0\}$. By construction, for all $x \in \mathrm{cl}\, K$, there is a subset I of $[n]$ such that $x_i = 0$ for all $i \in I$, and $\epsilon_i x_i \geq 0$ for all $i \notin I$. Hence, $(z_K^{(p)})_i = 0$, for all $i \in I$. Moreover, since $z_K^{(p)} \in \mathrm{cl}\,(K)$, it follows that if $z_K^{(p)}$ is a ϕ_p-extreme point of $Co^p(A) \cap K$, then there exists some $J \subset [m]$ and a uniqueness $t \in \Delta_J^{(p)}$ such that $z_K^{(p)} = \sum_{j \in J} t_j x^{(j)}$. Since $(z_K^{(p)})_i = 0$, for all $i \in I$, this ends the proof. □

In the following, for all finite subsets A of $\mathbb{R}^n$, let us denote $\mathcal{K}(A)$ the collection of all the n-dimensional orthant K such that $A \cap K \neq \emptyset$.

Lemma 4.10. *Let $A = \{x^{(1)}, ..., x^{(m)}\}$ be a finite subset of $\mathbb{R}^n$. For all natural numbers p, and all $K \in \mathcal{K}(A)$, we have $Co^p\big(\Xi^{(p)}(A) \cap K\big) = Co^p(A) \cap K$. Moreover*

$$Co^p(A) = \bigcup_{K \in \mathcal{K}(A)} Co^p\big(\Xi^{(p)}(A) \cap K\big).$$

Proof. For all $p \in \mathbb{N}$, let $\mathcal{E}_K^{(p)}$ be the set of all the extreme points of $Co^p(A) \cap K$. This implies that $Co^p\big(\mathcal{E}_K^{(p)}(A)\big) = Co^p(A) \cap K$. Since by definition $\Xi^{(p)}(A) \subset Co^p(A)$, for all p, we have $Co^p\big(\Xi^{(p)}(A) \cap K\big) \subset Co^p\big(\mathcal{E}_K^{(p)}(A)\big) = Co^p(A) \cap K$. Conversely, we know from Lemma 4.9 that $\mathcal{E}_K^{(p)} \subset \Xi^{(p)}(A) \cap K$.
Hence, $Co^p\big(\mathcal{E}_K^{(p)}(A)\big) \subset Co^p\big(\Xi^{(p)}(A) \cap K\big)$ and we deduce that:

$$Co^p\big(\Xi^{(p)}(A) \cap K\big) = Co^p\big(\mathcal{E}_K^{(p)}(A)\big) = Co^p(A) \cap K.$$

Moreover, we have, for all natural numbers p:

$$Co^p(A) = \bigcup_{K \in \mathcal{K}(A)} Co^p\big(\mathcal{E}_K^{(p)}(A)\big).$$

It follows that $Co^p(A) = \bigcup_{K \in \mathcal{K}(A)} Co^p\big(\Xi^{(p)} \cap K\big)$. □

Proposition 4.11. *Let $A = \{x^{(1)}, ..., x^{(m)}\}$ be a finite subset of $\mathbb{R}^n$. We have*

$$Co^\infty(A) = \mathop{Lim}_{p \to \infty} Co^p(A) = \bigcup_{K \in \mathcal{K}(A)} \mathbb{B}[\,\Xi^\infty(A) \cap K\,].$$

Proof. From Lemma 4.8, for all sequences $\xi_J\big(x^{(1)}, ..., x^{(m)}, t_{I,J}^{(p)}\big)$ of regular (I,J)-intermediate points of order p, there is an (I,J)-intermediate point in the limit $\zeta_{I,J} \in \Xi^\infty(A)$ such that $\lim_{p \to \infty} \xi_J\big(x^{(1)}, ..., x^{(m)}, t_{I,J}^{(p)}\big) = \zeta_{I,J}$. It follows that

$$\lim_{p \to \infty} Co^p(\Xi^{(p)}(A) \cap K) = Co^\infty\big(\Xi^\infty(A) \cap K\big).$$

Moreover, for all p, $Co^p\big(\Xi^{(p)} \cap K\big)$ is a compact subset of $\mathbb{R}^n$. Hence

$$Ls_{p \to \infty} Co^p(A) = Ls_{p \to \infty}\Big(\bigcup_{K \in \mathcal{K}(A)} Co^p\big(\Xi^{(p)} \cap K\big)\Big)$$

$$= \bigcup_{K \in \mathcal{K}(A)} Ls_{p \to \infty} Co^p(\Xi^{(p)} \cap K) = \bigcup_{K \in \mathcal{K}(A)} Co^\infty\big(\Xi^\infty(A) \cap K\big). \tag{40}$$

Since $Co^{\infty}\big(\Xi^{\infty}(A)\cap K\big) = \mathbb{B}[\,\Xi^{\infty}(A)\cap K\,]$, we deduce that $Ls_{p\to\infty}Co^p(A) = \bigcup_{K\in\mathcal{K}(A)}\mathbb{B}[\,\Xi^{\infty}(A)\cap K\,]$. To end the proof, note that

$$Co^{\infty}\big(\Xi^{\infty}(A)\cap K\big) = \underset{p\to\infty}{Lim}\, Co^p\big(\Xi^p(A)\cap K\big) = Li_{p\to\infty}Co^p\big(\Xi^p(A)\cap K\big).$$

Since for all K, $Li_{p\to\infty}\big(Co^p(A)\cap K\big) \subset \big(Li_{p\to\infty}Co^p(A)\big)\cap K$, it follows from equation (40) and Lemma 4.10 that

$$Ls_{p\to\infty}Co^p(A) = \bigcup_{K\in\mathcal{K}(A)} Ls_{p\to\infty}Co^p\big(\Xi^{(p)}(A)\cap K\big) = \bigcup_{K\in\mathcal{K}(A)} Li_{p\to\infty}Co^p\big(\Xi^{(p)}(A)\cap K\big)$$

$$= \bigcup_{K\in\mathcal{K}(A)} Li_{p\to\infty}\Big(Co^p(A)\cap K\Big) \subset \bigcup_{K\in\mathcal{K}(A)}\Big(Li_{p\to\infty}Co^p(A)\cap K\Big) = Li_{p\to\infty}Co^p(A),$$

which ends the proof. □

For all finite and non empty subsets A of $\mathbb{R}^n$, $Co^p(A)$ belongs to $\mathcal{K}(\mathbb{R}^n)$, the space of non empty compact subsets of $\mathbb{R}^n$, which is metrizable by the Hausdorff metric

$$D_H(K,L) = \inf\Big\{\varepsilon > 0 : K \subset \bigcup_{x\in L} B(x,\varepsilon[, \text{ and } L \subset \bigcup_{x\in K} B(x,\varepsilon[\Big\}.$$

Corollary 4.12. *For all finite non empty subsets A of $\mathbb{R}^n$, the sequence $\{Co^p(A)\}_{r\in\mathbb{N}}$ converges to $Co^{\infty}(A)$ in $\mathcal{K}(\mathbb{R}^n)$, with respect to the Hausdorff metric.*

Proof. Choose $\delta > 0$ such that $A \subset [-\delta,\delta]^n$; we have $\Phi_p(A) \subset [-\delta^{2p+1},\delta^{2p+1}]^n$, and therefore also $Co(\Phi_p(A)) \subset [-\delta^{2p+1},\delta^{2p+1}]^n$. Taking the inverse image by Φ_p yields $Co^p(A) \subset [-\delta,\delta]^n$; all the terms of the sequence $\{Co^p(A)\}_{r\in\mathbb{N}}$ are contained in the compact set $[-\delta,\delta]^n$. To conclude, recall that on compact metric spaces, Kuratowski-Painlevé convergence of a sequence of compact sets implies convergence in the Hausdorff metric. □

4.4. Caratheodory, Helly and Radon in limit for $\mathbb{B}$-polytopes

We first establish an analogue to the Caratheodory theorem fo $\mathbb{B}$-polytope. Notice that it is not clear so far if a $\mathbb{B}$-polytope is the $\mathbb{B}$-convex hull of a finite set of $\mathbb{R}^n$.

Lemma 4.13. *For all finite subsets $A = \{x^{(1)},...,x^{(m)}\}$ of $\mathbb{R}^n$ with $m \geq n+1$, and for all $x \in Co^{\infty}(A)$, there exists a subset A_{n+1} of A with* $\mathrm{Card}(A_{n+1}) = n+1$, *such that $x \in Co^{\infty}(A_{n+1})$. Moreover*

$$Co^{\infty}(A) = \bigcup_{A_{n+1}\in\mathcal{A}_{n+1}} Co^{\infty}(A_{n+1})$$

where $\mathcal{A}_{n+1}$ is the set of all the subsets of A having $n+1$ elements.

Proof. Suppose that $x \in Co^{\infty}(A)$. By definition, there exists an increasing sequence of natural numbers $\{p_q\}_{q\in\mathbb{N}}$, and a sequence $\{x^{(p_q)}\}_{q\in\mathbb{N}}$, with $x^{(p_q)} \in Co^{p_q}(A)$ and $\lim_{q\longrightarrow\infty} x^{(p_q)} = x$. For all natural numbers p, $Co^p(A)$ is homeomorphic to the usual convex polytope $Co(A)$. Therefore, for all q, there exists a subset

$A_{n+1}^{(p_q)}$ of A with $\mathrm{Card}(A_{n+1}) = n+1$ and such that $x^{(p_q)} \in Co^{p_q}(A_{n+1}^{(p_q)})$. Let $\mathcal{A}_{n+1}$ be the set of all the subsets of A having $n+1$ elements. Since $\mathcal{A}_{n+1}$ has a finite number of elements, there exists a subset A_{n+1} of A and a subsequence $\{p_{q_r}\}_{r\in\mathbb{N}}$ such that for all r $A_{n+1}^{(p_{q_r})} = A_{n+1}$. It follows that for all r, $x^{(p_{q_r})} \in Co^{p_{q_r}}(A_{n+1})$. Hence, since $\lim_{r\longrightarrow\infty} x^{(p_{q_r})} = x$, it follows that $x \in Co^{\infty}(A_{n+1})$, which proves the first part of the statement. Hence $Co^{\infty}(A) \subset \bigcup_{A_{n+1}\in\mathcal{A}_{n+1}} Co^{\infty}(A_{n+1})$. Since for all $A_{n+1} \in \mathcal{A}_{n+1}$, $A_{n+1} \subset A$, it follows that $Co^{\infty}(A)_{n+1} \subset Co^{\infty}(A)$, which proves the converse inclusion and ends the proof. □

In the following, we prove that the $\mathbb{B}$-polytopes satisfy an analogue of Radon's theorem.

Proposition 4.14. *Let $A = \{x^{(1)}, ..., x^{(m)}\}$ be a finite subset of $\mathbb{R}^n$ with $m \geq n+2$. Then there exists a partition $\{A_1, A_2\}$ of A such that $Co^{\infty}(A_1) \cap Co^{\infty}(A_2) \neq \emptyset$.*

Proof. For all $p \in \mathbb{N}$, $Co^p(A)$ is homeomorphic to the usual convex polytope $Co(A)$. Therefore, from the usual Radon's theorem, for all natural numbers p, there exists a partition $\{A_1^{(p)}, A_2^{(p)}\}$ of A such that $Co^p(A_1^{(p)}) \cap Co^p(A_2^{(p)}) \neq \emptyset$. Therefore, for all p, one can pick some $x^{(p)} \in Co^p(A_1^{(p)}) \cap Co^p(A_2^{(p)})$. Let $\mathcal{P}(A)$ the set of all the partitions of A. Since $\mathrm{Card}(\mathcal{P}(A))$ is finite, one can find an increasing subsequence $\{p_q\}_{q\in\mathbb{N}}$ and a partition (A_1, A_2) of $\mathcal{P}(A)$, such that for all q we have $A_1^{(p_q)} = A_1$ and $A_2^{(p_q)} = A_2$. It follows that for all q, $x^{(p_q)} \in Co^{p_q}(A_1) \cap Co^{p_q}(A_2)$. Moreover, there exists a compact subset K which contains $Co^p(A)$ for all p. Consequently, one can extract a subsequence $\{x^{(p_{q_r})}\}_{r\in\mathbb{N}}$ which converges to some $x \in \mathbb{R}^n$ and such that for all r, $x^{(p_{q_r})} \in Co^{p_{q_r}}(A_1) \cap Co^{p_{q_r}}(A_2)$. However, this implies that $x \in Co^{\infty}(A_1)$ and $x \in Co^{\infty}(A_2)$. Hence, $Co^{\infty}(A_1) \cap Co^{\infty}(A_2) \neq \emptyset$ which ends the proof. □

Similarly, an analogue of the Helly's theorem also holds for $\mathbb{B}$-polytopes.

Proposition 4.15. *Let $A = \{x^{(1)}, ..., x^{(n+2)}\}$ be a finite subset of $\mathbb{R}^n$. For all $i \in [n+2]$, let us denote $\Delta_i = Co^{\infty}(\{x^{(1)}, ..., x^{(n+2)}\}\backslash\{x^{(i)}\})$. Then there exists some $x \in \bigcap_{i\in[n+2]} \Delta_i$.*

Proof. The proof is similar to those established above. For all natural numbers p, $Co^p(A)$ is homeomorphic to the usual convex polytope $Co(A)$. Therefore, from the usual Hellys's theorem, for all natural numbers p, there is some $x^{(p)} \in \bigcap_{i\in[n+2]} \Delta_i^{(p)}$ where for all i, $\Delta_i^{(p)} = Co^p(\{x^{(1)}, ..., x^{(n+2)}\}\backslash\{x^{(i)}\})$. Moreover, there exists a compact subset K which contains $Co^p(A)$ for all p. Consequently, one can extract a subsequence $\{x^{(p_q)}\}_{q\in\mathbb{N}}$ which converges to some $x \in \mathbb{R}^n$ and such that for all q, $x^{(p_q)} \in \bigcap_{i\in[n+2]} \Delta_i^{(p_q)}$. However, this implies

$$x \in Ls_{p\longrightarrow\infty} \bigcap_{i\in[n+2]} \Delta_i^{(p)} \subset \bigcap_{i\in[n+2]} Ls_{p\longrightarrow\infty} \Delta_i^{(p)} = \bigcap_{i\in[n+2]} \Delta_i.$$ □

5. Dual structure of polytopes

In the following we introduce the operation $\langle \cdot, \cdot \rangle_\infty \colon \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$ defined for all $x, y \in \mathbb{R}^n$ by $\langle x, y \rangle_\infty = \boxplus_{i \in [n]} x_i y_i$. Let $\| \cdot \|_\infty$ be the Tchebychev norm defined by $\|x\|_\infty = \max_{i \in [n]} |x_i|$. It is established in [6] that for all $x, y \in \mathbb{R}^n$, we have: (i) $\sqrt{\langle x, x \rangle_\infty} = \|x\|_\infty$; (ii) $|\langle x, y \rangle_\infty| \leq \|x\|_\infty \|y\|_\infty$; (iii) For all $\alpha \in \mathbb{R}$, $\alpha \langle x, y \rangle_\infty = \langle \alpha x, y \rangle_\infty = \langle x, \alpha y \rangle_\infty$.

In the following, we say that a map $f \colon \mathbb{R}^n \longrightarrow \mathbb{R}$ is a $\mathbb{B}$-*form* if there exists some $a \in \mathbb{R}^n$ such that:

$$f(x) = \boxplus_{i \in [n]} a_i x_i = \langle a, x \rangle_\infty. \tag{41}$$

We intend to analyze the dual relation between $\mathbb{B}$-convex sets and the maps of the form $x \mapsto \langle a, x \rangle_\infty$, where $a \in \mathbb{R}^n$.

The function above is depicted in Figure 2.2.2. All the points such that $\langle a, x \rangle_\infty = 0$ are represented by the thick line. In the following, for all subsets A of $\mathbb{R}^n$ cl(A) and int(A) respectively stand for the closure and the interior of A.

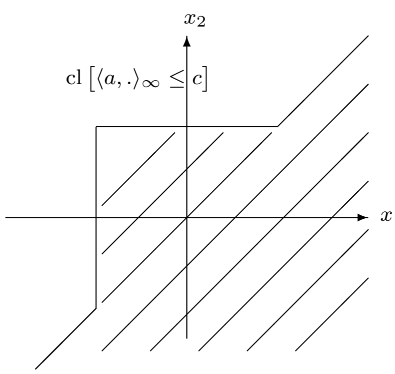

Figure 5.1: The domain cl $\left[\langle a, . \rangle_\infty \leq c\right]$

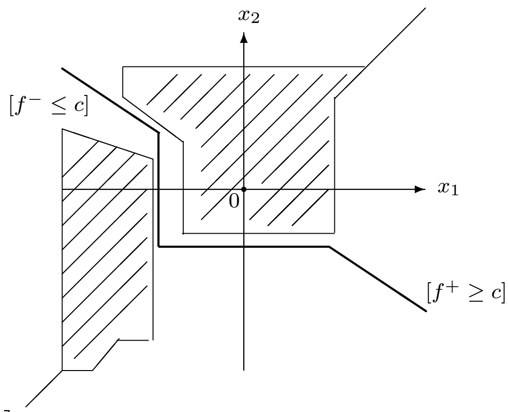

Figure 5.2: Separation of two polytopes

In the following, for all maps $f \colon \mathbb{R}^n \longrightarrow \mathbb{R}$ and all real numbers c, the notation $[\, f \leq c\,]$ stands for the set $f^{-1}(\,] - \infty, c]\,)$. Similarly, $[\, f < c\,]$ stands for $f^{-1}(\,] - \infty, c[\,)$ and $[\, f \geq c\,] = [\, -f \leq -c\,]$.

For all $u, v \in \mathbb{R}$, let us define the binary operation

$$u \stackrel{-}{\smile} v = \begin{cases} u & \text{if} \quad |u| > |v| \\ \min\{u, v\} & \text{if} \quad |u| = |v| \\ v & \text{if} \quad |u| < |v|. \end{cases}$$

An elementary calculus shows that $u \boxplus v = \frac{1}{2}\Big[u \stackrel{-}{\smile} v - \big[(-u) \stackrel{-}{\smile} (-v)\big]\Big]$.

Similarly one can introduce a symmetrical binary operation for all $u, v \in \mathbb{R}$ in the following way: $u \boxplus v = \frac{1}{2}\Big[(u \stackrel{-}{\smile} v) + (u \stackrel{+}{\smile} v)\Big]$.

Given m elements $u_1, ..., u_m$ of $\mathbb{R}$, not all of which are 0, let I_+, respectively I_-, be the set of indices for which $0 < u_i$, respectively $u_i < 0$. We can then write

$u_1 \overset{-}{\smile} \cdots \overset{-}{\smile} u_m = (\smile_{i\in I_+} u_i) \overset{-}{\smile} (\smile_{i\in I_-} u_i) = (\max_{i\in I_+} u_i) \overset{-}{\smile} (\min_{i\in I_-} u_i)$ from which we obtain

$$u_1 \overset{-}{\smile} \cdots \overset{-}{\smile} u_m = \begin{cases} \max_{i\in I_+} u_i & \text{if} \quad I_- = \emptyset \text{ or } \max_{i\in I_-} |u_i| < \max_{i\in I_+} u_i \\ \min_{i\in I_-} u_i & \text{if} \quad I_- = \emptyset \text{ or } \max_{i\in I_+} u_i < \max_{i\in I_-} |u_i| \\ \min_{i\in I_-} u_i & \text{if} \quad \max_{i\in I_-} |u_i| = \max_{i\in I_+} u_i. \end{cases} \tag{42}$$

We define a *lower* $\mathbb{B}$*-form* on $\mathbb{R}^n_+$ as a map $g\colon \mathbb{R}^n \to \mathbb{R}$ with the property that for all $(x_1, ..., x_n) \in \mathbb{R}^n_+$,

$$g(x_1, ..., x_n) = \langle a, x\rangle^-_\infty = a_1x_1 \overset{-}{\smile} \cdots \overset{-}{\smile} a_nx_n. \tag{43}$$

It was established in [6] that for all $c \in \mathbb{R}$, $g^{-1}\left(\,]-\infty, c]\right) = \{x \in \mathbb{R}^n : g(x) \leq c\}$ is closed. It follows that a $\mathbb{B}$-form is lower semi-continuous. It was established in [11] that $g^{-1}\left(\,]-\infty, c]\right) \cap \mathbb{R}^n_+$ is a $\mathbb{B}$-halfspace, that is a $\mathbb{B}$-convex subset of $\mathbb{R}^n_+$ whose the complement in $\mathbb{R}^n_+$ is also $\mathbb{B}$-convex.

Similarly, one can define an *upper* $\mathbb{B}$*-form* as a map $h\colon \mathbb{R}^n \to \mathbb{R}$ such that, for all $(x_1, ..., x_n) \in \mathbb{R}^n$,

$$h(x_1, ..., x_n) = \langle a, x\rangle^+_\infty = a_1x_1 \overset{+}{\smile} \cdots \overset{+}{\smile} a_nx_n. \tag{44}$$

For all $x \in \mathbb{R}^n$ we clearly have the following identities

$$\langle a, x\rangle^+_\infty = -\langle a, -x\rangle^-_\infty \text{ and } \langle a, x\rangle^-_\infty = -\langle a, -x\rangle^+_\infty. \tag{45}$$

The largest (smallest) lower (upper) semi-continuous minorant (majorant) of a map f is said to be the lower (upper) semi-continuous regularization of f. In the next statements it is shown that the lower (upper) $\mathbb{B}$-forms are the lower (upper) semi-continuous regularized of the $\mathbb{B}$-form.

It was shown in [6] that if g is a lower $\mathbb{B}$-form defined by $g(x_1, ..., x_n) = a_1x_1 \overset{-}{\smile} \cdots \overset{-}{\smile} a_nx_n$, for some $a \in \mathbb{R}^n$, then g is the lower semi-continuous regularization of the map $x \mapsto \langle a, x\rangle_\infty = \boxplus_{i\in[n]} a_ix_i$. Similarly, if h is an upper $\mathbb{B}$-form defined by $h(x_1, ..., x_n) = a_1x_1 \overset{+}{\smile} \cdots \overset{+}{\smile} a_nx_n$, for some $a \in \mathbb{R}^n$, then h is the upper semi-continuous regularization of the map $x \mapsto \langle a, x\rangle_\infty = \boxplus_{i\in[n]} a_ix_i$.

5.1. Halfspaces and separation of $\mathbb{B}$-polytopes over $\mathbb{R}^n$

In the following, the dual properties of a $\mathbb{B}$-polytope are analyzed. It is shown that a $\mathbb{B}$-polytope is $\mathbb{B}^\sharp$-convex. In [7], subsets of the form $[f^- \leq c]$ and $[f^+ \geq c]$, where c is a real number, were termed *pseudo-halfspaces*, and the following properties for all $\mathbb{B}$-forms f and all real numbers $c \in \mathbb{R}$ were established.

Proposition 5.1. *For all $\mathbb{B}$-forms f and all real numbers $c \in \mathbb{R}$, we have the following properties:*

(a) *The subset* $\operatorname{int}[f \leq c]$ *is $\mathbb{B}^\sharp$-convex;*

(b) *The subsets* $\operatorname{int}[f^- \leq c]$ *and* $\operatorname{int}[f^+ \geq c]$ *are $\mathbb{B}^\sharp$-convex;*

(c) *We have* $\operatorname{int}[f^- \leq c] = \mathbb{R}^n \backslash [f^+ \geq c]$ *and* $\operatorname{int}[f^+ \geq c] = \mathbb{R}^n \backslash [f^- \leq c]$.

In [7] the geometric deformation of a sequence of Φ_p-halfspace expressed in the form $H_p = [f_p \leq c_p]$ was analyzed. For each natural number p, $f_p\colon \mathbb{R}^n \longrightarrow \mathbb{R}$ is a φ_p-linear form defined by $f(x) = \langle a_p, x\rangle_p$, with $(a^{(p)}, c_p) \in \mathbb{R}^n \times \mathbb{R}$. It was shown that the Kuratowski-Paillevé limit of this sequence can be expressed with respect to the $\mathbb{B}$-forms.

Proposition 5.2. *Let f be a $\mathbb{B}$-form defined by $f(x) = \langle a, x\rangle_\infty$ for some point $a \in \mathbb{R}^n \backslash \{0\}$. For all natural numbers p let $f_p\colon \mathbb{R}^n \longrightarrow \mathbb{R}$ be a map defined by $f_p(x) = \langle a^{(p)}, x\rangle_p$ where $\{a_p\}_{p\in\mathbb{N}}$ is a sequence of $\mathbb{R}^n \backslash \{0\}$. If there exists a sequence $\{c_p\}_{p\in\mathbb{N}} \subset \mathbb{R}$ such that $\lim_{q\longrightarrow\infty}(a_p, c_p) = (a, c)$, then:*

$$\underset{p\to\infty}{Lim}\left[f_p \leq c_p\right] = \operatorname{cl}\left[f \leq c\right] = \left[f^- \leq c\right]$$

and

$$\underset{p\to\infty}{Lim}\left[f_p \geq c_p\right] = \operatorname{cl}\left[f \geq c\right] = \left[f^+ \geq c\right].$$

In [7], the following separation result was established.

Proposition 5.3. *Let C and D be two nonempty subsets of $\mathbb{R}^n$. Assume that $Co^\infty(C) \cap Co^\infty(D) = \emptyset$. There exists a $\mathbb{B}$-form $f\colon \mathbb{R}^n \longrightarrow \mathbb{R}$ and a real number c such that*

$$C \subset [f^- \leq c] \quad \text{and} \quad D \subset [f^+ \geq c],$$

where f^- and f^+ are the lower and upper semi-continuous regularized of f.

A subset S of $\mathbb{R}^n$ is a $\mathbb{B}$-polytope if there exists a finite part A of $\mathbb{R}^n$ such that $S = Co^\infty(A)$.

Proposition 5.4. *Let S and T be two $\mathbb{B}$-polytopes of $\mathbb{R}^n$. Assume that $S \cap T = \emptyset$. There exists a $\mathbb{B}$-form $f\colon \mathbb{R}^n \longrightarrow \mathbb{R}$ and a real number c such that*

$$S \subset [f^- \leq c] \text{ and } T \subset [f^+ \geq c],$$

where f^- and f^+ are the lower and upper semi-continuous regularized of f.

Proof. Suppose that S and T are two $\mathbb{B}$-polytopes of $\mathbb{R}^n$. By definition there exist two finite parts A and B of $\mathbb{R}^n$ such that $S = Co^\infty(A)$ and $T = Co^\infty(B)$. Since S and T are two disjoint sets, we have $Co^\infty(A) \cap Co^\infty(B) = \emptyset$. From Proposition 5.3, we obtain the result. □

In the following we explore the dual structure of a $\mathbb{B}$-polytope. A subset B of $\mathbb{R}^n$ is a box with respect to the usual partial order if there exist $u, v \in \mathbb{R}^n$ such that $u \leq v$ and $B = \{x \in \mathbb{R}^n : u \leq x \leq v\}$. It wa shown in [7] that a box B of $\mathbb{R}^n$ is a fixed $\mathbb{B}$-convex set, which means that $Co^{\infty}(B) = B$. The following result is an immediate consequence that will be extended in the next subsection.

Proposition 5.5. *A $\mathbb{B}$-polytope is the intersection of all lower pseudo-halfspaces containing it.*

Proof. Let S be a $\mathbb{B}$-polytope. We need to prove that if $y \notin C$, then there exist a lower $\mathbb{B}$-form f^- and a real number c such that

$$S \subset [f^- \leq c] \quad \text{and } f^-(y) > c.$$

By definition if S is a $\mathbb{B}$-polytope, there exists a finite subset A of $\mathbb{R}^n$ such that $S = Co^{\infty}(A)$. Hence S is closed. Therefore, if $y \notin C$, then there is some $\epsilon > 0$ such that $B_{\infty}(y, \epsilon] \cap S = \emptyset$. Let $E = \{y + \epsilon u : u \in \{-1, 1\}^n\}$. E is the set of all the extreme points of the polyhedron $B_{\infty}(y, \epsilon]$ that is a box. Thus, by construction, for all natural numbers p, $Co^p(E) = Co^{\infty}(E)$. From Proposition 5.4, there is a dual $\mathbb{B}$-form $f \colon \mathbb{R}^n \longrightarrow \mathbb{R}$ and some $c \in \mathbb{R}$ such that

$$S \subset [f^- \leq c] \quad \text{and } B_{\infty}(y, \epsilon] \subset [f^+ \geq c].$$

This implies that $y \in \operatorname{int}[f^+ \geq c]$. From Proposition 5.1, we deduce that $y \in \operatorname{int}[f \geq c] = \mathbb{R}^n \backslash \operatorname{cl}[f \leq c] = \mathbb{R}^n \backslash [f^- \leq c]$. Hence, $f^-(y) > c$ which ends the proof. $\square$

This result is extended in the the next section, where it is proven that there exists finite number of closed pseudo-halfspaces characterizing a $\mathbb{B}$-polytope.

5.2. External representation of a $\mathbb{B}$-polytope over $\mathbb{R}^n$

The implication of the results above for a the Painlevé-Kuratowski limit of a polytope are analyzed in this section. In the case where A is a subset of $\mathbb{R}^n$ then $Co^{\infty}(A)$ is the intersection of a finite number of pseudo-halfspaces.

The proof is based on the fact that there exists a configuration of facets for the Φ_p-convex hull of A which appears an infinity of times.

Lemma 5.6. *Let $A = \{x^{(1)}, ..., x^{(m)}\}$ be a finite part of $\mathbb{R}^n$. There exists an increasing subsequence $\{p_q\}_{q \in \mathbb{N}}$ and ℓ_A subsequences of linear forms $\{f_{j,p_q}\}_{q \in \mathbb{N}}$ respectively defined for all natural numbers q by $f_{j,p_q}(x) = \langle a_j^{(p_q)}, x \rangle_{p_q}$ such that*

$$Co^{p_q}(A) = \bigcap_{j \in [\ell_A]} [f_{j,p_q} \leq c_{j,p_q}],$$

where, for all $j \in [\ell_A]$, $\{c_{j,p_q}\}_{q \in \mathbb{N}}$ is a sequence of real numbers. Moreover, for all natural numbers q, $\bigcap_{j \in [\ell_A]} [f_{j,p_q} \leq c_{j,p_q}]$ is a minimal external representation of $Co^{p_q}(A)$.

Proof. For all natural numbers p, $Co^p(A)$ is a polytope. Therefore it can be written as the intersection of a finite number of $\ell(p)$ halfspaces. Hence, for each p, there exist $\ell(p)$ vectors $a_j^{(p)}$ and $\ell(p)$ real numbers $c_j^{(p)}$ such that:

$$Co^p(A) = \bigcap_{j \in [\ell(p)]} \big[f_{j,p} \leq c_{j,p}\big]$$

where for all j, and all $x \in \mathbb{R}^n$ $f_{j,p}(x) = \langle a_j^{(p)}, x\rangle_p$. The maximum number $\bar{\ell}$ of halfspaces characterizing the polytope $Co^p(A)$ is independent of p since for all natural numbers p $Co^p(A)$ is homeomorphic to $Co(A)$. Since $\ell(p)$ takes its values in $[\bar{\ell}]$ which is a finite set, there is some $\ell_A \in [\bar{\ell}]$ and an infinite subsequence $\{\ell(p_q)\}_{q\in\mathbb{N}}$ such that $\ell(p_q) = \ell_A$ for all natural numbers q. □

Proposition 5.7. *Let $A = \{x^{(1)}, ..., x^{(m)}\}$ be a finite part of $\mathbb{R}^n$. There exists a collection $\{a_j, c_j\}_{j\in[\ell_A]}$ of $\mathbb{R}^n \times \mathbb{R}$ such that*

$$Co^\infty(A) = \bigcap_{j \in [\ell_A]} \big[f_j^- \leq c_j\big],$$

where for all $j \in [\ell_A]$, f_j is a dual $\mathbb{B}$-form defined for all $x \in \mathbb{R}^n$ by $f_j(x) = \langle a_j, x\rangle_\infty$ and f_j^- is its lower continuous regularized.

Proof. From Lemma 5.6, there exists an increasing subsequence $\{p_q\}_{q\in\mathbb{N}}$ and ℓ_A subsequences of linear forms $\{f_{j,p_q}\}_{q\in\mathbb{N}}$ respectively defined for all natural numbers q by $f_{j,p_q}(x) = \langle a_j^{(p_q)}, x\rangle_{p_q}$ with $Co^{p_q}(A) = \bigcap_{j\in[\ell_A]} \big[f_{j,p_q} \leq c_{j,p_q}\big]$, where such an external representation is minimal. Hence for each j:

$$c_{j,p_q} = \sup_x \Big\{\langle a_j^{(p_q)}, x\rangle_{p_q} : x \in Co^{p_q}(A)\Big\} = \max_{k=1,\dots,m} \langle a_k^{(p_q)}, x^{(k)}\rangle_{p_q}.$$

For all vectors u of $\mathbb{R}^n$, we write $|u| = (|u_1|, ..., |u_n|)$. Fix $\bar{c} = n \max_{k=1,\dots,p} \|x^{(k)}\|_\infty$. For $j = 1, ..., \ell_A$, we have from the Hölder inequality:

$$-\bar{c} \leq -\max_{k=1,\dots,m} \langle |a_j|, |x^{(k)}|\rangle_{p_q} \leq c_{j,p_q} \leq \max_{k=1,\dots,m} \langle |a_j|, |x^{(k)}|\rangle_{p_q} \leq \bar{c}.$$

Since $C_\infty(0,1) \times [-\bar{c}, \bar{c}]$ is compact, $\Big(C_\infty(0,1) \times [-\bar{c}, \bar{c}]\Big)^{\ell_A}$ is also compact, and we deduce that it contains a subsequence

$$\Big\{\Big((a_1^{(p_{q_r})}, c_{1,p_{q_r}}), ..., (a_{\ell_A}^{(p_{q_r})}, c_{\ell_A,p_{q_r}})\Big)\Big\}_{r\in\mathbb{N}}$$

which converges to some $\Big((a_1, c_1), ..., (a_{\ell_A}, c_{\ell_A})\Big) \in \Big(C_\infty(0,1) \times [-\bar{c}, \bar{c}]\Big)^{\ell_A}$. Hence

$$\begin{aligned} Li_{r\longrightarrow\infty} Co^{p_{q_r}}(A) &= Li_{r\longrightarrow\infty} \bigcap_{j\in[\ell_A]} [f_{j,p_{q_r}} \leq c_{j,p_{q_r}}] \\ &\subset \bigcap_{j\in[\ell_A]} Li_{r\longrightarrow\infty}[\, f_{j,p_{q_r}} \leq c_{j,p_{q_r}}] \subset \bigcap_{j\in[\ell_A]} \mathrm{cl}\,[f_j \leq c_j]. \end{aligned}$$

However, by definition $Li_{p\to\infty} Co^p(A) \subset Li_{r\longrightarrow\infty} Co^{p_{q_r}}(A)$. Since we have established that $Co^\infty(A) = Lim_{p\to\infty} Co^p(A) = Li_{p\to\infty} Co^p(A)$ which yields the inclusion.

Let us show the converse inclusion. Assume that $y \notin Li_{p\to\infty}Co^p(A)$ and that

$$y \in \bigcap_{j\in[\ell_A]} \mathrm{cl}\,[f_j \leq c_j] \cap \mathbb{R}^n_+.$$

We can find some j, a subsequence $\{p_q\}_{q\in\mathbb{N}}$ and some $\epsilon > 0$ such that

$$B_\infty(y,\epsilon\,[\cap\Big[f_{j,p_q} \leq c_{j,p_q}\Big] = \emptyset.$$

Consequently, for all $q \in \mathbb{N}$ we have $B_\infty(y,\epsilon\,[\subset \Big[f_{j,p_q} \geq c_{j,p_q}\Big]$. This implies from Proposition 5.2 that $B_\infty(y,\epsilon\,[\subset \mathrm{cl}\,[f_j \geq c_j]$. Hence

$$y \in \mathrm{int}\Big(\mathrm{cl}\,\big[f_j \geq c_j\big]\Big) = \mathrm{int}\,\big[f_j^+ \geq c_j\big].$$

Thus, from Proposition 5.1.(c) $y \in \mathbb{R}^n \setminus \mathrm{cl}\,[f_j \leq c_j]$. Thus $y \notin \bigcap_{j\in[\ell_A]} \mathrm{cl}\,[f_j \leq c_j]$, a contradiction. We deduce that $\bigcap_{j\in[\ell_A]} \mathrm{cl}\,[f_j^- \leq c_j] \subset Li_{p\to\infty}Co^p(A)$. Therefore $\bigcap_{j\in[\ell_A]} \mathrm{cl}\,[f_j^- \leq c_j] = Li_{p\to\infty}Co^p(A) = Lim_{p\to\infty}\,Co^p(A) = Co^\infty(A)$, and the proof is complete. □

Proposition 5.8. *The interior of a $\mathbb{B}$-polytope is $\mathbb{B}^\sharp$-convex.*

Proof. Let $A = \{x^{(1)},...,x^{(m)}\}$ be a finite part of $\mathbb{R}^n$. From Proposition 5.8, there exists a collection $\{a_j,c_j\}_{j\in[\ell_A]}$ of $\mathbb{R}^n \times \mathbb{R}$ such that $Co_\infty(A) = \bigcap_{j\in[\ell_A]} \big[f_j^- \leq c_j\big]$, where, for all $j \in [\ell_A]$, f_j is a dual $\mathbb{B}$-form defined for all $x \in \mathbb{R}^n$ by $f_j(x) = \langle a_j, x\rangle_\infty$ and f_j^- is its lower continuous regularized. The finite intersection of several sets having a $\mathbb{B}^\sharp$-convex interior is $\mathbb{B}^\sharp$-convex. However, from Proposition 5.1.(b), for all $j \in [\ell_A]$, $\big[f_j^- \leq c_j\big]$ has a $\mathbb{B}^\sharp$-convex interior. It follows that the interior of a $\mathbb{B}$-polytope is $\mathbb{B}^\sharp$-convex. □

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