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Extreme Contractions on Finite-Dimensional Polygonal Banach Spaces*

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We explore extreme contractions on finite-dimensional polygonal Banach spaces, from the point of view of attainment of norm of a linear operator. We prove that if X is an n-dimensional polygonal Banach space and Y is any normed linear space and $T \in L(X, Y)$ is an extreme contraction, then T attains norm at n linearly independent extreme points of B_X. Moreover, if T attains norm at n linearly independent extreme points $x_1, x_2, \dots, x_n$ of B_X and does not attain norm at any other extreme point of B_X, then each Tx_i is an extreme point of B_Y. We completely characterize extreme contractions between a finite-dimensional polygonal Banach space and a strictly convex normed linear space. We introduce L-P property for a pair of Banach spaces and show that it has natural connections with our present study. We also prove that for any strictly convex Banach space X and any finite-dimensional polygonal Banach space Y, the pair (X, Y) does not have L-P property. Finally, we obtain a characterization of Hilbert spaces among strictly convex Banach spaces in terms of L-P property.

Keywords: Extreme contractions, polygonal Banach spaces, strict convexity, Hilbert spaces.

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1. Introduction.

The purpose of the present paper is to study the structure and properties of extreme contractions on finite-dimensional real polygonal Banach spaces. Characterization of extreme contractions between Banach spaces is a rich and intriguing area

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of research. It is worth mentioning that extreme contractions on a Hilbert space are well-understood [2, 4, 9, 10]. However, we are far from completely describing extreme contractions between general Banach spaces, although several mathematicians have studied the problem for particular Banach spaces [1, 3, 5, 12, 13]. Very recently, a complete characterization of extreme contractions between two-dimensional strictly convex and smooth real Banach spaces has been obtained in [11]. To the best of our knowledge, the characterization problem remains unsolved for higher dimensional Banach spaces. Lindenstrauss and Perles carried out a detailed investigation of extreme contractions on a finite-dimensional Banach space in [8]. It follows from their seminal work that extreme contractions on a finite-dimensional Banach space may have additional properties if the unit sphere of the space is a polytope. Motivated by this observation, we further explore extreme contractions between finite-dimensional polygonal Banach spaces. Without further ado, let us establish the relevant notations and terminologies to be used throughout the paper.

In this paper, the letters X, Y denote real Banach spaces. Let $B_X = \{x \in X : \|x\| \leq 1\}$ and $S_X = \{x \in X : \|x\| = 1\}$ denote the *unit ball* and the *unit sphere* of X respectively and let $L(X, Y)$ be the Banach space of all bounded linear operators from X to Y, endowed with the usual operator norm. Let E_X be the collection of all extreme points of the unit ball B_X. We say that a finite-dimensional Banach space X is a *polygonal* Banach space if S_X is a polytope, or equivalently, if B_X contains only finitely many extreme points. An operator $T \in L(X, Y)$ is said to be an *extreme contraction* if T is an extreme point of the unit ball $B_{L(X,Y)}$. We would like to note that extreme contractions are norm one elements of the Banach space $L(X, Y)$ but the converse is not necessarily true. For a bounded linear operator $T \in L(X, Y)$, let M_T be the collection of all unit vectors in X at which T attains norm, i.e.,

$$M_T = \{x \in S_X : \|Tx\| = \|T\|\}.$$

As we will see in due course of time, the notion of the norm attainment set M_T, corresponding to a linear operator T between the Banach spaces X and Y, plays a very important role in determining whether T is an extreme contraction or not. As a matter of fact, we prove that if X is an $n-$dimensional polygonal Banach space and Y is any normed linear space, then $T \in L(X, Y)$ is an extreme contraction implies that $span(M_T \cap E_X) = X$. Moreover, if $M_T \cap E_X$ contains exactly $2n$ elements then $T(M_T \cap E_X) \subset E_Y$. Indeed, this novel connection between extreme contractions and the corresponding norm attainment set is a major highlight of the present paper. As an application of this result, we completely characterize extreme contractions from a finite-dimensional polygonal Banach space to any strictly convex normed linear space. We would like to note that extreme contractions in $L(l_1, Y)$, where Y is any Banach space, have been completely characterized by Sharir in [13]. In this paper, we show that extreme contractions in $L(l_1^n, Y)$ can be characterized using our approach. We also illustrate that the nature of extreme contractions can change dramatically if we consider the domain space to be something other than finite-dimensional polygonal Banach spaces.

In [8], Lindenstrauss and Perles studied the set of extreme contractions in $L(X, X)$, for a finite-dimensional Banach space X. One of the main results presented in [8] is the following:

Theorem 1.1. *If X is a finite-dimensional Banach space, then the following statements are equivalent:*

(1) $T \in E_{L(X,X)}, \quad x \in E_X \Rightarrow Tx \in E_X$.

(2) $T_1, T_2 \in E_{L(X,X)} \ \Rightarrow \ T_1 \circ T_2 \in E_{L(X,X)}$.

(3) $\{T_i\}_{i=1}^m \subseteq E_{L(X,X)} \ \Rightarrow \ \|T_1 \circ \ldots \circ T_m\| = 1$, *for all m.*

Furthermore, they also proved in [8] that if X is a Banach space of dimension strictly less than 5 then X has the Properties (1), (2) and (3), as mentioned in the above theorem, if and only if either of the following is true:

(i) X is an inner product space.

(ii) B_X is a polytope with the property that for every facet (i.e., maximal proper face) K of B_X, B_X is the convex hull of $K \cup (-K)$.

In view of the results obtained in [8], it seems natural to introduce the following definition in the study of extreme contractions between Banach spaces:

Definition 1.2. Let X, Y be Banach spaces. We say that the pair (X, Y) has *L-P* (abbreviated form of *Lindenstrauss-Perles*) *property* if a norm one bounded linear operator $T \in L(X, Y)$ is an extreme contraction if and only if $T(E_X) \subseteq E_Y$.

We would like to remark that although the study conducted in [8] was only for the case $X = Y$, we have consciously formulated the above mentioned definition in a broader context, by not imposing the restriction that the domain space and the range space must be identical. In order to illustrate that our definition is meaningful, we provide examples of Banach spaces X, Y such that $X \neq Y$ and the pair (X, Y) has L-P property. As a matter of fact, we observe that the pair (l_∞^3, l_1^3) has L-P property. It is apparent that our study in this paper is motivated by the investigations carried out in [8]. We further give examples of finite-dimensional polygonal Banach spaces X, Y such that the pair (X, Y) does not have L-P property. We prove that if X is a strictly convex Banach space and Y is a finite-dimensional polygonal Banach space, then the pair (X, Y) does not have L-P property. We end the present paper with a characterization of Hilbert spaces among strictly convex Banach spaces in terms of L-P property.

2. Main results

Let us first notice the following easy proposition:

Proposition 2.1. *Let X be an n-dimensional Banach space and let $A \subseteq X$ be a bounded set. Suppose $\{x_1, x_2, \ldots, x_n\}$ is a basis of X. Then the following set $S = \{|\alpha_i| : z = \sum_{i=1}^n \alpha_i x_i \text{ and } z \in A\}$ is bounded.*

Using the above proposition, we obtain a necessary condition for a linear operator on a polygonal Banach space to be an extreme contraction, in terms of the norm attainment set.

Theorem 2.2. *Let X be an $n-$dimensional polygonal Banach space and let Y be any normed linear space. Let $T \in L(X,Y)$ be an extreme contraction. Then $span(M_T \cap E_X) = X$. Moreover, if $M_T \cap E_X$ contains exactly $2n$ elements then $T(M_T \cap E_X) \subset E_Y$.*

Proof. First, we prove that $span(M_T \cap E_X) = X$. Since X is an $n-$dimensional Banach space, an easy application of Krein-Milman theorem ensures that B_X has at least $2n$ extreme points out of which n are linearly independent. If M_T contains n linearly independent extreme points then we are done. Suppose that M_T does not contain n linearly independent extreme points. Let M_T contains at most k linearly independent extreme points, where $k < n$. Let $\{x_1, x_2, \ldots, x_k\}$ be one such linearly independent subset of M_T, consisting of extreme points of B_X. We extend it to a basis $\{x_1, x_2, \ldots, x_k, x_{k+1}, \ldots, x_n\}$ of X such that each x_i, $i = 1, 2, \ldots, n$, is an extreme point of B_X. Now, we can choose $w(\neq 0) \in B_Y$ such that $Tx_n \pm (w/j) \neq 0$ for every $j \in \mathbb{N}$. For each $j \in \mathbb{N}$, define linear operators $T_j, S_j \colon X \to Y$ as follows:

$$T_j(x_i) = Tx_i \ (i = 1, 2, \ldots, n-1) \qquad S_j(x_i) = Tx_i \ (i = 1, 2, \ldots, n-1)$$

$$T_j(x_n) = Tx_n + \frac{w}{j} \qquad S_j(x_n) = Tx_n - \frac{w}{j}.$$

Then $T \neq T_j$ and $T \neq S_j$ for all $j \in \mathbb{N}$. Also, we have $T = \frac{1}{2}(T_j + S_j)$ for all $j \in \mathbb{N}$. Since T is an extreme contraction, there exists a subsequence $\{j_k\}$ such that for all j_k, either $\|T_{j_k}\| > 1$ or $\|S_{j_k}\| > 1$. Without loss of generality, we may and do assume that $\|T_j\| > 1$ for all $j \in \mathbb{N}$.

Since X is finite-dimensional, each T_j attains norm at an extreme point of B_X. Suppose $\|T_j y_j\| = \|T_j\| > 1$ where each y_j is an extreme point of B_X. Let $z = a_1x_1 + \ldots + a_{n-1}x_{n-1} + a_nx_n \in S_X$ be arbitrary, then by Proposition 2.1, there exists $r > 0$ such that $|a_n| \leq r$, for each $z \in S_X$.

Now, $\|(T_j - T)z\| = \|(T_j - T)(a_1x_1 + \ldots + a_nx_n)\| = \|\frac{a_n w}{j}\| \leq \frac{r}{j} \to 0$ as $j \to \infty$.

So, $T_j \to T$ as $j \to \infty$. As each $y_j \in S_X$ and S_X is compact, we may assume without loss of generality that application of Krein-Milman theorem ensures $y_j \to y_0$ as $j \to \infty$, where $y_0 \in S_X$.

Noting that there are only finitely many extreme points and each y_j is an extreme point, we can further assume that $y_j = y_0$ for each j. So, $T_j y_j = T_j y_0 \to Ty_0$ as $j \to \infty$.

Therefore, $\|Ty_0\| \geq 1$. As $\|T\| = 1$, $\|Ty_0\| = 1$. So, $y_0 \in M_T$. Since y_0 is an extreme point and M_T has at most k linearly independent extreme points, the set $\{y_0, x_1, x_2, \ldots, x_k\}$ is linearly dependent. Let $y_0 = a_1x_1 + a_2x_2 + \ldots + a_kx_k$. Then, $T_j y_j = T_j y_0 = T_j(a_1x_1 + \ldots + a_kx_k) = a_1Tx_1 + \ldots + a_kTx_k = Ty_0$. Therefore, $\|T_j y_j\| = \|Ty_0\| = 1$, which is contradiction to the fact that $\|T_j\| > 1$ for all $j \in \mathbb{N}$. Thus, if T is an extreme contraction, then $span(M_T \cap E_X) = X$.

Now, we prove the next part of the theorem. As X is a finite-dimensional polygonal Banach space, E_X is a finite set. Let $\{\pm x_1, \pm x_2, \ldots, \pm x_m : \ \ m \geq n\} = E_X$ and $(M_T \cap E_X) = \{\pm x_i : i = 1, 2, \ldots, n\}$.

As T attains the norm only at $\{\pm\, x_i : i = 1, 2, \ldots, n\}$, $\|T\| = \|Tx_i\| = 1$ for all $i = 1, 2, \ldots, n$ and $\|Tx_j\| < \|T\| = 1$ for all $j = n+1, n+2, \ldots, m$. Therefore, there exists $\epsilon > 0$ such that $\|Tx_j\| < 1 - \epsilon$ for all $j = n+1, n+2, \ldots, m$.

Clearly, $\{x_1, x_2, ..., x_n\}$ is a basis of X. Therefore, there exist scalars $a_{1j}, a_{2j}, ..., a_{nj}$ such that $x_j = a_{1j}x_1 + a_{2j}x_2 + \ldots + a_{nj}x_n, \ \forall\ j = n+1, n+2, \ldots, m$. Now, using Proposition 2.1, there exists $r > 0$ such that $|a_{ij}| \leq r$ for all $i = 1, 2, \ldots, n$ and $j = n+1, n+2, \ldots, m$.

Suppose Tx_k is not an extreme point of B_Y for some $k \in \{1, 2, \ldots, n\}$. Then there exist $y, z \in S_Y \cap B[Tx_k, \frac{\epsilon}{2r}]$ such that $Tx_k = (1-t)y + tz$ for some $t \in (0, 1)$.

Let us define $T_1 \colon X \to Y$ by $T_1x_i = Tx_i$ for all $i = 1, 2, \ldots, n$ and $i \neq k$ and $T_1x_k = y, \ i = k$. Similarly, we define $T_2 \colon X \to Y$ by $T_2x_i = Tx_i$ for all $i = 1, 2, \ldots, n$ and $i \neq k$ and $T_2x_k = z, \ i = k$.

Clearly, $T_1 \neq T$ and $T_2 \neq T$. Also, we have $T = (1-t)T_1 + tT_2$. We claim that $\|T_1\| = \|T_2\| = 1$.

Clearly, $\|T_1x_i\| = \|Tx_i\| = 1$ for all $i = 1, 2, \ldots, n$ and $i \neq k$. Also, $\|T_1x_k\| = \|y\| = 1$ as $y \in S_Y \cap B[Tx_k, \frac{\epsilon}{2r}]$.

Furthermore, for each $j = n+1, \ldots, m$, we have,

$$\begin{aligned}
\|T_1x_j\| &= \|a_{1j}T_1x_1 + \ldots + a_{kj}T_1x_k + \ldots + a_{nj}T_1x_n\| \\
&= \|a_{1j}Tx_1 + \ldots + a_{kj}y + \ldots + a_{nj}Tx_n\| \\
&= \|a_{1j}Tx_1 + \ldots + a_{nj}Tx_n + a_{kj}(y - Tx_k)\| \\
&\leq \|Tx_j\| + |a_{kj}|\|y - Tx_k\| \leq 1 - \epsilon + r\frac{\epsilon}{2r} = 1 - \frac{\epsilon}{2} < 1.
\end{aligned}$$

Thus, for any extreme point x_i of B_X, $\|T_1x_i\| \leq 1$ and so $\|T_1\| = 1$. Similarly, $\|T_2\| = 1$. Therefore, $T = (1-t)T_1 + tT_2$ and $\|T_1\| = \|T_2\| = 1$. This contradicts the fact that T is an extreme contraction. Thus, if $M_T \cap E_X$ contains exactly $2n$ elements then $T(M_T \cap E_X) \subset E_Y$. This completes the proof of the theorem. □

As an immediate application of Theorem 2.2, it is possible to characterize extreme contractions from a finite-dimensional polygonal Banach space to a strictly convex normed linear space. We present the characterization in the form of the next theorem.

Theorem 2.3. *Let X be a finite-dimensional polygonal Banach space and let Y be a strictly convex normed linear space. Then $T \in L(X, Y)$, with $\|T\| = 1$, is an extreme contraction if and only if span$(M_T \cap E_X) = X$.*

Proof. The proof of the necessary part of the theorem follows directly from Theorem 2.2. Here, we prove the sufficient part of the theorem. Suppose that T is not an extreme contraction. Then there exist $T_1, T_2 \in L(X, Y)$ such that $T_1, T_2 \neq T$, $\|T_1\| = \|T_2\| = 1$ and $T = (1-t)T_1 + tT_2$ for some $t \in (0, 1)$. Let

$\{x_1, x_2, \ldots, x_n\} \subseteq M_T \cap E_X$ be a basis of X. Then $Tx_i = (1-t)T_1x_i + tT_2x_i$, for each $i \in \{1, 2, \ldots, n\}$. We also note that $T_1x_i, T_2x_i \in B_Y$, as $\|T_1\| = \|T_2\| = 1$. As Y is strictly convex, we have $E_Y = S_Y$. Therefore, each Tx_i is an extreme point of B_Y and it follows from that $Tx_i = T_1x_i = T_2x_i$, for each $i \in \{1, 2, \ldots, n\}$. However, this implies that T_1, T_2 agree with T on a basis of X and therefore, $T_1 = T_2 = T$. This contradicts our initial assumption that $T_1, T_2 \neq T$. Thus T is an extreme contraction and this completes the proof of the theorem. □

As another useful application of Theorem 2.2, it is possible to characterize extreme contractions from l_1^n to Y, where Y is any normed linear space. It is worth mentioning that the following corollary also follows from [13, Lemma 2.8].

Corollary 2.4. *Let $X = l_1^n$ and let Y be any normed linear space. Let $T \in L(X,Y)$ with $\|T\| = 1$. Then T is extreme contraction if and only if $M_T = E_X$ and $T(E_X) \subseteq E_Y$.*

Proof. Firstly, we know that $\{\pm e_1, \pm e_2, \ldots, \pm e_n\}$ is the set of all extreme points of the unit ball of l_1^n, where $e_i = (\underbrace{0, 0, \ldots, 0, 1}_{i}, 0, \ldots, 0)$ for each $i \in \{1, 2, \ldots, n\}$. Now, the proof of the necessary part of the theorem follows directly from Theorem 2.2. Here, we prove the sufficient part of the theorem. Suppose that T is not an extreme contraction. Then there exist $T_1, T_2 \in L(X,Y)$ such that $T_1, T_2 \neq T$, $\|T_1\| = \|T_2\| = 1$ and $T = (1-t)T_1 + tT_2$ for some $t \in (0,1)$. Then $Te_i = (1-t)T_1e_i + tT_2e_i$, for each $i \in \{1, 2, \ldots, n\}$. We also note that for each $i \in \{1, 2, \ldots, n\}$, $T_1e_i, T_2e_i \in B_Y$, as $\|T_1\| = \|T_2\| = 1$. Since $Te_i \in E_Y$, it follows that $Te_i = T_1e_i = T_2e_i$, for each $i \in \{1, 2, \ldots, n\}$. However, this implies that T_1, T_2 agree with T on a basis of X and therefore, $T_1 = T_2 = T$. This contradicts our initial assumption that $T_1, T_2 \neq T$. Thus T is an extreme contraction and this completes the proof of the corollary. □

Remark 2.5. If Y is a polygonal Banach space with p pair of extreme points then the number of extreme contractions in $L(l_1^n, Y)$ is $(2p)^n$. Moreover, since the number of extreme contractions in $L(l_1^n, Y)$ is finite, each extreme point of the unit ball of $L(l_1^n, Y)$ is also an exposed point of the unit ball of $L(l_1^n, Y)$. Therefore, the number of exposed points of the unit ball of $L(l_1^n, Y)$ is also $(2p)^n$. In particular,

(i) the number of extreme contractions in $L(l_1^n, l_1^n)$ is $2^n n^n$.

(ii) the number of extreme contractions in $L(l_1^n, l_\infty^n)$ is 2^{n^2}.

As illustrated in [8], one of the most intriguing aspects of the study of extreme contractions between Banach spaces is to explore the extremity of the images of the extreme points of the unit ball of the domain space under extreme contractions. Till now, we have considered only finite-dimensional polygonal Banach spaces as the domain space. However, if we choose the domain space to be something other than the polygonal Banach space then the scenario changes drastically. In fact, choosing the Euclidean space l_2^n as the domain space and l_∞^n as the co-domain space, we have the following proposition:

Proposition 2.6. *Let $T \in L(X,Y)$ with $\|T\| = 1$, where $X = l_2^n$ and $Y = l_\infty^n$. Then T is an extreme contraction if and only if corresponding matrix representation of T with respect to standard ordered basis is of the form*

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

where $\sum\limits_{j=1}^{n} {a_{ij}}^2 = 1$ *for all* $i = 1, 2, \ldots, n$.

Proof. We first note that a bounded linear operator T between reflexive Banach spaces is an extreme contraction if and only if T^* is an extreme contraction, which follows from the facts that $T = (1-t)T_1 + tT_2 \Leftrightarrow T^* = (1-t)T_1^* + tT_2^*$, $\|T\| = \|T^*\|$ and $(T^*)^* = T$ by reflexivity of domain and co-domain spaces. Now, the proof of the proposition directly follows from Corollary 2.4. $\square$

In the following two examples, we further illustrate the variation in the norm attainment set of extreme contractions, depending on the domain and the range space.

Example 2.7. In $L(l_2^2, l_\infty^2)$, $T = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$ is an extreme contraction though T attains norm only at $\pm(1,0)$.

Example 2.8. In $L(l_2^2, l_\infty^2)$, $T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is an extreme contraction. Here, T attains norm at $\pm(1,0)$ and $\pm(0,1)$ but $T(1,0) = (1,0)$ and $T(0,1) = (0,1)$ are not extreme points of unit ball of l_∞^2.

Let us now focus on the connections between the study carried out in [8] and some of the results obtained by us in the present paper, in light of the newly introduced notion of L-P property for a pair of Banach spaces (X,Y). It is easy to observe that l_1^n is a universal L-P space, in the sense that the pair (l_1^n, Y) has L-P property for every Banach space Y. We would also like to note that it follows from the works [6, 7] of Lima that there exist finite-dimensional polygonal Banach spaces X, Y, with $X \neq Y$, such that the pair (X,Y) has L-P property. Also, there are finite-dimensional polygonal Banach spaces X, Y with $X \neq Y$, such that the pair (X,Y) does not satisfy L-P property. As for example, the pair (l_∞^3, l_1^3) has L-P property, which follows from [7, Theorem 2.1], whereas the pair (l_∞^4, l_1^4) does not satisfy L-P property, which follows from Lemma 3.2 of [7].

Let us observe that Proposition 2.6 implies, in particular, that the pair (l_2^n, l_∞^n) does not have L-P property. Now, we prove that this observation is actually a consequence of the following general result.

Theorem 2.9. *Let X be any strictly convex Banach space and Y be a finite-dimensional polygonal Banach space. Then the pair (X, Y) does not have L-P property.*

Proof. Suppose that the pair (X, Y) satisfies L-P property. Then for any extreme contraction $T \in L(X, Y)$, we have $T(E_X) \subseteq E_Y$. As X is strictly convex, B_X has infinitely many extreme points, in fact, $E_X = S_X$. On the other hand, as Y is a finite-dimensional polygonal Banach space, B_Y has only finitely many extreme points. Thus, for any extreme contraction $T \in L(X, Y)$, T cannot be an one-to-one operator. Therefore, there exists a nonzero $x \in B_X$ such that $T(x) = 0$. So, $T(\frac{x}{\|x\|}) = 0$, which contradicts the fact that $T(E_X) = T(S_X) \subseteq E_Y$. This completes the proof of the fact that the pair (X, Y) does not have L-P property. $\square$

Remark 2.10. From Theorem 2.9, we can conclude that the pairs (l_p^n, l_1^n) and (l_p^n, l_∞^n) do not have L-P property for any $1 < p < \infty$. Furthermore, we already know that the pair (l_1^n, Y) has L-P property for every Banach space Y. This illustrates that there exist Banach spaces X, Y such that the pair (X, Y) has L-P property but the pair (Y, X) does not have L-P property.

Remark 2.11. The above Theorem 2.9 holds for any Banach space Y with countably many extreme points.

Now, we give a characterization of finite-dimensional Hilbert spaces among all finite-dimensional strictly convex Banach spaces in terms of L-P property.

Theorem 2.12. *Let X be a finite-dimensional strictly convex Banach space. Then X is a Hilbert space if and only if the pair (X, X) has L-P property.*

Proof. Firstly, we claim that for a strictly convex Banach space X, the pair (X, X) has L-P property if and only if every extreme contraction in $L(X, X)$ is an isometry. Clearly, if every extreme contraction in $L(X, X)$ is an isometry then the pair (X, X) has L-P property, as for a strictly convex space X, $S_X = E_X$. Conversely, since (X, X) satisfies L-P property, $T \in L(X, X)$ is an extreme contraction if and only if $T(E_X) \subseteq E_X$. As X is strictly convex, $E_X = S_X$. Thus, $T \in L(X, X)$ is extreme contraction if and only if $T(S_X) \subseteq S_X$. In other words, we must have $M_T = S_X$ and $\|T\| = 1$, i.e., $T \in L(X, X)$ is an isometry. This completes the proof of our claim. On the other hand, in [9], Navarro proved that a finite-dimensional Banach space X is a Hilbert space if and only if every extreme contraction in $L(X, X)$ is an isometry. The proof of the theorem follows directly from a combination of these two facts. $\square$

As an application of Theorem 2.12, we characterize Hilbert spaces among all strictly convex Banach spaces in terms of L-P property.

Corollary 2.13. *A strictly convex Banach space X is a Hilbert space if and only if for every two-dimensional subspace Y of X, the pair (Y, Y) has L-P property.*

Proof. Let us first prove the "only if" part. If X is a Hilbert space then every two-dimensional subspace Y of X is also a Hilbert space. Again, we know that for a finite-dimensional Hilbert space Y, isometries are the only extreme contractions in $L(Y, Y)$ and $E_Y = S_Y$. Thus, for every two-dimensional subspace Y of X, the pair (Y, Y) has L-P property.

Let us now prove the "if" part. If X is a strictly convex Banach space then every two-dimensional subspace Y of X is also a strictly convex Banach space. Since the pair (Y, Y) has L-P property, then by Theorem 2.12, Y is a Hilbert space. Thus every two-dimensional subspace of X is a Hilbert space which means that X itself has to be a Hilbert space. This establishes the theorem. □

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