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A Sandwich Theorem for Generalized Convexity and Applications

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In a recent paper [*A support theorem for generalized convexity and its applications*, J. Math. Anal. Appl. 458 (2018) 1044–1058] we have introduced the concept of (ω, t)-convexity and ω-convexity as a natural generalization of the concept of usual convexity, strong-convexity, approximate-convexity, Wright-convexity and many others. The main result of this paper gives a necessary and sufficient condition on ω under which we can separate an (ω, t)-convex (ω-convex) function and (ω, t)-concave (ω-concave) function by an (ω, t)-affine map (ω-affine map). It turns out that in both cases ω has to satisfy some functional equation. We solve these functional equations in the most frequently appearing form of the map ω. Several applications of the main results are also given.

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1. Introduction and terminology

Let us fix a number $t \in (0, 1)$ and denote by $\mathbb{Q}(t)$ the smallest field containing the singleton $\{t\}$. Throughout the present paper (unless explicitly stated otherwise) X stands for a linear space over the field $\mathbb{K}$, where $\mathbb{Q}(t) \subseteq \mathbb{K} \subseteq \mathbb{R}$ and D denotes a non-empty t-convex set i.e.

$$tD + (1 - t)D \subseteq D.$$

In the paper [24] we introduced a notion of (ω, t)-convexity in the following way:

If $\omega\colon D \times D \times [0, 1] \to \mathbb{R}$ is a given map, then we say that a function $f\colon D \to \mathbb{R}$ is:

- (ω, t)-*convex*, if $f(tx + (1 - t)z) \le tf(x) + (1 - t)f(z) + \omega(x, z, t),\ x, z \in D,$
- (ω, t)-*concave*, if $tf(x) + (1 - t)f(z) + \omega(x, z, t) \le f(tx + (1 - t)z),\ x, z \in D.$

If f is at the same time (ω, t)-*convex* and (ω, t)-*concave* then we say that it is (ω, t)-*affine*. In this case f satisfies the following functional equation

$$tf(x) + (1 - t)f(z) + \omega(x, z, t) = f(tx + (1 - t)z), \quad x, z \in D.$$

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If $t = \frac{1}{2}$ then f is said to be *ω-midpoint convex* (*ω-midpoint concave*, *ω-midpoint affine*, respectively). If the above inequalities are satisfied for all numbers $t \in [0,1]$ (where D stands for a convex set) then we say that f is *ω-convex* (*ω-concave*, *ω-affine*, respectively).

The following remark below shows that any function f can be represented as an ω-affine (in particular ω-convex) map with some function ω.

Remark 1.1. Since an arbitrary function $f\colon D \to \mathbb{R}$ satisfies the equality

$$f(tx + (1-t)y) = tf(x) + (1-t)f(y) + \omega(x,y,t)$$

with $\omega\colon D \times D \times [0,1] \to \mathbb{R}$ given by the formula

$$\omega(x,y,t) = f(tx + (1-t)y) - tf(x) - (1-t)f(y),$$

nothing can be said about the function f without additional assumptions on ω.

The notion of convex function was generalized by several authors. The main idea is based on modifying the right-hand side defining the inequality. The concept of *ω-convexity* is a common generalization of the notion of usual *convexity, strong-convexity, approximate-convexity, Wright-convexity* and many others. The term on the left-hand side of the inequality is the same in all definitions while the right-hand side of all inequalities has different form. Now, we recall briefly the definitions of the mentioned classes of functions.

Let D be a convex subset of a real normed space $(X, \|\cdot\|)$. A function $f\colon D \to \mathbb{R}$ is said to be *strongly t-convex* ($t \in (0,1)$) with modulus $c > 0$ if

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) - ct(1-t)\|x-y\|^2,$$

for all $x, y \in D$. If the above inequality is satisfied with $t = \frac{1}{2}$ then the function f is called *strongly midpoint convex.* If f is *strongly t-convex* for all $t \in (0,1]$ then we say that it is *strongly convex.*

Strongly convex functions were introduced by Polyak in [27]. They have useful properties in optimization theory. For instance, any strongly convex function $f\colon I \to \mathbb{R}$ (where $I \subseteq \mathbb{R}$ stands for an interval) has to be bounded from below, moreover, it attains a unique minimum on every closed subinterval of I. For more information about strongly convex functions see, for instance [3], [21], [26], [27], [37], [38] and the references therein. The classical notions of *t-convexity, midpoint-convexity, convexity* correspond to the case $c = 0$. The *strongly t-convex* functions are *(ω, t)-convex* with

$$\omega(x,y,t)\colon = -ct(1-t)\|x-y\|^2, \quad x, y \in D.$$

The notion of approximate convexity was introduced by Hyers and Ulam in [10]. A function $f\colon D \to \mathbb{R}$ defined on a convex subset D of a real normed vector space is called *ε-convex* (where $\varepsilon > 0$) if

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) + \varepsilon, \quad x, y \in D,\ t \in [0,1].$$

Further generalizations were made by Rolewicz who introduced in [31] the notion of *paraconvex* functions and *strongly paraconvex* functions (for more information the reader is refered to [32], [33]). The notion of approximate convexity has been recently successively generalized and intensively investigated (see [6], [9], [19], [34], [35], [39]). For fixed numbers $c > 0,\ p > 0,\ t \in (0,1)$ a real valued function f defined on convex subset D of a real normed space is called (c,t,p)-*convex* if it satisfies

$$f(tx + (1-t)y) \le tf(x) + (1-t)f(y) + ct(1-t)\|x-y\|^p, \quad x, y \in D.$$

The (c,t,p)-*convex* functions are (ω,t)-*convex* with

$$\omega(x,y,t) := ct(1-t)\|x-y\|^p, \quad x, y \in D.$$

In 1954 Wright [36] introduced a new convexity property. In Wright's honor a function $f\colon D \to \mathbb{R}$ defining on a convex subset D of a real linear space X, satisfying the inequality

$$f(tx + (1-t)y) + f((1-t)x + ty) \le f(x) + f(y), \quad x, y \in D$$

is said to be t-*Wright-convex*. If the above inequality is satisfied for all $t \in [0,1]$ then we say that f is a *Wright-convex* function. It is easy to see that any convex function is Wright-convex, on the other hand, if $a\colon X \to \mathbb{R}$ is additive, that is

$$a(x+y) = a(x) + a(y), \quad x, y \in X,$$

then it is also Wright-convex. The most important result concerning Wright-convex functions is the much more surprising statement that any Wright-convex function can be decomposed as the sum of additive and convex functions. This theorem was proved by Ng [18] (in the case where $X = \mathbb{R}^n$) and was generalized by many authors see for instance [12], [17], [23]. The t-Wright convex functions are (ω,t)-*convex* with

$$\omega(x,y,t) := f(x) + f(y) - f((1-t)x + ty), \quad x, y \in D.$$

Sandwich and separation theorems generalizing the celebrated Hahn-Banach theorem have a large number of applications in many branches of mathematic, mainly in functional analysis, optimization theory and convex programming. Several theorems of this type are known in the literature (see for instance [4], [5], [8], [13], [15], [22], [24], [25], [30] and the references therein). For convex functions the classical separation theorem states that if $f, -g\colon D \to \mathbb{R}$ are convex functions and $g \le f$ then there exists an affine function $a\colon D \to \mathbb{R}$ such that $g \le a \le f$. This theorem is a consequence of a separation theorem for convex sets. In this note we generalize a classical separation theorem to the case of (ω,t)-convex as well as ω-convex functions. The classical results are obtained from our main results by putting $\omega \equiv 0$. The applications of our main results for strongly-convex functions and approximately-convex functions are also given.

2. Results

We start our investigation with a lemma which will be useful in the proof of our main result.

Lemma 2.1. *Let $f\colon D\to\mathbb{R}$ be an (ω,t)-convex map, where $\omega\colon D\times D\times[0,1]\to\mathbb{R}$ satisfies the inequality*

$$\begin{aligned} &t\omega(z,u,t)+(1-t)\,\omega(z,v,t)-\omega(z,tu+(1-t)v,t)\\ &\le(1-t)\,\omega(u,v,t)-\omega((1-t)\,u+tz,(1-t)\,v+tz,t),\ \ u,v,z\in D. \end{aligned}\tag{1}$$

Then, for arbitrary $z\in D$, the function $f_z\colon D\to\mathbb{R}$ defined by the formula

$$f_z(x):=\frac{1}{1-t}\Big[f((1-t)x+tz)-tf(z)-\omega(z,x,t)\Big],$$

is (ω,t)-convex.

Proof. Fix $u,v\in D$ arbitrarily. Using the (ω,t)-convexity of f and (1) we get

$$\begin{aligned} f_z(tu+(1-t)v)&=\frac{1}{1-t}\Big[f((1-t)[tu+(1-t)v]+tz)-tf(z)-\omega(z,tu+(1-t)v,t)\Big]\\ &=\frac{1}{1-t}\Big[f(t[(1-t)u+tz]+(1-t)[(1-t)v+tz])-tf(z)-\omega(z,tu+(1-t)v,t)\Big]\\ &\le\frac{1}{1-t}\Big[tf((1-t)u+tz)+(1-t)f((1-t)v+tz)+\omega((1-t)u+tz,(1-t)v+tz,t)\\ &\qquad-tf(z)-\omega(z,tu+(1-t)v,t)\Big]\\ &=t\Big[\frac{1}{1-t}\big(f((1-t)u+tz)-tf(z)-\omega(z,u,t)\big)\Big]+\frac{t}{1-t}\omega(z,u,t)\\ &\qquad+(1-t)\Big[\frac{1}{1-t}\big(f((1-t)v+tz)-tf(z)-\omega(z,v,t)\big)\Big]+\omega(z,v,t)\\ &\qquad+\frac{1}{1-t}\omega((1-t)u+tz,(1-t)v+tz,t)-\frac{1}{1-t}\omega(z,tu+(1-t)v,t)\\ &\le tf_z(u)+(1-t)f_z(v)+\omega(u,v,t),\quad\text{which was to be proved.} \end{aligned}$$

□

Using similar arguments as in the previous lemma one can prove that for ω-convex maps the following counterpart of this lemma holds true.

Lemma 2.2. *Let D be a convex subset of a real linear space and let $f\colon D\to\mathbb{R}$ be an ω-convex map, where $\omega\colon D\times D\times[0,1]\to\mathbb{R}$ satisfies the inequality*

$$\begin{aligned} &s\omega(z,u,t)+(1-s)\omega(z,v,t)-\omega(z,su+(1-s)v,t)\\ &\le(1-t)\,\omega(u,v,s)-\omega((1-t)\,u+tz,(1-t)\,v+tz,s),\ \ u,v,z\in D,\ \ s,t\in[0,1]. \end{aligned}\tag{2}$$

Then for arbitrary $z\in D$ and any $t\in(0,1)$ the function $f_z\colon D\to\mathbb{R}$ defined by

$$f_z(x):=\frac{1}{1-t}\Big[f((1-t)x+tz)-tf(z)-\omega(z,x,t)\Big],$$

is ω-convex.

The main result of our paper reads as follows:

Theorem 2.3. *Let $f\colon D \to \mathbb{R}$ be an (ω, t)-convex function and let $g\colon D \to \mathbb{R}$ be an (ω, t)-concave function, where $\omega\colon D \times D \times [0, 1] \to \mathbb{R}$. Assume that*

$$g(x) \le f(x) \ \ \text{for all } x \in D.$$

Then there exists an (ω, t)-affine function $h_0\colon D \to \mathbb{R}$ such that

$$g(x) \le h_0(x) \le f(x) \ \ \text{for all } x \in D$$

if and only if for all $x, y, u, v \in D$ the map ω satisfies the functional inequality:

$$\begin{aligned} &t\omega(u, x, t) + (1-t)\omega(v, y, t) - \omega(tu + (1-t)v, tx + (1-t)y, t) \\ &\le t\omega(u, v, t) + (1-t)\omega(x, y, t) - \omega(tu + (1-t)x, tv + (1-t)y, t). \end{aligned} \tag{3}$$

Proof. Assume that the map ω satisfies the inequality (3). Without loss of generality we may assume that

$$\alpha = \inf_{x \in D}[f(x) - g(x)] = 0,$$

because we can take the map $f(x) - \alpha$ instead of $f(x)$ otherwise. We consider the family of maps

$$\mathcal{H} := \{h : D \to \mathbb{R} \mid h \text{ is an } (\omega, t)\text{-convex, } g(x) \le h(x) \le f(x),\ x \in D\}.$$

Clearly, $\mathcal{H} \neq \emptyset$, since $f \in \mathcal{H}$. Let define a partial order $\preceq$ in the following manner:

$$h_1, h_2 \in \mathcal{H},\ h_1 \preceq h_2 \Leftrightarrow h_1(x) \le h_2(x),\ x \in D.$$

We shall show that any chain in $\mathcal{H}$ has a lower bound in $\mathcal{H}$. To see it, take an arbitrary chain $\mathcal{L} \subseteq \mathcal{H}$. Let define the map $p\colon D \to \mathbb{R}$ via the formula

$$p(x) := \inf\{h(x) \mid \ h \in \mathcal{L}\}, \quad x \in D.$$

Obviously, directly from the definition

$$g(x) \le p(x) \le f(x), \quad x \in D,$$

moreover, p is a lower bound of $\mathcal{L}$. To see, that $p \in \mathcal{H}$ it suffices to show that p is (ω, t)-convex in D. To do this, fix $u, v \in D$ and $\varepsilon > 0$ arbitrarily. There exist $h_1, h_2 \in \mathcal{L}$ such that

$$p(u) + \varepsilon > h_1(u) \qquad \text{and} \qquad p(v) + \varepsilon > h_2(v).$$

Then putting $h_3 := \min_{\preceq}\{h_1, h_2\}$ we get

$$\begin{aligned} tp(u) + (1-t)p(v) + \varepsilon &> th_1(u) + (1-t)h_2(v) \ge th_3(u) + (1-t)h_3(v) \\ &\ge h_3(tu + (1-t)v) - \omega(u, v, t) \ge p(tu + (1-t)v) - \omega(u, v, t). \end{aligned}$$

Passing to the limit with $\varepsilon \to 0$, the desired (ω, t)-convexity property is obtained. By Kuratowski-Zorn's lemma there exists a minimal element $h_0 \in \mathcal{H}$. In particular, the inequalities

$$\begin{aligned} &tg(x) + (1-t)g(z) + \omega(x,z,t) \\ &\le g(tx + (1-t)z) \le h_0(tx + (1-t)z) \le th_0(x) + (1-t)h_0(z) + \omega(x,z,t) \end{aligned}$$

hold for all $x, z \in D$. Hence for all $x, z \in D$ we have

$$\begin{aligned} g(x) &\le \frac{1}{t}\Big[g(tx + (1-t)z) - (1-t)g(z) - \omega(x,z,t)\Big] \\ &\le \frac{1}{t}\Big[h_0(tx + (1-t)z) - (1-t)g(z) - \omega(x,z,t)\Big] \le h_0(x) + \frac{1-t}{t}[h_0(z) - g(z)]. \end{aligned}$$

Taking the infimum over all $z \in D$ and having in mind that

$$0 \le \inf_{z \in D}[h_0(z) - g(z)] \le \inf_{z \in D}[f(z) - g(z)] = 0,$$

we obtain, for all $x \in D$

$$g(x) \le \inf_{z \in D} \frac{1}{t}\Big[h_0(tx + (1-t)z) - (1-t)g(z) - \omega(x,z,t)\Big] \le h_0(x).$$

Therefore, the formula

$$k(x) := \inf_{z \in D} \frac{1}{t}\Big[h_0(tx + (1-t)z) - (1-t)g(z) - \omega(x,z,t)\Big]$$

correctly defines the map $k \colon D \to \mathbb{R}$, moreover, $g(x) \le k(x) \le h_0(x)$ for all $x \in D$.

To see that $k \in \mathcal{H}$ it remains to proof that k is an (ω, t)-convex. To do this, fix $\varepsilon > 0$ and $u, v \in D$ arbitrarily. There exist $z_1, z_2 \in D$ such that

$$k(u) + \varepsilon > \frac{1}{t}[h_0(tu + (1-t)z_1) - (1-t)g(z_1) - \omega(u, z_1, t)],$$

and
$$k(v) + \varepsilon > \frac{1}{t}[h_0(tv + (1-t)z_2) - (1-t)g(z_2) - \omega(v, z_2, t)].$$

As a consequence of the (ω, t)-convexity of f and the (ω, t)-concavity of g, and since ω satisfies the inequality (3), we get

$$\begin{aligned} &tk(u) + (1-t)k(v) + \omega(u,v,t) + \varepsilon \\ &> \frac{1}{t}\Big[th_0(tu + (1-t)z_1) - t(1-t)g(z_1) - t\omega(u, z_1, t) \\ &\qquad + (1-t)h_0(tv + (1-t)z_2) - (1-t)^2 g(z_2) - (1-t)\omega(v, z_2, t)\Big] + \omega(u,v,t) \\ &= \frac{1}{t}\Big[th_0(tu + (1-t)z_1) + (1-t)h_0(tv + (1-t)z_2) \\ &\qquad - (1-t)[tg(z_1) + (1-t)g(z_2)] - t\omega(u, z_1, t) - (1-t)\omega(v, z_2, t)\Big] + \omega(u,v,t) \end{aligned}$$

$$\begin{aligned}
&\geq \frac{1}{t}\Big[h_0(t[tu+(1-t)z_1]+(1-t)[tv+(1-t)z_2])-(1-t)g(tz_1+(1-t)z_2)\\
&\qquad -\omega(tu+(1-t)z_1,tv+(1-t)z_2,t)+(1-t)\omega(z_1,z_2,t)\\
&\qquad -t\omega(u,z_1,t)-(1-t)\omega(v,z_2,t)+t\omega(u,v,t)\Big]\\
&=\frac{1}{t}\Big[h_0(t[tu+(1-t)v]+(1-t)[tz_1+(1-t)z_2])\\
&\qquad -(1-t)g(tz_1+(1-t)z_2)-\omega(tu+(1-t)v,tz_1+(1-t)z_2,t)\Big]\\
&\qquad +\frac{1}{t}\Big[\omega(tu+(1-t)v,tz_1+(1-t)z_2,t)+(1-t)\omega(z_1,z_2,t)-t\omega(u,z_1,t)\\
&\qquad -(1-t)\omega(v,z_2,t)+t\omega(u,v,t)-\omega(tu+(1-t)z_1,tv+(1-t)z_2,t)\Big]\\
&\geq k(tu+(1-t)v).
\end{aligned}$$

Passing to the limit with $\varepsilon \to 0$, the desired (ω,t)-convexity property is obtained. We have shown that $k \in \mathcal{H}$ and $k \leq h_0$, so by the minimality of h_0 we get

$$h_0(x) \leq \inf_{z\in D}\frac{1}{t}\Big[h_0(tx+(1-t)z)-(1-t)g(z)-\omega(x,z,t)\Big], \quad x \in D.$$

Then for all $x, z \in D$

$$th_0(x)+(1-t)g(z)+\omega(x,z,t) \leq h_0(tx+(1-t)z).$$

Change the role of x and z to arrive at

$$th_0(z)+(1-t)g(x)+\omega(z,x,t) \leq h_0((1-t)x+tz), \quad x,z \in D,$$

which is equivalent to

$$g(x) \leq \frac{1}{1-t}[h_0((1-t)x+tz)-th_0(z)-\omega(z,x,t)] \leq h_0(x), \quad x \in D.$$

In order to apply Lemma 2.1 we shall show that the map ω satisfies the inequality (1). Indeed, by virtue of (ω,t)-convexity of f and (ω,t)-concavity of g for all $x, y \in D$ we have

$$f(tx+(1-t)y)-tf(x)-(1-t)f(y) \leq \omega(x,y,t) \leq g(tx+(1-t)y)-tg(x)-(1-t)g(y).$$

Putting $x = y$ in the above formula we obtain that $\omega(x,x,t) = 0,\ x \in D$. Now, the inequality (3) reduces to (1) by putting in (3) $u = v$. Therefore by Lemma 2.1 for arbitrary $z \in D$ the function

$$D \ni x \longrightarrow \frac{1}{1-t}\Big[h_0((1-t)x+tz)-th_0(z)-\omega(z,x,t)\Big],$$

is (ω,t)-convex and hence using again the minimality of h_0 we infer that

$$h_0(x) = \frac{1}{1-t}\Big[h_0((1-t)x+tz)-th_0(z)-\omega(z,x,t)\Big], \quad x,z \in D$$

or, equivalently, $\omega(z,x,t) = h_0(tz+(1-t)x) - th_0(z) - (1-t)h_0(x)$, for all $x, z \in D$, and this completes the proof of sufficiency.

For the proof of necessity assume that there is a function $h_0\colon D \to \mathbb{R}$ such that

$$\omega(x,z,t) = h_0(tx+(1-t)z) - th_0(x) - (1-t)h_0(z), \quad x, z \in D.$$

Then, for arbitrary $u, v, x, y \in D$, we get

$$\begin{aligned}
&\omega(tu+(1-t)x, tv+(1-t)y, t) + t\omega(u,x,t) + (1-t)\,\omega(v,y,t)\\
&= th_0(tu+(1-t)x) + (1-t)h_0(tv+(1-t)y) - h_0(t[tu+(1-t)x]\\
&\qquad + (1-t)[tv+(1-t)y]) + t\big[th_0(u) + (1-t)h_0(x) - h_0(tu+(1-t)x)\big]\\
&\qquad + (1-t)\big[th_0(v) + (1-t)h_0(y) - h_0(tv+(1-t)y)\big]\\
&= th_0(tu+(1-t)v) + (1-t)h_0(tx+(1-t)y) - h_0(t[tu+(1-t)v]\\
&\qquad + (1-t)[tx+(1-t)y]) + t\big[th_0(u) + (1-t)h_0(v) - h_0(tu+(1-t)v)\big]\\
&\qquad + (1-t)\big[th_0(x) + (1-t)h_0(y) - h_0(tx+(1-t)y)\big]\\
&= \omega(tu+(1-t)v, tx+(1-t)y, t) + t\omega(u,v,t) + (1-t)\,\omega(x,y,t). \qquad \square
\end{aligned}$$

A careful inspection of the proof of Theorem 2.3 (using Lemma 2.2 instead of Lemma 2.1) shows that for ω-convex functions the following theorem holds true

Theorem 2.4. *Let $D \subseteq X$ be a convex subset of a real linear space, let $f\colon D \to \mathbb{R}$ be an ω-convex function and let $g\colon D \to \mathbb{R}$ be an ω-concave function, where $\omega\colon D \times D \times [0,1] \to \mathbb{R}$. Assume that $g(x) \le f(x)$ for all $x \in D$.*

Then there exists an ω-affine function $h\colon D \to \mathbb{R}$ such that

$$g(x) \le h(x) \le f(x), \quad x \in D,$$

if and only if for all $x, y, u, v \in D$ and all $s, t \in [0,1]$ the map ω satisfies the functional inequality:

$$\begin{aligned}
&s\omega(u,x,t) + (1-s)\omega(v,y,t) - \omega(su+(1-s)v, sx+(1-s)y, t)\\
&\le t\omega(u,v,s) + (1-t)\omega(x,y,s) - \omega(tu+(1-t)x, tv+(1-t)y, s). \qquad (4)
\end{aligned}$$

Remark 2.5. Since the function $\omega \equiv 0$ satisfies the inequalities (1) and (3) ((2) and (4)) then the well-known classical separation theorem for t-convex functions as well as for convex functions are a consequence of our main results.

Remark 2.6. Let us observe that interchanging the role of x and v in the inequality (3) or (4) respectively we conclude that in fact ω satisfies the corresponding functional equation.

3. The corresponding functional equations

In this section we deal with functional equations appearing in our main results. Since, as we have seen, the most frequently form of ω occurring in the literature is the following one

$$\omega(x, y, t) = h(t)g(x - y), \tag{5}$$

where $h\colon [0,1] \to \mathbb{R}$ and $g\colon X \to \mathbb{R}$, then we solve the functional equations corresponding to the inequalities (3) and (4) in these cases.

Theorem 3.1. *A function $f\colon X \to \mathbb{R}$ satisfies the functional equation*

$$\begin{aligned} &tg(u - x) + (1 - t)g(v - y) - g(t(u - x) + (1 - t)(v - y)) \\ &= tg(u - v) + (1 - t)g(x - y) - g(t(u - v) + (1 - t)(x - y)), \end{aligned} \tag{6}$$

for all $u, v, x, y \in X$ if and only if g has the form

$$g(x) = a_0 + a_1(x) + a_2(x, x), \quad x \in X, \tag{7}$$

where a_0 is a constant, $a_1\colon X \to \mathbb{R}$ is an additive function, $a_2\colon X \times X \to \mathbb{R}$ is a bi-additive and symmetric function such that

$$a_2(tx, y) = ta_2(x, y), \quad x, y \in X. \tag{8}$$

Proof. Without loss of generality we may assume that $g(0) = 0$. (otherwise we replace the function g by $g - g(0)$). Put $x = y$ in (6) to get for all $x, u, v \in X$

$$tg(u-x)+(1-t)g(v-x)-g(t(u-x)+(1-t)(v-x)) = tg(u-v)-g(t(u-v)).$$

We can rewrite this equation as

$$tg(x) + (1 - t)g(y) - g(tx + (1 - t)y) = tg(x - y) - g(t(x - y)), \quad x, y \in X.$$

Using the well-known Daróczy and Páles identity of the mean $\frac{x+y}{2}$ (see [7])

$$\frac{x + y}{2} = t\Big[t\frac{x + y}{2} + (1 - t)y\Big] + (1 - t)\Big[tx + (1 - t)\frac{x + y}{2}\Big],$$

we get

$$\begin{aligned} tg\Big(t\frac{x + y}{2} + (1 - t)y\Big) + (1 - t)g\Big(tx + (1 - t)\frac{x + y}{2}\Big) - g\Big(\frac{x + y}{2}\Big) \\ = tg\Big(\frac{y - x}{2}\Big) - g\Big(t\frac{y - x}{2}\Big), \end{aligned}$$

for all $x, y \in X$. On the other hand,

$$tg\Big(\frac{x + y}{2}\Big) + (1 - t)g(y) - g\Big(t\frac{x + y}{2} + (1 - t)y\Big) = tg\Big(\frac{x - y}{2}\Big) - g\Big(t\frac{x - y}{2}\Big)$$

for all $x, y \in X$. Analogously,

$$tg(x) + (1 - t)g\Big(\frac{x + y}{2}\Big) - g\Big(tx + (1 - t)\frac{x + y}{2}\Big) = tg\Big(\frac{x - y}{2}\Big) - g\Big(t\frac{x - y}{2}\Big)$$

for all $x, y \in X$. Multiplying the second equation by t and the third by $1-t$ and summing up these three equalities, after simplification, we obtain for all $x, y \in X$

$$\begin{aligned} t(1-t)\Big[g(x)+g(y)-2g\Big(\frac{x+y}{2}\Big)\Big] \\ = t\Big[g\Big(\frac{y-x}{2}\Big)+g\Big(\frac{x-y}{2}\Big)\Big]-\Big[g\Big(t\frac{y-x}{2}\Big)+g\Big(t\frac{x-y}{2}\Big)\Big]. \end{aligned} \tag{9}$$

Now, consider an odd and an even part of g, say g_1 and g_2 respectively. Note that they both satisfy the equations (6) and (9), moreover, $g_1(0) = g_2(0) = 0$. For the odd part g_1 the equation (9) reduces to

$$g_1\Big(\frac{x+y}{2}\Big) = \frac{g_1(x)+g_1(y)}{2}, \quad x, y \in X.$$

Since g_1 is an affine function with condition $g_1(0) = 0$ then it is additive.

The even part g_2 satisfies the following equation for all $x, y \in X$:

$$t(1-t)\Big[g_2(x)+g_2(y)-2g_2\Big(\frac{x+y}{2}\Big)\Big] = 2\Big[tg_2\Big(\frac{x-y}{2}\Big)-g_2\Big(t\frac{x-y}{2}\Big)\Big]$$

Hence with $x+z \sim x$ and $y+z \sim y$ we can write the previous equation as

$$t(1-t)\Big[g_2(x+z)+g_2(y+z)-2g_2\Big(\frac{x+y}{2}+z\Big)\Big] = 2\Big[tg_2\Big(\frac{x-y}{2}\Big)-g_2\Big(t\frac{x-y}{2}\Big)\Big] \tag{10}$$

for all $x, y, z \in X$. Note that the right-hand side of (10) is independent of z. Therefore the left-hand side is unchanged upon setting $z = -\frac{x+y}{2}$ and next $z = 0$. That is, we have

$$g_2(x)+g_2(y)-2g_2\Big(\frac{x+y}{2}\Big) = 2g_2\Big(\frac{x-y}{2}\Big), \quad x, y \in X.$$

Setting $u = \frac{x+y}{2}, v = \frac{x-y}{2}$ we obtain

$$g_2(u+v)+g_2(u-v) = 2g_2(u)+2g_2(v), \quad u, v \in X.$$

Since g_2 is a quadratic functional then there exists a bi-additive and symmetric map $a_2\colon X\times X \to \mathbb{R}$ such that $g_2(x) = a_2(x,x)$, for all $x \in X$ (see [1] Proposition 1, p. 166). Inserting this representation into (6) and simplifying we get the desired form $a_2(tx,y) = ta_2(x,y)$, for all $x, y \in X$.

Conversely, it is easy to check that the maps of the form (7) and (8) are solutions to the equation (6). □

Now, we solve the functional equation corresponding to the inequality (4) for ω of the form (5).

Theorem 3.2. *Let X be a real linear space, and let $h\colon [0,1] \to \mathbb{R}$ be a function such that $h(t) \neq 0$ for all $t \in (0,1)$. Then a function $g\colon X \to \mathbb{R}$ satisfies the functional equation*

$$\begin{aligned} &h(t)[sg(u-x)+(1-s)g(v-y)-g(s(u-x)+(1-s)(v-y))] \\ &= h(s)[tg(u-v)+(1-t)g(x-y)-g(t(u-v)+(1-t)(x-y))], \end{aligned} \tag{11}$$

for all $u, v, x, y \in X$ and $s, t \in [0,1]$ if and only if g has the form

$$g(x) = a_0 + a_1(x) + a_2(x,x), \quad x \in X, \tag{12}$$

where a_0 is a constant, $a_1\colon X \to \mathbb{R}$ is a linear function, $a_2\colon X \times X \to \mathbb{R}$ is a bi-additive and symmetric function with the property

$$a_2(tx, y) = ta_2(x,y), \quad x, y \in X,\ t \in \mathbb{R}. \tag{13}$$

Moreover, if the function a_2 is not identically equal to 0 then

$$h(t) = ct(1-t), \quad t \in (0,1),$$

where $c \in \mathbb{R}$ is a constant.

Proof. Analogously as in the proof of the previous theorem we may assume that $g(0) = 0$. Putting $s = t \in (0,1)$ into (11) and dividing by $h(t)$ we obtain the equation (6). By Theorem 7 (together with the condition $g(0) = 0$) the solution of (6) is of the form

$$g(x) = a_1(x) + a_2(x,x), \quad x \in X,$$

where $a_1\colon X \to \mathbb{R}$ is additive, $a_2\colon X \times X \to \mathbb{R}$ is a bi-additive and symmetric map. Obviously, an odd part of g is equal to a_1 and an even part is equal to the diagonalization of a_2 and they both satisfy equation (11).

For an additive function a_1 equation (11) reduces to

$$h(s)[ta_1(x) - a_1(tx)] = h(t)[sa_1(x) - a_1(sx)], \quad x \in X,\ s, t \in [0,1].$$

Putting $s = \alpha \in \mathbb{Q} \cap (0,1)$ in the above equation and using a $\mathbb{Q}$-homogeneity of a_1 we obtain

$$h(\alpha)[ta_1(x) - a_1(tx)] = 0, \quad x \in X,\ t \in [0,1],$$

and because $h(\alpha) \neq 0$ this implies that

$$a_1(tx) = ta_1(x), \quad x \in X,\ t \in (0,1). \tag{14}$$

Substituting $\frac{1}{t}x$ in the place of x and dividing by t we get

$$a_1(\frac{1}{t}x) = \frac{1}{t}a_1(x), \quad x \in X,\ t \in (0,1).$$

This together with (14) gives

$$a_1(sx) = sa_1(x), \quad x \in X,\ s \in [0, \infty).$$

Because of the oddness of a_1 we infer that it is linear.

On the other hand one can check that the diagonalization of a_2 satisfies the equation (11) with $t = s$ if and only if for all $x, y, u, v \in X$ and $t \in [0, 1]$ we have

$$\begin{gathered} h(t)[a_2(tu, x) - ta_2(u, x) + ta_2(x, x) - a_2(tx, x) + a_2(v, tv) - ta_2(v, v) \\ + ta_2(v, y) - a_2(v, ty) + ta_2(u, v) - a_2(tu, v) + a_2(tx, v) - a_2(tv, x) \\ + a_2(x, ty) - ta_2(x, y)] = 0. \end{gathered}$$

Putting $x = y = 0$ in the above equation we get

$$h(t)[a_2(t(v - u), v) - ta_2(v - u, v)] = 0,$$

which means that $a_2(tx, y) = ta_2(x, y)$ for all $x, y \in X$, $t \in [0, 1]$. Since a_2 is additive in the first variable then $a_2(tx, y) = ta_2(x, y)$ for all $x, y \in X$, $t \in \mathbb{R}$.

Substituting a_2 back into (11) and simplifying, we obtain

$$h(t)s(1 - s)a_2(x, x) = h(s)t(1 - t)a_2(x, x), \quad x \in X,\ s, t \in (0, 1).$$

Now, if a_2 is not identically equal to 0 then

$$h(t)s(1 - s) = h(s)t(1 - t), \quad s, t \in (0, 1),$$

putting $c := \frac{h(s)}{s(1-s)}$ we get $h(t) = ct(1 - t)$ for all $t \in (0, 1)$.

Conversely, it is easy to check that the maps of the form (12) and (13) are solutions to the equation (11). The proof of our theorem is finished. □

4. Applications

Now, we apply our main result to the proof of a separation theorem for strongly t-convex as well as for $(c, t, 2)$-approximately convex functions.

Theorem 4.1. *Let $(X, (\cdot|\cdot))$ be a real inner product space, let $D \subseteq X$ be a t-convex set, where $t \in (0, 1)$ and let $c \in \mathbb{R}$. Assume that the functions $f, p\colon D \to \mathbb{R}$ satisfy, for all $x, y \in D$, the inequalities*

$$f(tx + (1 - t)y) \le tf(x) + (1 - t)f(y) + ct(1 - t)\|x - y\|^2, \quad \textit{and}$$

$$tp(x) + (1 - t)p(y) + ct(1 - t)\|x - y\|^2 \le p(tx + (1 - t)y).$$

If, moreover, $p(x) \le f(x)$ for all $x \in D$, then there exists a function $h\colon D \to \mathbb{R}$ such that $p(x) \le h(x) \le f(x)$ for all $x \in D$, and,

$$ct(1 - t)\|x - y\|^2 = h(tx + (1 - t)y) - th(x) - (1 - t)h(y), \quad x, y \in D.$$

Proof. Following Theorem 2.3 it is enough to check whether the function $\omega\colon D \times D \times [0,1] \to \mathbb{R}$ given by the formula

$$\omega(x,y,t) := ct(1-t)\|x-y\|^2,$$

satisfies the functional inequality (3). Obviously ω is of the form (5), where $g\colon X \to \mathbb{R}$ is defined via the formula

$$g(x) = c\|x\|^2 = c(x|x), \quad x \in X.$$

Since the map $a_2(x,y) := c(x|y),\ x,y \in X$ is bi-additive and symmetric with the property

$$a_2(tx,y) = c(tx|y) = tc(x|y) = ta_2(x|y), \quad x,y \in X$$

hence our result follows from Theorem 2.3 and Theorem 3.1. □

Let us observe that using similar arguments and Theorem 2.4 and Theorem 3.2 instead of Theorem 2.3 and Theorem 3.1 one can prove the following theorem.

Theorem 4.2. *Let $(X,(\cdot|\cdot))$ be a real inner product space, let $D \subseteq X$ be a convex set, and let $c \in \mathbb{R}$. Assume that the functions $f,p\colon D \to \mathbb{R}$ satisfy the inequalities*

$$f(tx+(1-t)y) \le tf(x) + (1-t)f(y) + ct(1-t)\|x-y\|^2, \quad \textit{and}$$

$$tp(x) + (1-t)p(y) + ct(1-t)\|x-y\|^2 \le p(tx+(1-t)y)$$

for all $x,y \in D$ and $t \in [0,1]$. If, moreover, $p(x) \le f(x)$ for all $x \in D$, then there exists a function $h\colon D \to \mathbb{R}$ such that $p(x) \le h(x) \le f(x)$ for all $x \in D$, and

$$ct(1-t)\|x-y\|^2 = h(tx+(1-t)y) - th(x) - (1-t)h(y),\ x,y \in D,\ t \in [0,1].$$

As a last consequence of our main result we present the theorem which gives a characterization of an inner product space (see also [24]). There are several papers which give conditions under which the norm in a real-linear space can be defined from an inner product. The most famous result of this type was given by Jordan and von Neumann [11] who proved that a linear metric space X is an inner product space if and only if it satisfies the parallelogram low i.e.

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2, \qquad x,y \in X.$$

A survey of characterizations of inner product spaces can be found in the famous book of Amir [2].

Theorem 4.3. *Let $(X,\|\cdot\|)$ be a real normed space. The following conditions are equivalent:*

(i) *For an arbitrary number $t \in (0,1)$ there exists a function $p\colon X \to \mathbb{R}$ such that*

$$\|x-y\|^2 = tp(x) + (1-t)p(y) - p(tx+(1-t)y), \quad x,y \in X.$$

(ii) *There exist a number $t \in (0,1)$ and a function $p\colon X \to \mathbb{R}$ such that*

$$\|x-y\|^2 = tp(x) + (1-t)p(y) - p(tx+(1-t)y), \quad x,y \in X.$$

(iii) $(X,\|\cdot\|)$ *is an inner product space.*

Proof. (i) $\Rightarrow$ (ii): trivial.

(ii) $\Rightarrow$ (iii): note that because of (ii) and the proof of our main result the function $\omega\colon X \times X \times [0,1] \to \mathbb{R}$ given by

$$\omega(x,y,s) = \|x-y\|^2, \quad x,y \in X,\ s \in [0,1],$$

satisfies equation (6), where $g = \|\cdot\|^2$. It follows from Theorem 3.1 that g has the form

$$g(x) = a_0 + a_1(x) + a_2(x,x), \quad x \in X, \tag{15}$$

where a_0 is a constant, $a_1\colon X \to \mathbb{R}$ is an additive function, $a_2\colon X \times X \to \mathbb{R}$ is a bi-additive and symmetric function such that

$$a_2(tx,y) = ta_2(x,y), \quad x,y \in X. \tag{16}$$

Since g is an even function and $g(0)=0$ then $g(x) = a_2(x,x),\ x \in X$. Therefore for all $x,y \in X$ we obtain

$$\begin{aligned}\|x+y\|^2 + \|x-y\|^2 &= a_2(x+y,x+y) + a_2(x-y,x-y)\\ &= a_2(x,x) + 2a_2(x,y) + a_2(y,y) + a_2(x,x) - 2a_2(x,y) + a_2(y,y)\\ &= 2a_2(x,x) + 2a_2(y,y) = 2\|x\|^2 + 2\|y\|^2.\end{aligned}$$

The norm satisfies the parallelogram low, which implies that $(X, \|\cdot\|)$ is an inner product space.

For the proof of implication (iii) $\Rightarrow$ (i) take an arbitrary $t \in (0,1)$ and define $p(x) := \|x\|^2/(t(1-t))$, $x \in X$. Because the norm is defined from an inner product then

$$tp(x) + (1-t)p(y) - p(tx+(1-t)y) = \|x-y\|^2, \quad x,y \in X,$$

which completes our proof. □

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