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"On the Figure of Columns" of Lagrange Revisited*

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In the design of clamped-clamped circular columns that can accommodate the largest vertical load before buckling, the moment of inertia of the horizontal section is assumed to be proportional to the p-th power of the cross-sectional area for some $p > 0$: $p = 2$ (solid column) and $p = 1$ (hollow column). Existence of maximizing profiles has been established by Cox and Overton [SIAM J. Math. Anal. 23 (1992) 297–325] under the assumption of strictly positive lower and upper bounds on the profiles. Yet, their numerical computations indicate that the bounds are not necessary to get maximizing profiles.

In this paper, we revisit some aspects of the *unconstrained problem* with emphasis on the cases $0 < p \le 1$. The existence of a maximizing profile in $L^1(0,1)$ is established for $0 < p < 1$ without bounds. The existence theorem of Cox and Overton is extended to continuous profiles in $C[0,1]$. In addition, their strictly positive lower bound assumption is relaxed for $p \le 1$ in both $L^\infty(0,1)$ and $C[0,1]$. The dual inf sup problem is shown to be bounded above by 48 for $p \le 1$, but is $+\infty$ for $p > 1$. Additional results are given for *degenerate profiles*, that is, profiles whose lower bound is zero at some points.

Keywords: Lagrange problem, column, buckling, eigenvalue, degenerate differential operator.

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1. Introduction

Lagrange [11] around 1770-1773 considered the design of vertical columns that can accommodate the largest vertical load before buckling. Olhoff and Niordson [12] introduced in 1979 the family of problems (1) for which the quadratic moment of inertia $I(x)$ of the horizontal cross section $S(x)$ at height x is proportional to the power p, $0 < p < \infty$, of the cross-sectional area $A(x)$ of $S(x)$ for a normalized column of height 1 and volume 1 ($p = 2$ for the solid column and $p = 1$ for the

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hollow column – a hollow cylinder of large radius with a very thin wall). For a circular clamped-clamped column

$$\begin{gathered}
\sup_{A\in\mathcal{A}} \lambda_p(A), \quad \lambda_p(A) \stackrel{\text{def}}{=} \inf_{v\in W_0^{2,\infty}(0,1)} \frac{\int_0^1 A^p\,|v''|^2\,dx}{\int_0^1 |v'|^2\,dx} = \inf_{v\in E} \int_0^1 A^p\,|v''|^2\,dx, \\
\mathcal{A} \stackrel{\text{def}}{=} \left\{ A \in L^1(0,1) : 0 \le A(x) \text{ in } [0,1] \text{ and } \int_0^1 A\,dx = 1 \right\},
\end{gathered} \tag{1}$$

where $W_0^{2,\infty}(0,1)$ is the Sobolev space of functions v such that $v(0) = v(1) = 0$ and $v'(0) = v'(1) = 0$ and $E \stackrel{\text{def}}{=} \{v \in W_0^{2,\infty}(0,1) : \|v'\|_{L^2(0,1)} = 1\}$.

For the purpose of this paper we start from and rely on the paper of Cox and Overton [2] in 1992 where the reader can find a well-documented account of the history of this challenging problem. In a first part, they proved existence of a maximizing profile for a family of A's that are a priori bounded below and above by some strictly positive constants (*constrained problem*). In a second part, they present numerical computations including computations without the lower and upper bounds (*unconstrained problem*). In the *unconstrained case*, their assumption is in good agreement with their numerical computations for the solid column. For the hollow column, their numerical computations give a load around 47.9898 and show the possibility that the horizontal cross sectional area $A(x)$ might be zero in two points x along the vertical axis as for the piecewise quadratic profile of Olhoff and Niordson [12] in 1979 and Seiranian [14] in 1984. The problem was revisited by Egorov who proposed a new approach for both the solid [3, 6] in 2002 and the hollow column [4, 5] in 2003: from [6, p. 3574]

> Note that Cox and Overton proved in [2] the existence theorem under the supplementary conditions $0 < \alpha \le A(x) \le \beta < +\infty$. This does not imply the existence theorem for the Lagrange problem.

In 2017 Egorov [7] provided a numerical figure of 41.48 for the critical load of the hollow column and a lower bound of 0.1754 for his maximizing profile compared to 47.9898 and 0.03 obtained by Cox and Overton [2] and 47.9920 and 0.0263 by us [9].

In this paper we revisit the *unconstrained problem* for the clamped-clamped circular column with emphasis on the region $0 < p \le 1$ and the hollow column. In addition to the fact that the first eigenvalue $\lambda_p(A)$ might not be simple which results in a non-differentiability of the mapping $A \mapsto \lambda_p(A)$, the problem presents at least two additional technical difficulties. Firstly, for the maximization problem, the unit ball in $L^1(0,1)$ and, in particular, the convex subset $\mathcal{A}$, are not weakly compact. Secondly, for the minimization problem, if A is not bounded below by a strictly positive constant, the corresponding fourth order differential operator is degenerate.

The computations of Cox and Overton [2] come up to the following conclusions. For $0 < p < 1$ the lowest eigenvalue $\lambda_p(A)$ is simple and for $p > 1$ it is double. The bifurcation occurs at $p = 1$. Specifically (Cox and Overton [2, p. 316]):

> Regarding the behaviour as p tends to 1 from above, we have found that the minimum of the optimal design tends to zero, and, though the optimal buckling load remains double, the least eigenvalue of the corresponding dual matrix tends to zero. Below $p = 1$ we found optimal designs with simple buckling loads.

They plot the minimum $\inf_{x\in[0,1]} A(x)$ and the maximum $\sup_{x\in[0,1]} A(x) = \|A\|_\infty$ of the maximizing profiles as a function of p, $1 < p < 4.5$ in [2, Fig. 7, p. 317]:

> Our analysis of (4.5)–(4.7) led us to believe that, for $p > 1$, the minimum (maximum) of the optimal design increases (decreases) with p. This is reinforced by Fig. 7, whose lower (upper) curve traces the minimum (maximum) of the optimal design as a function of p.

We made the following additional observations (see [9]):

(1) As p goes to zero the first two eigenvalues converge to the first two eigenvalues of the uniform column.

(2) The upper bound on the maximizing profiles seems to go to 2 as p goes to zero.

(3) For $p > 1$, the lower bound on the profiles is monotonically decreasing towards 0 as $p > 1$ decreases to 1 as in [2, Fig. 7, p. 317]. Between 0 and 1 it is very small but numerically strictly positive. It is not clear whether it is mathematically zero or not.

We first prove that the dual inf sup problem is bounded above by 48 for $p \le 1$, but is $+\infty$ for $p > 1$ ruling out its use in characterizing a maximizing profile. In order to avoid dealing with a possible degenerate eigenvalue equation (a profile with a zero lower bound), convex analysis techniques (cf., for instance, Ekeland and Temam [8]) are used on the function $A \mapsto \lambda_p(A)$. We first extend from $L^\infty(0, 1)$ to continuous profiles in $C[0, 1]$ the existence theorem of Cox and Overton [2] for profiles bounded above and below by strictly positive constants. Secondly, for $0 < p < 1$, we prove the existence of $L^1(0, 1)$ maximizing profiles without a priori bounds (Thm. 3.2). Finally, for $0 < p \le 1$, we prove the existence of both $L^\infty(0, 1)$ and $C[0, 1]$ maximizing profiles without the strictly positive lower bound (Thm. 3.4). This opens the possibility of the occurence of degenerate maximizing profiles in the region $0 < p \le 1$ similar to the piecewise quadratic profile of Olhoff and Niordson [12] and Seiranyan [14]. Finally, Theorem 4.1 shows that if $u \in W^2_0(0, 1)$ is a non-zero function such that $|u''(x)| = \|u''\|_\infty$ a.e. in $[0, 1]$, then, for each $p > 0$, u is the eigenfunction of a degenerate profile. So non-degenerate maximizing profiles cannot be associated with such u's but they are not ruled out by our new existence theorems for $0 < p \le 1$.

2. Preliminary results

2.1. Choice of families of profiles

Denote by $\|A\|_p$ the $L^p(0, 1)$ norm and

$$L^p_+(0, 1) \overset{\text{def}}{=} \{A \in L^p(0, 1) : A(x) \ge 0 \text{ a.e. in } [0, 1]\}, \quad 0 < p \le \infty. \tag{2}$$

For $A \in L^1_+(0,1)$, the function $f_p(A,v)$ is bounded below by 0. Hence, its infimum with respect to $v \in W^{2,\infty}_0(0,1)\backslash\{0\}$

$$\lambda_p(A) \overset{\text{def}}{=} \inf_{0\neq v\in W^{2,\infty}_0(0,1)} f_p(v,A), \quad f_p(v,A) \overset{\text{def}}{=} \frac{\int_0^1 A^p(x)\,|v''(x)|^2\,dx}{\int_0^1 |v'(x)|^2\,dx} \tag{3}$$

is well-defined and belongs to $[0,+\infty]$. We first show that $\lambda_p(A)$ cannot be $+\infty$ even if there exist functions $A \in L^1_+(0,1)$ such that $\|A\|_p = +\infty$ for all $p > 1$.[1]

Lemma 2.1. *For each p, $0 < p < +\infty$, the function $A \mapsto \lambda_p(A) \colon L^1_+(0,1) \to [0,+\infty)$ is well-defined. Moreover, $\mathcal{A} \cap L^p(0,1) = \mathcal{A}$ and $\lambda_p(A) \le 48\,\|A\|_1^p$ for $0 < p \le 1$; $\lambda_p(A) \le 48\,\|A\|_p^p$ for $p > 1$.*

Remark 2.2. For $0 < p \le 1$ and $A \in \mathcal{A}$, $\lambda_p(A) \le 48$ and $\sup_{A\in\mathcal{A}} \lambda_p(A) \le 48$, but it is not clear whether $\sup_{A\in\mathcal{A}} \lambda_p(A)$ is finite or not for all $p > 1$. □

Proof of Lemma 2.1. (i) For $0 < p \le 1$, $v \in W^{2,\infty}_0(0,1)$, and $A \in L^1_+(0,1)$

$$f_p(v,A) \le \int_0^1 A^p\,dx\,\frac{\|v''\|^2_\infty}{\|v'\|_2^2} \le \left[\int_0^1 A\,dx\right]^p \frac{\|v''\|^2_\infty}{\|v'\|_2^2} \le \|A\|_1^p\,\frac{\|v''\|^2_\infty}{\|v'\|_2^2}.$$

Choose the following $W^{2,\infty}_0(0,1)$ function

$$u(x) = \frac{1}{2}\begin{cases} x^2, & 0 \le x < 1/4, \\ \frac{1}{8} - (x-\frac{1}{2})^2, & 1/4 \le x < 3/4, \\ (x-1)^2, & 3/4 \le x \le 1, \end{cases} \quad u'(x) = \begin{cases} x, & 0 \le x \le 1/4 \\ \frac{1}{2} - x, & 1/4 \le x < 3/4 \\ x-1, & 3/4 \le x \le 1, \end{cases} \tag{4}$$

$u''(x) = -1$ in $(1/4,3/4)$ and $+1$ elsewhere so that $|u''(x)| = 1$ a.e. in $[0,1]$. Moreover, $\|u'\|_2^2 = 1/48$. Therefore, for $0 < p \le 1$, $\lambda_p(A) \le \|A\|_1^p 48$, and the function $A \mapsto \lambda_p(A) \colon L^1_+(0,1) \to [0,+\infty)$ is well-defined.

(ii) For $p > 1$ and $A \in L^1_+(0,1)$, $p \mapsto \|A\|_p$ is monotonically increasing[2] and $\|A\|_1 \le \|A\|_p \le \|A\|_\infty$. If $\|A\|_\infty < +\infty$, $\lambda_p(A) \le \lambda_p(1)\,\|A\|_\infty^p \le (2\pi)^2\,\|A\|_\infty^p < +\infty$. Given $0 \ne A \in L^1_+(0,1)$ such that $\|A\|_p = +\infty$ for some $p > 1$, then $\|A\|_\infty = +\infty$. The set $S = \{x \in [0,1] : A(x) \ge 2\|A\|_1\}$ is non-empty with measure $0 < |S| \le 1/2$. By Lusin's Theorem, for $\varepsilon > 0$, there exists a compact $K \subset S$ such that $|S\backslash K| < \varepsilon$ and A is continuous on K. So $K^c \cap (0,1)$ is open, $|K^c \cap (0,1)| = 1 - |K| = 1 - |S| + |S\backslash K| \ge 1/2 + |S\backslash K| \ge 1/2$, and the space $W^{2,\infty}_0(K^c \cap (0,1))$ contains non-zero functions. By setting $v = 0$ on K, $W^{2,\infty}_0(K^c \cap (0,1))$ can be viewed as a subspace of $W^{2,\infty}_0(0,1)$ and

$$\lambda_p(A) \le \inf_{\substack{v\in W^{2,\infty}_0(K^c\cap(0,1)) \\ v\neq 0}} f_p(v,A) \le (2\,\|A\|_1)^p \inf_{\substack{v\in W^{2,\infty}_0(K^c\cap(0,1)) \\ v\neq 0}} \frac{\int_0^1 |v''|^2\,dx}{\int_0^1 |v'|^2\,dx} < +\infty$$

[1] For instance, consider the function $A(x) = (\log 2)\,[[(x/2)(\log(x/2))^2]^{-1}$ which belongs to $\mathcal{A}$. For all $0 < p \le 1$, $\|A\|_p \le 1$. But for all $p > 1$, $\|A\|_p = +\infty$ and $A(x) \to +\infty$ as $x \to 0$.

[2] The case $A(x) = 1$ corresponds to the uniform column for which $\lambda_p(1) = (2\pi)^2$.

and the map $A \mapsto \lambda_p(A)\colon L^1_+(0,1) \to [0,+\infty)$ is well-defined. However, the infimum on the right-hand side depends on A and, with this estimate, we cannot get a uniform upper bound independent of $A \in \mathcal{A}$ as in the cases $0 < p \le 1$. □

A complementary approach to working in $L^1_+(0,1)$ and $\mathcal{A}$ is to work in the spaces $L^\infty_+(0,1)$ and $\mathcal{A}^\infty \overset{\text{def}}{=} \mathcal{A} \cap L^\infty(0,1)$ for which $f_p(v,A)$ is finite for all $v \in H^2_0(0,1)$ and the function $A \mapsto \lambda_p(A)\colon L^\infty_+(0,1) \to \mathbb{R}$ is well-defined. By density of $L^\infty(0,1)$ in $L^1(0,1)$, the three suprema

$$\sup_{A\in\mathcal{A}} \lambda_p(A) \ge \sup_{A\in\mathcal{A}\cap L^p(0,1)} \lambda_p(A) \ge \sup_{A\in\mathcal{A}^\infty} \lambda_p(A) \tag{5}$$

are equal and the uniform column provides a lower bound for the suprema

$$\sup_{A\in\mathcal{A}} \lambda_p(A) = \sup_{A\in\mathcal{A}\cap L^p(0,1)} \lambda_p(A) = \sup_{A\in\mathcal{A}^\infty} \lambda_p(A) \ge \lambda_p(1) = (2\pi)^2. \tag{6}$$

The maximization problems (6) with respect to $\mathcal{A}$, $\mathcal{A} \cap L^p(0,1)$, or $\mathcal{A}^\infty$ will be referred to as the *unconstrained problems* as opposed to the cases where finite non-zero bounds are imposed on the profiles: $0 < \alpha \le A(x) \le \beta$.

We immediately make the following observation: if the function A is zero almost everywhere on an interval (a,b) or $[a,b] \subset [0,1]$, $a < b$, then $\lambda_p(A) = 0$. To see this, choose a function $w \in W^{2,\infty}_0(a,b)$ such that $\|w'\|_{L^2(a.b)} = 1$. Extending w by zero on the remainder of the interval $[0,1]$, we obtain a function $v \in W^{2,\infty}_0(0,1)$ such that $f_p(A,v) = 0$. So, we can *a priori* eliminate all the profiles A's that are not strictly positive almost everywhere on a non trivial interval. This amounts to work with functions that are strictly positive almost everywhere on $[0,1]$ and for which A^{-1} exists and is also strictly positive almost everywhere.

2.2. Upper bound on the dual or inf sup problem

It is often useful when dealing with a sup inf problem to study the dual inf sup problem. From the general inequality sup inf $\le$ inf sup

$$\sup_{A\in\mathcal{A}} \inf_{\substack{v\in W^{2,\infty}_0(0,1)\\ v\neq 0}} \frac{\int_0^1 A^p(x)\,|v''(x)|^2\,dx}{\int_0^1 |v'(x)|^2\,dx} \le \inf_{\substack{v\in W^{2,\infty}_0(0,1)\\ v\neq 0}} \sup_{A\in\mathcal{A}} \frac{\int_0^1 A^p(x)\,|v''(x)|^2\,dx}{\int_0^1 |v'(x)|^2\,dx}. \tag{7}$$

An important change occurs at $p = 1$ and the dual problem cannot be used to extract information or characterize a maximizing profile for $p > 1$.

Theorem 2.3. (i) *For* $0 < p \le 1$,

$$\inf_{\substack{v\in W^{2,\infty}_0(0,1)\\ v\neq 0}} \sup_{A\in\mathcal{A}} \frac{\int_0^1 A(x)\,|v''(x)|^2\,dx}{\int_0^1 |v'(x)|^2\,dx} \le \inf_{\substack{v\in W^{2,\infty}_0(0,1)\\ v\neq 0}} \frac{\|v''\|^2_{L^\infty(0,1)}}{\|v'\|^2_{L^2(0,1)}} \le 48. \tag{8}$$

(ii) *For* $p = 1$,

$$\inf_{\substack{v \in W_0^{2,\infty}(0,1)\\ v \neq 0}} \sup_{A \in \mathcal{A}} \frac{\int_0^1 A(x)\,|v''(x)|^2\,dx}{\int_0^1 |v'(x)|^2\,dx} = \inf_{\substack{v \in W_0^{2,\infty}(0,1)\\ v \neq 0}} \frac{\|v''\|^2_{L^\infty(0,1)}}{\|v'\|^2_{L^2(0,1)}} \leq 48. \tag{9}$$

(iii) *For* $p > 1$,

$$\inf_{\substack{v \in W_0^{2,\infty}(0,1)\\ v \neq 0}} \sup_{A \in \mathcal{A}^{1,\infty}} \frac{\int_0^1 A^p\,|v''|^2\,dx}{\int_0^1 |v'|^2\,dx} = +\infty, \quad \mathcal{A}^{1,\infty} \stackrel{\text{def}}{=} \mathcal{A} \cap W^{1,\infty}(0,1), \tag{10}$$

$$\inf_{\substack{v \in W_0^{2,\infty}(0,1)\\ v \neq 0}} \sup_{A \in \mathcal{A}} \frac{\int_0^1 A^p\,|v''|^2\,dx}{\int_0^1 |v'|^2\,dx} = \inf_{\substack{v \in W_0^{2,\infty}(0,1)\\ v \neq 0}} \sup_{A \in \mathcal{A}^\infty} \frac{\int_0^1 A^p\,|v''|^2\,dx}{\int_0^1 |v'|^2\,dx} = +\infty.$$

where $\mathcal{A}^{1,\infty} \subset \mathcal{A} \cap C[0,1]$ *is the family of Lipschitzian profiles.*

Proof. (i) For $0 < p \leq 1$ and $A \in \mathcal{A}$, $\int_0^1 A^p\,dx \leq [\int_0^1 A\,dx]^p = 1$, then

$$\frac{\int_0^1 A^p(x)\,|v''(x)|^2\,dx}{\int_0^1 |v'(x)|^2\,dx} \leq \int_0^1 A^p(x)\,dx\,\frac{\|v''\|^2_{L^\infty(0,1)}}{\|v'\|^2_{L^2(0,1)}} \leq \frac{\|v''\|^2_{L^\infty(0,1)}}{\|v'\|^2_{L^2(0,1)}}.$$

As a result

$$\inf_{\substack{v \in W_0^{2,\infty}(0,1)\\ v \neq 0}} \sup_{A \in \mathcal{A}} \frac{\int_0^1 A^p(x)\,|v''(x)|^2\,dx}{\int_0^1 |v'(x)|^2\,dx} \leq \inf_{\substack{v \in W_0^{2,\infty}(0,1)\\ v \neq 0}} \frac{\|v''\|^2_{L^\infty(0,1)}}{\|v'\|^2_{L^2(0,1)}}.$$

By using the function u given by (4), we get a finite upper bound

$$\inf_{\substack{v \in H_0^2(0,1)\\ v \neq 0}} \sup_{A \in \mathcal{A}} \frac{\int_0^1 A^p(x)\,|v''(x)|^2\,dx}{\int_0^1 |v'(x)|^2\,dx} \leq \inf_{\substack{v \in W_0^{2,\infty}(0,1)\\ v \neq 0}} \frac{\|v''\|^2_{L^\infty(0,1)}}{\|v'\|^2_{L^2(0,1)}} \leq 48.$$

(ii) For $p = 1$ part (i) can be sharpened. Recall that

$$\frac{\int_0^1 A\,|v''|^2\,dx}{\int_0^1 |v'|^2\,dx} \leq \int_0^1 A\,dx\,\frac{\|v''\|^2_{L^\infty(0,1)}}{\|v'\|^2_{L^2(0,1)}} = \frac{\|v''\|^2_{L^\infty(0,1)}}{\|v'\|^2_{L^2(0,1)}} \leq 48.$$

For $v \in W_0^{2,\infty}(0,1)$ there are two cases. If $|v''(x)| = \|v''\|_\infty$, $0 \leq x \leq 1$,

$$\frac{\int_0^1 A\,|v''|^2\,dx}{\int_0^1 |v'|^2\,dx} = \frac{\|v''\|^2_{L^\infty(0,1)}}{\|v'\|^2_{L^2(0,1)}} \geq \sup_{A \in \mathcal{A}} \frac{\int_0^1 A\,|v''|^2\,dx}{\int_0^1 |v'|^2\,dx}.$$

If $v \in W_0^{2,\infty}(0,1)$ such that $|v''(x)| < \|v''\|_\infty$ on a set S of non-zero measure, $0 < |S| < 1$ and $|[0,1] \backslash S| > 0$. Choose an $\bar{A} \in \mathcal{A}$ such that $\bar{A} = 0$ on S and $\int_{[0,1]\backslash S} \bar{A}\,dx = 1$

$$\frac{\int_0^1 \bar{A}\,|v''|^2\,dx}{\int_0^1 |v'|^2\,dx} = \frac{\int_{[0,1]\backslash S} \bar{A}\,|v''|^2\,dx}{\int_0^1 |v'|^2\,dx} = \frac{\|v''\|^2_{L^\infty(0,1)}}{\|v'\|^2_{L^2(0,1)}} \geq \sup_{A \in \mathcal{A}} \frac{\int_0^1 A\,|v''|^2\,dx}{\int_0^1 |v'|^2\,dx}$$

and we again get the equality

$$\frac{\int_0^1 \bar{A}\,|v''|^2\,dx}{\int_0^1 |v'|^2\,dx} = \frac{\|v''\|^2_{L^\infty(0,1)}}{\|v'\|^2_{L^2(0,1)}} = \sup_{A\in\mathcal{A}} \frac{\int_0^1 A\,|v''|^2\,dx}{\int_0^1 |v'|^2\,dx} = \frac{\int_0^1 \bar{A}\,|v''|^2\,dx}{\int_0^1 |v'|^2\,dx}$$

$$\Rightarrow \inf_{\substack{v\in H_0^2(0,1)\\ v\neq 0}} \sup_{A\in\mathcal{A}} \frac{\int_0^1 A(x)\,|v''(x)|^2\,dx}{\int_0^1 |v'(x)|^2\,dx} = \inf_{0\neq v\in W^{2,\infty}(0,1)} \frac{\|v''\|^2_{L^\infty(0,1)}}{\|v'\|^2_{L^2(0,1)}}.$$

(iii) Since $0 \neq v \in W_0^{2,\infty}(0,1)$ implies that $0 \neq v'' \in L^\infty(0,1)$, there exists a Lebesgue point a in $(0,1)$ such that

$$\lim_{\varepsilon\searrow 0} \frac{1}{2\varepsilon}\int_{a-\varepsilon}^{a+\varepsilon} |v''|^2\,dx = |v''|^2(a) > 0.$$

For $p > 1$ and $0 < 1/n < \min\{a, 1-a\}$, consider the sequence $\{A_n\} \subset \mathcal{A}^{1,\infty}$

$$A_n(x) \stackrel{\text{def}}{=} \frac{2}{3}\begin{cases} 2n^2\left(x + \dfrac{1}{n} - a\right), & a - \dfrac{1}{n} \leq x < a - \dfrac{1}{2n} \\ n, & a - \dfrac{1}{2n} \leq x < a + \dfrac{1}{2n} \\ 2n^2\left(a + \dfrac{1}{n} - x\right), & a + \dfrac{1}{2n} \leq x \leq a + \dfrac{1}{n} \\ 0, & \text{otherwise} \end{cases}$$

$$\Rightarrow \int_0^1 A_n\,dx = 1, \quad \|A_n\|_\infty = n, \quad \|A_n'\|_\infty = \frac{4}{3}n^2, \quad \|A_n\|_p \leq \frac{2}{3}\,n^{1/p}\frac{2}{n}.$$

Evaluate the integral for $p > 1$,

$$\begin{aligned}\int_0^1 A^p\,|v''|^2\,dx &\geq \int_{a-1/(2n)}^{a+1/(2n)} A^p\,|v''|^2\,dx = \left(\frac{2}{3}\right)^p n^p \int_{a-1/(2n)}^{a+1/(2n)} |v''|^2\,dx \\ &\geq \left(\frac{2}{3}\right)^p \underset{+\infty}{\underset{\downarrow}{n^{p-1}}}\ \underbrace{\frac{1}{1/n}\int_{a-1/(2n}^{a+1/(2n)} |v''|^2\,dx}_{\to|v''|^2(a)>0} \to +\infty \qquad \text{as } n\to\infty.\end{aligned}$$

We conclude that for $p>1$ and $0\neq v''\in L^2(0,1)$, $\displaystyle\sup_{A\in\mathcal{A}^{1,\infty}} \int_0^1 A^p\,|v''|^2\,dx = +\infty$. $\square$

2.3. Family of even profiles with respect to 1/2

Given $0 \leq \alpha \leq \beta$, consider the following family of profiles

$$\mathcal{A}_{\alpha,\beta} \stackrel{\text{def}}{=} \left\{A \in L^1(0,1) : \alpha \leq A(x) \leq \beta \text{ and } \int_0^1 A(x)\,dx = 1\right\} \subset \mathcal{A}^\infty. \tag{11}$$

We shall need the definition of an even and an odd function with respect to 1/2.

Definition 2.4. A function $f\colon [0,1] \to \mathbb{R}$ is *even* with respect to 1/2 if, for all $x \in [0,1]$, $f(1-x) = f(x)$. A function $f\colon [0,1] \to \mathbb{R}$ is *odd* if, for all $x \in [0,1]$, $f(1-x) = -f(x)$.

Theorem 2.5. (i) *For* $0 < p \le 1$,

$$\sup_{A\in\mathcal{A}_{\mathrm{even}}} \lambda_p(A) = \sup_{A\in\mathcal{A}} \lambda_p(A), \quad \mathcal{A}_{\mathrm{even}} \overset{\mathrm{def}}{=} \{A \in \mathcal{A} : A(1-x) = A(x) \text{ a.e. in } [0,1]\}.$$

(ii) *For* $p > 1$ *and* $0 < \alpha \le \beta$, $\sup_{A\in\mathcal{A}_{\alpha,\beta,\mathrm{even}}} \lambda_p(A) = \sup_{A\in\mathcal{A}_{\alpha,\beta}} \lambda_p(A)$, *where* $\mathcal{A}_{\alpha,\beta,\mathrm{even}} = \mathcal{A}_{\alpha,\beta} \cap \mathcal{A}_{\mathrm{even}}$.

Proof. (i) For $0 < p \le 1$ the function $y \mapsto y^p$ is concave and, hence, $A \mapsto f_p(A,v)\colon \mathcal{A} \to \mathbb{R}$ is concave, This property is sufficient to conclude.

(ii) For $p > 1$ and $0 < \alpha \le \beta$, see [2, p. 294]. □

2.4. Monotony of the suprema as a function of $p \ge 0$

Define

$$p \mapsto \lambda(p) \overset{\mathrm{def}}{=} \sup_{A\in\mathcal{A}} \lambda_p(A)\colon [0,+\infty) \to [0,+\infty], \tag{12}$$

where $\lambda(0) = (2\pi)^2$ is the lower bound on $\lambda(p)$ for all $p > 0$. Numerical experimentation in Figure 2.1 indicates that $\lambda(p)$ is finite, monotone increasing, continuous, and possibly concave in the region $0 < p < 4.5$. By Theorem 2.3, $\lambda(p)$ is bounded by 48 for $p \le 1$, but there is no proof so far that $\lambda(p)$ be even finite for $p > 1$. As p goes to zero, the first $\lambda(p) = \lambda_p^{(1)}$ and second $\lambda_p^{(2)}$ eigenvalues converge to the first and second eigenvalues 39.4784176... and 80.76291... of the uniform column. Moreover, it seems that $\lambda_p^{(1)} < \lambda_p^{(2)}$ (first eigenvalue simple) for $p < 1$, and $\lambda_p^{(1)} = \lambda_p^{(2)}$ (first eigenvalue double) for $p > 1$.

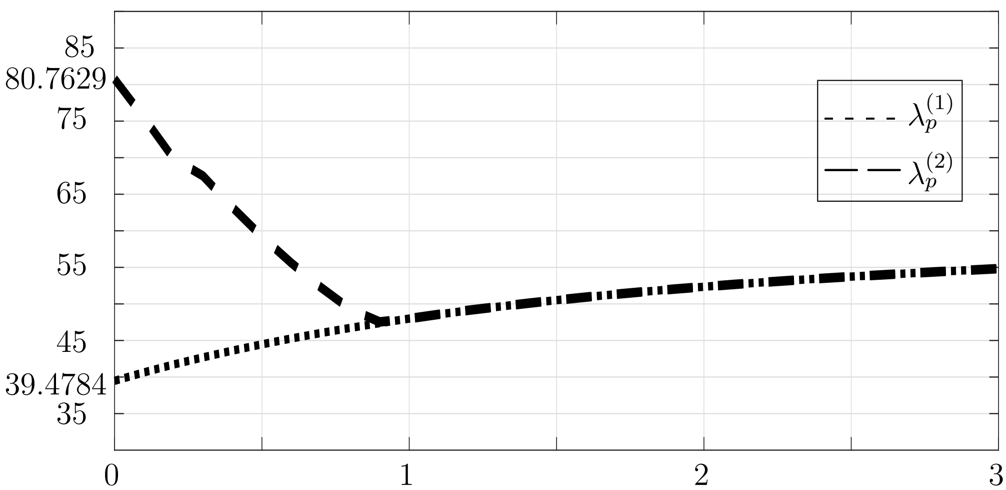

Figure 2.1: First ($\lambda_p^{(1)}$) and second ($\lambda_p^{(2)}$) eigenvalues of the maximizing profile versus p.

The monotony property is easy to verify but the concavity is not obvious.

Theorem 2.6. *The function $\lambda(p)$ is monotone increasing, that is,*

$$\text{for all pairs } 0 \le p \le p', \quad \lambda(0) = (2\pi)^2 \le \sup_{A \in \mathcal{A}} \lambda_p(A) \le \sup_{A \in \mathcal{A}} \lambda_{p'}(A).$$

In addition, for each $\beta > 0$,

$$p \mapsto \sup_{A \in \mathcal{A}_{0,\beta}} \lambda_p(A) \colon (0, +\infty) \to [(2\pi)^2, +\infty) \tag{13}$$

is continuous.

We need a small lemma.

Lemma 2.7. *For $A \in \mathcal{A}^\infty$, $0 \ne u \in W_0^{2,\infty}(0,1)$, and $0 < p \le p' < +\infty$,*

$$f_p(A,u) \le f_{p'}\left(\tilde{A}, u\right), \qquad \textit{where} \qquad \tilde{A} = A^{p/p'} / \int_0^1 A^{p/p'}\, dx.$$

Proof. Given $0 < p \le p'$ and $A \in \mathcal{A}^\infty$, then $p'/p \ge 1$ and

$$f_p(A,u) = \frac{\int_0^1 A^p\, |u''|^2\, dx}{\int_0^1 |u'|^2\, dx} = \left[\int_0^1 A^{p/p'}\, dx\right]^{p'} f_{p'}(\tilde{A}, u),$$

where $\tilde{A} \overset{\text{def}}{=} A^{p/p'} / \|A^{p/p'}\|_1 \ge 0$ and $\|\tilde{A}\|_1 = 1$. But for $p/p' \le 1$,

$$\int_0^1 A^{p/p'}\, dx \le \left[\int_0^1 (A^{p/p'})^{p'/p}\, dx\right]^{p/p'} = \left[\int_0^1 A\, dx\right]^{p/p'} = 1$$

which yields the expected inequality. $\square$

Proof of Theorem 2.6. (i) *Monotonicity.* From Lemma 2.7 for $v \in W_0^{2,\infty}(0,1)$ and $0 < p \le p'$, $f_p(A,v) \le f_{p'}\left(\tilde{A}, v\right)$. Take the infimum with respect to $v \in W_0^{2,\infty}(0,1)$,

$$\lambda_p(A) \le \lambda_{p'}(\tilde{A}) \le \sup_{A \in \mathcal{A}^\infty} \lambda_{p'}(A) \quad \Rightarrow \sup_{A \in \mathcal{A}^\infty} \lambda_p(A) \le \sup_{A \in \mathcal{A}^\infty} \lambda_{p'}(A).$$

(ii) *Continuity.* We have seen that if $\lambda_p(A) \ge (2\pi)^2$, then $A(x) > 0$ almost everywhere in $[0,1]$. If $A(x) > 0$, then $1 = A(x)\, A^{-1}(x) \le \beta\, A^{-1}(x)$. Hence $A^{-1}(x) \ge 1/\beta$ a.e. in $[0,1]$. For $p' > p$

$$\int_0^1 A^p |u''|^2 dx = \int_0^1 A^{-(p'-p)} A^{p'} |u''|^2 dx \ge \beta^{p-p'} \int_0^1 A^{p'} |u''|^2 dx$$

$$\Rightarrow \lambda_p(A) \ge \beta^{p-p'}\, \lambda_{p'}(A)$$

$$\Rightarrow 0 \le \lambda(p') - \lambda(p) \le (\beta^{p'-p} - 1)\, \lambda_p(A) = |\beta^{p'-p} - 1|\, \lambda_p(A).$$

Similarly, for $p' < p$, $0 \le \lambda(p) - \lambda(p') \le (1 - \beta^{p'-p})\, \lambda_p(A) = |\beta^{p'-p} - 1|\, \lambda_p(A)$ which implies $|\lambda(p') - \lambda(p)| \le |\beta^{p'-p} - 1|\, \lambda_p(A)$ and yields the continuity at p. $\square$

3. Existence of maximizing profiles

3.1. Extending Cox and Overton [2] from $\mathcal{A}^\infty$ to $\mathcal{A}^\infty \cap C[0,1]$

We first extend the existence theorem of Cox and Overton [2, p. 296] to continuous profiles.

Theorem 3.1. *Let $0 < \alpha \le \beta$ and $0 < p < \infty$.*

(i) *There exists an even function $\hat{A} \in \mathcal{A}_{\alpha,\beta}$ such that $\lambda_p(A) \le \lambda_p(\hat{A})$, for all $A \in \mathcal{A}_{\alpha,\beta}$.*

(ii) *There exists an even function $\hat{A} \in \mathcal{A}_{\alpha,\beta} \cap C[0,1]$ such that $\lambda_p(A) \le \lambda_p(\hat{A})$, for all $A \in \mathcal{A}_{\alpha,\beta} \cap C[0,1]$.*

Proof. (i) See Cox and Overton [2, p. 296]. (ii) All the arguments remain the same for $\mathcal{A}_{\alpha,\beta} \cap C[0,1]$. Since $0 < \alpha \le A(x) \le \beta,$ the inverse $A^{-1}(x)$ is continuous and $0 < \beta^{-1} \le A^{-1}(x) \le \alpha^{-1}$. They use the argument that the convex, bounded and strongly closed set

$$\mathcal{B} \overset{\text{def}}{=} \{B \in L^\infty 0,1) : 0 < \beta^{-1} \le B(x) \le \alpha^{-1}\},$$

is compact in $L^\infty(0,1)$ weak star. But, since the uniform norm on $C[0,1]$ coincides with the L^∞ norm, the subset

$$\mathcal{B} \cap C[0,1] = \{B \in C[0,1] : 0 < \beta^{-1} \le B(x) \le \alpha^{-1}\},$$

is also convex, bounded, and closed in $L^\infty(0,1)$-strong. As a result, it is compact in $L^\infty(0,1)$ weak star. So from each sequence in $\mathcal{B} \cap C[0,1]$ we can extract a subsequence that weakly star converges to some element of $\mathcal{B} \cap C[0,1]$. With this last argument the result follows from their proof. $\square$

3.2. Cases $0 < p < 1$ in $\mathcal{A}$ without bounds

For $0 < p < 1$ the initial problem (1) can be reformulated by introducing a number $r > 1$ such that $p < rp < 1$ and making the change of variable $B = A^{1/r}$ for which

$$\int_0^1 |B|^r\, dx = \int_0^1 A\, dx = 1 \quad \Rightarrow \|B\|_r = 1 \le 1.$$

For $r > 1$ the unit ball $\mathcal{B} = \{B \in L^r(0,1) : \|B\|_r \le 1\}$ in $L^r(0,1)$ is convex and weakly compact. Consider the new problem

$$\sup_{B\in\mathcal{B}} \lambda_p(|B|^r) \quad \Rightarrow \sup_{B\in\mathcal{B}} \lambda_p(|B|^r) \ge \sup_{A\in\mathcal{A}} \lambda_p(A) \ge (2\pi)^2. \tag{14}$$

To get an existence result, we need the weak upper semicontinuity of the mapping $B \mapsto \lambda_p(|B|^r) \colon \mathcal{B} \to \mathbb{R}$. If there exists a maximizing $B^* \in \mathcal{B}$, then $\|B^*\|_r \le 1$ and $B^* \ne 0$ since the supremum is greater or equal to $(2\pi)^2$. If $\|B^*\|_r < 1$, choose $\hat{B}(x) = |B^*(x)|\; \|B^*\|_r$ for which $\|\hat{B}\|_r = 1$ and, a fortiori, $\hat{B} \in \mathcal{B}$. We get

$$\sup_{B\in\mathcal{B}} \lambda_p(|B|^r) = \lambda_p(|B^*|^r) = (\|B^*\|_r^r)^p\, \lambda_p(|\hat{B}|^r) < \lambda_p(|\hat{B}|^r)$$

and this contradicts the fact that B^* is a maximizer in $\mathcal{B}$. Therefore, $\|B^*\|_r = 1$. Finally, by choosing $\hat{A} = |B^*|^r$, $\|\hat{A}\|_1 = \|B^*\|_r^r = 1$,

$$\lambda_p(\hat{A}) = \lambda_p(|B^*|^r) = \sup_{B\in\mathcal{B}} \lambda_p(|B|^r) \geq \sup_{A\in\mathcal{A}} \lambda_p(A) \geq \lambda_p(\hat{A})$$

and $\hat{A} \in \mathcal{A}$ is a maximizer of the initial problem.

Theorem 3.2. *Let* $0 < p < 1$.

(i) *Let* $r > 1$ *be such that* $p < rp < 1$. *The function* $B \mapsto \lambda_p(|B|^r)\colon L^r(0,1) \to \mathbb{R}$ *is concave and upper semicontinuous.*

(ii) *There exists an even function* $\hat{A} \in \mathcal{A}$ *such that* $\lambda_p(\hat{A}) = \sup_{A\in\mathcal{A}} \lambda_p(A) \leq 48$.

Proof. (i) (a) *Concavity.* For $0 < rp \leq 1$, $y \mapsto y^{rp}\colon \mathbb{R}_+ \to \mathbb{R}$ is concave. For $\alpha \in [0,1]$ and B_1, $B_2 \in L^r(0,1)$,

$$\begin{aligned}
&\int_0^1 (\alpha|B_1| + (1-\alpha)|B_2|)^{rp}\ |v''|^2\,dx \\
&\qquad\qquad \geq \alpha \int_0^1 |B_1|^{rp}\,|v''|^2\,dx + (1-\alpha)\int_0^1 |B_2|^{rp}\,|v''|^2\,dx \\
&\Rightarrow\ \lambda_p((\alpha|B_1| + (1-\alpha)|B_2|)^r) \geq \alpha\,\lambda_p(|B_1|^r) + (1-\alpha)\,\lambda_p(|B_2|^r).
\end{aligned}$$

(b) *Strong continuity of* $B \mapsto f_p(|B|^r, u)\colon L^r(0,1) \to \mathbb{R}$. For $0 < pr \leq 1$, $y \mapsto y^{rp}\colon \mathbb{R}_+ \to \mathbb{R}$ is concave, Hölder continuous, and $|y_2^{rp} - y_1^{rp}| \leq |y_2 - y_1|^{rp}$. As a result for $u \in E = \{v \in W_0^{2,\infty}(0,1) : \|v'\|_{L^2(0,1)} = 1\}$,

$$\begin{aligned}
&|f_p(|B_2|^r, u) - f_p(|B_1|^r, u)| \\
&\quad \leq \int_0^1 |B_2^{rp} - B_1^{rp}|\ |u''|^2\,dx \leq \|u''\|^2_{L^\infty(0,1)} \int_0^1 |B_2^{rp} - B_1^{rp}|\ dx \\
&\quad \leq \|u''\|^2_{L^\infty(0,1)} \int_0^1 |B_2 - B_1|^{rp}\ dx \\
&\quad \leq \|u''\|^2_{L^\infty(0,1)} \left[\int_0^1 |B_2 - B_1|^r\ dx\right]^p = \|u''\|^2_{L^\infty(0,1)}\,\|B_2 - B_1\|_r^{rp}.
\end{aligned}$$

(c) *Upper semicontinuity.* Given B and h such that $\lambda_p(|B|^r) < h$. there exists $0 \neq u \in W_0^{2,0}(0,1)$ such that $\lambda_p(|B|^r) \leq f_p(|B|^r, u) < h$. By continuity of $f_p(|B|^r, u)$ at $B \in L^r(0,1)$, there exists a neighbourhood $V(AB)$ of B in $L^r(0,1)$ such that

$$\forall B' \in V(B)), \quad f_p(|B'|^r, u) < h \quad \Rightarrow\ \lambda_p(|B'|^r) = \inf_{0\neq v\in W_0^{2,\infty}(0,1)} f_p(|B'|^r, v) < h.$$

Thence, the upper semicontinuity of $\lambda_p(|B|^r)$ at B.

(ii) Since $L^r(0,1)$ is reflexive for $r > 1$, $\mathcal{B}$ is weakly compact and the concave strongly upper semicontinuous function $B \mapsto \lambda_p(|B|^r)$ is weakly upper semicontinuous. Hence, from the remarks preceeding the theorem, there exists $B^* \in \mathcal{B}$, $\|B^*\|_r = 1$, that maximizes $\lambda_p(|B|^r)$ and there exists $\hat{A} = |B^*|^r \in \mathcal{A}$ that maximizes $\lambda_p(A)$. By Theorem 2.5, $\hat{A}$ can be chosen even with respect to $1/2$. $\square$

3.3. Cases $0 < p \leq 1$ for profiles in $\mathcal{A}^\infty$ and $\mathcal{A}^\infty \cap C[0,1]$

3.3.1. Properties of the mapping $A \mapsto \lambda_p(A)$

Theorem 3.3. (i) *Let $0 < p \leq 1$. The function $A \mapsto \lambda_p(A)\colon L^1_+(0,1) \to \mathbb{R}$ is concave and upper semicontinuous in $L^1(0,1)$. Hence, it is also concave and upper semicontinuous in $L^\infty(0,1)$ and in $C[0,1]$.*

(ii) *Let $p > 1$. The function $A \mapsto \lambda_p(A)\colon L^\infty_+(0,1) \to \mathbb{R}$ is upper semicontinuous in $L^\infty(0,1)$. The function $A \mapsto \lambda_p(A)\colon L^\infty_+(0,1) \cap C[0,1] \to \mathbb{R}$ is also upper semicontinuous in $L^\infty(0,1)$ and in $C[0,1]$.*

Proof. (i) (a) *Concavity.* For $0 < p \leq 1$, $y \mapsto y^p\colon \mathbb{R}_+ \to \mathbb{R}$ is concave. For $\alpha \in [0,1]$, A_1, $A_2 \in L^1_+(0,1)$, and $v \in W^{2,\infty}_0(0,1)$

$$\int_0^1 (\alpha A_1 + (1-\alpha)A_2)^p\, |v''|^2\, dx \geq \alpha \int_0^1 A_1^p\, |v''|^2\, dx + (1-\alpha) \int_0^1 A_2^p\, |v''|^2\, dx$$

$$\geq \alpha \inf_{v\in E} \int_0^1 A_1^p\, |v''|^2\, dx + (1-\alpha) \inf_{v\in E} \int_0^1 A_2^p\, |v''|^2\, dx = \alpha\, \lambda_p(A_1) + (1-\alpha)\, \lambda_p(A_2)$$

$$\Rightarrow \lambda_p(\alpha A_1 + (1-\alpha)A_2) \geq \alpha\, \lambda_p(A_1) + (1-\alpha)\, \lambda_p(A_2).$$

(b) *Strong continuity of* $A \mapsto f_p(A,u)\colon L^1_+(0,1) \to \mathbb{R}$. For $0 < p \leq 1$, $y \mapsto y^p\colon \mathbb{R}_+ \to \mathbb{R}$ is concave, Hölder continuous, and $|y_2^p - y_1^p| \leq |y_2 - y_1|^p$. As a result for $u \in E = \{v \in W^{2,\infty}_0(0,1) : \|v'\|_{L^2(0,1)} = 1\}$,

$$|f_p(A_2,u) - f_p(A_1,u)| = \left| \int_0^1 (A_2^p - A_1^p)\, |u''|^2\, dx \right| \leq \|u''\|^2_{L^\infty(0,1)}\, \|A_2^p - A_1^p\|_{L^1(0,1)}$$

$$\leq \|u''\|^2_{L^\infty(0,1)}\, \|A_2 - A_1\|^p_{L^1(0,1)}$$

$$\Rightarrow \forall A_1, A_2 \in L^1_+(0,1), \quad |f_p(A_2,u) - f_p(A_1,u)| \leq \|u''\|^2_{L^\infty(0,1)}\, \|A_2 - A_1\|^p_{L^1(0,1)}.$$

(c) *Upper semicontinuity.* Given A and h such that $\lambda_p(A) < h$. there exists $0 \neq u \in W^{2,\infty}_0(0,1)$ such that $\lambda_p(A) \leq f_p(A,u) < h$. By continuity of $f_p(A,u)$ at $A \in L^1_+(0,1)$, there exists a neighbourhood $V(A)$ of A in $L^1(0,1)$ such that

$$\forall B \in V(A) \cap L^1_+(0,1), \quad f_p(B,u) < h$$

$$\Rightarrow \lambda_p(B) = \inf_{0\neq v \in W^{2,\infty}_0(0,1)} f_p(B,v) \leq f_p(B,u) < h.$$

Thence, the upper semicontinuity of $\lambda_p(A)$ at A.

(d) The statements for $L^\infty_+(0,1) \cap C[0,1]$ follow from the fact that the uniform norm on $C[0,1]$ coincides with the $L^\infty(0,1)$ norm on $C[0,1]$.

(ii) Same technique as in (b) except that we use the fact that $A \mapsto f_p(A,u)$ is locally Lipschitz. Indeed, for $p > 1$, the function $y \mapsto y^p$ is differentiable at x and

$$\forall \varepsilon > 0,\ \exists\, \delta > 0, \text{ such that for all } y,\ |y - x| < \delta, \quad \left| \frac{y^p - x^p}{y - x} - p\, x^{p-1} \right| < \varepsilon.$$

For $A_1,\ A_2 \in L^\infty(0,1)$,

$$\|A_2 - A_1\|_{L^\infty(0,1)} < \delta, \quad |A_2^p(x) - A_1^p(x)| \le (p\,\|A_1\|_{L^\infty(0,1)}^{p-1} + \varepsilon)|A_2(x) - A_1(x)|$$

and for all A_2 such that $\|A_2 - A_1\|_{L^\infty(0,1)} < \delta$,

$$\|A_2^p - A_1^p\|_{L^\infty(0,1)} \le \underbrace{\left(p\,\|A_1\|_{L^\infty(0,1)}^{p-1} + \varepsilon\right)}_{=c(A_1,\varepsilon)} \|A_2 - A_1\|_{L^\infty(0,1)}.$$

Hence, in a δ-neighborhood of A_1

$$|f_p(A_2,u) - f_p(A_1,u)| = \left| \int_0^1 (A_2^p - A_1^p)\ |u''|^2\,dx \right|$$
$$\le \|u''\|_{L^2(0,1)}^2\,\|A_2^p - A_1^p\|_{L^\infty(0,1)} \le \|u''\|_{L^2(0,1)}^2\,c(A_1,\varepsilon)\,\|A_2 - A_1\|_{L^\infty(0,1)}.$$

The remainder of the proof is the same. □

3.3.2. Existence theorem

For $0 < p \le 1$, the lower bound can be dropped in the existence Theorem 3.1.

Theorem 3.4. *Let $0 < p \le 1$ and $\beta > 0$.*

(i) *There exists an even function $\hat{A} \in \mathcal{A}_{0,\beta}$ such that*

$$\lambda_p(\hat{A}) = \sup_{A \in \mathcal{A}_{0,\beta}} \lambda_p(A).$$

(ii) *There exists an even function $\hat{A} \in \mathcal{A}_{0,\beta} \cap C[0,1]$ such that*

$$\sup_{A \in \mathcal{A}_{0,\beta} \cap C[0,1]} \lambda_p(A) = \lambda_p(\hat{A}). \tag{15}$$

Proof. (i) Since $L^\infty(0,1) = (L^1(0,1))^*$, $(L^1(0,1))^*$ can be endowed with its *weak star topology* $\sigma((L^1(0,1))^*, L^1(0,1))^*$. The set $\mathcal{A}_{0,\beta}$ is convex, closed and bounded in $(L^1(0,1))^*$-strong. Hence, it is compact in the weak star topology. Moreover, for $0 < p \le 1$, the function $A \mapsto \lambda_p(A)$ is concave and strongly upper semicontinuous by Theorem 3.3. Hence, it is upper semi-continuous for the weak star topology and there exists a maximizing profile in the compact $\mathcal{A}_{0,\beta}$. Moreover, by Theorem 2.5 (i), the profile can be chosen even.

(ii) The set $\mathcal{A}_{0,\beta} \cap C[0,1]$ is a closed convex subset of $\mathcal{A}_{0,\beta}$ in $L^\infty(0,1)$-strong since the norm $L^\infty(0,1)$ coincides with the norm of the supremum on $C[0,1]$. So, it is also compact for the weak star topology on $L^\infty(0,1)$. □

3.3.3. Function of Olhoff-Niordson for the hollow column ($p = 1$)

The available published computations are given in Table 3.1. It is not clear whether the lower bound on A is mathematically strictly positive or not. The maximizing profile of Cox and Overton [2] (without bounds) and ours are very close to the piecewise quadratic profile introduced by Olhoff and Niordson [12] and Seiranyan [14]

$$A(x) \overset{\text{def}}{=} \frac{3}{2}\begin{cases} 1-16\,x^2 & 0 \le x \le 1/4, \\ 16\,x-16\,x^2-3 & 1/4 \le x \le 3/4, \\ 32\,x-16\,x^2-15 & 3/4 \le x \le 1, \end{cases} \tag{16}$$

with zeros in 1/4 and 3/4. Its eigenvalue of 48 corresponds to the $W_0^{2,\infty}(0,1)$ eigenfunction (4). Theorem 3.4 (ii) does not a priori rule out this profile or degenerate profiles.

eigenvalues	[9]	Cox-Overton [2]	Seiranyan [14]	Egorov [7]
$\lambda_1^{(1)}$	47.9920	47.9898	48	41.48
$\lambda_1^{(2)}$	47.9927	n/a	n/a	41.48
$\inf_{x\in[0,1]} A(x)$	0.0263	$\approxeq 0.03$	0	0.1754
$\|A\|_\infty$	1.4984	$\approxeq 1.50$	1.5	1.5655

Table 3.1: Eigenvalues $\lambda_1^{(1)}$ and $\lambda_1^{(2)}$, $\inf_{x\in[0,1]} A(x)$, and $\sup_{x\in[0,1]} A(x)$ for $p=1$.

The function (16) belongs to $\mathcal{A}^{1,\infty} = \mathcal{A} \cap W^{1,\infty}(0,1)$, is strictly positive except in $x = 1/4$ and $x = 3/4$, and A^{-1} exists a.e. So, it is possible to make sense of an eigenfunction in a subspace of $H_0^1(0,1)$ larger than $W_0^{2,\infty}(0,1)$. Given $A \in \mathcal{A}^{1,\infty}$ and

$$v \in V^2(A) \overset{\text{def}}{=} \left\{ v \in H_0^1(0,1) : Av' \in H_0^1(0,1) \right\}, \tag{17}$$

the distribution $A\,v''$ belongs to the space $L^2(0,1)$ since

$$A\,v'' = (Av')' - A'\,v' \in L^2(0,1). \tag{18}$$

This suggests to minimize with respect to the space

$$\hat{V}^1(A) \overset{\text{def}}{=} \left\{ v \in V^2(A)) : A^{-(1/2)}\,[(Av')' - A'v'] \in L^2(0,1) \right\}, \tag{19}$$

where $A^{1/2}v'' = A^{-(1/2)}\ [(Av')' - A'v'] \in L^2(0,1)$ and hence

$$f_1(A,v) = \frac{\int_0^1 A\,|v''|^2\,dx}{\int_0^1 |v'|^2\,dx}$$

makes sense.

By changing the sign of the central part of the function A between 1/4 and 3/4 in (16), we obtain a function $\tilde{A} \in W^{2,\infty}(0,1)$ such that $3/2 - \tilde{A} \in W_0^{2,\infty}(0,1)$:

$$\tilde{A}(x) \overset{\text{def}}{=} \frac{3}{2}\begin{cases} 1-16\,x^2 & 0 \le x < 1/4, \\ 16\,(x-1/2)^2-1 & 1/4 \le x < 3/4, \\ 1-16\,(x-1)^2 & 3/4 \le x \le 1, \end{cases} \tag{20}$$

with first and second derivatives

$$\tilde{A}'(x) = 48 \begin{cases} -x, & 0 \le x < 1/4 \\ -(1/2-x), & 1/4 \le x < 3/4 \\ -(x-1), & 3/4 \le x \le 1 \end{cases} \qquad \tilde{A}''(x) = 48 \begin{cases} -1, & 0 \le x < 1/4 \\ +1, & 1/4 \le x < 3/4 \\ -1, & 3/4 < x \le 1 \end{cases}$$

$$\Rightarrow |\tilde{A}| = A, \quad |\tilde{A}'| = |A'|, \quad \tilde{A}\,\tilde{A}' = A\,A', \quad \|\tilde{A}'\|_2 = \sqrt{48}, \tag{21}$$

$$\Rightarrow \boxed{A\,\tilde{A}'' = -48\,\tilde{A}, \quad \tilde{A}\,\tilde{A}'' = -48\,A.} \tag{22}$$

We summarize and sharpen the known results.

Theorem 3.5. *Let $p = 1$ and A be the function defined by (16). $\lambda = 48$ is an eigenvalue that corresponds to the eigenfunctions*

$$w_1 = \tilde{A} - 3/2 \in W_0^{2,\infty}(0,1) \quad \textit{and} \quad w_2 = A - 3/2 \in \hat{V}^1(A), \tag{23}$$

$$Aw_i'' = -48\,w_i - 72 \in W^{1,\infty}(0,1),\ w_i \in W_0^{1,\infty}(0,1),\ Aw_i' \in W_0^{1,\infty}(0,1), \tag{24}$$

where $\tilde{A}$ is given by (20). In both cases $|w_i''(x)|$ is constant almost everywhere in $[0,1]$, but $w_2''(x)$ has a Dirac mass of magnitude 24 at $x = 1/4$ and $x = 3/4$:

$$w_2'' = -48 + 24\,\delta(1/4) + 24\,\delta(3/4) \quad \Rightarrow \int_0^1 w_2''\,dx = 0. \tag{25}$$

Proof. We have already proved part (i). It remains to prove part (ii).

(a) First consider w_1. In the light of the identities (22)

$$A(x)\,[\tilde{A}(x) - 3/2]'' + 48\,\left[\tilde{A}(x) - 3/2\right] = 48\,(-3/2) = -72. \tag{26}$$

It is readily seen that $\tilde{A} \in W^{2,\infty}(0,1)$ and that $\tilde{A}^2 \in W^{2,\infty}(0,1)$. The function $u = \tilde{A} - 3/2 \in W_0^{2,\infty}(0,1)$ that we have introduced is an eigenfunction for $\lambda = 48$.

(b) There is a second eigenfunction $w_2 = A - 3/2$ that does not belong to $H_0^2(0,1)$, but to $\hat{V}^1(A)$. Indeed, $Aw_2' = AA' = \tilde{A}\tilde{A}' \in H_0^1(0,1)$ and

$$\begin{aligned} &(Aw_2')' - A'w_2' = \tilde{A}'\tilde{A}' + \tilde{A}\tilde{A}'' - A'A' = \tilde{A}\tilde{A}'' = -48\,A \in W^{1,\infty}(0,1) \\ &\Rightarrow A^{-1/2}\,((Aw_2')' - A'w_2') = -48\,A^{1/2} \in C[0,1] \subset L^2(0,1). \end{aligned}$$

Hence, by the definition (19) of $V^1(A)$, $w_2 \in \hat{V}^1(A)$ and, from equation (18), $AA'' \in L^2(0,1)$. Since $(A')^2 = (\tilde{A}')'^2$

$$\begin{aligned} A\,A'' = (A\,A')' - A'\,A' = 1/2\,(A^2)' - (A')^2 = 1/2\,(\tilde{A}^2)' - (\tilde{A}')^2 = \tilde{A}\,\tilde{A}'' = -48\,A \\ \Rightarrow A\,[A - 3/2]'' + 48\,[A - 3/2] = 48\,(-3/2) = -72 \end{aligned}$$

and $w_2 \in \hat{V}^1(A)$ is an eigenfunction for 48. □

Remark 3.6. The function w_2 is the one given by Olhoff and Niordson [12, sec. 4, equation (49), p. 25] in 1979. They discard this solution after substitution in the Rayleigh quotient, since they do not obtain the expected 48. Seiranyan [14] in 1984 pointed out that the computation is incorrect since they have not taken into account the Dirac masses at $x = 1/4$ and $x = 3/4$ in the second derivative. $\square$

Remark 3.7. As a complementaty information on the function (16), we show that $A^{-1/2} \in L^1(0,1)$, $A^{-1/2} \notin L^2(0,1)$, and $A^{1/2} \notin W^{1,\infty}(0,1)$. By factorizing the polynomials in the expression (16) of A, we obtain

$$A(x) = 24 \begin{cases} \left(\frac{1}{4}-x\right)\left(\frac{1}{4}+x\right), & 0 \leq x \leq \frac{1}{4} \\ \left(x-\frac{1}{4}\right)\left(\frac{3}{4}-x\right), & \frac{1}{4} \leq x \leq \frac{3}{4}, \\ \left(x-\frac{3}{4}\right)\left(\frac{5}{4}-x\right), & \frac{3}{4} \leq x \leq 1 \end{cases} \quad A'(x) = 48 \begin{cases} -x, & 0 \leq x < \frac{1}{4} \\ \left(\frac{1}{2}-x\right), & \frac{1}{4} < x < \frac{3}{4} \\ -(x-1), & \frac{3}{4} \leq x \leq 1. \end{cases}$$

So $A \in W^{1,\infty}(0,1)$. On the other hand, $(A^{1/2})' = (1/2)\, A'\, A^{-1/2} \notin L^2(0,1)$. Actually, the zeros of each polynomial in A and in A' are simple and distinct, but the zeros of A' cannot cancel those of A and, even less, those of $A^{1/2}$. In the interval $[0,1/4]$,

$$\int_0^{1/4} |A^{-1/2}|^2\, dx = \int_0^{1/4} \frac{1}{\left(\frac{1}{4}-x\right)\left(\frac{1}{4}+x\right)}\, dx \geq 2 \int_0^{1/4} \frac{1}{\left(\frac{1}{4}-x\right)}\, dx = +\infty.$$

It is also readily checked that

$$A^{-1/2} \in L^1(0,1), \quad A^{-1/2} \notin L^2(0,1) \quad \text{and} \quad A^{1/2} \notin W^{1,\infty}(0,1),$$

since, for instance, in the interval $[0,1/4]$,

$$\int_0^{1/4} A^{-1/2}\, dx = \int_0^{1/4} \frac{1}{\sqrt{(1/4-x)\,(1/4+x)}}\, dx \leq 2 \int_0^{1/4} \frac{1}{\sqrt{(1/4-x)}}\, dx = 2. \qquad \square$$

3.3.4. Spaces $V^p(A)$, $0 < p \leq 1$, for $A \in L^1_+(0,1)$

We can enlarge the space $W_0^{2,\infty}(0,1)$ to the following larger Hilbert space.

Definition 3.8. Given $A \in L^1_+(0,1)$ and $0 < p \leq 1$, denote by $V^p(A)$ the *completion* of $W_0^{2,\infty}(0,1)$ with respect to the norm and scalar product

$$\|u\|_{V^p(A)} \stackrel{\text{def}}{=} [(u,u)_{V^p(A)}]^{1/2}, \quad (u,v)_{V^p(A)} \stackrel{\text{def}}{=} \int_0^1 u'\, v' + A^p\, u''\, v''\, dx.$$

By definition, $V^p(A)$ is a non-trivial Hilbert space, $W_0^{2,\infty}(0,1) \subset V^p(A) \subset H_0^1(0,1)$, and $W_0^{2,\infty}(0,1)$ is dense in $V^p(A)$. If there exists $0 < \alpha \leq \beta$ such that $\alpha \leq A(x) \leq \beta$ almost everywhere in $[0,1]$, then $V^p(A) = H_0^2(0,1)$. In addition, since

$$\|v'\|_2^2 \le \|v'\|_2^2 + \|A^{p/2}v''\|_2^2 \le \max\{1, \|A\|_1^p\}\|v'\|_\infty^2 + \|v''\|_2^2,$$

the injections $W_0^{2,\infty}(0,1) \subset V^p(A) \subset H_0^1(0,1)$ are continuous. Moreover,

$$\forall\, 0 < p < p' \le 1, \quad \|v'\|_{L^2}^2 + \|A^{p'/2}v''\|_2^2 \le \max\{1, \|A\|_1^{p'-p}\}\|v'\|_2^2 + \|A^{p/2}v''\|_2^2$$

and hence the injections $W_0^{2,\infty}(0,1) \subset V^p(A) \subset V^{p'}(A) \subset H_0^1(0,1)$ are also continuous for $0 < p < p' \le 1$.

Remark 3.9. Spaces with weights have been used in the literature but with the additional assumption that $A^{-1} \in L^1(0,1)$ (see, for instance, [10], [1]). This condition is verified for the strictly positive bounds of [2]. □

$V^p(A)$ is the natural space associated with the minimization problem. Since the numerator and the denominator of the function $f_p(A,v)$ are continuous with respect to the norm of $V^p(A)$, $f_p(A,v)$ is continuous at every point $0 \ne v \in V^p(A)$.

Theorem 3.10. *Given $A \in L^1_+(0,1)$ and $0 < p \le 1$,*

$$\lambda_p(A) = \inf_{0 \ne v \in V^p(A)} \frac{\int_0^1 A^p\, |v''|^2\, dx}{\int_0^1 |v'|^2\, dx} = \inf_{0 \ne v \in W_0^{2,\infty}(0,1)} \frac{\int_0^1 A^p\, |v''|^2\, dx}{\int_0^1 |v'|^2\, dx}. \tag{27}$$

Proof. The infimum with respect to $W_0^{2,\infty}(0,1)$ exists and is finite since the quotient is bounded below by 0. We proceed by contradiction. Assume that

$$\inf_{0 \ne v \in V^p(A)} \frac{\int_0^1 A^p\, |v''|^2\, dx}{\int_0^1 |v'|^2\, dx} < m \overset{\text{def}}{=} \inf_{0 \ne v \in W_0^{2,\infty}(0,1)} \frac{\int_0^1 A^p\, |v''|^2\, dx}{\int_0^1 |v'|^2\, dx}.$$

By definition of the infimum, there exists $0 \ne u \in V^p(A)$ such that

$$\inf_{0 \ne v \in V^p(A)} \frac{\int_0^1 A^p\, |v''|^2\, dx}{\int_0^1 |v'|^2\, dx} \le \frac{\int_0^1 A^p\, |u''|^2\, dx}{\int_0^1 |u'|^2\, dx} < m \overset{\text{def}}{=} \inf_{0 \ne v \in W_0^{2,\infty}(0,1)} \frac{\int_0^1 A^p\, |v''|^2\, dx}{\int_0^1 |v'|^2\, dx}.$$

By density of $W_0^{2,\infty}(0,1)$ in $V^p(A)$, there exists a sequence $\{u_n\}$, $u_n \ne 0$, in $W_0^{2,\infty}(0,1)$ such that $u_n \to u$ in $V^p(A)$, leading to the contradiction

$$\begin{aligned}\frac{\int_0^1 A^p\, |u''|^2\, dx}{\int_0^1 |u'|^2\, dx} < m \overset{\text{def}}{=} \inf_{0 \ne v \in W_0^{2,\infty}(0,1)} \frac{\int_0^1 A^p\, |v''|^2\, dx}{\int_0^1 |v'|^2\, dx} &\le \frac{\int_0^1 A^p\, |u_n''|^2\, dx}{\int_0^1 |u_n'|^2\, dx} \\ &\to \frac{\int_0^1 A^p\, |u''|^2\, dx}{\int_0^1 |u'|^2\, dx}.\end{aligned}$$

□

Remark 3.11. If the maximizing A is not bounded below by some strictly positive constant, we have not been able to prove the existence of a minimizing solution in $V^p(A)$. Choose a minimizing sequence $\{u_n\}$ in $V^p(A)$ such that

$$\|u_n'\|_2 = 1, \quad (2\pi)^2 \le \int_0^1 A^p\, |u_n''|^2\, dx \le \lambda_p(A) + 1, \quad \int_0^1 A^p\, |u_n''|^2\, dx \to \lambda_p(A).$$

The sequence $\{u_n\}$ is bounded in $V^p(A)$ and there exists $u \in V^p(A)$ and a subsequence, still denoted $\{u_n\}$, such that $u_n \to u$ in $V^p(A)$-weak and

$$\int_0^1 A^p\,|u''|^2\,dx \le \lambda_p(A), \quad \|u'\|_2 \le 1. \tag{28}$$

To conclude we would need to show that $\|u'\|_2 = 1$. So, if the maximizing A is degenerate, it might very well be possible that no element of $V^p(A)$ achieves the infimum. □

If there is a minimizer $u \in V^p(A)$, it is an *eigenfunction* corresponding to $\lambda_p(A)$ which is solution of the equation

$$u \in V^p(A), \quad A^p\,u'' + \lambda_p(A)\,u = q \in P^1(0,1), \tag{29}$$

for some $q \in P^1(0,1)$, the space of polynomials de degree less or equal to 1. The condition $u \in V^p(A)$ is important since this equation may have other parasitic solutions in a larger space. Condition (29) is equivalent to

$$u \in V^p(A), \quad (A^p\,u'')' + \lambda_p(A)\,u' = c, \tag{30}$$

for some constant c. Since u does not necessarily belong to $H^2_0(0,1)$, the equation

$$u \in V^p(A), \quad (A^p\,u'')'' + \lambda_p(A)\,u'' = 0 \tag{31}$$

is to be interpreted in the distributional sense.

Remark 3.12. For $A \in L^1_+(0,1)$ and $A(x) > 0$ a.e., the space $V^1(A)$ is a subset of $\hat{V}^1(A)$ in (19), but they do not seem to be equal. So, the next question is whether $w^2 \in \hat{V}^1(A)$ in Theorem 3.5 is a *parasitic solution* of the eigenfunction equation or not. If this is the case, then $w^1 \in W^2_0(0,1)$ in Theorem 3.5 could be the unique eigenfunction in $V^1(A)$ making $\lambda = 48$ a simple eigenvalue. But is 48 the smallest eigenvalue? In that context the counterexample given by Olhoff and Niordson [12, *Remark*, p. 25] in 1979

$$u(x) \overset{\text{def}}{=} \begin{cases} x^2, & 0 \le x < 1/4, \\ (1-2x)/8, & 1/4 \le x < 3/4, \\ -(1-x)^2, & 3/4 \le x \le 1, \end{cases} \quad u'(x) = \begin{cases} 2x, & 0 \le x < 1/4 \\ -1/4, & 1/4 \le x < 3/4 \\ 2(1-x), & 3/4 \le x \le 1 \end{cases} \tag{32}$$

gives a Rayleigh quotient equal to $\frac{4}{7}\,48 \approxeq 27.4285 < (2\pi)^2 \approxeq 39.4784 < 48$. This is the figure they obtained to exclude the profile (16):

> "Substituting (32) into the well-known expression for the Rayleigh quotient, this quotient attains the value 27.43... This result not only implies that $\hat{\lambda} = 48$ is not the fundamental eigenvalue of the design (16), but also shows that this design is not optimal".

However, this function u belongs to $\hat{V}^1(A)$, but we don't know if it belongs to $V^1(A)$. If it does not then their argument would not hold. □

4. Eigenfunctions and degenerate profiles

For $0 < p \le 1$ our new existence Theorems 3.2 and 3.4 do not rule out the possibility of a degenerate maximizing profile, that is, a profile whose lower bound is zero at some points. Spaces with weights have been used in the literature but with the additional assumption that $A^{-1} \in L^1(0,1)$ (see, for instance, [10], [1]). Such a condition is verified for the strictly positive bounds of [2].

We first give a technical result in part (i) followed by a general result in part (ii) that shows that if $u \in W_0^2(0,1)$ is a non-zero function such that $|u''(x)| = \|u''\|_\infty$ a.e. in $[0,1]$, then, for each $p > 0$, u is the eigenfunction of a continuous degenerate profile. Thence, if the maximizing profile is not degenerate, the corresponding eigenfunction cannot be of this type.

Theorem 4.1. *Let $0 < p < \infty$.*

(i) *Given a function $0 \ne u \in W_0^{2,\infty}(0,1)$ such that $|u''(x)| = 1$ a.e. in $[0,1]$, then $u'' = 2\chi_U - 1$, where χ_U is the characteristic function of*

$$U \stackrel{\text{def}}{=} \{x \in [0,1] : u''(x) = 1\} \subset [0,1] \tag{33}$$

that satisfies the conditions

$$\int_0^1 \chi_U \, dx = 1/2, \quad \int_0^1 (x - 1/2)\, \chi_U \, dx = 0, \quad \partial U \ne \varnothing, \quad \int_{\partial U} dx = 0. \tag{34}$$

Conversely, given a measurable subset U of $[0,1]$ that satisfies conditions (34)*, it generates a function $0 \ne u \in W^{2,\infty}(0,1)$ such that $u'' = 2\chi_U - 1$ and $|u''(x)| = 1$ a.e. in $[0,1]$.*

(ii) *Given $p > 0$ and a measurable $U \subset [0,1]$ that satisfies conditions* (34)*, the associated function $0 \ne u \in W_0^{2,\infty}(0,1)$ such that $u'' = 2\chi_U - 1$ a.e. in $[0,1]$ can only be an eigenfunction for the eigenvalue*

$$\mu_p(A) = \frac{1}{\left[\int_0^1 |u(x_0) - u(x)|^{1/p} \, dx\right]^p}, \tag{35}$$

corresponding to the following degenerate profile in $\mathcal{A}^\infty \cap C[0,1]$

$$A(x) = \frac{|u(x_0) - u(x)|^{1/p}}{\int_0^1 |u(x_0) - u(x)|^{1/p} \, dx}, \quad x_0 \in \partial U, \tag{36}$$

where $\partial U \ne \varnothing$ has zero measure, u is constant on ∂U, and $A = 0$ on ∂U .

Proof. (i) Consider a function $v \in W_0^{2,\infty}(0,1)$ such that $|v''(x)| = 1$, $v''(x) = \pm 1$ almost everywhere. Up to a set of zero measure define the following subset of $[0,1]$,

$$U \stackrel{\text{def}}{=} \{x \in [0,1] : v''(x) = 1\} \quad \text{and} \quad [0,1]\backslash U = \{x \in [0,1] : v''(x) = -1\} \tag{37}$$

and the corresponding characteristic function of U: $\chi_U(x) = 1$ if $x \in U$ and $\chi_U(x) = 0$ if $x \notin U$, The second order derivative can now be written as

$$v'' = \chi_U - [1 - \chi_U] = 2\,\chi_U - 1 \text{ a.e. in } [0,1]. \tag{38}$$

In order to have v in $H_0^2(0,1)$, v'' must be orthogonal to $P^1[0,1]$. This is equivalent to the following two conditions

$$\int_0^1 [2\,\chi_U - 1]\; dx = 0 \quad \text{and} \quad \int_0^1 x\;[2\,\chi_U - 1]\; dx = 0 \tag{39}$$

which are themselves equivalent to conditions (34). This eliminates all subsets U of measure different from $1/2$ and consequently $\partial U \neq \varnothing$ and the measure of ∂U is zero.

Conversely, given a Lebesgue measurable subset U of $[0,1]$ that verifies the conditions (34), it generates a unique $W_0^{2,\infty}(0,1)$ function such that $u'' = 2\,\chi_U - 1$.

(ii) Given a Lebesgue measurable subset U of $[0,1]$ that satisfies the conditions (34), we want to construct an even function $A \in \mathcal{A}^\infty \cap C[0,1]$ for which u is an eigenfunction for some $\mu = \mu_p(A)$, that is,

$$A(x)^p\, u''(x) + \mu\, u(x) = c$$

for some $c \in \mathbb{R}$. Since $|u''(x)| = 1$ almost everywhere

$$0 \leq A(x)^p = \frac{c - \mu\, u(x)}{u''(x)} = \begin{cases} c - \mu\, u(x), & \text{if } x \in U, \\ \mu\, u(x) - c, & \text{if } x \notin U. \end{cases} \tag{40}$$

Since $\partial U \neq \varnothing$, by continuity of A at all points $x_0 \in \partial U$, $c = \mu\, u(x_0)$, that is,

$$\forall x_0 \in \partial U, \quad u(x_0) = c/\mu \;\text{ and }\; A(x_0) = 0, \quad A(x)^p = \mu\,|u(x_0) - u(x)| \tag{41}$$

$$\int_0^1 A\, dx = 1 \quad \Rightarrow \; A(x) = \frac{|u(x_0) - u(x)|^{1/p}}{\int_0^1 |u(x_0) - u(x)|^{1/p}\, dx}. \tag{42}$$

Finally, since the measure of U and its complement are $1/2$,

$$1 = \int_0^1 A(x)\, dx = \mu^{1/p} \int_0^1 |u(x_0) - u(x)|^{1/p}\, dx$$

$$\mu = \frac{1}{\left[\int_0^1 |u(x_0) - u(x)|^{1/p}\, dx\right]^p}, \qquad c = \frac{u(x_0)}{\left[\int_0^1 |u(x_0) - u(x)|^{1/p}\, dx\right]^p}. \qquad \square$$

If U is the union of a finite number of intervals, the function u is piecewise a polynomial of degree 2 on each interval. For instance, choose $U = [0,a] \cup [1-a,1]$ for some a, $0 \leq a \leq 1/2$. From conditions (34):

$$1/2 = \int_0^a dx + \int_{1-a}^1 dx = 2a \quad \Rightarrow \; \boxed{a = 1/4,}$$

$$0 = \int_0^{1/4} (x - 1/2)\, dx + \int_{3/4}^1 (x - 1/2)\, dx = 0.$$

The associated $W_0^{2,\infty}(0,1)$ function is

$$u(x) = \frac{1}{2}\begin{cases} x^2, & 0 \le x < 1/4, \\ \dfrac{1}{8} - (x - \dfrac{1}{2})^2, & 1/4 < x < 3/4, \\ (x-1)^2, & 3/4 \le x \le 1. \end{cases} \qquad u'(x) = \begin{cases} x, & 0 \le x < 1/4, \\ \dfrac{1}{2} - x, & 1/4 < x < 3/4, \\ x-1, & 3/4 \le x \le 1. \end{cases}$$

which is even with respect to 1/2 and

$$\|u'\|^2 = \int_0^{1/4} x^2\,dx + \int_{1/4}^{3/4} (1/2 - x)^2\,dx + \int_{3/4}^{1} (x-1)^2\,dx = \frac{1}{3}\left[\frac{1}{4^3} + 2\frac{1}{4^3} + \frac{1}{4^3}\right] = \frac{1}{48}.$$

The bound of 48 cannot be improved by dividing each subinterval into two as follows

$$U = [0, 1/8] \cup [2/8, 3/8] \cup [5/8, 6/8] \cup [7/8, 8/8]$$

since $|u''(x)| = 1$ remains constant, but the norm $\|u'\|_{L^2}$ goes to zero as intervals are subdivided and the resulting bound would be larger than 48.

Here, $\partial U = \{1/4, 3/4\}$ and $u(1/4) = u(3/4) = 1/32$.

$$A(x) = \frac{|1/32 - u(x)|^{1/p}}{\int_0^1 |1/32 - u(x)|^{1/p}\,dx}, \quad \mu_p(A) = \frac{1}{\left[\int_0^1 |1/32 - u(x)|^{1/p}\,dx\right]^p},$$

$$|1/32 - u(x)| = \frac{1}{2}\left\{\begin{array}{ll} 1/16 - x^2, & 0 \le x < 1/4, \\ 3/16 - (x - 1/2)^2, & 1/4 < x < 3/4, \\ 1/16 - (x-1)^2, & 3/4 \le x \le 1, \end{array}\right\} = \frac{1}{48} A_s(x),$$

where A_s is the profile of Olhoff and Niordson [12, p. 25] in 1979 and Seiranyan [14]

$$A_s(x) \stackrel{\text{def}}{=} \frac{3}{2}\begin{cases} 1 - 16\,x^2, & 0 \le x < 1/4, \\ 16\,x - 16\,x^2 - 3, & 1/4 \le x < 3/4, \\ 32\,x - 16\,x^2 - 15, & 3/4 \le x \le 1. \end{cases} \tag{43}$$

$$\Rightarrow \; A(x) = \frac{A_s^{1/p}}{\int_0^1 A_s^{1/p}\,dx}, \quad \mu_p(A) = 48\,\frac{1}{\left[\int_0^1 A_s^{1/p}\,dx\right]^p}. \tag{44}$$

For $p = 1$, $A = A_s$ and $\mu_1(A) = 48$. As p goes to zero, $\|A_s\|_{1/p} \to \|A_s\|_\infty = 3/2$ and $48\,\|A_s\|_{1/p} \to 32$ which is below the lower bound $(2\pi)^2$ on $\sup_{A \in \mathcal{A}} \lambda_p(A)$.

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