

On the $\mathbb{H}$ -Cone-Functions for H-Convex Sets

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Given a compact and H-convex subset K of the Heisenberg group $\mathbb{H}$, with the origin e in its interior, we are interested in finding a homogeneous H-convex function f such that $f(e) = 0$ and $f|_{\partial K} = 1$; we will call this function f the $\mathbb{H}$ -cone-function of vertex e and base ∂K . While the equivalent version of this problem in the Euclidean framework has an easy solution, in our context this investigation turns out to be quite entangled, and the problem can be unsolvable. The approach we follow makes use of an extension of the notion of convex family introduced by Fenchel. We provide the precise, even if awkward, condition required to K so that ∂K is the base of an $\mathbb{H}$ -cone-function of vertex e . Via a suitable employment of this condition, we prove two interesting binding constraints on the shape of the set K , together with several examples.

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1. Introduction

Given a compact and convex set K in $\mathbb{R}^n$, with the origin e in its interior, the problem of constructing the homogeneous convex function $f: K \rightarrow [0, +\infty)$ so that $f(e) = 0$ and $f|_{\partial K} = 1$ has always a unique solution; this function is called *cone-function* with vertex e and base ∂K , and it is the biggest convex function with these constraints in e and in ∂K . If we move the vertex from the origin to a generic point x_0 in the interior of K , and require that the convex function has some positive value on the boundary of K , then, by translation and multiplication by a constant, the problem is strictly related to the initial one.

Among the several applications of the cone-functions, our attention is drawn by the Alexandrov Maximum Principle (see [14]) and the study of the engulfing property for convex functions in order to obtain some regularity for the solution of the Monge-Ampère equation in the spirit of Caffarelli (see [6], [7]). Homogeneity and convexity play both a fundamental role (see Section 3).

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In principle, the construction of the cone-function f with vertex e and base ∂K can be undertaken following two equivalent approaches. The first one consists in giving directly the graph of f : this is the surface obtained by taking the half lines with the origin of $\mathbb{R}^{n+1}$ as endpoint, and passing through every point of $\partial K \times \{1\} \subset \mathbb{R}^{n+1}$. Actually, this approach cannot be carried on in the Heisenberg case, due to its sub-Riemannian structure.

The second approach relies on the remark that every function is characterized via its level sets (see the definition below). Thereby, one is led to consider the family of sets $\mathcal{F} = \{K_\tau\}_{\tau \geq 0}$, where K_τ is obtained by dilating K (i.e., $K_\tau = \tau K$), and study whether the function f defined by the property $f|_{\partial K_\tau} = \tau$, for every $\tau \geq 0$, turns out to be convex.

The study of the families of sets $\mathcal{F}$ giving rise to a convex function dates back to the pioneering work of Fenchel (see [13]). Since we will follow this approach in the framework of the Heisenberg group, let us illustrate briefly the main ideas in a very general setting.

Given a nonempty set Ω , a real number τ , and a function $f: \Omega \rightarrow \mathbb{R}$, we define the *level set* $\text{lev}_{\leq \tau}(f)$ of f , for the level τ , as follows:

$$\text{lev}_{\leq \tau}(f) = \{x \in \Omega : f(x) \leq \tau\}.$$

If $\tau < \inf f$, then $\text{lev}_{\leq \tau}(f) = \emptyset$. If we denote by I the smallest interval containing the range of f , then the interval I has some interior points, unless f is a constant function. It is easy to show that the family of subsets $\{\text{lev}_{\leq \tau}(f)\}_{\tau \in I}$ enjoys some properties, that we collect in the following

Definition 1.1. A family $\{K_\tau\}_{\tau \in I}$ of subsets of Ω is said to be a *family of level sets* in Ω if

- (I) $\bigcup_{\tau \in I} K_\tau = \Omega$;
- (II) $K_{\tau_1} \subseteq K_{\tau_2}$, if $\tau_1 < \tau_2$;
- (III) $\bigcap_{\tau > \tau_0} K_\tau = K_{\tau_0}$, and $\bigcap_{\tau \in I} K_\tau = \emptyset$ if I is open to the left.

It is interesting to point out that any family of sets $\{K_\tau\}_{\tau \in I}$ of Ω indexed on an interval I , and satisfying (I)–(III) in the definition above, gives rise to a unique function $f: \Omega \rightarrow \mathbb{R}$ via the formula

$$f(x) = \inf\{\tau \in I : x \in K_\tau\} = \min\{\tau \in I : x \in K_\tau\}; \quad (1)$$

in particular, $\text{lev}_{\leq \tau}(f) = K_\tau$, thereby establishing a one-to-one correspondence between the functions f defined on Ω and the families of level sets of Ω .

As a matter of fact, if $\Omega \subseteq \mathbb{R}^n$, and $\mathcal{K} = \{K_\tau\}_{\tau \in I}$ is a family of convex subsets of Ω , then the function f defined via (1) is not convex, in general, but it satisfies the weaker condition

$$f((1 - \theta)x + \theta y) \leq \max\{f(x), f(y)\}, \quad \forall x, y \in \Omega, \theta \in [0, 1],$$

i.e., f is always a quasi-convex function. In order to get a convex function starting from a family of level sets, the assumption of convexity of each set K_τ is too weak, and the following stronger condition interconnecting the nested sets $\{K_\tau\}$ should be taken into account:

$$(C) \quad (1 - \theta)K_{\tau_0} + \theta K_{\tau_1} \subseteq K_{\tau_\theta},$$

for every $\tau_0, \tau_1 \in I$, $\theta \in [0, 1]$, $\tau_\theta = (1 - \theta)\tau_0 + \theta\tau_1$.

In [13] Fenchel showed that condition (C) does indeed characterize the sets of a family $\mathcal{F}$ giving rise to a convex function f via (1); such a family was named *convex family in Ω* . He proved the following:

Theorem 1.2. *Let Ω be a nonempty convex subset of $\mathbb{R}^n$ and let $\mathcal{F} = \{K_\tau\}_{\tau \in I}$ be a family of level sets in Ω . Then, $\mathcal{F}$ is a convex family if and only if condition (C) is fulfilled.*

In the particular case of a family $\mathcal{F} = \{\tau K\}_{\tau \geq 0}$, where K is a compact and convex set with e in the interior, the theorem above applies and gives an answer to the problem raised in the beginning of this introduction. As a matter of fact, the function f defined via (1) is exactly the cone-function of vertex e and base ∂K .

Our aim is to investigate whether, and under what conditions, these ideas can be fruitfully employed in order to construct an $\mathbb{H}$ -cone-function, that is, a cone-function in the Heisenberg group $\mathbb{H}$. In this context, we deal with a notion of convexity, the so-called H-convexity, both for sets and functions, which is related to the sub-Riemannian structure of $\mathbb{H}$ (see Definitions 2.1, 2.3). We recall only that $\mathbb{R}^3$ -convexity implies H-convexity.

In this paper, we will follow the Fenchel approach. First, we study the families $\mathcal{F} = \{K_\tau\}_{\tau \in I}$ of level sets in $\mathbb{H}$ giving rise to H-convex functions. In the spirit of Theorem 1.2, we provide an apparently simple but quite unhandy characterization for $\mathcal{F}$ to be an H-convex family: condition (C_H) in Proposition 2.6 turns out to be the Heisenberg version of the previous condition (C).

The second step of the paper is to consider a compact, H-convex set $K \subset \mathbb{H}$, with e in the interior, and therefore deal with the family $\mathcal{F} = \{K_\tau\}_{\tau \geq 0}$, where $K_\tau = \delta_\tau K$ and δ_τ is the non-isotropic dilation of $\mathbb{H}$. Our purpose is to detect what conditions must be fulfilled by K in order that $\mathcal{F}$ is an H-convex family. This function, homogeneous with respect to the dilations δ_λ by construction, will be called *$\mathbb{H}$ -cone-function*, and K will be said to generate an *H-convex family* via dilations. It is worthwhile noting that in $\mathbb{H}$, contrary to the Euclidean case, the notion of H-convexity and the dilations δ_λ have a good mutual behaviour only along the horizontal directions.

Let us devote a few lines to make a delicate point clearer. Suppose that K is compact and $\mathbb{R}^3$ -convex, with e in the interior. Then, the cone-function f with vertex e and base ∂K does exist; it is an $\mathbb{R}^3$ -convex function, thereby H-convex. Starting from the same set K , and working with the non isotropic dilations of $\mathbb{H}$, it may happen that we can construct the $\mathbb{H}$ -cone-function ϕ with vertex e and base

∂K ; in this case, this function ϕ is H-convex, but obviously it is different from the previous Euclidean cone-function f (see Examples 3.3 and 3.9). However, it may be that such $\mathbb{H}$ -cone-function ϕ does not exist (see Examples 3.11).

More generally, let K be H-convex, not $\mathbb{R}^3$ -convex; in this case, undoubtedly the cone-function with vertex e and base ∂K does not exist (otherwise, its level sets would be $\mathbb{R}^3$ -convex); but it may happen that we can construct an $\mathbb{H}$ -cone-function with vertex e and base ∂K .

Our interest is essentially motivated by the attempt to approach the problem of the engulfing properties for H-convex functions using the Alexandrov Maximum Principle (see [2], [11]). The reader could be surprised that there exist H-convex sets that do not admit $\mathbb{H}$ -cone-functions. However, in order to study the engulfing property for H-convex functions, whose level sets are H-convex, this seems to be the right line of investigation. In this initial study about the existence of $\mathbb{H}$ -cone-functions, we try to shed some light on the “shape” of a set K that generates an H-convex family. Whenever $K_\tau = \delta_\tau K$, this condition can be written in terms of closedness of K with respect to suitable parabolic arcs with ending points in K (Proposition 3.10). Taking advantage of this characterization, and with the focus on radial sets, i.e., sets invariant with respect to a rotation around the t axis, we are able to prove our main result, that concerns some binding constraints that K must fulfill:

Theorem 1.3. *Let $K \subset \mathbb{H}$ be a radial and compact set which generates an H-convex family via dilations. The following statements hold:*

- (i) *if $\xi_0 = (x_0, y_0, 0) \in \partial K$, then $B_G(e, \|\xi_0\|_G) \subset K$, where $B_G(\xi, r)$ denotes the Korányi ball with center ξ and radius r ;*
- (ii) *the projection of K onto the xy -plane is contained in K .*

This result is interesting and surprising, and it shows that the Heisenberg geometry imposes some unexpected obstacles for the existence of convex cone-type functions.

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2. H-convex sets, families and functions in $\mathbb{H}$

2.1. Preliminaries on the Heisenberg group $\mathbb{H}$

In this short subsection we collect some essential notions concerning the Heisenberg group that will be useful for the sequel. For a thorough discussion on this topic, see, for instance, [10].

The *Heisenberg group* $\mathbb{H}$ is the Lie group given by the underlying manifold $\mathbb{R}^3$ with the non commutative group law

$$\xi \circ \xi' = (x, y, t) \circ (x', y', t') = (x + x', y + y', t + t' + 2(x'y - xy')),$$

unit element $e = (0, 0, 0)$, and $\xi^{-1} = (-x, -y, -t)$. Left translations and anisotropic dilations are, in this setup, $L_{\xi_0}(\xi) = \xi_0 \circ \xi$ and $\delta_\lambda(x, y, t) = (\lambda x, \lambda y, \lambda^2 t)$. A function $f: \mathbb{H} \rightarrow \mathbb{R}$ is said to be homogeneous if

$$f(\delta_\lambda \xi) = \lambda f(\xi), \quad \forall \xi \in \mathbb{H}, \lambda > 0.$$

In the whole paper, the homogeneity, unless otherwise specified, will be related to the dilation δ_λ .

The differentiable structure on $\mathbb{H}$ is determined by the left invariant vector fields

$$X = \partial_x + 2y\partial_t, \quad Y = \partial_y - 2x\partial_t, \quad Z = \partial_t, \quad \text{with } [X, Y] = -4Z.$$

The vector field Z commutes with the vector fields X and Y ; X and Y are called *horizontal vector fields*. The Lie algebra $\mathfrak{h}$ of $\mathbb{H}$ is the stratified algebra $\mathfrak{h} = \mathbb{R}^3 \simeq V_1 \oplus V_2$, where $V_1 = \text{span}\{X, Y\}$, $V_2 = \text{span}\{Z\}$; $\langle \cdot, \cdot \rangle$ will denote the inner product in $V_1 \simeq \mathbb{R}^2$. Via the exponential map $\exp: \mathfrak{h} \rightarrow \mathbb{H}$ we identify the vector $\alpha X + \beta Y + \gamma Z$ in $\mathfrak{h}$ with the point (α, β, γ) in $\mathbb{H}$.

The sub-Riemannian structure of the Heisenberg group gives rise, among other things, to some “horizontal” notions. We say that r is a *horizontal line* through ξ_0 if $r = \{\xi = L_{\xi_0}(\exp(sw)) \in \mathbb{H}, s \in \mathbb{R}\}$, for some $w \in V_1$ fixed. We denote by $\mathcal{R}_{\xi_0}$ the set of all the horizontal lines through ξ_0 . A *horizontal segment* is a convex and bounded subset of a horizontal line.

Given a point $\xi_0 = (x_0, y_0, t_0) \in \mathbb{H}$, the *horizontal plane* H_{ξ_0} associated to ξ_0 is the plane in $\mathbb{H}$ defined by

$$H_{\xi_0} = L_{\xi_0}(\exp(V_1)) = \{\xi = (x, y, t) : t = t_0 + 2y_0x - 2x_0y\}.$$

Note that $\xi' \in H_\xi$ if and only if $\xi \in H_{\xi'}$.

It is well known that a homogeneous norm $N: \mathbb{H} \rightarrow [0, +\infty)$ is a homogeneous function satisfying the classical properties of the norm, with respect to the law of the group. As a particular case, the *Korányi norm* is defined by

$$\|\xi\|_G = ((x^2 + y^2)^2 + t^2)^{1/4}, \quad \forall \xi = (x, y, t) \in \mathbb{H}. \quad (2)$$

Furthermore, $\|\xi\|_q = \max\{\|(x, y)\|_E, |t|^{1/2}\}$ is the *homogeneous quasi-norm* for every $\xi = (x, y, t) \in \mathbb{H}$.

We will denote by $B_G(\xi_0, r)$ the *closed Korányi ball* with center ξ_0 and radius $r \geq 0$, obtained by a left translation: more precisely,

$$B_G(\xi_0, r) = L_{\xi_0}(\{\xi \in \mathbb{H} : \|\xi\|_G \leq r\}).$$

Similarly, $B_q(\xi_0, r)$ denotes the closed ball with center ξ_0 and radius $r \geq 0$ associated to $\|\cdot\|_q$. We denote by $B_E(\xi_0, r)$ the usual closed Euclidean ball with centre $\xi_0 \in \mathbb{R}^n$ and radius $r \geq 0$, and by $\|\cdot\|_E$ the usual Euclidean norm in $\mathbb{R}^n$. The topology induced in $\mathbb{H}$ by all these balls are equivalent.

2.2. $\mathbb{H}$ -convex sets, families and functions

Let us start with the definition of $\mathbb{H}$ -convex set (see [12]):

Definition 2.1. A subset Ω of $\mathbb{H}$ is *H-convex* if it contains every horizontal segment with endpoints in Ω , i.e., if $\xi \in \Omega$ and $\xi' = \xi \circ \exp v \in \Omega$ for some $v \in V_1$, then

$$\xi \circ \exp(\theta v) \in H_\xi \cap \Omega, \quad \forall \theta \in [0, 1]. \quad \square$$

It is clear, by the definition, that an $\mathbb{R}^3$ -convex set Ω is $\mathbb{H}$ -convex in the Heisenberg group, since every horizontal segment is a particular $\mathbb{R}^3$ -segment.

The regularity of these $\mathbb{H}$ -convex sets was studied in [1] by means of intrinsic cones at non-characteristic points. Let us devote a few words to one of the results therein. For a set $\Omega \subset \mathbb{H}$, a point $\xi_0 \in \partial\Omega$ is said to be non-characteristic for Ω if there exists a horizontal line r through ξ_0 so that r enters Ω at ξ_0 , i.e., $r \cap \text{int}(\Omega) \cap B_q(\xi_0, \rho) \neq \emptyset$, for all $\rho > 0$.

For every horizontal line $r \in \mathcal{R}_e$, let $\pi_r: \mathbb{H} \rightarrow r$ be the usual (Euclidean) orthogonal projection onto r , and let us define the operator $\pi_r^\perp: \mathbb{H} \rightarrow \mathbb{H}$ to be the $\mathbb{H}$ -projection via the identity

$$\xi = \pi_r^\perp(\xi) \circ \pi_r(\xi), \quad \forall \xi \in \mathbb{H}.$$

We emphasize that $\pi_r^\perp$ is strictly related to the non commutative group operation in $\mathbb{H}$. The open cone $C(e, r, \alpha, h)$ with vertex $e \in \mathbb{H}$, axis $r \in \mathcal{R}_e$, aperture $\alpha > 0$, and height $h > 0$ is given by

$$C(e, r, \alpha, h) = \{\xi \in \mathbb{H} : \|\pi_r^\perp(\xi)\|_q < \alpha \|\pi_r(\xi)\|_q < \alpha h\}.$$

The open cone $C(\xi_0, L_{\xi_0}(r), \alpha, h)$ is defined by a left translation, i.e.,

$$C(\xi_0, L_{\xi_0}(r), \alpha, h) = L_{\xi_0}(C(e, r, \alpha, h)).$$

In [1], the authors prove the following result, that provides useful information about the shape of an $\mathbb{H}$ -convex set:

Theorem 2.2. *Let Ω be an $\mathbb{H}$ -convex set. Let ξ_0 be a non-characteristic point for Ω , with $L_{\xi_0}(r) \in \mathcal{R}_{\xi_0}$ entering Ω at ξ_0 . Then, there exist $\rho > 0$, $\alpha > 0$, and $h > 0$ so that, for all $\xi \in B_q(\xi_0, \rho) \cap \partial\Omega$,*

$$C(\xi, L_\xi(r), \alpha, h) \subset \text{int}(\Omega).$$

For more recent results on this topic, see also [15].

Let us now deal with the notion of $\mathbb{H}$ -convexity for functions:

Definition 2.3. Let $\Omega \subset \mathbb{H}$ be $\mathbb{H}$ -convex. A function $f: \Omega \rightarrow \mathbb{R}$ is said to be *H-quasiconvex* if and only if

$$f(\xi \circ \exp(\theta v)) \leq \max\{f(\xi), f(\xi')\},$$

for every $\xi \in \Omega$, $\xi' = \xi \circ \exp(v) \in H_\xi \cap \Omega$ and $\theta \in [0, 1]$; equivalently, all the level sets $\text{lev}_{\leq \tau}(f)$ are $\mathbb{H}$ -convex, for every $\tau \in \mathbb{R}$.

A function $f: \Omega \rightarrow \mathbb{R}$ is said to be *H-convex* if it is $\mathbb{R}$ -convex along every horizontal segment within Ω , i.e.,

$$f(\xi \circ \exp(\theta v)) \leq (1 - \theta)f(\xi) + \theta f(\xi'), \quad (3)$$

for every $\xi \in \Omega$, $\xi' = \xi \circ \exp v \in H_\xi \cap \Omega$ and $\theta \in [0, 1]$. $\square$

It follows trivially that the $\mathbb{R}^3$ -convexity implies the H-convexity in the Heisenberg group, since every horizontal segment is a particular $\mathbb{R}^3$ -segment; nevertheless, there exist H-convex functions that are not $\mathbb{R}^3$ -convex function. An example is provided by the Korányi gauge, i.e., the function $\xi \mapsto \|\xi\|_G$.

The definition of H-convexity implies that the level sets of an H-convex function are H-convex sets. However, like in the Euclidean case, if all the sets $\text{lev}_{\leq \tau}(f)$ are H-convex, one cannot infer that f is H-convex. Let us consider, for instance, the function $f: \mathbb{H} \rightarrow \mathbb{R}$ defined by $f(x, y, t) = \ln(x^2 + y^2 + t^2 + 1)$, and note that all the level sets are Euclidean convex, in particular H-convex. Since the horizontal Hessian is not positive semidefinite everywhere, we deduce that this function is not H-convex (see [8] for details and a thorough analysis of the H-quasiconvex functions, named *weakly H-quasiconvex* functions therein).

Likewise the Euclidean framework, given a function $f: \mathbb{H} \rightarrow \mathbb{R}$ and a point $\xi_0 \in \mathbb{H}$, one defines the so-called horizontal subdifferential $\partial_H f(\xi_0)$ as the (possibly empty) subset of V_1 given by

$$\partial_H f(\xi_0) = \{p \in V_1 : f(\xi_0 \circ \exp(v)) \geq f(\xi_0) + \langle p, v \rangle, \forall v \in V_1\}.$$

In [9], it is shown that the nonemptiness of $\partial_H f(\xi)$ at every $\xi \in \mathbb{H}$ completely characterizes the H-convexity of f . In addition, via an integration technique, the function f can be built up starting from the horizontal subdifferential.

Despite the notion of H-convexity is affected by the horizontal structure, the regularity of H-convex functions is high. In fact, in [4] the authors prove that any H-convex function is locally Lipschitz continuous with respect to the sub-Riemannian Carnot-Carathéodory metric. This is weaker than the Euclidean Lipschitz continuity, but it implies that every H-convex function is, at least, continuous.

Motivated by the previous notions of H-convexity and H-quasiconvexity, and following the ideas of Fenchel [13], we give the following definition:

Definition 2.4. Let $\mathcal{K} = \{K_\tau\}_{\tau \in I}$ be a family of level sets in $\mathbb{H}$ (see Definition 1.1). We say that $\mathcal{K}$ is an *H-quasiconvex family* (*H-convex family*) in $\mathbb{H}$ if the sets $\{K_\tau\}_{\tau \in I}$ are the level sets of an H-quasiconvex (H-convex) function $f: \mathbb{H} \rightarrow \mathbb{R}$, i.e., $\text{lev}_{\leq \tau}(f) = K_\tau$, for $\tau \in I$. $\square$

As mentioned in the introduction, there is a unique function $f: \mathbb{H} \rightarrow \mathbb{R}$ associated to $\mathcal{K}$ as in the previous definition, and it is given by the formula

$$f(\xi) = \inf\{\tau \in I : \xi \in K_\tau\} = \min\{\tau \in I : \xi \in K_\tau\}. \quad (4)$$

The following remark trivially holds:

Remark 2.5. If $\{K_\tau\}_{\tau \in I}$ is an H-convex family, then, for every $\tau \in I$, the set K_τ is H-convex, and closed. $\square$

In the next result condition $(\mathbf{C}_H)$ will come into play; this condition forces the nested sets $\{K_\tau\}$ to have a mutual “convex” behaviour when a horizontal movement is involved. Despite being quite unhandy, its role will be prominent in the sequel: this is, in fact, the precise condition that an H-convex family must fulfill, just like a convex family in the Euclidean case is identified by Theorem 1.2.

Proposition 2.6. Let $\{K_\tau\}_{\tau \in I}$ be a family of level sets in $\mathbb{H}$, and denote by f the related function defined in (4). Then, f is H-convex if and only if, for every $\tau_0, \tau_1 \in I$, for every $v \in V_1$, and for every $\theta \in [0, 1]$,

$$(\mathbf{C}_H) \quad K_{\tau_0} \cap (K_{\tau_1} \circ \exp v) \subseteq K_{\tau_\theta} \circ \exp(\theta v),$$

where $\tau_\theta = (1 - \theta)\tau_0 + \theta\tau_1$.

Proof. Let us prove the “if” part. Take any $\xi_0 \in \mathbb{H}$, $v \in V_1$, and set $\xi_1 = \xi_0 \circ \exp v$. Let $f(\xi) = \min_{\xi \in K_\tau} \tau$, $\tau_0 = f(\xi_0)$, and $\tau_1 = f(\xi_1)$. We have that $\xi_0 \in K_{\tau_0}$, and $\xi_1 \in K_{\tau_1}$. In particular,

$$\xi_0 \in K_{\tau_0} \cap (K_{\tau_1} \circ \exp(-v)).$$

From the assumption, $\xi_0 \circ \exp(\theta v) \in K_{\tau_\theta}$, where $\tau_\theta = (1 - \theta)\tau_0 + \theta\tau_1$. Therefore,

$$f(\xi_0 \circ \exp(\theta v)) = \min_{\xi_0 \circ \exp(\theta v) \in K_\tau} \tau \leq \tau_\theta = (1 - \theta)f(\xi_0) + \theta f(\xi_1),$$

thereby proving the inequality (3), with $\xi = \xi_0$.

The “only if” part: take $\xi_0 \in K_{\tau_0} \cap (K_{\tau_1} \circ \exp v)$. This implies that $\xi_0 \circ \exp(-v) \in K_{\tau_1}$, i.e., $f(\xi_0 \circ \exp(-v)) \leq \tau_1$. From

$$f(\xi_0 \circ \exp(-\theta v)) \leq (1 - \theta)f(\xi_0) + \theta f(\xi_0 \circ \exp(-v)) \leq \tau_\theta,$$

we get that $\xi_0 \circ \exp(-\theta v) \in K_{\tau_\theta}$, and thus the assertion follows. $\square$

Remark 2.7. Condition $(\mathbf{C}_H)$ can be restated as follows:

$$\xi_0 \in K_{\tau_0}, \quad \xi_0 \circ \exp v \in K_{\tau_1} \implies \xi_0 \circ \exp(\theta v) \in K_{\tau_\theta},$$

for all $\tau_0, \tau_1 \in I$, $\theta \in [0, 1]$, and $\tau_\theta = (1 - \theta)\tau_0 + \theta\tau_1$. $\square$

It is easy to prove, but crucial for the sequel, the following:

Remark 2.8. Let $\mathcal{K} = \{K_\tau\}_{\tau \in I}$ be a family of level sets in $\mathbb{H}$.

- (i) If $\mathcal{K}$ satisfies condition $(\mathbf{C}_H)$, then K_τ is H-convex and closed, for every $\tau \in I$;
- (ii) condition $(\mathbf{C}_H)$ is invariant with respect to the dilations of $\mathbb{H}$, that means, $\mathcal{K} = \{K_\tau\}_{\tau \in I}$ satisfies condition $(\mathbf{C}_H)$ if and only if, for every $\alpha > 0$, the family of level sets $\mathcal{K}^\alpha = \{\delta_\alpha K_\tau\}_{\tau \in I}$ in $\mathbb{H}$ satisfies condition $(\mathbf{C}_H)$.

Note that, setting $\tau_0 = \tau_1$ in $(\mathbf{C}_H)$, we obtain that if $\xi_0 \in K_{\tau_0} \cap (K_{\tau_0} \circ \exp v)$, i.e. ξ_0 and $\xi_0 \circ \exp(-v)$ belong to K_{τ_0} , then $\xi_0 \circ \exp(-\theta v) \in K_{\tau_0}$ for every $\theta \in [0, 1]$. Hence, in order to obtain the $\mathbb{H}$ -convexity in (i) of Remark 2.8, the fact that $\mathcal{K}$ is a family of level sets is not necessary.

3. $\mathbb{H}$ -convex families via dilations and $\mathbb{H}$ -cone-functions

In this section we focus on the particular families of sets obtained by dilation of a fixed set K , with the purpose of identifying those sets K that give rise to a convex ($\mathbb{H}$ -convex) family. This problem is strictly related to the existence of the so called cone-functions ($\mathbb{H}$ -cone-functions). Let us clarify briefly the problem and its motivation. In order to do that, we recall the following Alexandrov Maximum Principle:

Theorem 3.1. *Let $\Omega \subset \mathbb{R}^n$ be bounded, open and convex. Let $u \in C(\overline{\Omega})$ be convex with $u = 0$ on $\partial\Omega$. Then, for every $x_0 \in \Omega$,*

$$|u(x_0)|^n \leq C_n \text{dist}(x_0, \partial\Omega) \text{diam}(\Omega)^{n-1} \mathcal{L}^n(\partial u(\Omega)), \quad (5)$$

where $C_n > 0$ is a constant depending only on the dimension n .

Clearly, $\mathcal{L}^n$ denotes the Lebesgue measure in $\mathbb{R}^n$.

We recall that the *subdifferential* of the function $u: \mathbb{R}^n \rightarrow \mathbb{R}$ at the point x_0 is defined by

$$\partial u(x_0) = \{p \in \mathbb{R}^n : u(x) \geq u(x_0) + \langle p, x - x_0 \rangle, \forall x \in \mathbb{R}^n\},$$

and we define the *normal mapping* of the function u as the multivalued function $\partial u: \mathbb{R}^n \rightarrow \mathcal{P}(\mathbb{R}^n)$ by $x \mapsto \partial u(x)$. In particular,

$$\partial u(A) = \bigcup_{x \in A} \partial u(x),$$

for every set $A \subset \mathbb{R}^n$. The main idea of the proof of Theorem 3.1 can be summarized in these inclusions:

$$\partial v(x_0) \subset \partial v(\Omega) \subset \partial u(\Omega). \quad (6)$$

Here, the function v is precisely the cone-function with vertex x_0 and base $\partial\Omega$. The “shape” of the set $\partial v(x_0)$ contains indeed all the information about $\text{dist}(x_0, \partial\Omega)$, $\text{diam}(\Omega)$ and $|u(x_0)|$ that are needed in (5). The first inclusion in (6) is clear by definition, but it is an equality, in fact, since the cone-function is homogeneous on $\mathbb{R}^n$ with respect to the center x_0 . The second inclusion arises from the comparison principle for the normal mapping, which works taking into account that $u(x_0) = v(x_0)$ and $v = u$ on $\partial\Omega$. For all the details, the reader can give a look at the first chapter of [14], but we hope that these hints help to clarify the role of the cone-function, with its enjoyed properties of homogeneity and convexity.

Moving to the Heisenberg case, some of these ideas can be adapted, even if with quite hard proofs, if we replace in (6) ∂u with $\partial_H u$. Again, the homogeneity of

a possible $\mathbb{H}$ -cone-function is crucial, but now the $\mathbb{H}$ -convexity of v is needed, since the second inclusion in (6) holds only for $\mathbb{H}$ -convex functions (see [2] for all details).

This section is devoted to the construction of these cone-functions and $\mathbb{H}$ -cone-functions. Let us consider, first, the simpler Euclidean case, and then pass to the Heisenberg setting. In the following, we will assume that $K \subset \mathbb{H} = \mathbb{R}^3$ satisfies the condition

(a) K is compact and $e \in \text{int}(K)$.

3.1. The Euclidean case

The problem of constructing a cone-function for a compact and convex set K has a simple solution; the following result holds:

Proposition 3.2. *Suppose that $K \subset \mathbb{R}^3$ is a convex set satisfying condition (a). Then, the family $\mathcal{K} = \{K_\tau\}_{\tau \in [0, +\infty)}$, defined by $K_\tau = \tau K$, is an $\mathbb{R}^3$ -convex family. Moreover, the (convex) function $f: \mathbb{R}^3 \rightarrow [0, +\infty)$ associated to $\mathcal{K}$ by equation (1) is homogeneous w.r.t. the Euclidean dilations, i.e., $f(\lambda\xi) = \lambda f(\xi)$, for every $\lambda \geq 0$ and $\xi \in \mathbb{R}^3$.*

Proof. First, let us prove that $\mathcal{K}$ is a family of level sets in $\mathbb{R}^3$ according to Definition 1.1. It is trivial to show that (a) implies (I). The convexity of K implies (II). The proof of (III) will be presented hereafter, when dealing with the Heisenberg setting.

Let us now show that (C) is fulfilled. Take $\tau_0, \tau_1 \in [0, +\infty)$, and $\theta \in [0, 1]$, and set $\tau_\theta = (1 - \theta)\tau_0 + \theta\tau_1$. Let $\xi \in (1 - \theta)\tau_0 K + \theta\tau_1 K$; then, there exist $\xi_1, \xi_2 \in K$ so that

$$\xi = (1 - \theta)\tau_0\xi_1 + \theta\tau_1\xi_2 = \tau_\theta \left(\frac{(1 - \theta)\tau_0}{\tau_\theta}\xi_1 + \frac{\theta\tau_1}{\tau_\theta}\xi_2 \right).$$

Since $\xi = \tau_\theta\xi'$, where ξ' is a convex combination of points of K , we conclude that $\xi \in \tau_\theta K$. It is an easy exercise to prove that f is homogeneous with respect to the Euclidean dilations. $\square$

We will say that the function f in Proposition 3.2 is the *cone-function* with vertex e and base ∂K . This definition is motivated by the shape of the graph of f in $\mathbb{R}^4$, which is a cone with vertex $(e, 0) \in \mathbb{R}^3 \times \mathbb{R}$ and base ∂K , having “value” 1 on ∂K .

The following simple example of cone-function in $\mathbb{R}^3$ is interesting when compared to the forthcoming Example 3.9 in $\mathbb{H}$, where we start from the same set K , but we deal with different dilations:

Example 3.3. (K unit ball and Euclidean dilations) Let $K = B_E(0, 1)$. Trivially, K satisfies the assumptions of Proposition 3.2; $K_\tau = \tau K = B_E(0, \tau)$ for every $\tau > 0$, and the cone-function $f: \mathbb{R}^3 \rightarrow \mathbb{R}$ defined by (1) is $f(x, y, t) = \|(x, y, t)\|_E$.

3.2. The Heisenberg case

In this subsection we deal with the problem of constructing an H-convex family arising by dilations of a fixed set $K \subset \mathbb{H}$, and, as a byresult, identifying a cone-function in $\mathbb{H}$ with vertex e and base ∂K . Let us now consider the family of sets $\mathcal{K} = \{K_\tau\}_{\tau \in [0, +\infty)}$ defined by

$$K_\tau = \delta_\tau K, \quad \tau \geq 0. \quad (7)$$

For these type of families $\mathcal{K}$ the following proposition holds:

Proposition 3.4. *Suppose that K satisfies assumption (a) and*

(b) $\delta_\tau K \subset K$, for every $\tau \in [0, 1)$.

Then, the family $\mathcal{K} = \{K_\tau\}_{\tau \in [0, +\infty)}$, defined via dilations of the set K as in (7), is a family of level sets in $\mathbb{H}$. Moreover, if $f: \mathbb{H} \rightarrow [0, +\infty)$ is the function associated to $\mathcal{K}$ by (4), then f is homogeneous.

Proof. First, let us prove that $\mathcal{K}$ is a family of level subsets of $\mathbb{H}$. Again (a) implies (I). Now, take $\tau_0 < \tau_1$, and $\xi \in \delta_{\tau_0} K$, i.e., $\xi = \delta_{\tau_0} \xi'$ for some $\xi' \in K$. Since $\delta_{\tau_0} \xi' = \delta_{\tau_1}(\delta_{\tau_0/\tau_1} \xi')$, it follows from (b). that $\delta_{\tau_0/\tau_1} \xi' \in K$, therefore (II) holds.

In order to prove (III), note that, from (II), $\cap_{\tau > \tau_0} \delta_\tau K \supseteq \delta_{\tau_0} K$. To prove the opposite inclusion, let us now take any ξ from the closed set $\cap_{\tau > \tau_0} \delta_\tau K$. Since $\xi = \delta_\tau(\delta_{1/\tau} \xi)$, this implies that $\delta_{1/\tau} \xi \in K$, for every $\tau > \tau_0$. By the closedness of K , and by the continuity of the map $\tau \mapsto \delta_{1/\tau} \xi$, we have $\lim_{\tau \rightarrow \tau_0} \delta_{1/\tau} \xi = \delta_{1/\tau_0} \xi \in K$, and therefore, $\xi \in \delta_{\tau_0} K$.

Denote by $f: \mathbb{H} \rightarrow \mathbb{R}$ the function whose level sets are given by $\text{lev}_{\leq \tau}(f) = \delta_\tau K$. Then,

$$f(\delta_\lambda \xi) = \inf\{\tau \in [0, +\infty) : \xi \in K_{\tau/\lambda}\} = \lambda \inf\{\tau \in [0, +\infty) : \xi \in K_\tau\} = \lambda f(\xi),$$

thereby proving that f is homogeneous. $\square$

The proposition above and its proof highlight that, while in the Euclidean case the assumption of convexity of K gives (II). in Definition 1.1, in the Heisenberg setting the requirement of H-convexity has nothing to do with the dilations, and hence with (II). Moreover, as we will see, the H-convexity assumption on K does not guarantee that $\mathcal{K} = \{\delta_\tau K\}_{\tau \in [0, +\infty)}$ is an H-convex family (see Example 3.11).

Let us point out a property that is fulfilled by any homogeneous function on $\mathbb{H}$:

Proposition 3.5. *Let $f: \mathbb{H} \rightarrow [0, +\infty)$ be a homogeneous function.*

If $p \in \partial_H f(\xi)$, then $p \in \partial_H f(\delta_\lambda \xi)$, for every $\lambda > 0$. Moreover, for every set K such that $e \in \text{int}(K)$, we have

$$\partial_H f(K) := \bigcup_{\xi \in K} \partial_H f(\xi) = \partial_H f(e).$$

Proof. Suppose that $p \in \partial_H f(\xi)$. Then,

$$\begin{aligned} f(\delta_\lambda \xi \circ \exp v) &= f(\delta_\lambda(\xi \circ \exp(v/\lambda))) = \lambda f(\xi \circ \exp(v/\lambda)) \\ &\geq \lambda(f(\xi) + \langle p, v/\lambda \rangle) = f(\delta_\lambda \xi) + \langle p, v \rangle. \end{aligned}$$

The second part of the claim follows easily from the first part and from the upper semicontinuity of the multivalued function $\partial_H f: \mathbb{H} \rightrightarrows V_1$ (see [9] for details). $\square$

With the aim of generating H-convex families via dilations of a set $K \subset \mathbb{H}$, we will assume that the set K satisfies conditions **(a)**, **(b)**, and

(c) K is an H-convex set of $\mathbb{H}$.

An interesting and natural question is the following: suppose that K satisfies properties **(a)**–**(c)**; is there any chance that the family $\mathcal{K}$ defined by (7) is an H-convex family? In this line of investigation we give the following definition:

Definition 3.6. We say that a compact set $K \subset \mathbb{H}$ *generates an H-convex family via dilations* if the family $\mathcal{K}$ defined by (7) is an H-convex family. In this case, we say that the function $f: \mathbb{H} \rightarrow [0, +\infty)$ associated to $\mathcal{K}$ by (4) is the $\mathbb{H}$ -cone-function with vertex e and base ∂K .

Let us emphasize that the $\mathbb{H}$ -cone-functions defined above are homogeneous w.r.t. the dilations δ_λ , and are different from the cone-functions of the Euclidean case (compare, for instance, Example 3.3 and Example 3.9). Moreover, let us mention that in [5] the authors introduce, for other purposes with respect to our investigations, two different notions of cone-function.

Remark 3.7. If a compact set $K \subset \mathbb{H}$ generates an H-convex family via dilations, then K is H-convex and $e \in \text{int}(K)$.

Proof. The set K is H-convex by Remark 2.5. Suppose, by contradiction, that $e \notin \text{int}(K)$; then, there exists a sequence $\{\xi_n\}_n \subset \mathbb{H} \setminus K$, with $\xi_n \rightarrow e$. If $\xi_n \notin K$, then

$$\lim_{n \rightarrow \infty} f(\xi_n) \geq 1;$$

however, since f is H-convex (by the result in [4]), it is continuous, and $f(e) = 0$. This leads to a contradiction. $\square$

Let us now spend a few words on the compactness assumption in Definition 3.6: if a family $\mathcal{K}$ defined by K via (7) is an H-convex family, then the compactness of K is not an outcome. For example, the set $K = \{(x, y, t) \in \mathbb{H} : |x| \leq 1\}$ generates an H-convex family via dilations, and the unique function associated to such family via (4) is the $\mathbb{R}^3$ -convex, H-convex and homogeneous function $f(x, y, t) = |x|$. However, since we are interested in the construction of cone-functions having as a base the boundary of a compact set, we require in Definition 3.6 that K is compact.

Let us provide two examples of $\mathbb{H}$ -cone-functions:

Example 3.8. (K unit Korányi ball) Let $K = B_G(0, 1)$; trivially, K generates an H-convex family, since the function $f(\xi) = \|\xi\|_G$, associated to $\{\delta_\tau K\}$, is H-convex and homogeneous, thereby it is the $\mathbb{H}$ -cone-function of K .

Example 3.9. (K unit Euclidean ball) Let $K = B_E(0, 1)$. For every $\tau > 0$, we have

$$K_\tau = \delta_\tau K = \left\{ (x, y, t) \in \mathbb{H} : \frac{x^2}{\tau^2} + \frac{y^2}{\tau^2} + \frac{t^2}{\tau^4} \leq 1 \right\} \quad (8)$$

Denote by f the function defined via (4); clearly, $f(\xi) = \tau$ if and only if $\xi \in \partial K_\tau$. Since $\xi = (x, y, t) \in \partial K_\tau$ if and only if $\tau^4 - \tau^2(x^2 + y^2) - t^2 = 0$, solving for τ gives $\tau = f(x, y, t)$, where

$$f(x, y, t) = \frac{1}{\sqrt{2}} \sqrt{x^2 + y^2 + \sqrt{(x^2 + y^2)^2 + 4t^2}}. \quad (9)$$

We show that f is H-convex. In order to do this, we recall first a property of the composition that preserves convexity. We consider a function $F: \mathbb{R}_+^2 \rightarrow [0, +\infty)$, $\mathbb{R}^2$ -convex and monotone in the sense that

$$x_1 \leq x'_1 \text{ and } x_2 \leq x'_2 \quad \implies \quad F(x_1, x_2) \leq F(x'_1, x'_2),$$

and the functions $f_i: \mathbb{H} \rightarrow [0, +\infty)$, $i = 1, 2$, both H-convex. For every $\xi_0 \in \mathbb{H}$, $v \in V_1$ and $s \in [0, 1]$, we have

$$\begin{aligned} & F\left(f_1(\xi_0 \circ \exp(sv)), f_2(\xi_0 \circ \exp(sv))\right) \\ & \leq F\left((1-s)f_1(\xi_0) + sf_1(\xi_0 \circ \exp(v)), (1-s)f_2(\xi_0) + sf_2(\xi_0 \circ \exp(v))\right) \\ & = F\left((1-s)(f_1(\xi_0), f_2(\xi_0)) + s(f_1(\xi_0 \circ \exp(v)), f_2(\xi_0 \circ \exp(v)))\right) \\ & \leq (1-s)F\left(f_1(\xi_0), f_2(\xi_0)\right) + sF\left(f_1(\xi_0 \circ \exp(v)), f_2(\xi_0 \circ \exp(v))\right) \end{aligned}$$

Since the function f in (9) is so that

$$f(\xi) = \frac{1}{\sqrt{2}} F(f_1(\xi), f_2(\xi)),$$

where $F(x_1, x_2) = \|(x_1, x_2)\|_E$, $f_1(x, y, t) = \|(x, y)\|_E$ and $f_2(x, y, t) = \|(x, y, 2t)\|_G$, and F is convex and monotone on $\mathbb{R}_+^2$, f_1 is $\mathbb{R}^3$ -convex and, thereby, H-convex, and f_2 is H-convex (see Example 3.8), then, by the previous argument, we get that f is H-convex. Therefore, $K = B_E(0, 1)$ generates an H-convex family via dilations. In particular, f is the $\mathbb{H}$ -cone-function with vertex e and base the boundary of the Euclidean ball $B_E(0, 1)$. Finally, we mention that in Example 5.2 in [3] the authors prove that f is a homogeneous norm, called Lee-Naor norm. $\square$

Let us go back to the initial question about the construction of the $\mathbb{H}$ -cone-function. In order to provide an answer, we write more explicitly condition $(\mathbf{C}_H)$ in the particular case of families obtained by dilations:

Proposition 3.10. *Let $K \subset \mathbb{H}$ such that assumptions (a)–(c) hold, and let us consider the family $\mathcal{K} = \{K_\tau\}_{\tau \in [0, +\infty)}$ defined by K via dilations as in (7). Then, $\mathcal{K}$ satisfies condition $(\mathbf{C}_H)$ if and only if, for every ξ_0 , $\tau > 0$ and (α, β) so that*

$$\xi_0 = (x_0, y_0, t_0) \in K, \quad (10)$$

$$\delta_{1/\tau}(\xi_0 \circ \exp(\alpha, \beta)) = \left(\frac{x_0 + \alpha}{\tau}, \frac{y_0 + \beta}{\tau}, \frac{t_0 + 2(y_0\alpha - x_0\beta)}{\tau^2} \right) \in K,$$

the curve $s \mapsto \tilde{\xi}(\theta(s))$ takes its values within K , where $\tilde{\xi}(\theta(\cdot))$ is defined as follows: setting $A = x_0\beta - y_0\alpha$,

- (i) if $x_0(\tau - 1) \neq \alpha$, then $s \in [x_0, \frac{x_0 + \alpha}{\tau}]$ (or $[\frac{x_0 + \alpha}{\tau}, x_0]$) and
- $$\tilde{\xi}(\theta(s)) = \left(s, \frac{(y_0(\tau - 1) - \beta)s + A}{(\tau - 1)x_0 - \alpha}, \frac{t_0((\tau - 1)s - \alpha)^2 - 2A(x_0 - s)((\tau - 1)s - \alpha)}{((\tau - 1)x_0 - \alpha)^2} \right);$$
- (ii) if $x_0(\tau - 1) = \alpha$ and $y_0(\tau - 1) \neq \beta$, then $s \in [y_0, \frac{y_0 + \beta}{\tau}]$ (or $[\frac{y_0 + \beta}{\tau}, y_0]$) and
- $$\tilde{\xi}(\theta(s)) = \left(x_0, s, \frac{t_0((\tau - 1)s - \beta)^2 - 2x_0(\beta - (\tau - 1)y_0)(y_0 - s)((\tau - 1)s - \beta)}{((\tau - 1)y_0 - \beta)^2} \right);$$
- (iii) if $x_0(\tau - 1) = \alpha$ and $y_0(\tau - 1) = \beta$, then $A = 0$ and we get, for $s \in [0, 1]$,
- $$\tilde{\xi}(\theta(s)) = \left(x_0, y_0, \frac{t_0}{(1 + s(\tau - 1))^2} \right).$$

Proof. First of all, due to Remark 2.8(i), we can set $\tau_0 = 1$ in $(\mathbf{C}_H)$. Let us now consider $\xi_0 \in K \cap (K_\tau \circ \exp(-v))$, where $v = (\alpha, \beta)$. This is equivalent to $\xi_0 \in K$ and $\delta_{1/\tau}(\xi_0 \circ \exp v) \in K$, that is exactly (10). As in Remark 2.7, condition $(\mathbf{C}_H)$ can restated as:

$$\tilde{\xi}(\theta) = \delta_{1/\tau_\theta}(\xi_0 \circ \exp(\theta v)) = \left(\frac{x_0 + \theta\alpha}{\tau_\theta}, \frac{y_0 + \theta\beta}{\tau_\theta}, \frac{t_0 + 2(y_0\alpha - x_0\beta)\theta}{\tau_\theta^2} \right) \in K,$$

for all $\theta \in [0, 1]$, where $\tau_\theta = 1 + \theta(\tau - 1)$. We will discuss in detail case (i), and will leave the other two cases to the interested reader. Set

$$s = \frac{x_0 + \theta\alpha}{1 + \theta(\tau - 1)}.$$

Since $\theta \in [0, 1]$, and $x_0(\tau - 1) \neq \alpha$, we get that $s \in [x_0, \frac{x_0 + \alpha}{\tau}]$ (or $[\frac{x_0 + \alpha}{\tau}, x_0]$). Solving for θ gives

$$\theta = \frac{x_0 - s}{s(\tau - 1) - \alpha}.$$

Replacing this expression of θ in the second and third component of $\tilde{\xi}(\theta)$, and taking into account the expression of A , we get, respectively,

$$\tilde{y}(\theta(s)) = \frac{(y_0(\tau - 1) - \beta)s + A}{(\tau - 1)x_0 - \alpha},$$

$$\tilde{t}(\theta(s)) = \frac{t_0((\tau - 1)s - \alpha)^2 - 2A(x_0 - s)((\tau - 1)s - \alpha)}{((\tau - 1)x_0 - \alpha)^2}. \quad \square$$

The following example shows that the problem of finding the $\mathbb{H}$ -cone-function can be unsolvable. More precisely, it shows that there exist sets K satisfying **(a)**–**(c)** that do not fulfill **(C_H)**; thus, there are no $\mathbb{H}$ -cone-functions with vertex e and with the boundary of these sets as bases.

Example 3.11. (K cylinder) Let $K = \{(x, y, t) \in \mathbb{H} : \|(x, y)\|_E \leq 1, |t| \leq 1\}$. Clearly, K satisfies **(a)**–**(c)**, and it is $\mathbb{R}^3$ -convex; nevertheless, **(C_H)** does not hold. In fact, let us fix $(x_0, \beta) \in \mathbb{R}^2$, with $x_0 > 0$, $\beta < 0$ and $\|(x_0, \beta)\|_E < 1$, and consider

$$\xi_0 = (x_0, 0, 1), \quad v = (0, \beta), \quad \tau = \sqrt{1 - 2x_0\beta} > 1.$$

Under these choices, assumption (10) becomes

$$\xi_0 = (x_0, 0, 1) \in K, \quad \text{and} \quad \left(\frac{x_0}{\tau}, \frac{\beta}{\tau}, 1\right) \in K,$$

and it is obviously fulfilled. Since $x_0(\tau - 1) \neq 0$, by Proposition 3.10(i), condition **(C_H)** requires that

$$\tilde{\xi}(\theta(s)) = \left(s, \frac{\beta(x_0 - s)}{(\tau - 1)x_0}, \frac{s^2(\tau - 1 + 2x_0\beta) - 2x_0^2\beta s}{(\tau - 1)x_0^2}\right) \in K;$$

for $s \in [x_0/\tau, x_0]$. Taking into account that $2x_0\beta = 1 - \tau^2$, we obtain

$$\frac{(\tau - 1 + 2x_0\beta)}{(\tau - 1)x_0^2} = -\frac{\tau}{x_0^2} < 0.$$

Therefore the third component of the curve $\tilde{\xi}(\theta(s))$ is greater than 1 for every $s \in (x_0/\tau, x_0)$. This implies that $\tilde{\xi}(\theta(s)) \notin K$. $\square$

4. $\mathbb{H}$ -cone-functions with base on the boundary of radial H-convex sets

In order to better understand the role played by condition **(C_H)** in relation to the shape of K , and with the aim of constructing an $\mathbb{H}$ -cone-function for a compact, H-convex set K having e in the interior, let us focus on a particular class of sets, the radial ones. We say that $K \subset \mathbb{H}$ is *radial* if

$$\xi_0 = (x_0, y_0, t_0) \in K \iff (x, y, t_0) \in K, \quad \forall (x, y) \in \mathbb{R}^2 : \|(x, y)\|_E = \|(x_0, y_0)\|_E.$$

The section investigates, in several steps, the radial sets that are candidates to have an $\mathbb{H}$ -cone-function associated.

It is clear that, if K is a compact and radial set, then the $\mathbb{H}$ -cone-function f with vertex e and base ∂K (when it exists) is radial, by default, which means that $f(x, y, t) = g(\sqrt{x^2 + y^2}, t)$ for some function $g: [0, +\infty) \times \mathbb{R} \rightarrow \mathbb{R}$ (equivalently, $f(x, y, t) = f(O(x, y), t)$ for every $O \in O(2)$). The H-subdifferential $\partial_H f$ of this kind of functions enjoys nice properties, as we show:

Proposition 4.1. *Let $f: \mathbb{H} \rightarrow \mathbb{R}$ be a radial function. If $p \in \partial_H f(x, y, t)$, then $Op \in \partial_H f(O(x, y), t)$.*

Proof. Denote by $\pi: \mathbb{H} \rightarrow \mathbb{R}^2$ the projection $\pi(\xi) = z$, if $\xi = (z, t)$, and by $J: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ the operator defined as $J(x, y) = (y, -x)$. For every $p \in \partial_H f(\xi)$, and $O \in O(2)$, we have, for every $v \in V_1 \simeq \mathbb{R}^2$,

$$\begin{aligned} f(Oz + v, t - 2\langle Oz, Jv \rangle) &= f(O(z + O^T v), t - 2\langle Oz, JOO^T v \rangle) \\ &= f(O(z + O^T v), t - 2\langle Oz, OJO^T v \rangle) = f(O(z + O^T v), t - 2\langle z, JO^T v \rangle) \\ &= f(z + O^T v, t - 2\langle z, JO^T v \rangle) \geq f(z, t) + \langle p, O^T v \rangle = f(Oz, t) + \langle Op, v \rangle, \end{aligned}$$

thereby proving the claim. $\square$

By collecting the results of the previous proposition and Proposition 3.5, we have the following

Corollary 4.2. *Let $f: \mathbb{H} \rightarrow [0, +\infty)$ be the $\mathbb{H}$ -cone-function with vertex e and base ∂K , where $K \subset \mathbb{H}$ is radial. If $p \in \partial_H f(\xi)$, then $Op \in \partial_H f(O(\delta_\lambda \xi))$, for every $\lambda > 0$ and $O \in O(2)$.*

Let us now start our investigation on the shape of the radial set K with the following

Proposition 4.3. *Let $K \subset \mathbb{H}$ be a radial and compact set which generates an H -convex family. Let us define $t_{\min}$ and $t_{\max}$ as follows:*

$$t_{\min} = \min\{t : (0, 0, t) \in K\}, \quad t_{\max} = \max\{t : (0, 0, t) \in K\}.$$

If $\xi_0 = (x_0, y_0, t_0) \in \partial K$ and $t_0 \in (t_{\min}, t_{\max})$, then ξ_0 is a non-characteristic point.

Proof. From Remark 3.7, the set K is H -convex, and $e \in \text{int}(K)$, i.e., there exists $\rho > 0$ so that $B_G(e, \rho) \subset K$. In particular, $t_{\min} < 0 < t_{\max}$.

First, let us suppose that $\xi_0 = (x_0, 0, 0) \in \partial K$, with $x_0 > 0$. It is clear that the horizontal segment with endpoints e and ξ_0 is contained in K , but this does not imply that ξ_0 is non-characteristic. Let t be small enough, such that $\xi_t = (0, -t/2, x_0 t) \in B_G(e, \rho)$. Since $\xi_t = \xi_0 \circ \exp(-x_0, -t/2) \in H_{\xi_0}$, the H -convexity of K implies that, for every $s \in [0, 1]$,

$$\xi_0 \circ \exp(s(-x_0, -t/2)) = ((1-s)x_0, -st/2, x_0 ts) \in K.$$

Furthermore, K is radial; then, for every $s \in [0, 1]$ and $\theta \in [0, 2\pi)$,

$$(r_s \cos \theta, r_s \sin \theta, x_0 ts) \in K, \quad r_s = \|((1-s)x_0, -st/2)\|_E;$$

in particular, $(r_s, 0, x_0 ts) \in K$. Note that $(0, 0, x_0 ts) \in K$, as $\|(0, 0, x_0 ts)\|_G < \rho$; hence, all the horizontal segments with endpoints $(r_s, 0, x_0 ts)$ and $(0, 0, x_0 ts)$ belong to K , for every $s \in [0, 1]$.

The radially of K entails that, if we consider the horizontal half line r through ξ_0 given by

$$r = \{((1-s)x_0, 0, 0) \in \mathbb{H} : s \geq 0\}, \quad (11)$$

then it enters K , thereby showing that ξ_0 is non-characteristic.

Now let us consider the case $\xi_0 = (x_0, 0, t_0) \in \partial K$, with $x_0 > 0$ and $t_0 \in (t_{\min}, t_{\max})$. The set K is radial and $\mathbb{H}$ -convex, thus $(0, 0, t_0) \in K$. Hence, there exists $\rho > 0$ so that $B_G((0, 0, t_0), \rho) \subset K$. A similar argument as before applies, and proves the assertion. $\square$

The previous result reveals that a radial set generating an $\mathbb{H}$ -convex family enjoys very nice regularity properties, and we are in the position to apply Theorem 2.2 to the points $\xi_0 \in \partial K$, where K and ξ_0 are as in Proposition 4.3. In what follows, we will see that a sharper regularity result can be proved for such K . To do this, we need the next

Lemma 4.4. *Let $K \subset \mathbb{H}$ be a radial set so that the family $\mathcal{K} = \{K_\tau\}_{\tau \in [0, +\infty)}$, defined by K via dilations as in (7), satisfies condition $(\mathbf{C}_H)$. If $\xi_0 = (x_0, y_0, t_0) \in K$ with $t_0 > 0$ and $r_0 = \|(x_0, y_0)\|_E > 0$, then K contains the solid of revolution S_{ξ_0} , defined as follows:*

$$S_{\xi_0} = \left\{ (x, y, \varphi(r)) \in \mathbb{H} : r \in [r_0/\sqrt{2}, r_0], \|(x, y)\|_E \leq r \right\},$$

$$\text{where } \varphi(r) = \frac{1}{2} \left(t_0 + \sqrt{t_0^2 + 4r_0^2 r^2 - 4r^4} \right).$$

Proof. Without loss of generality, let $\xi_0 = (x_0, 0, t_0) \in K$, with x_0 and t_0 positive, and fix $\eta \in (0, \pi/2)$. The point $(x_0 \cos \eta, x_0 \sin \eta, t_0) \in K$ can be obtained as $\delta_{1/\tau}(\xi_0 \circ \exp v)$, with $v = (\alpha, \beta)$, by solving the system

$$x_0 \cos \eta = \frac{x_0 + \alpha}{\tau}, \quad x_0 \sin \eta = \frac{\beta}{\tau}, \quad t_0 = \frac{t_0 - 2x_0\beta}{\tau^2}.$$

$$\text{This gives} \quad \alpha = x_0(\tau \cos \eta - 1), \quad \beta = \tau x_0 \sin \eta,$$

while τ is the positive solution of the equation

$$t_0 \tau^2 + 2x_0^2 \sin \eta \tau - t_0 = 0, \tag{12}$$

$$\text{i.e.,} \quad \tau = \frac{\sqrt{x_0^4 \sin^2 \eta + t_0^2} - x_0^2 \sin \eta}{t_0}.$$

Since $\tau < 1$, the curve $\tilde{\xi}$ can be expressed as in Proposition 3.10(i):

$$\begin{aligned} \tilde{\xi}(\theta(s)) = & \left(s, \frac{(x_0 - s) \sin \eta}{(1 - \cos \eta)}, \right. \\ & \left. \frac{t_0((\tau - 1)s - x_0(\tau \cos \eta - 1))^2 - 2x_0^2 \tau \sin \eta (x_0 - s)((\tau - 1)s - x_0(\tau \cos \eta - 1))}{\tau^2 x_0^2 (1 - \cos \eta)^2} \right), \end{aligned}$$

for $s \in [x_0 \cos \eta, x_0]$. Note that the third component of the previous expression has the same value for $s = x_0 \cos \eta$ and $s = x_0$.

This implies that, setting $\bar{s} = x_0(\cos \eta + 1)/2$, the vertex of the parabola $\tilde{\xi}$ is

$$(x_V, y_V, t_V) = \left(\frac{x_0(\cos \eta + 1)}{2}, \frac{x_0 \sin \eta}{2}, \frac{(\tau + 1)(t_0(\tau + 1) - 2x_0^2 \tau \sin \eta)}{4\tau^2} \right).$$

Let us concentrate on t_V . To obtain its final expression, we will make repeatedly use of the relation (12):

$$\begin{aligned} t_V &= \frac{(\tau + 1)(t_0(\tau + 1) - 2x_0^2 \tau \sin \eta)}{4\tau^2} \\ &= \frac{t_0\tau^2 + 2t_0\tau + t_0 - 2x_0^2\tau^2 \sin \eta - 2x_0^2\tau \sin \eta}{4\tau^2} \stackrel{(12)}{=} \frac{2t_0\tau^2 + 2t_0\tau - 2x_0^2\tau^2 \sin \eta}{4\tau^2} \\ &= \frac{t_0}{2} + \frac{t_0 - x_0^2\tau \sin \eta}{2\tau} \stackrel{(12)}{=} \frac{t_0}{2} + \frac{2x_0^2\tau \sin \eta + t_0\tau^2 - x_0^2\tau \sin \eta}{2\tau} \\ &= \frac{t_0}{2} + \frac{x_0^2 \sin \eta + t_0\tau}{2} = \frac{t_0}{2} + \frac{\sqrt{x_0^4 \sin^2 \eta + t_0^2}}{2}. \end{aligned}$$

Note that $t_V > t_0$. Condition $(\mathbf{C}_H)$ implies that $(x_V, y_V, t_V) \in K$ for every $\eta \in (0, \pi/2)$. Furthermore, since K is radial and $\|(x_V, y_V)\|_E = x_0\sqrt{\frac{1+\cos \eta}{2}}$, we have that

$$\left(x_0\sqrt{\frac{1+\cos \eta}{2}} \cos \eta', x_0\sqrt{\frac{1+\cos \eta}{2}} \sin \eta', \frac{t_0 + \sqrt{t_0^2 + x_0^4 \sin^2 \eta}}{2} \right) \in K,$$

for every η' , and $\eta \in (0, \pi/2)$. The previous set of points can be described as a surface of revolution, specifically, as the surface obtained by the rotation of a curve $r \rightarrow (r, 0, \varphi(r))$ around the t -axis, where $\varphi: [x_0/\sqrt{2}, x_0] \rightarrow \mathbb{R}$. With a change of variable, we obtain

$$\varphi(r) = \frac{1}{2} \left(t_0 + \sqrt{t_0^2 + 4x_0^2 r^2 - 4r^4} \right).$$

Hence, $\left\{ (x, y, \varphi(r)) \in \mathbb{H} : r \in [r_0/\sqrt{2}, r_0], \|(x, y)\|_E = r \right\} \subset K$;

since K is H -convex (see Remark 2.8(i)), we have the claim. $\square$

The previous lemma provides an “inner constraint” at a point of the boundary of a set K that generates an H -convex family via dilation. One may wonder whether it is possible to provide global information on this set K . Let us start with an example:

Example 4.5. (K cylinder with a Korányi ball as hat) Let

$$K = \{(x, y, t) \in \mathbb{H} : \|(x, y)\|_E \leq 1, t^2 \leq 15\} \cap B_G(e, 2).$$

Let us verify that the set above generates an H-convex family via dilations. Let $\xi = (x, y, t)$, and consider the function $f: \mathbb{H} \rightarrow \mathbb{R}$ defined as follows:

$$f(\xi) = \max \left\{ \|(x, y)\|_E, \frac{1}{2} \|(x, y, t)\|_G \right\}.$$

This function is H-convex, as the maximum of two H-convex functions. Moreover, it is homogeneous, and $\text{lev}_{\leq 1}(f) = K$. Hence, K generates an H-convex family via dilation, and f is the $\mathbb{H}$ -cone-function of vertex e and base ∂K . $\square$

It is clear that the $\mathbb{H}$ -cone-function f in Example 4.5 is a homogeneous norm. In [3] the authors prove that, if N is a homogeneous norm, then $N(x, y, t) \leq N(x, y, 0)$, for every $(x, y, t) \in \mathbb{H}$ (see Lemma 2.10); this inequality provides information about the unitary ball w.r.t. the norm N . Specifically, the following remark holds:

Remark 4.6. Let $N: \mathbb{H} \rightarrow [0, +\infty)$ be a homogeneous norm. Moreover, let us suppose that N is a radial H-convex function. Let $B_N(e, 1) = \{\xi \in \mathbb{H} : N(\xi) \leq 1\}$. Then,

- (i) $B_N(e, 1)$ generates an H-convex family via dilation and the $\mathbb{H}$ -cone-function with vertex e and base $\partial B_N(e, 1)$ is exactly N ;
- (ii) the previous mentioned lemma in [3] implies that

$$\text{Pr}_{\{t=0\}}(B_N(e, 1)) \subset (B_N(e, 1) \cap \{(x, y, 0) \in \mathbb{H}\}),$$

where $\text{Pr}_{\{t=0\}}$ denotes the projection onto the xy -plane. $\square$

As a matter of fact, there exist H-convex, homogeneous and radial functions that are not homogeneous norms (see, for instance, Example 4.9). In the following, we try to shed some light on the properties that a radial set K satisfying (a)–(c) should enjoy in order to have some chance to generate an $\mathbb{H}$ -cone-function.

Lemma 4.7. Let $K \subset \mathbb{H}$, and let $\mathcal{K}$ be as in Lemma 4.4. If $\xi_0 = (x_0, y_0, t_0) \in K$, $t_0 > 0$, and $r_0 = \|(x_0, y_0)\|_E > 0$, then K contains the spherical cap of the Korányi ball B_G^+ , that is given by:

$$B_G^+ = \{\xi = (x, y, t) \in \mathbb{H} : \xi \in B_G(e, \|\xi_0\|_G), t \geq t_0\}.$$

If $t_0 < 0$, then, symmetrically, $B_G^- \subset K$, where B_G^- is defined by setting $t \leq t_0$ in the previous line.

Proof. By the radially of K , we restrict our attention to the profile of K , i.e., the set $\partial K \cap \{y = 0\}$. If $\xi_0 = (x_0, y_0, t_0) \in K$, then $\xi'_0 = (r_0, 0, t_0) \in K$, where $r_0 = \sqrt{x_0^2 + y_0^2}$, and Lemma 4.4 implies that $S_{\xi_0} \subset K$. In particular, K contains the image of the curve

$$r \mapsto \Phi(r) = (r, 0, \varphi(r)), \quad r \in [r_0/\sqrt{2}, r_0],$$

where φ is given in Lemma 4.4. An easy computation shows that, at $(r_0, 0, \varphi(r_0))$, the following relation holds:

$$\varphi'(r_0) = -\frac{2r_0^3}{\varphi(r_0)}.$$

This equality provides a pointwise relation linking r_0 , $\varphi(r_0)$ and $\varphi'(r_0)$. This argument can be repeated for every point in K . As a matter of fact, we are looking for a curve $r \mapsto \Psi(r) = (r, 0, \psi(r)) \in K$, defined on a left neighbourhood $\mathcal{U}$ of r_0 , and satisfying

$$\begin{cases} \psi'(r) = -\frac{2r^3}{\psi(r)}, & r \in \mathcal{U} \\ \psi(r_0) = t_0. \end{cases}$$

The solution of the Cauchy problem above can be extended to $[0, r_0]$ and is given by

$$\psi(r) = \sqrt{-r^4 + r_0^4 + t_0^2},$$

i.e. $(r^4 + \psi^2(r))^{1/4} = (r_0^4 + t_0^2)^{1/4}$. Since the curve Ψ belongs to K , and K is $\mathbb{H}$ -convex and radial, we have the claim. $\square$

We are now ready to prove the first part of Theorem 1.3:

Proof of Theorem 1.3(i). Without loss of generality, let us assume that $\xi_0 = (x_0, 0, 0) \in \partial K$, with $x_0 > 0$. By Proposition 4.3 the point ξ_0 is non-characteristic and the horizontal line r through ξ_0 in (11) enters K . Clearly, there exists $\{\xi_n^+\}_{n>0} \subset K$, with $\xi_n^+ = (x_n, 0, t_n)$, $t_n > 0$, $\xi_n^+ \rightarrow \xi_0$, and there exists $\{\xi_n^-\}_{n>0} \subset K$, with $\xi_n^- = (x_n, 0, t_n)$, $t_n < 0$, $\xi_n^- \rightarrow \xi_0$. Finally, from Lemma 4.7, taking the limit as $n \rightarrow \infty$, by means of a continuity argument and the closedness of K , we prove the claim. $\square$

The previous result has an immediate and nice application to homogeneous norms:

Remark 4.8. Let N be a homogeneous and radial norm. Since $B_N(e, 1)$ generates an $\mathbb{H}$ -convex family via dilations, then $B_G(e, 1) \subset B_N(e, 1)$. $\square$

Now we are ready to prove the second part of Theorem 1.3, that generalizes Remark 4.6:

Proof of Theorem 1.3(ii). We argue by contradiction. Let $\bar{\xi} = (\bar{x}, 0, 0) \in \partial K$, with $\bar{x} > 0$, and suppose that $\xi_0 = (x_0 \bar{x}, 0, t_0) \in \partial K$ for some $x_0 > 1$ and $t_0 > 0$. Since $\delta_{1/\bar{x}} \bar{\xi} = (1, 0, 0) \in \partial(\delta_{1/\bar{x}} K)$ and $\delta_{1/\bar{x}} \xi_0 = (x_0, 0, t_0/\bar{x}^2) \in \partial(\delta_{1/\bar{x}} K)$, then, by Remark 2.8(ii), there is no loss of generality if we take $\bar{x} = 1$.

We will prove that the condition $\bar{\xi} = (1, 0, 0) \in \partial K$ and $\xi_0 = (x_0, 0, t_0) \in \partial K$, for some $x_0 > 1$ and $t_0 > 0$, leads to a contradiction. From Theorem 1.3 we get that

$$\xi_{t,\eta} = (r \cos \eta, r \sin \eta, t) \in \partial B_G(e, 1) \subset K,$$

with $r = \sqrt[4]{1 - t^2}$, and for some fixed $-1 < t < 0$ and η . Taking in (10) $\xi_0 = (x_0, 0, t_0) \in K$, the point $\xi_{t,\eta}$ can be obtained as the second point in (10) by considering

$$\alpha = r\tau \cos \eta - x_0, \quad \beta = r\tau \sin \eta,$$

and $\tau > 0$ is a solution of the equation

$$-t\tau^2 - 2x_0 r\tau \sin \eta + t_0 = 0 \tag{13}$$

(we recall that $t < 0$). In order to do that, let us choose $\eta \in (0, \pi)$ so that $\sin \eta = \frac{\sqrt{-t_0 t}}{x_0 r}$; for this value of η we obtain $\tau = \sqrt{\frac{t_0}{-t}}$. In this case, the curve $\tilde{\xi}$ in Proposition 3.10(i) is given by

$$\tilde{\xi}(\theta(s)) = \left(s, \frac{r(x_0 - s) \sin \eta}{x_0 - r \cos \eta}, \frac{t_0((\tau - 1)s - r\tau \cos \eta + x_0)^2 - 2x_0 r \tau \sin \eta (x_0 - s)((\tau - 1)s - r\tau \cos \eta + x_0)}{\tau^2(x_0 - r \cos \eta)^2} \right),$$

where $s \in [r \cos \eta, x_0]$. A tedious calculation gives that the curve $s \mapsto \tilde{\xi}(\theta(s))$ intersects the xy -plane if and only if $s = \bar{s} = \frac{t_0 r \cos \eta - \tau t x_0}{t_0 - \tau t}$. Let us prove that the third component of $\xi_{t,\eta}$ can be chosen so that $\|\tilde{\xi}(\theta(\bar{s}))\|_G > 1$. In fact, taking into account our choice of r and η (and hence of τ defined by (13)), we have that

$$\begin{aligned} \|\tilde{\xi}(\theta(\bar{s}))\|_G^2 &= \left(\frac{t_0 r \cos \eta - \tau t x_0}{t_0 - \tau t} \right)^2 + \left(\frac{t_0 r \sin \eta}{t_0 - \tau t} \right)^2 \\ &= \frac{t_0 r^2 + 2\sqrt{t_0} \sqrt{-t} \sqrt{x_0^2 r^2 + t_0 t} - t x_0^2}{(\sqrt{t_0} + \sqrt{-t})^2} \\ &= \frac{t_0 \sqrt{1 - t^2} + 2\sqrt{t_0} \sqrt{-t} \sqrt{x_0^2 \sqrt{1 - t^2} + t_0 t} - t x_0^2}{(\sqrt{t_0} + \sqrt{-t})^2}. \end{aligned}$$

If $t \rightarrow 0^-$ we have

$$\|\tilde{\xi}(\theta(s_0))\|_G^2 = \frac{t_0 + 2x_0 \sqrt{t_0} \sqrt{-t} + o(\sqrt{-t})}{t_0 + 2\sqrt{t_0} \sqrt{-t} + o(\sqrt{-t})} \rightarrow 1^+,$$

since $t_0 > 0$ and $x_0 > 1$ are fixed. Hence, if t is negative and small enough, we find a point of the parabola $\tilde{\xi}$ that belongs to the xy -plane, but is not in K . This contradicts condition $(\mathbf{C}_H)$ and the proof is finished. $\square$

The next example highlights that there exist $\mathbb{H}$ -convex radial homogeneous functions that are not homogeneous norms.

Example 4.9. Let us consider the function $f: \mathbb{H} \rightarrow \mathbb{R}$ defined as follows:

$$f(x, y, t) = \begin{cases} f_1(x, y, t), & t \geq 0 \\ f_2(x, y, t), & t < 0, \end{cases}$$

where $f_1, f_2: \mathbb{H} \rightarrow \mathbb{R}$ are the $\mathbb{H}$ -convex radial homogeneous functions given by

$$f_1(x, y, t) = \max \left\{ \|(x, y)\|_E, \frac{1}{2} \|(x, y, t)\|_G \right\}, \quad f_2(x, y, t) = \|(x, y, t)\|_G.$$

The function f is trivially $\mathbb{H}$ -convex on the half-space $\overline{\mathbb{H}^+} = \{(x, y, t) : t \geq 0\}$, as well as $\mathbb{H}^- = \{(x, y, t) : t < 0\}$. Let us prove that f is $\mathbb{H}$ -convex on the whole $\mathbb{H}$,

i.e., $f|_r$ is a convex function for every r , where r denotes a horizontal line. Note that, if $r \subset \overline{\mathbb{H}^+}$, or $r \subset \mathbb{H}^-$, then there is nothing to prove. Thus, we can consider the horizontal lines r having the parametric form

$$s \mapsto (x_0 + vs, y_0 + v's, ws), \quad s \in \mathbb{R},$$

where $x_0, y_0, v, v', w \in \mathbb{R}$, $(x_0, y_0) \neq (0, 0)$, $(v, v') \neq (0, 0)$, $w \neq 0$. Since the function f is radial, without loss of generality we can assume that $y_0 = 0$, and set $x_0 > 0$. Let us consider the restriction $f|_r: \mathbb{R} \rightarrow \mathbb{R}$; if $w > 0$, then

$$\begin{aligned} f|_r(s) &= f(x_0 + vs, v's, ws) \\ &= \begin{cases} \max \left\{ \|(x_0 + vs, v's)\|_E, \frac{1}{2} \|(x_0 + vs, v's, ws)\|_G \right\}, & s \geq 0 \\ \|(x_0 + vs, v's, ws)\|_G, & s < 0. \end{cases} \end{aligned}$$

It is a simple exercise to prove that, if $g_1, g_2: \mathbb{R} \rightarrow \mathbb{R}$ are convex functions, and $g: \mathbb{R} \rightarrow \mathbb{R}$ is the function given by

$$g(s) = \begin{cases} g_1(s), & s < 0 \\ g_2(s), & s \geq 0, \end{cases}$$

then, under the assumptions that g is continuous at $s = 0$, $D_-g(0)$, $D_+g(0)$ exist and $D_-g(0) \leq D_+g(0)$, the function g is convex on the whole $\mathbb{R}$; this is true, in particular, if $Dg(0)$ is well defined.

In our case, the function f is $\mathbb{H}$ -convex on $\overline{\mathbb{H}^+}$ and $\mathbb{H}^-$, therefore $f|_r$ is convex on $(-\infty, 0)$, and $[0, +\infty)$. Furthermore, $f|_r$ is differentiable at $s = 0$. Indeed, simple computations show that, whenever $s \rightarrow 0$, then, if $v > 0$, $f|_r(s) = x_0 + sv + o(s)$, while, if $v = 0$, $f|_r(s) = x_0 + o(s)$. In both cases, $Df|_r(0)$ does exist. This proves that $f|_r$ is convex on $\mathbb{R}$. The case $w < 0$ can be treated in the same way. $\square$

The following example is crucial: it shows that there exists a compact and radial set, $\mathbb{H}$ -convex, but not $\mathbb{R}^3$ -convex, whose boundary is not the base of an $\mathbb{H}$ -cone-function with vertex e .

Example 4.10. Let us consider the function $f_1: \mathbb{H} \rightarrow \mathbb{R}$ defined by

$$f_1(x, y, t) = \left((x^2 + y^2)^2 + 2 + \frac{1}{2} \sin t \right)^{1/4} - 2^{1/4}.$$

This is an $\mathbb{H}$ -convex function studied in [4]. Now let us define the constants c and c_0 by

$$0 < c_0 = \sqrt[4]{(1 + 2^{1/4})^4 - 2} < c = \sqrt[4]{(1 + 2^{1/4})^4 - 3/2},$$

and consider the point $\bar{\xi} = (c, 0, 3\pi/2)$. Define $f: \mathbb{H} \rightarrow \mathbb{R}$ by

$$f(\xi) = \max \left\{ f_1(\xi), \frac{\|\xi\|_G}{\|\bar{\xi}\|_G} \right\},$$

and consider the set $K = \text{lev}_{\leq 1}(f)$. The function f is trivially $\mathbb{H}$ -convex, $f(e) = 0$ and $f|_{\partial K} = 1$. Clearly, $e \in \text{int}(K)$; moreover, K is $\mathbb{H}$ -convex and compact. It is easy to see that

$$\xi_0 = (c_0, 0, 0) \in \partial K, \quad \xi_1 = (c_0, 0, \pi/2) \notin K, \quad \xi_2 = (c_0, 0, \pi) \in \partial K;$$

therefore K is not $\mathbb{R}^3$ -convex. Finally, we note that $\bar{\xi} \in \partial K$ and

$$\text{Pr}_{\{t=0\}}(\bar{\xi}) = (c, 0, 0) \notin (K \cap \{(x, y, 0) \in \mathbb{H}\}) = \{(x, y, 0) \in \mathbb{H} : \|(x, y)\|_E \leq c_0\};$$

hence, by Theorem 1.3, K does not generate an $\mathbb{H}$ -convex family via dilation. The previous arguments imply that there is neither an $\mathbb{H}$ -cone-function, nor a cone-function, with vertex e and base ∂K . $\square$

References

- [1] G. Arena, A. Caruso, R. Monti: *Regularity properties of H -convex sets*, J. Geom. Anal. 22 (2012) 583–602.
- [2] Z. M. Balogh, A. Calogero, and A. Kristály: *Sharp comparison and Aleksandrov-type maximum principles in Heisenberg groups*, J. Functional Analysis 269 (2015) 2669–2708.
- [3] Z. M. Balogh, K. Fässler, H. Sobrino: *Isometric embeddings into Heisenberg groups*, Geom. Dedicata 195 (2019) 163–192.
- [4] Z. M. Balogh, M. Rickly: *Regularity of convex functions on Heisenberg groups*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. 2 (2003) 847–868.
- [5] T. Bieske, F. Dragoni, J. Manfredi: *The Carnot-Carathéodory distance in infinite Laplacian*, J. Geom. Anal. 19 (2009) 737–754.
- [6] L. A. Caffarelli: *Interior $W^{2,p}$ -estimates for solutions of the Monge-Ampère equation*, Ann. Math. 131 (1990) 135–150.
- [7] L. A. Caffarelli: *Some regularity properties of solutions of Monge-Ampère equation*, Comm. Pure Appl. Math. 44 (1991) 965–969.
- [8] A. Calogero, G. Carcano, R. Pini: *On weakly H -quasiconvex functions on the Heisenberg group*, J. Convex Analysis 15 (2008) 753–766.
- [9] A. Calogero, R. Pini: *Horizontal normal map on the Heisenberg group*, J. Nonlinear Convex Analysis 12(2) (2011) 287–307.
- [10] L. Capogna, D. Danielli, S. Pauls, J. T. Tyson: *An Introduction to the Heisenberg Group and the Sub-Riemannian Isoperimetric Problem*, Birkhäuser, Basel (2007).
- [11] L. Capogna, D. Maldonado: *A note on the engulfing property and the $\Gamma^{1+\alpha}$ -regularity of convex functions in Carnot groups*, Proc. Amer. Math. Soc. 134 (2006) 3191–3199.
- [12] D. Danielli, N. Garofalo, D. M. Nhieu: *Notions of convexity in Carnot groups*, Comm. Anal. Geom. 11 (2003) 263–341.
- [13] W. Fenchel: *Convex cone, sets, and functions*, Princeton University Press, Princeton (1953).
- [14] C. E. Gutiérrez: *The Monge-Ampère Equation*, Birkhäuser, Boston (2001).
- [15] D. Morbidelli: *On the inner cone property for convex sets in two-step Carnot groups, with applications to monotone sets*, (2018) arXiv:1808.06513.