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On Convexity and ψ-Uniform Convexity of G-Invariant Functions on an Eaton Triple

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An Eaton triple is an algebraic system related to a decomposition statement for vectors of an inner product space V and to some special inner product inequality connected with this decomposition. The Spectral Decomposition for the space of Hermitian matrices associated with Fan-Theobald's trace inequality is a typical example of such a situation. In this paper, for a given Eaton triple (V, G, D) and for a function $F: V \to \mathbb{R}$, invariant with respect to the group G acting on V, we study the problem of extending convexity of F from the convex cone $D \subset V$ to the space V. In our approach we reduce the problem from E-system (V, G, D) to its subsystem (W, H, E). Thus we obtain some results related to theorems due to J. von Neumann, C. Davis, A. S. Lewis and T.-Y. Tam et al. Analogous problems are discussed for ψ-uniform convex functions and c-strongly convex functions. Finally, applications are given for matrix spaces endowed with the structure of Eaton triple.

Keywords: Convex function, eigenvalues, singular value, G-invariant function, G-majorization, Eaton triple, normal decomposition system, normal map, ψ-uniformly convex function.

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1. Introduction

In this paper we deal with relationships between the ψ-convexity of a group invariant real function on a linear space and the ψ-convexity of this function on its subspace. In particular, we study corresponding problems for c-strongly convex functions. To do so, we utilize the theory of group majorizations and of normal decomposition systems and Eaton triples (E-systems).

In this expository section we present notation, relevant definitions and theorems. We also demonstrate a brief history of similar problems for convex functions.

Throughout, the symbol $\mathbb{M}_n(\mathbb{C})$ stands for the space of $n \times n$ complex matrices, and $\mathbb{U}_n$ stands for the group of $n \times n$ unitary matrices.

By $s(X) = (s_1(X), \dots, s_n(X))$ we denote the n-vector of *singular values* of a matrix $X \in \mathbb{M}_n(\mathbb{C})$, where $s_i(X)$ is the ith largest singular value of X, $i = 1, \dots, n$, i.e., $s_i(X) = \lambda_i(X^*X)^{1/2}$ is the ith largest eigenvalue of the hermitian matrix $(X^*X)^{1/2}$, $i = 1, \dots, n$.

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We say that a norm $|||\cdot|||\colon \mathbb{M}_n(\mathbb{C}) \to [0,\infty)$ is *unitarily invariant*, if

$$|||U_1 X U_2||| = |||X||| \quad \text{for all } X \in \mathbb{M}_n(\mathbb{C}) \text{ and } U_1, U_2 \in \mathbb{U}_n \tag{1}$$

(cf. [1, p. 91]). Equation (1) can be interpreted as the G-invariance of the norm $|||\cdot|||$, where G is the group of unitary equivalences $U_1(\cdot)U_2$ with $U_1, U_2 \in \mathbb{U}_n$.

A norm $\|\cdot\|\colon \mathbb{R}^n \to [0,\infty)$ is said to be a *symmetric gauge function* if $\|\cdot\|$ is invariant with respect to the group $\mathbb{GP}_n$ of $n \times n$ *generalized permutation matrices* (cf. [1, [p. 44]). Recall that each row and column of an $n \times n$ generalized permutation matrix contains unique nonzero element equal to 1 or -1.

By $\mathbb{H}_n$ we denote the linear space of $n \times n$ hermitian matrices.

By $\lambda(X) = (\lambda_1(X), \ldots, \lambda_n(X))$ we denote the n-vector of *eigenvalues* of an hermitian matrix $X \in \mathbb{H}_n$, where $\lambda_i(X)$ is the ith largest eigenvalue of X, $i = 1, \ldots, n$.

A function $F\colon \mathbb{H}_n \to \mathbb{R}$ is said to be *weakly unitarily invariant* on $\mathbb{H}_n$, if

$$F(UXU^*) = F(X) \quad \text{for all } X \in \mathbb{H}_n \text{ and } U \in \mathbb{U}_n. \tag{2}$$

Property (2) says that F is G-invariant, where G is the group of unitary similarities $U(\cdot)U^*$ with $U \in \mathbb{U}_n$.

Below we review some relevant results related to the problem of characterizations of group invariant functions.

Theorem 1.1. (von Neumann [24]) *Let $|||\cdot|||\colon \mathbb{M}_n(\mathbb{C}) \to [0,\infty)$ be a unitarily invariant norm on $\mathbb{M}_n(\mathbb{C})$. Then*

$$|||X||| = \|s(X)\| \quad \textit{for } X \in \mathbb{M}_n(\mathbb{C}),$$

for some symmetric gauge function $\|\cdot\|$ on $\mathbb{R}^n$.

Theorem 1.2. (Davis [2]) *Let $F\colon \mathbb{H}_n \to \mathbb{R}$ be a weakly unitarily invariant convex function on $\mathbb{H}_n$. Then F is of the form*

$$F(X) = f(\lambda(X)) \quad \textit{for } X \in \mathbb{H}_n,$$

for some convex function f on $\mathbb{R}^n$ invariant under coordinate permutations.

Let $\mathbf{g} = \mathbf{t} \oplus \mathbf{p}$ be a Cartan decomposition of a real semisimple Lie algebra $\mathbf{g}$. Here $\oplus$ means the direct sum with respect to the Killing form of $\mathbf{g}$, and $\mathbf{p}$ is the Cartan subspace of $\mathbf{g}$. By K we denote a maximal compact subgroup of the adjoint group $\mathrm{Int}\,(\mathbf{g})$. Here $\mathbf{t}$ is a subalgebra of $\mathbf{g}$ tangent to K. $\mathrm{Ad}\,(K)$ is a subgroup of the orthogonal group on $\mathbf{p}$ (with respect to the Killing form of $\mathbf{g}$). By $\mathbf{a}$ we denote a maximal Abelian subalgebra in $\mathbf{p}$. Now, D is the closed Weyl chamber and W is the Weyl group acting on $\mathbf{a}$. For $x \in \mathbf{p}$, by $\gamma(x)$ we denote the unique element of the set $D \cap (\mathrm{Ad}\,K)x$.

Theorem 1.3. (Lewis [11, Theorem 2.5]) *Let $f\colon \mathbf{a} \to [-\infty,\infty]$ be a W-invariant function. Then f is convex on $\mathbf{a}$ iff $f \circ \gamma$ is convex on $\mathbf{p}$.*

Theorem 1.4. (Lewis [11, Theorem 3.7]) *AdK-invariant norms on $\mathbf{p}$ are those functions F of the form $f \circ \gamma$, where f is a W-invariant norm on $\mathbf{a}$.*

Definition 1.5. Let $(V, \langle \cdot, \cdot \rangle)$ be a real inner product space with $\dim V < \infty$. Let G be a closed subgroup of the orthogonal group $O(V)$ acting on V. Let D be a closed convex cone in V. We say that the system (V, G, D) is an *Eaton triple* (for short, E-system), if

(A1) $D \cap Gx$ is nonempty for each $x \in V$, where $Gx = \{gx : g \in G\}$,

(A2) $\langle gx, y \rangle \leq \langle x, y \rangle$ for all $x, y \in D$ and $g \in G$.

It is known (see [13, p. 14]) that for an E-system (V, G, D), the intersection $D \cap Gx$ is a singleton set for each $x \in V$.

Definition 1.6. If axioms **(A1)-(A2)** are met then the unique element of $D \cap Gx$ is denoted by $x_\downarrow$. Thus the map $(\cdot)_\downarrow \colon V \to D$, $V \ni x \to x_\downarrow \in D$, is defined. The triple $(V, G, (\cdot)_\downarrow)$ is called a *normal decomposition system*, and $(\cdot)_\downarrow$ is called the *normal map* of E-system (V, G, D) (see [11]).

It is easy to verify that for each $x \in V$,

$$x = gx_\downarrow \quad \text{for some } g \in G, \tag{3}$$

$$(x_\downarrow)_\downarrow = x_\downarrow, \tag{4}$$

$$(gx)_\downarrow = x_\downarrow \quad \text{for } g \in G. \tag{5}$$

Property (3) says that each $x \in V$ has decomposition $gx_\downarrow$. And (4) is the idempotence of the normal map $(\cdot)_\downarrow$. while (5) asserts that $(\cdot)_\downarrow$ is G-invariant map.

We define the preorder of G-majorization $\prec_G$ on V, as follows.

Definition 1.7. For the group $G \subset O(V)$ and $x, y \in V$, we say that y is G-*majorized* by x, and write $y \prec_G x$, if $y \in \operatorname{conv} Gx$, i.e., y belongs to the convex hull of the G-orbit $Gx = \{gx : g \in G\}$.

Definition 1.8. A real function $F \colon I \to \mathbb{R}$ is said to be *G-increasing* (resp. *G-decreasing*) on a set $I \subset V$, if for $x, y \in I$,

$$y \prec_G x \quad \text{implies} \quad F(y) \leq F(x) \ \ (\text{resp. } F(y) \geq F(x)).$$

Under the axioms **(A1)–(A2)**, for $x, y \in V$,

$$y \prec_G x \quad \text{iff} \quad y_\downarrow \prec_G x_\downarrow \quad \text{iff} \quad \langle y_\downarrow, z \rangle \leq \langle x_\downarrow, z \rangle \quad \text{for all } z \in D. \tag{6}$$

Moreover,

$$\langle x_\downarrow, z \rangle = \max_{g \in G} \langle gx, z \rangle \quad \text{for } x \in V \text{ and } z \in D. \tag{7}$$

It follows from (6)–(7) that

$$(x + y)_\downarrow \prec_G x_\downarrow + y_\downarrow \quad \text{for } x, y \in V, \tag{8}$$

$$(px)_\downarrow = px_\downarrow \quad \text{for } x \in V \text{ and } 0 \leq p \in \mathbb{R}. \tag{9}$$

Example 1.9. We define the notion of *majorization* $\prec$ on $\mathbb{R}^n$ [12]. For $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$, we say that y is *majorized* by x, and write $y \prec x$, if the sum of k largest entries of y does not exceed the sum of k largest entries of x for each $k = 1, \ldots, n$, with equality for $k = n$. We take

$$\begin{aligned} V &= \mathbb{R}^n \text{ with the inner product } \langle x, y\rangle = \sum_{i=1}^n x_i y_i \text{ for } x = (x_1, \ldots, x_n) \in \mathbb{R}^n \text{ and } \\ &\quad y = (y_1, \ldots, y_n) \in \mathbb{R}^n, \\ G &= \mathbb{P}_n = \text{ the group of all } n \times n \text{ permutation matrices}, \\ D &= \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_1 \geq \ldots \geq x_n\} \end{aligned}$$

(see [3, 4]). Then (V, G, D) is an E-system, and for $x, y \in \mathbb{R}^n$: $y \prec_G x$ iff $y \prec x$.

Example 1.10. We introduce the notion of *absolutely weak majorization* $\prec_{aw}$ on $\mathbb{R}^n$. Let $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$. We say that y is *absolutely weakly majorized* by x, and write $y \prec_{aw} x$, if the sum of k largest entries of $|y| = (|y_1|, \ldots, |y_n|)$ does not exceed the sum of k largest entries of $|x| = (|x_1|, \ldots, |x_n|)$ for each $k = 1, \ldots, n$.

If the entries of x and y are nonnegative then we write $y \prec_w x$ in place of $y \prec_{aw} x$. The preorder $\prec_w$ is called *weak majorization* [12]. We consider

$$\begin{aligned} V &= \mathbb{R}^n \text{ with the inner product } \langle x, y\rangle = \sum_{i=1}^n x_i y_i \text{ for } x = (x_1, \ldots, x_n) \in \mathbb{R}^n \text{ and } \\ &\quad y = (y_1, \ldots, y_n) \in \mathbb{R}^n, \\ G &= \mathbb{GP}_n = \text{ the group of all } n \times n \text{ generalized permutation matrices}, \\ D &= \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_1 \geq \ldots \geq x_n \geq 0\} \end{aligned}$$

(see [3, 4]). Then (V, G, D) is an E-system, and for $x, y \in \mathbb{R}^n$: $y \prec_G x$ iff $y \prec_{aw} x$.

Definition 1.11. For a given E-system $\mathcal{E} = (V, G, D)$, we define $V_0 = \operatorname{span} D$ and $G_0 = \{g \in G : gV_0 = V_0\}$. If in addition $\mathcal{E}_0 = (V_0, G_{0|V_0}, D)$ is E-system, then $\mathcal{E}_0$ is called *reduced system (triple)* of $\mathcal{E}$.

For notation simplicity, we write (V_0, G_0, D) in place of $(V_0, G_{0|V_0}, D)$.

In the previous context of a semisimple Lie algebra $\mathbf{g} = \mathbf{t} \oplus \mathbf{p}$, the triple $\mathcal{E} = (\mathbf{p}, \operatorname{Ad} K, D)$ is an E-system. In addition, $(\mathbf{a}, W, D)$ is the reduced system of $\mathcal{E}$ (cf. [11, [Propositions 3.3-3.4]).

Theorem 1.12. (Tam and Hill [20, Theorem 4]) *Assume that (V, G, D) is an E-system and (V_0, G_0, D) is its reduced system. A G-invariant function $F\colon V \to \mathbb{R}$ is convex on V iff the restriction $F_{|V_0}$ is convex on V_0.*

Theorem 1.13. (Tam and Hill [20, Theorem 6]) *Assume that (V, G, D) is an E-system and (V_0, G_0, D) is its reduced system. A G-invariant function $F\colon V \to \mathbb{R}$ is a norm on V iff the restriction $F_{|V_0}$ is a norm on V_0.*

Definition 1.14. Assume (V, G, D) is an E-system. Let H be a closed subgroup of G, W be an H-invariant subspace of V (with the inherited inner product), and $E \subset W$ be a closed convex cone included in D. The triple (W, H, E) is called a *subsystem* of (V, G, D), if (W, H, E) is an E-system.

Hereafter we write (W, H, E) instead of $(W, H|_W, E)$. The reduced systems of E-systems are special examples of subsystems.

Below we quote some characterization of subsystems of E-systems (see [14, Theorem 3] and [13, Theorem 3.2], cf. [15, Theorem 3.6]).

Theorem 1.15. ([14, Theorem 3]) *Suppose (V, G, D) is an Eaton system. Let H be a closed subgroup of G, let W be an H-invariant subspace of V, and let $E \subset W$ be a closed convex subcone of D. Let P and Q be the orthoprojectors, respectively, from V onto W and from* $\operatorname{span} D$ *onto* $\operatorname{span} E$. *Assume $QD = E$. Then the following statements are mutually equivalent:*

(a) (W, H, E) *is a subsystem of* (V, G, D).

(b) $$Px \preceq_H Qx_\downarrow \quad \text{for } x \in V. \tag{10}$$

(c) $$W \subset \bigcup_{g \in G} gE, \tag{11}$$

and, in addition, the orderings $\preceq_H$ and $\preceq_G$ coincide on W, i.e.

$$y \preceq_H x \ \text{ iff } \ y \preceq_G x \ \text{ for } \ x, y \in W. \tag{12}$$

(d) $$W = \bigcup_{h \in H} hE. \tag{13}$$

On account of Theorem 1.15, given a subsystem $\mathcal{F} = (W, H, E)$ of an E-system $\mathcal{E} = (V, G, D)$, the normal map $(\cdot)_\downarrow$ for $\mathcal{F}$ is the restriction to W of the one for $\mathcal{E}$.

Remark 1.16. In Theorem 1.15, if $D = E$, then $\operatorname{span} D = \operatorname{span} E$ and the operator Q is the identity on $\operatorname{span} D$. Therefore the triple (W, H, E) becomes (W, H, D), and inequality (10) takes the form:

$$Px \preceq_H x_\downarrow \quad \text{for } x \in V. \tag{14}$$

2. E-systems and convex functions

In this section, for a given G-invariant function F we study the problem of expansion of the (usual) convexity of F from a linear space W to a linear space V such that $W \subset V$. To this end, we equip both the spaces with the structure of E-systems $\mathcal{E} = (V, G, D)$ and $\mathcal{F} = (W, H, D)$, so that $\mathcal{F}$ is a subsystem of $\mathcal{E}$. We *do not assume* that $\mathcal{F}$ is the reduced triple of $\mathcal{E}$ as in Theorems 1.12 and 1.13.

We begin our discussions with two basic lemmas. The former shows a relationship between G-invariant functions and the normal map of an E-system (V, G, D). The latter gives the equivalence of the G-increasity and G-convexity.

Lemma 2.1. *Let (V, G, D) be an E-system with its normal map $(\cdot)_\downarrow$. Let $F \colon V \to \mathbb{R}$ be a function. Then the following statements are equivalent:*

(i) *F is G-invariant on V,*

(ii) *F is of the form $F(x) = f(x_\downarrow)$ for $x \in V$, where $f \colon D \to \mathbb{R}$ is a real function on D.*

(iii) *$F(x) = F(x_\downarrow)$ for $x \in V$.*

Proof. (i)$\Rightarrow$ (ii). Assume that F is G-invariant on V. We define $f = F_{|D}$.

Take any $x \in V$. There exists $g_0 \in G$ such that $x = g_0 x_\downarrow$ (see (3)). Therefore $F(x) = F(g_0 x_\downarrow) = F(x_\downarrow) = F_{|D}(x_\downarrow) = f(x_\downarrow)$, because $x_\downarrow \in D$. Thus we have $F(x) = f(x_\downarrow)$. The arbitrariness of $x \in V$ gives $F = f((\cdot)_\downarrow)$, as required.

(ii) $\Rightarrow$ (iii). Suppose that $F(x) = f(x_\downarrow)$ for $x \in V$, where $f \colon D \to \mathbb{R}$ is a real function on D. In particular, for $x \in D \subset V$, we have $x_\downarrow = x$, and therefore $F(x) = f(x_\downarrow) = f(x)$. This means $F_{|D} = f$. In consequence, $F(x) = F_{|D}(x_\downarrow) = F(x_\downarrow)$ for $x \in V$, as claimed.

(iii) $\Rightarrow$ (i). Assume $F(x) = F(x_\downarrow)$ for $x \in V$. Then we get $F(gx) = F((gx)_\downarrow) = F(x_\downarrow) = F(x)$ for any $x \in V$ and $g \in G$, because $(\cdot)_\downarrow$ is G-invariant. Hence F is G-invariant. This completes the proof. □

Definition 2.2. We say that a function $F \colon V \to \mathbb{R}$ is *G-convex on a set $I \subset V$*, if

$$F(p_1 g_1 x + \ldots + p_n g_n x) \le p_1 F(g_1 x) + \ldots + p_n F(g_n x)$$

for $x \in I$, $n \in \mathbb{N}$, $g_1, \ldots, g_n \in G$ and $p_1, \ldots, p_n \ge 0$ with $p_1 + \ldots + p_n = 1$.

Evidently, (usual) convexity of a real function on V implies its G-convexity on V.

Lemma 2.3. *Let (V, G, D) be an E-system and $F \colon V \to \mathbb{R}$ be a G-invariant function. The following statements are equivalent:*

(i) *F is G-increasing on V.*

(ii) *F is G-convex on V.*

Proof. (i) $\Rightarrow$ (ii). Assume that F is G-increasing on V. Fix any $x \in V$, $n \in \mathbb{N}$, $g_1, \ldots, g_n \in G$ and $p_1, \ldots, p_n \ge 0$ with $p_1 + \ldots + p_n = 1$. Denote $y = p_1 g_1 x + \ldots + p_n g_n x$. Then $y \prec_G x$. Hence $F(y) \le F(x)$, i.e.,

$$F(p_1 g_1 x + ... + p_n g_n x) \le F(x) = p_1 F(x) + ... + p_n F(x) = p_1 F(g_1 x) + ... + p_n F(g_n x).$$

The last equality is due to the G-invariance of F. Thus F is G-convex on V.

(ii) $\Rightarrow$ (i). Assume that F is G-convex on V. We shall show that F is G-increasing on V. Let $x, y \in V$ be such that $y \prec_G x$. Then $y = p_1 g_1 x + \ldots + p_n g_n x$ for some $n \in N$, $g_1, \ldots, g_n \in G$ and $p_1, \ldots, p_n \ge 0$ with $p_1 + \ldots + p_n = 1$. Therefore

$$\begin{aligned} F(y) &= F(p_1 g_1 x + \ldots + p_n g_n x) \le p_1 F(g_1 x) + \ldots + p_n F(g_n x) \\ &= p_1 F(x) + \ldots + p_n F(x) = F(x), \end{aligned}$$

which completes the proof. □

In the next theorem, for a given E-system (V, G, D), we expand the property of (usual) convexity of a G-invariant function F from the convex cone $D \subset V$ to the linear space V.

Theorem 2.4. *Let (V, G, D) be an E-system, and $F\colon V \to \mathbb{R}$ be a G-invariant function. Then the following four statements are mutually equivalent:*

(i) *F is convex on V,*

(ii) *F is convex on D, and F is G-convex on V,*

(iii) *F is convex on D, and F is G-increasing on V,*

(iv) *F is convex on D, and F is G-increasing on D.*

Proof. (i) $\Rightarrow$ (ii). Assume that F is convex on V. Since $D \subset V$, it is evident that F is convex on D. Additionally, the convexity of F on V ensures the G-convexity of F on V.

(ii) $\Rightarrow$ (iii). We appeal to Lemma 2.3, to obtain the required result.

(iii) $\Rightarrow$ (iv). This is clear by the inclusion $D \subset V$.

(iv) $\Rightarrow$ (i). Let F be convex on D, and F be G-increasing on D. We shall prove that F is convex on V. Let $x_1, \ldots, x_n \in V$ be arbitrarily fixed.

Take any $p_1 \geq 0, \ldots, p_n \geq 0$ such that $\sum\limits_{i=1}^{n} p_i = 1$. Since

$$\sum_{i=1}^{n} p_i x_i = g_0 \left(\sum_{i=1}^{n} p_i x_i \right)_{\downarrow} \quad \text{for some } g_0 \in G$$

(see (3)), in light of the G-invariance of F, we see that

$$F\left(\sum_{i=1}^{n} p_i x_i\right) = F\left(g_0\left(\sum_{i=1}^{n} p_i x_i\right)_{\downarrow}\right) = F\left(\left(\sum_{i=1}^{n} p_i x_i\right)_{\downarrow}\right). \tag{15}$$

However, by (8) and (9) we have

$$\left(\sum_{i=1}^{n} p_i x_i\right)_{\downarrow} \prec_G \sum_{i=1}^{n} p_i (x_i)_{\downarrow} \tag{16}$$

with $\left(\sum\limits_{i=1}^{n} p_i x_i\right)_{\downarrow} \in D$ and $\sum\limits_{i=1}^{n} p_i (x_i)_{\downarrow} \in D$. In addition, F is G-increasing on D.

Therefore we get
$$F\left(\left(\sum_{i=1}^{n} p_i x_i\right)_{\downarrow}\right) \leq F\left(\sum_{i=1}^{n} p_i (x_i)_{\downarrow}\right). \tag{17}$$

But F is convex on D, so we have

$$F\left(\sum_{i=1}^{n} p_i (x_i)_{\downarrow}\right) \leq \sum_{i=1}^{n} p_i F\left((x_i)_{\downarrow}\right). \tag{18}$$

By combining (15), (17) and (18) and using the G-invariance of F, we obtain

$$F\left(\sum_{i=1}^{n} p_i x_i\right) \leq \sum_{i=1}^{n} p_i F\left((x_i)_{\downarrow}\right) = \sum_{i=1}^{n} p_i F\left(x_i\right). \tag{19}$$

This proves that F is convex on V. $\square$

Corollary 2.5. *Let (V, G, D) be an E-system and $f\colon D \to \mathbb{R}$ be a function. Then the following four statements are mutually equivalent:*

(i) $f((\cdot)_\downarrow)$ *is convex on V,*

(ii) $f((\cdot)_\downarrow)$ *is convex on D, and $f((\cdot)_\downarrow)$ is G-convex on V,*

(iii) $f((\cdot)_\downarrow)$ *is convex on D, and $f((\cdot)_\downarrow)$ is G-increasing on V,*

(iv) $f((\cdot)_\downarrow)$ *is convex on D, and $f((\cdot)_\downarrow)$ is G-increasing on D.*

Proof. Combine Lemma 2.1 and Theorem 2.4 with $F = f((\cdot)_\downarrow)$. □

We are now in a position to establish a general analogue to Theorems 1.1–1.13 in the context of E-systems.

Theorem 2.6. *Let $\mathcal{E} = (V, G, D)$ be an E-system and $\mathcal{F} = (W, H, D)$ be a subsystem of $\mathcal{E}$. Let $F\colon V \to \mathbb{R}$ be a G-invariant function. Then the following two statements are equivalent:*

(i) *F is convex on V,*

(ii) *F is convex on W.*

Proof. (i) ⇒ (ii). This is obvious by $W \subset V$.

(ii) ⇒ (i). Assume F is convex on W.

Since $\mathcal{F} = (W, H, D)$ is a subsystem of $\mathcal{E} = (V, G, D)$, we have

$$W \subset V \quad \text{and} \quad H \subset G. \tag{20}$$

By Theorem 1.15, the group majorizations $\prec_G$ and $\prec_H$ coincide on W, i.e., for $w, u \in W$,

$$w \prec_G u \quad \text{if and only if} \quad w \prec_H u. \tag{21}$$

By (20), the G-invariance of F on V implies the H-invariance of F on W.

Now, by Theorem 2.4, item **(iv)**, applied to the E-system (W, H, D), we deduce that F is convex on D and F is H-increasing on D. On account of (21) via the inclusion $D \subset W$, we infer that F is G-increasing on D.

Therefore the condition **(iv)** of Theorem 2.4 holds valid for (V, G, D). In consequence, condition **(i)** of Theorem 2.4 for (V, G, D) is met, too. That is, the function F is convex on V. This finishes the proof of Theorem 2.6. □

Remark 2.7. It is worth to observe that Theorem 2.6 applied to the reduced triple $(W, H, D) = (V_0, G_0, D)$ leads to Theorem 1.12. If in addition F is a G-invariant norm then Theorem 2.6 becomes Theorem 1.13. In light of Lemma 2.1, Theorem 2.6 extends Theorems 1.1–1.4.

Example 2.8. We consider E-systems (V, G, D) and (W, H, D), where

$V \;=\;$ $\mathbb{H}_n$ = the linear space of $n \times n$ hermitian matrices
with the inner product $\langle x, y\rangle = \operatorname{tr} xy$ for $x, y \in \mathbb{H}_n$,

G = the group of unitary similarities $U(\cdot)U^*$ with $U \in \mathbb{U}_n$,

D = $\{\operatorname{diag}(\lambda_1, \lambda_2, \dots, \lambda_n) : \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n\}$ = the convex cone of $n \times n$ real diagonal matrices with decreasingly ordered diagonal,

W = $\mathbb{S}_n(\mathbb{R})$ = the linear space of $n \times n$ real symmetric matrices with the inner product $\langle x, y\rangle = \operatorname{tr} xy$ for $x, y \in \mathbb{S}_n(\mathbb{R})$,

H = the group of orthogonal similarities $O(\cdot)O^T$ with $O \in \mathbb{O}_n$,

$y \prec_G x$ iff $\lambda(y) \prec \lambda(x)$ for $x, y \in \mathbb{H}_n$,

$y \prec_H x$ iff $\lambda(y) \prec \lambda(x)$ for $x, y \in \mathbb{S}_n(\mathbb{R})$

(see [3, 4]). Here axiom **(A1)** holds by the Spectral Theorem for hermitian (real symmetric) matrices, and **(A2)** is met by Fan-Theobald's inequality [6, 23].

Let $F\colon V \to \mathbb{R}$ be a weakly unitarily invariant function (i.e., G-invariant function).

By Theorem 2.6 we conclude that F is convex on $V = \mathbb{H}_n$ if and only if F is convex on $W = \mathbb{S}_n(\mathbb{R})$ (cf. Theorem 1.2).

Example 2.9. We take (V, G, D) and (W, H, D) to be E-systems with

V = $\mathbb{M}_n(\mathbb{C})$ = the linear space of $n \times n$ complex matrices with the inner product $\langle x, y\rangle = \operatorname{Re} \operatorname{tr} xy^*$ for $x, y \in \mathbb{M}_n(\mathbb{C})$,

G = the group of unitary equivalences $U_1(\cdot)U_2^*$ with $U_1, U_2 \in \mathbb{U}_n$,

D = $\{\operatorname{diag}(s_1, s_2, \dots, s_n) : s_1 \geq s_2 \geq \dots \geq s_n \geq 0\}$ = the convex cone of $n \times n$ real diagonal matrices with decreasingly ordered nonnegative diagonal,

W = $\mathbb{M}_n(\mathbb{R})$ = the linear space of $n \times n$ real symmetric matrices with the inner product $\langle x, y\rangle = \operatorname{tr} xy^T$ for $x, y \in \mathbb{M}_n(\mathbb{R})$,

H = the group of orthogonal equivalences $O_1(\cdot)O_2^T$ with $O_1, O_2 \in \mathbb{O}_n$,

$y \prec_G x$ iff $s(y) \prec_w s(x)$ for $x, y \in \mathbb{M}_n(\mathbb{C})$,

$y \prec_H x$ iff $s(y) \prec_w s(x)$ for $x, y \in \mathbb{M}_n(\mathbb{R})$

(see [3, 4]). In this situation, **(A1)** is satisfied by the Singular Value Decomposition of complex (real) matrices, and **(A2)** is fulfilled by von Neumann's trace inequality.

Suppose that $F\colon \mathbb{M}_n(\mathbb{C}) \to \mathbb{R}$ is a unitarily invariant function (i.e., G-invariant function). Then, by Theorem 2.6, F is convex on $V = \mathbb{M}_n(\mathbb{C})$ if and only if F is convex on $W = \mathbb{M}_n(\mathbb{R})$ (cf. Theorem 1.1).

3. E-systems and uniformly convex functions

We introduce functions of class $\mathcal{C}_\psi$ in the following manner.

Definition 3.1. Let (V, G, D) be an E-system with normal map $(\cdot)_\downarrow\colon V \to D$. Let $\psi\colon D \to \mathbb{R}$. By $\mathcal{C}_\psi(I)$ we denote the collection of all functions $F\colon I \to \mathbb{R}$ defined on a convex set $I \subset V$ such that for $x, y \in I$ and $p \in [0, 1]$

$$F(px + (1-p)y) + p(1-p)\psi((x_\downarrow - y_\downarrow)_\downarrow) \leq pF(x) + (1-p)F(y). \tag{22}$$

If (22) is satisfied then the function $F\colon I \to \mathbb{R}$ is said to be *of class* $\mathcal{C}_\psi$ on I.

If (22) is satisfied with $\psi \geq 0$, then the function $F\colon I \to \mathbb{R}$ is said to be *ψ-uniformly convex* on I with modulus $\psi\colon D \to \mathbb{R}$. □

By putting $x = y$ and $p = \frac{1}{2}$ into (22), one gets

$$\psi(0) \leq 0 \tag{23}$$

(provided that the class $\mathcal{C}_\psi$ is not empty). Observe that if $\psi \equiv 0$ then $\mathcal{C}_0(I)$ is the class of all real (usual) convex functions on I.

If $\psi \geq 0$ then $\mathcal{C}_\psi(I) \subset \mathcal{C}_0(I)$, i.e., each function of class $\mathcal{C}_\psi$ is necessarily convex.

If $\psi \leq 0$ then $\mathcal{C}_0(I) \subset \mathcal{C}_\psi(I)$, i.e., each convex function is necessarily of class $\mathcal{C}_\psi$.

If $F\colon I \to \mathbb{R}$ is differentiable on a convex set $I \subset V$, then condition (22) gives

$$F(x) - F(y) \geq \langle \nabla F(y), x - y \rangle + \psi((x_\downarrow - y_\downarrow)_\downarrow) \tag{24}$$

for $x, y \in I$, where $\langle \cdot, \cdot \rangle$ is the inner product on V. In fact, (22) amounts to

$$\frac{F(y + p(x - y)) - F(y)}{p} + (1 - p)\psi((x_\downarrow - y_\downarrow)_\downarrow) \leq F(x) - F(y) \tag{25}$$

for $x, y \in I$, $0 < p \leq 1$. By letting $p \to 0^+$, we get (24) from (25).

We are interested in sufficient conditions for G-invariant functions to be of class $\mathcal{C}_\psi$ on the whole space V.

Theorem 3.2. *Let (V, G, D) be an E-system with normal map $(\cdot)_\downarrow\colon V \to D$, and $F\colon V \to \mathbb{R}$ be a G-invariant function. If F is of class $\mathcal{C}_\psi$ on D, and F is G-increasing on D, then F is of class $\mathcal{C}_\psi$ on V, i.e.,*

$$F\left(px + (1 - p)y\right) \leq pF(x) + (1 - p)F(y) - p(1 - p)\psi((x_\downarrow - y_\downarrow)_\downarrow) \tag{26}$$

for $x, y \in V$ and $p \in [0, 1]$.

Proof. Fix arbitrarily $x, y \in V$ and $p \in [0, 1]$. Because

$$px + (1 - p)y = g_0 \left(px + (1 - p)y\right)_\downarrow \quad \text{for some } g_0 \in G$$

(see (3)), the G-invariance of F guarantees that

$$F\left(px + (1 - p)y\right) = F\left(g_0 \left(px + (1 - p)y\right)_\downarrow\right) = F\left(\left(px + (1 - p)y\right)_\downarrow\right). \tag{27}$$

In addition, by (8) and (9) we have

$$\left(px + (1 - p)y\right)_\downarrow \prec_G px_\downarrow + (1 - p)y_\downarrow \tag{28}$$

with $\left(px + (1 - p)y\right)_\downarrow \in D$ and $px_\downarrow + (1-p)y_\downarrow \in D$. Additionally, F is G-increasing on D, so we obtain

$$F\left(\left(px + (1 - p)y\right)_\downarrow\right) \leq F\left(px_\downarrow + (1 - p)y_\downarrow\right). \tag{29}$$

Since F is of class $\mathcal{C}_\psi$ on D and $x_\downarrow, y_\downarrow \in D$, we find that

$$F\left(px_\downarrow + (1-p)y_\downarrow\right) \le pF(x_\downarrow) + (1-p)F(y_\downarrow) - p(1-p)\psi((x_\downarrow - y_\downarrow)_\downarrow). \tag{30}$$

Taking into account (27), (29) and (30) via the G-invariance of F we get

$$\begin{aligned} F\left(px + (1-p)y\right) &\le pF(x_\downarrow) + (1-p)F(y_\downarrow) - p(1-p)\psi((x_\downarrow - y_\downarrow)_\downarrow) \\ &= pF(x) + (1-p)F(y) - p(1-p)\psi((x_\downarrow - y_\downarrow)_\downarrow). \end{aligned} \tag{31}$$

This proves inequality (26). □

We now demonstrate a Jensen type inequality for functions of class $\mathcal{C}_\psi$ on an E-system (cf. [16, Proposition 6.1]).

Lemma 3.3. *Let (V, G, D) be an E-system with normal map $(\cdot)_\downarrow \colon V \to D$. Let $\psi \colon D \to \mathbb{R}$. Assume $F \colon I \to \mathbb{R}$ is a differentiable function of class $\mathcal{C}_\psi$ on a convex set $I \subset V$. Let $x_i \in I$ and $p_i \ge 0$, $i = 1, \dots, m$, with $\sum_{i=1}^m p_i = 1$. Denote $\bar{x} = \sum_{i=1}^m p_i x_i$. Then the following Jensen type inequality holds:*

$$F\left(\sum_{i=1}^m p_i x_i\right) + \sum_{i=1}^m p_i \psi(((x_i)_\downarrow - \bar{x}_\downarrow)_\downarrow) \le \sum_{i=1}^m p_i F(x_i). \tag{32}$$

If in addition the composition function $\psi((\cdot)_\downarrow)$ is convex (e.g., ψ is convex and G-increasing), then

$$F\left(\sum_{i=1}^m p_i x_i\right) + \psi\left(\left(\sum_{i=1}^m p_i (x_i)_\downarrow - \bar{x}_\downarrow\right)_\downarrow\right) \le \sum_{i=1}^m p_i F(x_i). \tag{33}$$

Moreover, if $x_i \in D$ then (33) *reduces to*

$$F\left(\sum_{i=1}^m p_i x_i\right) + \psi(0) \le \sum_{i=1}^m p_i F(x_i). \tag{34}$$

Proof. It follows from (22) and (24) that

$$F(x_i) - F(\bar{x}) \ge \langle \nabla F(\bar{x}), x_i - \bar{x}\rangle + \psi(((x_i)_\downarrow - \bar{x}_\downarrow)_\downarrow) \quad \text{for } i = 1, \dots, m, \tag{35}$$

where $\langle \cdot, \cdot \rangle$ is the inner product on V. Hence

$$\begin{aligned} &\sum_{i=1}^m p_i (F(x_i) - F(\bar{x})) \ge \sum_{i=1}^m p_i \langle \nabla F(\bar{x}), x_i - \bar{x}\rangle + \sum_{i=1}^m p_i \psi(((x_i)_\downarrow - \bar{x}_\downarrow)_\downarrow) \\ &= \langle \nabla F(\bar{x}), \sum_{i=1}^m p_i (x_i - \bar{x})\rangle + \sum_{i=1}^m p_i \psi(((x_i)_\downarrow - \bar{x}_\downarrow)_\downarrow) = \sum_{i=1}^m p_i \psi(((x_i)_\downarrow - \bar{x}_\downarrow)_\downarrow). \end{aligned} \tag{36}$$

The last inequality holds, because $\sum\limits_{i=1}^m p_i (x_i - \bar{x}) = 0$. This proves inequality (32).

To see the second part of Lemma 3.3, notice that the (usual) Jensen inequality for the convex function $\psi((\cdot)_\downarrow)$ implies

$$\psi\left(\left(\sum_{i=1}^m p_i(x_i)_\downarrow - \bar{x}_\downarrow\right)_\downarrow\right) = \psi\left(\left(\sum_{i=1}^m p_i((x_i)_\downarrow - \bar{x}_\downarrow)\right)_\downarrow\right)$$
$$\leq \sum_{i=1}^m p_i\psi(((x_i)_\downarrow - \bar{x}_\downarrow)_\downarrow). \tag{37}$$

This and (32) give (33), as wanted.

Furthermore, if $x_i \in D$ then $\sum_{i=1}^m p_i x_i \in D$, and $(x_i)_\downarrow = x_i$ and

$$\bar{x}_\downarrow = \left(\sum_{i=1}^m p_i x_i\right)_\downarrow = \sum_{i=1}^m p_i x_i.$$

Therefore (37) can be restated as

$$\psi(0) = \psi\left(\left(\sum_{i=1}^m p_i x_i - \bar{x}\right)_\downarrow\right) = \psi\left(\left(\sum_{i=1}^m p_i(x_i - \bar{x})\right)_\downarrow\right)$$
$$\leq \sum_{i=1}^m p_i\psi((x_i - \bar{x})_\downarrow). \tag{38}$$

In consequence, (33) leads to (34), which completes the proof of Lemma 3.3. □

Remark 3.4. From (23) we know that if F is of class $\mathcal{C}_\psi$, then $\psi(0) \leq 0$. If moreover $\psi(0) = 0$, then (34) says that the (usual) Jensen's inequality holds for F on D. So, in this event F must be convex on D.

Definition 3.5. Let $I \subset V$ be a set. By $\mathcal{C}_\psi(G, I)$ we denote the collection of all functions $F\colon V \to \mathbb{R}$ such that

$$F(pg_1x + (1-p)g_2x) + p(1-p)\psi(0) \leq pF(g_1x) + (1-p)F(g_2x) \tag{39}$$

for $x \in I$, $g_1, g_2 \in G$ and $p \in [0, 1]$.

If (39) holds, then we say that F is *of class* $\mathcal{C}_\psi(G)$ on I.

If (39) holds with $\psi \geq 0$, then we say that F is ψ*-uniformly* G*-convex* on I. □

Note that $(g_1x)_\downarrow - (g_2x)_\downarrow = x_\downarrow - x_\downarrow = 0$ for $x \in V$ and $g_1, g_2 \in G$. Therefore property (39) is a special case of property (22). That is, if $F\colon V \to \mathbb{R}$ is of class $\mathcal{C}_\psi$ on I, then F is of class $\mathcal{C}_\psi(G)$ on I.

Definition 3.6. Let (V, G, D) be an E-system with normal map $(\cdot)_\downarrow\colon V \to D$. A function $F\colon I \to \mathbb{R}$ is said to be (ψ, G)*-increasing* on a set $I \subset V$, if for $x, y \in I$,

$$y \prec_G x \quad \text{implies} \quad F(y) \leq F(x) - \psi((x_\downarrow - y_\downarrow)_\downarrow).$$

Definition 3.7. Let (V, G, D) be an E-system with normal map $(\cdot)_\downarrow \colon V \to D$. A function $F \colon I \to \mathbb{R}$ is said to be (ψ, G)-*convex* on a set $I \subset V$, if for $x \in I$, $g_i \in G$ and $p_i \in [0,1]$, $i = 1, \dots, m$, with $\sum\limits_{i=1}^{m} p_i = 1$,

$$F(p_1 g_1 x + \ldots + p_n g_n x) \le p_1 F(g_1 x) + \ldots + p_n F(g_n x) - \psi((x_\downarrow - y_\downarrow)_\downarrow).$$

Lemma 3.8. *Let (V, G, D) be an E-system with normal map $(\cdot)_\downarrow$. Assume that $F \colon V \to \mathbb{R}$ is a G-invariant function. The following two statements are equivalent:*

(i) *F is (ψ, G)-increasing on V.*

(ii) *F is (ψ, G)-convex on V.*

Proof. (i) $\Rightarrow$ (ii). Let F be (ψ, G)-increasing on V. We fix arbitrarily any $x \in V$, $m \in \mathbb{N}$, $g_1, \dots, g_m \in G$ and $p_1, \dots, p_m \ge 0$ with $p_1 + \ldots + p_m = 1$. We put $y = p_1 g_1 x + \ldots + p_m g_m x$. This means $y \prec_G x$. Consequently, $F(y) \le F(x) - \psi((x_\downarrow - y_\downarrow)_\downarrow)$, i.e.,

$$\begin{aligned} F(p_1 g_1 x + \ldots + p_m g_m x) &\le F(x) - \psi((x_\downarrow - y_\downarrow)_\downarrow) \\ &= p_1 F(x) + \ldots + p_m F(x) - \psi((x_\downarrow - y_\downarrow)_\downarrow) \\ &= p_1 F(g_1 x) + \ldots + p_m F(g_m x) - \psi((x_\downarrow - y_\downarrow)_\downarrow). \end{aligned}$$

The last equality follows from the G-invariance of F, so that F is (ψ, G)-convex on V.

(ii) $\Rightarrow$ (i). Let F be (ψ, G)-convex on V. We have to show that F is (ψ, G)-increasing on V. For this end, assume that $x, y \in V$ are such that $y \prec_G x$. Then $y = p_1 g_1 x + \ldots + p_m g_m x$ for some $m \in N$, $g_1, \dots, g_m \in G$ and $p_1, \dots, p_m \ge 0$ with $p_1 + \ldots + p_m = 1$. For this reason

$$\begin{aligned} F(y) &= F(p_1 g_1 x + \ldots + p_m g_m x) \le p_1 F(g_1 x) + \ldots + p_m F(g_m x) - \psi((x_\downarrow - y_\downarrow)_\downarrow) \\ &= p_1 F(x) + \ldots + p_m F(x) - \psi((x_\downarrow - y_\downarrow)_\downarrow) = F(x) - \psi((x_\downarrow - y_\downarrow)_\downarrow). \end{aligned}$$

This shows the assertion. □

In the next theorem, we present necessary and sufficient conditions for differentiable G-invariant functions to be ψ-uniformly convex on the whole space V.

Theorem 3.9. *Let (V, G, D) be an E-system with its normal map $(\cdot)_\downarrow$, $F \colon V \to \mathbb{R}$ be a differentiable G-invariant function, and $\psi \colon D \to \mathbb{R}_+$ be a nonnegative function. Then the following statements are mutually equivalent:*

(i) *F is ψ-uniformly convex on V,*

(ii) *F is ψ-uniformly convex on D, and F is (ψ, G)-increasing on V,*

(iii) *F is ψ-uniformly convex on D, and F is (ψ, G)-increasing on D.*

Proof. (i) $\Rightarrow$ (ii). Suppose that F is ψ-uniformly convex on V. It is clear that F is ψ-uniformly convex on D, because $D \subset V$. Evidently, F is of class C_ψ on V.

We shall show that F is (ψ, G)-increasing on V. To do so, fix arbitrarily $x, y \in V$ such that $y \prec_G x$. Then $y = \sum_{i=1}^m p_i g_i x$ for some $g_i \in G$ and $p_i \geq 0$, $i = 1, \ldots, m$, with $\sum_{i=1}^m p_i = 1$. By the Jensen type inequality (32) applied for $x_i = g_i x$ (see Lemma 3.3) and by the G-invariance of $(\cdot)_\downarrow$ and of F, we find that

$$\begin{aligned} F(y) + \psi((x_\downarrow - y_\downarrow)_\downarrow) &= F\left(\sum_{i=1}^m p_i g_i x\right) + \sum_{i=1}^m p_i \psi((x_\downarrow - y_\downarrow)_\downarrow) \\ &= F\left(\sum_{i=1}^m p_i g_i x\right) + \sum_{i=1}^m p_i \psi(((g_i x)_\downarrow - y_\downarrow)_\downarrow) \\ &\leq \sum_{i=1}^m p_i F(g_i x) = \sum_{i=1}^m p_i F(x) = F(x), \end{aligned}$$

as desired.

(ii) $\Rightarrow$ (iii). Obvious by the inclusion $D \subset V$.

(iii) $\Rightarrow$ (i). Let F be ψ-uniformly convex on D, and F be (ψ, G)-increasing on D. We shall establish the ψ-uniform convexity of F on V. Fix arbitrarily $x, y \in V$ and $p \in [0, 1]$. Thanks to (3) we can see that

$$px + (1-p)y = g_0 \left(px + (1-p)y\right)_\downarrow \quad \text{for some } g_0 \in G.$$

The G-invariance of F leads to

$$F\left(px + (1-p)y\right) = F\left(g_0 \left(px + (1-p)y\right)_\downarrow\right) = F\left(\left(px + (1-p)y\right)_\downarrow\right). \tag{40}$$

Moreover, by (8) and (9) we have

$$\left(px + (1-p)y\right)_\downarrow \prec_G px_\downarrow + (1-p)y_\downarrow \tag{41}$$

with $\left(px + (1-p)y\right)_\downarrow \in D$ and $px_\downarrow + (1-p)y_\downarrow \in D$. Furthermore, F is (ψ, G)-increasing on D. Therefore

$$\begin{aligned} & F\left(\left(px + (1-p)y\right)_\downarrow\right) \leq \\ \leq F\left(px_\downarrow + (1-p)y_\downarrow\right) &- \psi\left(\left(px_\downarrow + (1-p)y_\downarrow - (px + (1-p)y)_\downarrow\right)_\downarrow\right). \end{aligned} \tag{42}$$

Since F is ψ-uniformly convex on D and $x_\downarrow, y_\downarrow \in D$, we deduce that

$$F\left(px_\downarrow + (1-p)y_\downarrow\right) \leq pF(x_\downarrow) + (1-p)F(y_\downarrow) - p(1-p)\psi((x_\downarrow - y_\downarrow)_\downarrow). \tag{43}$$

By combining (40), (42) and (43) with the G-invariance of F, we conclude that

$$\begin{aligned} F\left(px + (1-p)y\right) &\leq pF(x) + (1-p)F(y) - p(1-p)\psi((x_\downarrow - y_\downarrow)_\downarrow) \\ &\quad -\psi\left(\left(px_\downarrow + (1-p)y_\downarrow - (px + (1-p)y)_\downarrow\right)_\downarrow\right) \\ &\leq pF(x) + (1-p)F(y) - p(1-p)\psi((x_\downarrow - y_\downarrow)_\downarrow). \end{aligned} \tag{44}$$

The last inequality holds, because $\psi \geq 0$. This finishes the proof that F is ψ-uniformly convex on V. $\square$

We are now ready to state, for ψ-uniformly convex functions, an analogue to Theorems 1.1–1.12 (designed for convex functions).

Theorem 3.10. *Let $\mathcal{E} = (V, G, D)$ be an E-system and $\mathcal{F} = (W, H, D)$ be a subsystem of $\mathcal{E}$. Let $F\colon V \to \mathbb{R}$ be a differentiable G-invariant function, and $\psi\colon D \to \mathbb{R}_+$ be a nonnegative function. Then the following two statements are equivalent:*

(i) *F is ψ-uniformly convex on V,*

(ii) *F is ψ-uniformly convex on W.*

Proof. (i) $\Rightarrow$ (ii). Clear by $W \subset V$.

(ii) $\Rightarrow$ (i). Suppose that F is ψ-uniformly convex on W. We have

$$W \subset V \quad \text{and} \quad H \subset G, \tag{45}$$

because $\mathcal{F} = (W, H, D)$ is a subsystem of $\mathcal{E} = (V, G, D)$. Moreover, by virtue of Theorem 1.15, the group majorizations $\prec_G$ and $\prec_H$ coincide on W, i.e., for $w, u \in W$,

$$w \prec_G u \quad \text{if and only if} \quad w \prec_H u. \tag{46}$$

Since F is G-invariant on V, also F is H-invariant on W (see (45)). Utilizing Theorem 3.9(iii), applied to the E-system (W, H, D), we obtain that F is ψ-uniformly convex on D and F is (ψ, H)-increasing on D. Statement (46) and the inclusion $D \subset W$ imply that F is (ψ, G)-increasing on D.

In summary, the condition (iii) of Theorem 3.9 is fulfilled for (V, G, D). Therefore condition (i) of Theorem 3.9 for (V, G, D) is satisfied, which means that the function F is ψ-uniformly convex on V. This is what we wanted to prove. $\square$

We introduce c-strongly convex functions on an E-system, as follows.

Definition 3.11. Let (V, G, D) be an E-system with its normal map $(\cdot)_\downarrow$. Let $c \in \mathbb{R}$. By $\mathcal{C}_c(I)$ we denote the collection of all functions $F\colon I \to \mathbb{R}$ defined on a convex set $I \subset V$ such that, for $x, y \in I$, $p \in [0, 1]$,

$$F(px + (1-p)y) + \frac{c}{2} p(1-p) \|((x_\downarrow - y_\downarrow)_\downarrow)\|^2 \leq pF(x) + (1-p)F(y), \tag{47}$$

where $\|\cdot\| = \langle \cdot, \cdot \rangle^{1/2}$ is the norm induced by the inner product on V.

If (22) is satisfied then the function $F\colon I \to \mathbb{R}$ is said to be *of class $\mathcal{C}_c$* on I.

If (22) is satisfied with $c \geq 0$, then the function $F\colon I \to \mathbb{R}$ is said to be *c-strongly convex* on I.

Corollary 3.12. *Let (V, G, D) be an E-system, $F\colon V \to \mathbb{R}$ be a differentiable G-invariant function, and c be a nonnegative number. Then the following statements are mutually equivalent:*

(i) *F is c-strongly convex on V,*

(ii) *F is c-strongly convex on D, and F is (ψ, G)-increasing on V,*

(iii) *F is c-strongly convex on D, and F is (ψ, G)-increasing on D.*

Proof. Apply Theorem 3.9 for the function $\psi\colon D \to \mathbb{R}_+$ given by $\psi(x) = \frac{c}{2}\|x\|^2$ for $x \in D$, where $\|\cdot\| = \langle\cdot,\cdot\rangle^{1/2}$ is the norm induced by the inner product on V. □

Corollary 3.13. *Let $\mathcal{E} = (V, G, D)$ be an E-system and $\mathcal{F} = (W, H, D)$ be a subsystem of $\mathcal{E}$. Let $F\colon V \to \mathbb{R}$ be a differentiable G-invariant function, and c be a nonnegative number. Then the following two statements are equivalent:*

(i) *F is c-strongly convex on V,*

(ii) *F is c-strongly convex on W.*

Proof. Use Theorem 3.10 for the function $\psi\colon D \to \mathbb{R}_+$ given by $\psi(x) = \frac{c}{2}\|x\|^2$ for $x \in D$, where $\|\cdot\| = \langle\cdot,\cdot\rangle^{1/2}$ is the norm induced by the inner product on V. □

Example 3.14. (Continued Example 2.8) We study the two E-systems (V, G, D) and (W, H, D) defined in Example 2.8.

Let $F\colon \mathbb{H}_n \to \mathbb{R}$ be a differentiable weakly unitarily invariant function (i.e., G-invariant function). Let $\psi\colon D \to \mathbb{R}_+$, where $D = \{\mathrm{diag}(\lambda_1, ..., \lambda_n) : \lambda_1 \geq ... \geq \lambda_n\}$.

By Theorem 3.10 we infer that F is ψ-uniformly convex on $V = \mathbb{H}_n$ if and only if F is ψ-uniformly convex on $W = \mathbb{S}_n(\mathbb{R})$ (cf. Theorem 1.2).

Similarly, by Corollary 3.13, if $c \geq 0$ then F is c-strongly convex on $V = \mathbb{H}_n$ if and only if F is c-strongly convex on $W = \mathbb{S}_n(\mathbb{R})$ (cf. Theorem 1.2). □

Example 3.15. (Continued Example 2.9) Let (V, G, D) and (W, H, D) be the two E-systems defined in Example 2.9.

Assume that $F\colon \mathbb{M}_n(\mathbb{C}) \to \mathbb{R}$ is a differentiable unitarily invariant function. Let $\psi\colon D \to \mathbb{R}_+$, where $D = \{\mathrm{diag}(s_1, \dots, s_n) : s_1 \geq \dots \geq s_n \geq 0\}$.

Then, by Theorem 3.10, F is ψ-uniformly convex on $V = \mathbb{M}_n(\mathbb{C})$ if and only if F is ψ-uniformly convex on $W = \mathbb{M}_n(\mathbb{R})$ (cf. Theorem 1.1).

Likewise, by Corollary 3.13, if $c \geq 0$ then F is c-strongly convex on $V = \mathbb{M}_n(\mathbb{C})$ if and only if F is c-strongly convex on $W = \mathbb{M}_n(\mathbb{R})$ (cf. Theorem 1.1).

Example 3.16. We consider the E-systems (V, G, D) and (W, H, D), where $n = 2r$ is even, and

V = $\mathbb{K}_n(\mathbb{C})$ = the linear space of $n \times n$ complex skew-symmetric matrices with the inner product $\langle x, y\rangle = \mathrm{Re}\,\mathrm{tr}\, xy^*$ for $x, y \in \mathbb{K}_n(\mathbb{C})$,

G = the group of unitary congruences $U(\cdot)U^T$ with $U \in \mathbb{U}_n$,

$$D = \left\{ \begin{pmatrix} 0 & s_1 \\ -s_1 & 0 \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} 0 & s_r \\ -s_r & 0 \end{pmatrix} : s_1 \geq \dots \geq s_r \geq 0 \right\},$$

W = $\mathbb{K}_n(\mathbb{R})$ = the linear space of $n \times n$ real skew-symmetric matrices with the inner product $\langle x, y\rangle = \operatorname{tr} xy^T$ for $x, y \in \mathbb{K}_n(\mathbb{R})$,

H = the group of orthogonal congruences $O(\cdot)O^T$ with $O \in \mathbb{O}_n$,

$y \prec_G x$ iff $\theta(y) \prec_w \theta(x)$ for $x, y \in \mathbb{K}_n(\mathbb{C})$,

$y \prec_H x$ iff $\theta(y) \prec_w \theta(x)$ for $x, y \in \mathbb{K}_n(\mathbb{R})$,

where $\theta(x) = (s_1(x), \dots, s_r(x))$, and $s_1(x) = s_1(x) \geq \dots \geq s_r(x) = s_r(x)$ are the all singular values of $x \in \mathbb{K}_n(\mathbb{C})$ (see [3, 4, 7, 22]). Here axiom **(A1)** holds by Hua Decomposition for skew symmetric matrices, and **(A2)** holds by von Neumann's trace inequality.

Let $F\colon \mathbb{K}_n(\mathbb{C}) \to \mathbb{R}$ be a differentiable G-invariant function. Let $\psi\colon D \to \mathbb{R}_+$.

Theorem 3.10 guarantees that F is ψ-uniformly convex on $V = \mathbb{K}_n(\mathbb{C})$ if and only if F is ψ-uniformly convex on $W = \mathbb{K}_n(\mathbb{R})$.

Assume $c \geq 0$. In light of Corollary 3.13, F is c-strongly convex on $V = \mathbb{K}_n(\mathbb{C})$ if and only if F is c-strongly convex on $W = \mathbb{K}_n(\mathbb{R})$.

By putting $c = 0$, one gets that F is convex on $V = \mathbb{K}_n(\mathbb{C})$ if and only if F is convex on $W = \mathbb{K}_n(\mathbb{R})$. □

Example 3.17. Let (V, G, D) and (W, H, D) be E-systems with

V = $\mathbb{M}_n(\mathbb{C})$ = the linear space of $n \times n$ complex matrices with the inner product $\langle x, y\rangle = \operatorname{Re} \operatorname{tr} xy^*$ for $x, y \in \mathbb{M}_n(\mathbb{C})$,

G = the group of unitary equivalences $U_1(\cdot)U_2^*$ with $U_1, U_2 \in \mathbb{U}_n$,

D = $\{\operatorname{diag}(s_1, s_2, \dots, s_n) : s_1 \geq s_2 \geq \dots \geq s_n \geq 0\}$,

W = $\mathbb{S}_n(\mathbb{C})$ = the linear space of $n \times n$ complex symmetric matrices with the inner product $\langle x, y\rangle = \operatorname{Re} \operatorname{tr} xy^*$ for $x, y \in \mathbb{S}_n(\mathbb{C})$,

H = the group of unitary congruences $U(\cdot)U^T$ with $U \in \mathbb{U}_n$,

$y \prec_G x$ iff $s(y) \prec_w s(x)$ for $x, y \in \mathbb{M}_n(\mathbb{C})$,

$y \prec_H x$ iff $s(y) \prec_w s(x)$ for $x, y \in \mathbb{S}_n(\mathbb{C})$

(see [3, 4, 9]). Here **(A1)** for (W, H, D) is satisfied by Autonne Decomposition of complex symmetric matrices [9, p. 136], and **(A2)** is fulfilled by von Neumann's trace inequality.

Suppose that $F\colon \mathbb{M}_n(\mathbb{C}) \to \mathbb{R}$ is a differentiable unitarily invariant function. Let $\psi\colon D \to \mathbb{R}_+$. Then, by Theorem 3.10, F is ψ-uniformly convex on $V = \mathbb{M}_n(\mathbb{C})$ if and only if F is ψ-uniformly convex on $W = \mathbb{S}_n(\mathbb{C})$.

Let $c \geq 0$. On account of Corollary 3.13, F is c-strongly convex on $V = \mathbb{M}_n(\mathbb{C})$ if and only if F is c-strongly convex on $W = \mathbb{S}_n(\mathbb{C})$.

Under $c = 0$, we see that F is convex on $V = \mathbb{M}_n(\mathbb{C})$ if and only if F is convex on $W = \mathbb{S}_n(\mathbb{C})$. □

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