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Erratum to: Isoperimetric Inequalities for the Principal Eigenvalue of a Membrane and the Energy of Problems with Robin Boundary Conditions

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A correct version of Lemma 2.3 and Theorem 2.4 in *Isoperimetric Inequalities for the Principal Eigenvalue of a Membrane and the Energy of Problems with Robin Boundary Conditions* [J. Convex Analysis 22(3) (2015) 627–640] is presented.

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1. Introduction

We are indebted to Richard Laugesen who drew our attention to some problems in the proof of Lemma 2.3 in [2]. This lemma is crucial for the inequality of the principal eigenvalue derived in Theorem 2.4 in [2]. Both results cannot hold in this generality. We present here a new proof of Lemma 2.3 for a restricted class of strictly convex domains and for a different constant. It relies on arguments from convex analysis and potential theory and it raises an interesting question. For the sake of completeness we start with the description of the harmonic transplantation before proving the corrected lemma. At the end we indicate the changes in Theorem 2.4.

1.1. Harmonic transplantation

Consider the Green's function with Dirichlet boundary condition given by

$$G_\Omega(x,y) = c_n(S(|x-y|) - H(x,y)), \tag{1}$$

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where
$$c_n = \begin{cases} \frac{1}{2\pi} & \text{if } n=2 \\ \frac{1}{(n-2)|\partial B_1|} & \text{if } n>2 \end{cases} \tag{2}$$
and
$$S(t) = \begin{cases} -\ln(t) & \text{if } n=2 \\ t^{2-n} & \text{if } n>2. \end{cases}$$

For fixed $y \in \Omega$ the function $H(\cdot, y)$ is harmonic. Set $B_R = \{x : |x| < R\}$. Then

$$G_{B_R}(0,x) = \begin{cases} c_n\,(r^{2-n} - R^{2-n}) & \text{if } n>2, \\ c_2 \log \frac{R}{r} & \text{if } n=2. \end{cases}$$

Definition 1.1. The *harmonic radius* at a point $y \in \Omega$ is given by

$$r(y) = \begin{cases} e^{-H(y,y)} & \text{if } n=2, \\ H(y,y)^{-\frac{1}{n-2}} & \text{if } n>2. \end{cases} \qquad \square$$

The harmonic radius vanishes on the boundary $\partial\Omega$ and takes its maximum r_Ω at the harmonic center y_h. For convex domains the center is unique (see [1, 4]).

The harmonic radius satisfies the isoperimetric inequality [5],[1]

$$|B_{r_\Omega}| \le |\Omega|. \tag{3}$$

Note that $G_{B_R}(x,0)$ is a monotone function in $r = |x|$. Consider any radial function $\phi\colon B_{r_\Omega} \to \mathbb{R}$ thus $\phi(x) = \phi(r)$. Then there exists a function $\omega\colon \mathbb{R} \to \mathbb{R}$ such that
$$\phi(x) = \omega(G_{B_{r_\Omega}}(x,0)).$$
To $\phi(x)$ we associate the transplanted function $U\colon \Omega \to \mathbb{R}$ defined by $U(x) = \omega(G_\Omega(x, y_h))$. Then for any positive function $f(s)$, cf. [5] or [1], the following inequalities hold true
$$\int_\Omega |\nabla U|^2\,dx = \int_{B_{r_\Omega}} |\nabla\phi|^2\,dx \tag{4}$$
$$\int_\Omega f(U)\,dx \ge \int_{B_{r_\Omega}} f(\phi)\,dx. \tag{5}$$
For our purpose we need an estimate of $\int_\Omega f(U)\,dx$ from above.

1.2. Auxiliary lemmas for Green's functions

We collect some results which will be used for proving (5) with the opposite inequality sign. Let

$$\Omega^t := \{x \in \Omega : G_\Omega(x, y_h) > t\}, \quad B^t := \{x \in B_{r_\Omega} : G_{B_{r_\Omega}}(x,0) > t\},$$

We write for convenience $m_\Omega(t) = |\Omega^t|$ and $m_B(t) := |B^t|$.

Since the harmonic radii of the Green's functions $G_{\Omega^t}(x, y_h)$ and $G_{B^t}(x, 0)$ are the same we can conclude from (3) that

$$m_\Omega(t) \geq m_B(t) \quad \text{for all } t \geq 0. \tag{6}$$

Next we show that the function

$$Q(t) := \frac{m_B(t)}{m_\Omega(t)}$$

is monotone increasing. This will give a bound for $m_\Omega(t)$ from above. The coarea formula implies

$$m_\Omega(t) = \int_t^\infty \int_{\partial\Omega^s} \frac{1}{|\nabla G_\Omega(x, y_h)|} \, dS \, ds.$$

For $\mathcal{L}^1$ a.a. t this implies
$$\dot{m}_\Omega(t) = -\int_{\partial\Omega^t} \frac{1}{|\nabla G_\Omega(x, y_h)|} \, dS. \tag{7}$$

Then by Schwarz's inequality the (n-1)-dimensional Hausdorff measure can be estimated as follows.

$$\begin{aligned}(\mathcal{H}^{n-1}(\partial\Omega^t))^2 &= \left(\int_{\partial\Omega^t} \frac{\sqrt{|\nabla G_\Omega(x_t, y_h)|}}{\sqrt{|\nabla G_\Omega(x_t, y_h)|}} \, dS \right)^2 \\ &\leq \int_{\partial\Omega^t} |\nabla G_\Omega(x_t, y_h)| \, dS \int_{\partial\Omega^t} \frac{1}{|\nabla G_\Omega(x_t, y_h)|} \, dS = -\dot{m}_\Omega(t).\end{aligned}$$

Here we used the normalization of $G_\Omega(x, y)$ (cf. (2)) which implies

$$\int_{\partial\Omega^t} |\nabla G_\Omega(x_t, y_h)| \, dS = \int_{\partial\Omega^t} -\partial_\nu S(x_t, y_h) \, dS = 1.$$

Equality holds for the ball. Then

$$\dot{Q}(t) = \frac{\dot{m}_B(t)}{m_\Omega(t)} - \frac{m_B(t)\dot{m}_\Omega(t)}{m_\Omega^2(t)} \geq -\frac{(\mathcal{H}^{n-1}(\partial B^t))^2}{m_\Omega(t)} + \frac{m_B(t)}{m_\Omega^2(t)} (\mathcal{H}^{n-1}(\partial\Omega^t))^2.$$

If we replace $\mathcal{H}^{n-1}(\partial\Omega^t)$ by its lower bound given by the isoperimetric inequality

$$\frac{(\mathcal{H}^{n-1}(\partial\Omega^t))^{\frac{1}{n-1}}}{m_\Omega^{1/n}(t)} \geq \frac{(\mathcal{H}^{n-1}(\partial B^t))^{\frac{1}{n-1}}}{m_B^{1/n}(t)}$$

we obtain

$$\begin{aligned}\frac{m_B(t)}{m_\Omega^2(t)} (\mathcal{H}^{n-1}(\partial\Omega^t))^2 &\geq \frac{m_B(t)}{m_\Omega^{2/n}(t) m_B(t)^{2\frac{n-1}{n}}} (\mathcal{H}^{n-1}(\partial B^t))^2 \\ &= \frac{m_\Omega^{1-2/n}(t)}{m_\Omega(t) m_B^{1-2/n}(t)} (\mathcal{H}^{n-1}(\partial B^t))^2.\end{aligned}$$

The claim follows now from (6). The monotonicity of $Q(t)$ leads to the estimate

$$\frac{|\Omega|}{|B_{r_\Omega}|} m_B(t) \geq m_\Omega(t) \text{ for all } t > 0. \tag{8}$$

We shall also need the following well-known result, namely the strict convexity of Ω implies the strict convexity of Ω^t for all $t > 0$. In this case ∇G_Ω never vanishes and (7) holds for all $t > 0$.

1.3. Corrected Lemma 2.3

Define $$C(\partial\Omega)^{-1} := \min\left\{|\nabla G_\Omega(x, y_h)| : x \in \partial\Omega\right\}.$$

The constant $C(\partial\Omega)$ depends on the curvature of $\partial\Omega$. We shall also need the constant

$$R := \max\{|y - y_h| : y \in \partial\Omega\}.$$

The ball $B_R(y_h)$ centered at y_h and of radius R contains Ω. Moreover let

$$\tilde{\gamma}_\Omega := \max\left\{C(\partial\Omega)\left(\frac{R}{r_\Omega}\right)^{n-1}, \frac{m_\Omega(0)}{m_B(0)}\right\}. \tag{9}$$

Lemma 1.2. *Let Ω be a bounded strictly convex domain. Suppose that $f(s)$ is positive and monotone increasing. Let $\phi(x)\colon B_{r_\Omega} \to \mathbb{R}$ be radial and monotone increasing. Then*

$$\int_\Omega f(U)\, dx \leq \tilde{\gamma}_\Omega \int_{B_{r_\Omega}} f(\phi)\, dx. \tag{10}$$

Proof. Integration along level surfaces implies

$$\begin{aligned}\int_\Omega f(U)\, dx &= -\int_0^\infty f(\omega(t))\, dm_\Omega(t)\\ &= -f(\omega(t))\, m_\Omega(t)\Big|_0^\infty + \int_0^\infty f'(\omega(t))\, \omega'(t)\, m_\Omega(t)\, dt.\end{aligned}$$

If $f(\omega(0))$ is bounded

$$\int_\Omega f(U)\, dx = f(\omega(0))|\Omega| + \int_0^\infty f'(\omega(t))\, \omega'(t)\, m_\Omega(t)\, dt.$$

The claim is thus proved if we find some $\tilde{\gamma}_\Omega > 1$ such that

$$\begin{aligned}&f(\omega(0))m_\Omega(0) + \int_0^\infty f'(\omega(t))\, \omega'(t)\, m_\Omega(t)\, dt\\ &\quad\leq \tilde{\gamma}_\Omega f(\omega(0))m_B(0) + \tilde{\gamma}_\Omega \int_0^\infty f'(\omega(t))\, \omega'(t)\, m_B(t)\, dt.\end{aligned}$$

This condition can be rewritten as

$$-\int_0^\infty f'(\omega(t))\,\omega'(t)\,(\tilde\gamma_\Omega\, m_B(t) - \; m_\Omega(t))\; dt \le f(\omega(0))\,(\tilde\gamma_\Omega\, m_B(0) - m_\Omega(0))\,.$$

Note that $\quad -f'(\omega(t))\,\omega'(t) > 0\;$ for all $t > 0$.

Thus it is sufficient to find $\tilde\gamma_\Omega$ such that

$$\begin{aligned} P(t) := \tilde\gamma_\Omega\, m_B(t) - \; m_\Omega(t) \le \tilde\gamma_\Omega\, m_B(0) - \; m_\Omega(0) \\ \text{and} \quad \tilde\gamma_\Omega\, m_B(0) - \; m_\Omega(0) > 0. \end{aligned} \tag{11}$$

Our goal is to determine $\tilde\gamma_\Omega$ such that $\dot P(t)$ is nonpositive. By (7) and the subsequent arguments

$$\dot P(t) = -\tilde\gamma_\Omega(\mathcal{H}^{n-1}(\partial B^t))^2 + \int\limits_{\partial\Omega^t} \frac{1}{|\nabla G_\Omega(x,y_h)|}\, dS.$$

For any point $x_t \in \partial\Omega^t$ the mean curvature $H(x_t)$ of $\partial\Omega^t$ is given by

$$H(x_t) = \frac{\nabla G_\Omega(x_t,y_h)\cdot(D^2 G_\Omega(x_t,y_h)\nabla G_\Omega(x_t,y_h))}{|\nabla G_\Omega(x_t,y_h)|^3} > 0$$

In view of the convexity of $\partial\Omega^t$ this is equivalent to

$$\nabla G_\Omega \cdot \nabla\left(|\nabla G_\Omega|^2\right) > 0 \qquad \text{in}\ \Omega. \tag{12}$$

A direct consequence of (12) is

$$\min\left\{|\nabla G_\Omega(x,y_h)| : x \in \overline{\Omega^t}\right\} = \min\left\{|\nabla G_\Omega(x,y_h)| : x \in \partial\Omega^t\right\}. \tag{13}$$

Hence $\quad \min\left\{|\nabla G_\Omega(x,y_h)| : x \in \overline{\Omega}\right\} = C(\partial\Omega)^{-1}$. Then

$$\int\limits_{\partial\Omega^t} \frac{1}{|\nabla G_\Omega(x,y_h)|}\, dS_t \le \frac{\mathcal{H}^{n-1}(\partial\Omega^t)}{\min\{|\nabla G_\Omega(x,y_h)| : x \in \partial\Omega\}} \quad = \quad C(\partial\Omega)\,\mathcal{H}^{n-1}(\partial\Omega^t).$$

Next we want to estimate $\mathcal{H}^{n-1}(\partial\Omega^t)$ from above. Since $B_R(y_h) \supset \Omega$ the maximum principle for harmonic function implies

$$G_{B_R}(x,y_h) \ge G_\Omega(x,y_h).$$

Consequently Ω^t is contained in B_R^t where

$$B_R^t := \{x \;:\; G_{B_R}(x,y_h) > t\}.$$

From the Cauchy surface area formula it follows that if $C_1 \subset C_2$ are two convex sets in $\mathbb{R}^n$, then $\mathcal{H}^{n-1}(\partial C_1) \le \mathcal{H}^{n-1}(\partial C_2)$. Thus we deduce that

$$\mathcal{H}^{n-1}(\partial\Omega^t) \le \mathcal{H}^{n-1}(\partial B_R^t).$$

Finally we obtain

$$\dot{P}(t) \leq -\tilde{\gamma}_\Omega(\mathcal{H}^{n-1}(\partial B^t))^2 + \mathcal{H}^{n-1}(\partial B^t_R)C(\partial\Omega). \tag{14}$$

Set $\mathcal{H}^{n-1}(\partial B^t) = \mathcal{H}^{n-1}(\partial B^t_R)\, q(t)$ where

$$q(t) = \begin{cases} \frac{\mathcal{H}^{n-1}(\partial B^t)}{\mathcal{H}^{n-1}(\partial B^t_R)} = \left(\frac{t\, c_n^{-1} + R^{2-n}}{t\, c_n^{-1} + r_\Omega^{2-n}}\right)^{\frac{n-1}{n-2}} & \text{if } n > 2, \\ \frac{r_\Omega}{R} & \text{if } n = 2. \end{cases}$$

Since $R \geq r_\Omega$ this gives $q(t) \geq \left(\frac{r_\Omega}{R}\right)^{n-1}$.

Thus by (14): $\dot{P}(t) \leq C(\partial\Omega)\, \mathcal{H}^{n-1}(\partial B^t_R) - \tilde{\gamma}_\Omega \left(\frac{r_\Omega}{R}\right)^{n-1} \mathcal{H}^{n-1}(\partial B^t_R)$.

For any
$$\tilde{\gamma}_\Omega > \max\left\{ C(\partial\Omega)\left(\frac{R}{r_\Omega}\right)^{n-1}, \frac{m_\Omega(0)}{m_B(0)} \right\}$$

we have $P(0) \geq P(t)$. This completes the proof of (11) and the lemma. □

Open problem. The constant $\tilde{\gamma}_\Omega$ could be improved considerably if a more precise estimate of

$$\oint_{\partial\Omega^t} \frac{1}{|\nabla G_{\Omega^t}|}\, dS_t = -\dot{m}_\Omega(t)$$

could be given. This requires mainly more information on the curvature of $\partial\Omega^t$.

1.4. Application: upper bound for the principal eigenvalue of a membrane with Robin boundary conditions of opposite sign

Consider the eigenvalue defined by

$$\lambda(\Omega) = \inf_{W^{1,2}(\Omega)} \left\{ \int_\Omega |\nabla u|^2\, dx - \alpha \oint_{\partial\Omega} u^2\, dS \right\} \text{ with } \int_\Omega u^2\, dx = 1 \text{ and } \alpha > 0\,. \tag{15}$$

The minimizer u is a solution of

$$\Delta u + \lambda(\Omega) u = 0 \text{ in } \Omega,\ \partial_\nu u = \alpha u \text{ on } \partial\Omega,\ \partial_\nu \text{ outer normal derivative.} \tag{16}$$

If we choose a constant as a trial function in (15), we see immediately that $\lambda(\Omega)$ is negative. Theorem 2.4 of our paper has to be replaced by

Theorem 1.3. *If $\Omega \subset \mathbb{R}$ is strictly convex with maximal harmonic radius r_Ω then*

$$\lambda(\Omega) < \tilde{\gamma}_\Omega\, \lambda(B_{r_\Omega}),$$

where $\tilde{\gamma}_\Omega$ is defined in (9). The equality sign is never attained.

The proof is the same as in [2], but uses the new version of Lemma 2.3.

Remark Lemma 2.3 was also used in [3]. There too it has to be replaced by the correct version given in this Note. In particular the estimates concerning the fundamental frequencies have to be adapted.

References

[1] C. Bandle, M. Flucher: *Harmonic radius and concentration of energy; hyperbolic radius and Liouville's equations* $\Delta U = e^U$ *and* $\Delta U = U^{(n+2)/(n-2)}$, SIAM Review 38 (1969) 191–238.

[2] C. Bandle, A. Wagner: *Isoperimetric inequalities for the principal eigenvalue of a membrane and the energy of problems with Robin boundary conditions*, J. Convex Analysis 22(3) (2015) 627–640.

[3] C. Bandle, A. Wagner: *Shape optimization for an elliptic operator with infinitely many positive and negative eigenvalues*, Adv. Nonlinear Analysis 7 (2018) 49–66.

[4] P. Cardaliaguet, R. Tahraoui: *On the strict concavity of the harmonic radius in dimension* $N \geq 3$, J. Math. Pures Appl. (9) 81(2) (2002) 223–240.

[5] J. Hersch: *Transplantation harmonique, transplantation par modules, et théorèmes isopérimétriques*, Comment. Math. Helv. 44 (1969) 354–366.