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Multicones, Duality and Matrix Invariance

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We analyze the geometric structure and properties of a certain class of subsets of $\mathbb{R}^d$, here called *multicones*, which are natural generalizations of the classical cones. For them we introduce and investigate a suitable extension of the concept of duality, which allows us to treat in a convenient way many issues related to their invariance and strict invariance for real matrices.

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1. Introduction

The observation that a real positive $d \times d$ matrix corresponds to an endomorphism of $\mathbb{R}^d$ mapping the positive orthant into itself has naturally led to investigate endomorphisms admitting an *invariant cone* (the natural generalization of the orthant). In this context, we mention the Perron-Frobenius Theorems (1907 and 1912, see [9, 16, 17]), which prove the existence of a *leading* eigenvalue equal to the spectral radius, and its generalization due to Birkhoff [2] and the work by Vandergraft [22], where necessary and sufficient conditions on a matrix to have an invariant cone are given.

Still concerning the analysis of particular spectral properties of matrices and families of matrices, recently a generalization of cone, called *multicone*, has been introduced by Avila, Bochi and Yoccoz [1] in dimension 2 and, successively, extended to any dimension by Bochi and Gourmelon [3], who defined the concept of *multicone of index k*. Substantially, this structure is a homogeneous symmetric subset of $\mathbb{R}^d$, (strictly) invariant for certain families of nonsingular matrices and containing a k-dimensional subspace.

Since classical cones are connected components of multicones of index 1 (in the above terminology), we focus our attention on, roughly speaking, symmetric sets consisting in the union of a finite number of cones, and we call these structures

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simply *multicones*. Here they are defined in an intrinsically geometric way, independently of matrix invariance issues, and, in our opinion, deserve their own attention in themselves. In this framework, we introduce a suitable extension of the notion of duality which, as in the cone case, confirms to be a useful tool to carry on the analysis of matrix invariance issues.

The structure of this paper takes into account both the targets to be a self-contained study and to provide an "elementary" treatment.

In Section 2 we recall the basic notions of cone theory, the concept of duality and many related properties. We also review the main spectral properties of matrices admitting an invariant cone. In particular, we stress the strictly invariant case, which selects a particular class of matrices with only one simple *leading eigenvalue* (here called *asymptotically rank-one*). Most of this section material is well-known. However, a detailed treatment is available in our survey [5], where some additional specific results used in this paper are introduced.

In Section 3 we present the definition of multicone discussing its basic peculiarities and, in Section 4, we extend to it, suitably adapted, the concept of duality.

Finally, we analyze the properties fulfilled by those matrices which admit an invariant (Section 5) or strictly invariant multicone (Section 6). In doing this, we often use duality as a convenient tool. At the same time, we are able to transfer many of the invariance issues (with respect to a matrix A) from a multicone to its *dual multicone* (with respect to the transpose matrix A^T).

Our overall study reveals to be a fundamental background for our further paper [6], where the existence of an invariant multicone for a family of matrices is investigated.

2. Notation and preliminaries

We refer to $\mathbb{R}^d$ as a real vector space endowed with the Euclidean scalar product, denoted by x^Ty for any $x, y \in \mathbb{R}^d$. The metric and topological structures of this Euclidean space are induced by this pairing.

In this framework, if U is a nonempty subset of $\mathbb{R}^d$, we denote by $\mathrm{cl}(U)$ its *closure*, by $\mathrm{conv}(U)$ its *convex hull*, by $\mathrm{int}(U)$ its *interior* and by ∂U its *boundary* as a subset of $\mathbb{R}^d$. We also denote by $\mathrm{span}(U)$ the smallest vector subspace containing U. Finally, we set

$$\mathbb{R}_+U := \left\{\alpha x \;\middle|\; \alpha \geq 0 \text{ and } x \in U\right\},$$

and

$$U^\perp := \left\{h \in \mathbb{R}^d \;\middle|\; h^Tx = 0 \text{ for all } x \in U\right\}$$

denotes the *orthogonal set* of U.

In particular, if H is a (vector) hyperplane of $\mathbb{R}^d$ (i.e., a linear subspace of $\mathbb{R}^d$ of dimension $d-1$), then $H = \{h\}^\perp$ for a suitable vector $h \in \mathbb{R}^d \setminus \{0\}$, unique up to a scalar.

The hyperplane H splits $\mathbb{R}^d$ into two parts, say the *positive* and the *negative* semi-space

$$S_+^h := \{x \in \mathbb{R}^d \mid h^T x \geq 0\} \quad \text{and} \quad S_-^h := \{x \in \mathbb{R}^d \mid h^T x \leq 0\},$$

respectively. Clearly,

$$\operatorname{int}\left(S_+^h\right) = \{x \in \mathbb{R}^d \mid h^T x > 0\} \quad \text{and} \quad \operatorname{int}\left(S_-^h\right) = \{x \in \mathbb{R}^d \mid h^T x < 0\}.$$

Now let A be a real $d \times d$ matrix. We identify it with the corresponding endomorphism

$$f_A \colon \mathbb{R}^d \to \mathbb{R}^d$$

defined by $f_A(x) = Ax$. Hence, the kernel and the image of f_A will be simply denoted by $\ker(A)$ and $\operatorname{range}(A)$, respectively, and, if U is a subset of $\mathbb{R}^d$, its image will be denoted by $A(U)$.

Nevertheless, the preimage of a subset V of $\mathbb{R}^d$ will be explicitly denoted by $f_A^{-1}(V)$.

Definition 2.1. A subset U of $\mathbb{R}^d$ is said to be *invariant under the action of the matrix* A on $\mathbb{R}^d$ (in short, *invariant for* A) if $A(U) \subseteq U$. □

Assumption 2.2. In order to avoid trivial cases, from now on we assume that A is a nonzero matrix.

If $\lambda \in \mathbb{C}$ and a nonzero vector $v \in \mathbb{C}^d$ are such that $Av = \lambda v$, then they are called *eigenvalue* and *eigenvector* of A, respectively.

The set $V_\lambda(A)$, or simply V_λ, consisting of such eigenvectors and of the zero vector, is a linear subspace called the *eigenspace* corresponding to λ. Obviously, V_λ is invariant under the action of A.

Denoting by $\mu_a(\lambda)$ the *algebraic multiplicity* of λ (as root of the characteristic polynomial $\det(A - \lambda I)$) and by $\mu_g(\lambda)$ the *geometric multiplicity* of λ (i.e., $\dim_{\mathbb{C}}(V_\lambda)$), it is also well-known that $\mu_g(\lambda) \leq \mu_a(\lambda)$. If the equality holds, then λ is called *nondefective*. Otherwise, it is called *defective*.

Definition 2.3. Let λ be an eigenvalue of A and $k = \mu_a(\lambda)$. Then the linear space

$$W_\lambda(A) := \ker((A - \lambda I)^k) \subseteq \mathbb{C}^d$$

is called *generalized eigenspace* corresponding to λ and each of its nonzero elements which does not belong to V_λ is called *generalized eigenvector*.

If no misunderstanding occurs, we shall simply write W_λ instead of $W_\lambda(A)$. □

It is clear that W_λ is a linear subspace invariant for A and it is well-known that $\dim_{\mathbb{C}}(W_\lambda) = \mu_a(\lambda)$ (see, e.g., Lax [15], Theorem 11). Therefore, $V_\lambda = W_\lambda$ if and only if λ is nondefective.

If $\lambda \in \mathbb{R}$, then W_λ is a linear subspace of $\mathbb{R}^d$ and $\dim_{\mathbb{R}}(W_\lambda) = \mu_a(\lambda)$.

Otherwise, if $\lambda \in \mathbb{C} \setminus \mathbb{R}$, since the conjugate of λ is an eigenvalue as well, we set $U_{\mathbb{C}}(\lambda, \bar{\lambda}) := W_\lambda \oplus W_{\bar{\lambda}} \subseteq \mathbb{C}^d$. With $k := \mu_a(\lambda) = \dim_{\mathbb{C}}(W_\lambda)$, it is clear that $\dim_{\mathbb{C}}(U_{\mathbb{C}}(\lambda, \bar{\lambda})) = 2k$. Setting also $U_{\mathbb{R}}(\lambda, \bar{\lambda}) := U_{\mathbb{C}}(\lambda, \bar{\lambda}) \cap \mathbb{R}^d$, it turns out that $\dim_{\mathbb{R}}(U_{\mathbb{R}}(\lambda, \bar{\lambda})) = 2k$ and that this linear space is spanned by the real and the imaginary parts of the vectors of W_λ. Clearly, $U_{\mathbb{R}}(\lambda, \bar{\lambda})$ is invariant for A.

Therefore, if $\lambda_1, \ldots, \lambda_r \in \mathbb{R}$ and $\mu_1, \bar{\mu}_1, \ldots, \mu_s, \bar{\mu}_s \in \mathbb{C} \setminus \mathbb{R}$ are the distinct roots of the characteristic polynomial, then

$$\mathbb{R}^d = \bigoplus_{i=1}^{r} W_{\lambda_i} \oplus \bigoplus_{i=1}^{s} U_{\mathbb{R}}(\mu_i, \bar{\mu_i}). \tag{1}$$

Finally, recall that the set $\sigma(A)$ of the (real or complex) eigenvalues is called the *spectrum* of A and the nonnegative real number

$$\rho(A) := \max_{\lambda \in \sigma(A)} |\lambda|$$

is called the *spectral radius* of A. It is well-known that either $\rho(A) > 0$ or $A^d = 0$.

The eigenvalues whose modulus is $\rho(A)$ are called *leading eigenvalues* and the corresponding eigenvectors are called *leading eigenvectors*. The remaining eigenvalues and eigenvectors are called *secondary eigenvalues* and *secondary eigenvectors*, respectively.

Remark 2.4. If the matrix A admits a real leading eigenvalue λ_1, we can write

$$\mathbb{R}^d = W_A \oplus H_A,$$

where $$W_A := W_{\lambda_1} \quad \text{and} \quad H_A := \bigoplus_{i=2}^{r} W_{\lambda_i} \oplus \bigoplus_{i=1}^{s} U_{\mathbb{R}}(\mu_i, \bar{\mu_i}). \tag{2}$$

Observe that both W_A and H_A are linear subspaces invariant for A.

The proof of the following property is given in [5].

Proposition 2.5. *Let A be a matrix admitting a real leading eigenvalue $\lambda_1 > 0$ and let $x \in \mathbb{R}^d$. Then*

$$Ax \in H_A \implies x \in H_A.$$

2.1. Cones and duality

The notion of *proper cone* is standard enough in the literature (see, e.g., Tam [21], Schneider and Tam [19] and Rodman, Seyalioglu and Spitkovsky [18]). The more general notion of *cone* is, instead, not universally shared: accordingly to the various authors, it involves a variable subset (or even all, see Schneider and Vidyasagar [20]) of the requirements for proper cones.

In this paper, we deal with proper cones, as defined in the standard way, and with cones that verify a particular subset of the possible properties. We also find it useful to consider a weaker instance of our definition of cone, that we refer to as *quasi-cone*.

Definition 2.6. Let K be a nonempty closed and convex set of $\mathbb{R}^d$ and consider the following conditions:

(c1) $\mathbb{R}_+ K \subseteq K$ (i.e., K is *positively homogeneous*);

(c2) $K \cap -K = \{0\}$ (i.e., K is *pointed* or *salient*);

(c3) $\operatorname{span}(K) = \mathbb{R}^d$ (i.e., K is *full* or *solid*).

We say that K is a *quasi-cone* if it satisfies (c1). If, in addition, it satisfies (c2), we say that K is a *cone*. Finally, if it satisfies all the above properties, we say that K is a *proper cone*.

If a quasi-cone K is not solid, we also say that it is a *degenerate quasi-cone*. □

In this section we recall a minimal set of basic notions and properties which are necessary to develop the theory of multicones. Most is well-known and may be found, e.g., in Fenchel [8], Schneider and Vidyasagar [20] and Tam [21]. In any case, for a systematic and more detailed treatment of this preliminary material about quasi-cones, including the proofs, we refer the reader to [5].

Definition 2.7. Given a hyperplane H, we say that a nonempty positively homogeneous set $U \subset \mathbb{R}^d$ is *supported by* H (or, briefly, H*-supported*) if

$$U \subseteq S_+^h \quad \text{or} \quad U \subseteq S_-^h.$$

We say that U is *strictly supported by* H (or, briefly, *strictly* H*-supported*) if

$$U \setminus \{0\} \subseteq \operatorname{int}\left(S_+^h\right) \quad \text{or} \quad U \setminus \{0\} \subseteq \operatorname{int}\left(S_-^h\right). \qquad \square$$

Definition 2.8. Given a nonempty set $U \subset \mathbb{R}^d$, the intersection of all the quasi-cones containing U (i.e., the smallest quasi-cone containing U) is called the *quasi-cone generated by* U and we denote it by $\operatorname{qcone}(U)$. □

The quasi-cone generated by U can be represented explicitly in formula.

Proposition 2.9. *For any nonempty set* $U \subset \mathbb{R}^d$*, we have*

$$\operatorname{qcone}(U) = \operatorname{cl}\left(\operatorname{conv}\left(\mathbb{R}_+ U\right)\right) = \operatorname{cl}\left(\mathbb{R}_+ \operatorname{conv}\left(U\right)\right).$$

Note that, while $\operatorname{qcone}(U)$ is defined for any set U, the smallest cone containing U may well not exist. Anyway, if it does exist, then it coincides with $\operatorname{qcone}(U)$.

Definition 2.10. Let $U \subset \mathbb{R}^d$ be nonempty and assume that $\operatorname{qcone}(U)$ is a cone. Then we denote it by $\operatorname{cone}(U)$ and call it the *cone generated by* U. □

In the particular case of cones, Proposition 2.9 assumes the following stronger formulation.

Proposition 2.11. *Let* $U \subset \mathbb{R}^d$ *be nonempty and assume that* $\operatorname{qcone}(U)$ *is a cone. Then*

$$\operatorname{cone}(U) = \operatorname{conv}\left(\operatorname{cl}\left(\mathbb{R}_+ U\right)\right) = \operatorname{cl}\left(\operatorname{conv}\left(\mathbb{R}_+ U\right)\right) = \operatorname{cl}\left(\mathbb{R}_+ \operatorname{conv}\left(U\right)\right).$$

The notion of *duality* is essential in the study of cones.

Definition 2.12. Let U be a nonempty set of $\mathbb{R}^d$. Then

$$U^\star := \{h \in \mathbb{R}^d \mid h^T x \geq 0 \quad \forall x \in U\}$$

is called he *dual set* of U. By convention, we also define $\emptyset^\star := \mathbb{R}^d$. □

Note that, if U is a subset of $\mathbb{R}^d$, then $U \subseteq S_+^h$ if and only if $h \in U^\star \setminus \{0\}$.

Proposition 2.13. *Let U and V be nonempty sets of $\mathbb{R}^d$. Then we have:*

(i) $U \subseteq U^{\star\star}$;

(ii) $U \subseteq V$ *implies* $U^\star \supseteq V^\star$;

(iii) $(U \cup V)^\star = U^\star \cap V^\star$;

(iv) $(U \cap V)^\star \supseteq U^\star \cup V^\star$.

Definition 2.14. If K is a quasi-cone of $\mathbb{R}^d$, the set

$$K^\star = \{h \in \mathbb{R}^d \mid h^T x \geq 0 \quad \forall x \in K\}$$

is called the *dual quasi-cone* of K. □

We recall that, for any quasi-cone K, it holds

$$K^{\star\star} = K \tag{3}$$

and that, for any pair $K^{(1)}$ and $K^{(2)}$ of quasi-cones,

$$K^{(1)} \subseteq K^{(2)} \quad \Longleftrightarrow \quad (K^{(1)})^\star \supseteq (K^{(2)})^\star. \tag{4}$$

Corollary 2.15. *Let K be a quasi-cone. Then K is pointed if and only if $K^\star$ is solid and, dually, $K^\star$ is pointed if and only if K is solid. In particular, K is a proper cone if and only if $K^\star$ is a proper cone.*

Lemma 2.16. *If K is a cone and $h \in K^\star \setminus \{0\}$, then*

$$h \in \operatorname{int}(K^\star) \iff K \cap \{h\}^\perp = \{0\}.$$

Proposition 2.17. *Let K be a cone of $\mathbb{R}^d$. Then*

(i) $$\operatorname{int}(K^\star) = \{h \in \mathbb{R}^d \mid h^T x > 0 \quad \forall x \in K \setminus \{0\}\};$$

(ii) *if K is proper, then*

$$K^\star \setminus \{0\} = \{h \in \mathbb{R}^d \mid h^T x > 0 \quad \forall x \in \operatorname{int}(K)\}.$$

Finally, if $K^{(1)}$ and $K^{(2)}$ are proper cones, then

$$K^{(1)} \setminus \{0\} \subseteq \operatorname{int}(K^{(2)}) \quad \Longrightarrow \quad \operatorname{int}((K^{(1)})^\star) \supseteq (K^{(2)})^\star \setminus \{0\}.$$

The last part of this subsection is devoted to some properties concerning the quasi-cone generated by a finite union of quasi-cones.

Lemma 2.18. *Let $K^{(1)}, \dots, K^{(r)}$ be quasi-cones of $\mathbb{R}^d$ and $U := \bigcup_{i=1}^r K^{(i)}$. Then*

$$(\mathrm{qcone}\,(U))^\star = U^\star = \bigcap_{i=1}^{r} \left(K^{(i)}\right)^\star .$$

Moreover, the above set, which is a quasi-cone itself, is $\neq \{0\}$ if and only if U is supported by some hyperplane H.

The notion of separatedness of two closed convex subsets of $\mathbb{R}^d$ has to be slightly modified (e.g., following Klee [11]) to adapt it to the case of cones.

Definition 2.19. Two cones $K^{(1)}$ and $K^{(2)}$ of $\mathbb{R}^d$ are said to be *separated* if there exists a hyperplane $H = \{h\}^\perp$ such that

$$K^{(1)} \setminus \{0\} \subseteq \mathrm{int}\left(S_+^h\right) \quad \text{and} \quad K^{(2)} \setminus \{0\} \subseteq \mathrm{int}\left(S_-^h\right).$$

Moreover, we say that such an H is a *separating hyperplane* for $K^{(1)}$ and $K^{(2)}$. □

Let us mention two well know results, the first of which is the “cone version” of a “separation-type” theorem.

Theorem 2.20. *Two cones $K^{(1)}$ and $K^{(2)}$ of $\mathbb{R}^d$ are separated if and only if $K^{(1)} \cap K^{(2)} = \{0\}$.*

Proposition 2.21. *Let $K^{(1)}, \dots, K^{(r)}$ be cones of $\mathbb{R}^d$, H a hyperplane and*

$$K := \mathrm{qcone}(K^{(1)} \cup \ldots \cup K^{(r)}).$$

Then the following statements are equivalent:

(i) *K is strictly H-supported ;*

(ii) *$K^{(1)} + \cdots + K^{(r)}$ is strictly H-supported and, hence, closed;*

(iii) *$K^{(1)} \cup \ldots \cup K^{(r)}$ is strictly H-supported .*

In this case, $K = K^{(1)} + \cdots + K^{(r)}$ is a cone, too.

2.2. Matrices with an invariant cone

Let A be a real $d \times d$ matrix. It is clear that, if $\lambda > 0$ is a real eigenvalue and $\dim(V_\lambda) = 1$, both the two half-lines which constitute V_λ are invariant for A. Therefore, it makes sense to extend the search of invariant sets from linear subspaces to cones.

In the case of cones the notion of invariance is the general one (see Definition 2.1), but it is useful to recall the following refinement.

Definition 2.22. We say that a proper cone K is *strictly invariant under the action of the matrix A* on $\mathbb{R}^d$ (in short, *strictly invariant for A*) if

$$A(K \setminus \{0\}) \subseteq \mathrm{int}(K). \qquad \square$$

For example, the positive orthant $\mathbb{R}_+^d$ is invariant for a real matrix with nonnegative entries, whereas it is strictly invariant for a matrix with all strictly positive entries.

We recall that A and the *transpose matrix* A^T have the same eigenvalues with the same multiplicities. More precisely, for any eigenvalue $\lambda \in \mathbb{C}$ it holds that $\dim(V_\lambda(A)) = \dim(V_\lambda(A^T))$ and $\dim(W_\lambda(A)) = \dim(W_\lambda(A^T))$.

Proposition 2.23. *A proper cone K is invariant (respectively, strictly invariant) for a matrix A if and only if the dual cone $K^\star$ is invariant (respectively, strictly invariant) for the transpose matrix A^T.*

The next Theorems 2.24 and 2.26 and their corollaries resume the well-known Perron-Frobenius theorems (see, for instance, Vandergraft [22]) and some other related results collected in [5] (see also Krasnosel'skiĭ, Lifshits and Sobolev [12]).

Theorem 2.24. *Let a proper cone K be invariant for a nonzero matrix A. Then the following facts hold:*

(i) *the spectral radius $\rho(A)$ is an eigenvalue of A;*

(ii) *the cone K contains an eigenvector v corresponding to $\rho(A)$;*

(iii) $\mathrm{int}(K) \cap H_A = \emptyset$.

Corollary 2.25. *If $\rho(A) > 0$ in the assumptions of the previous theorem we have*

$$\mathrm{int}(K) \cap \ker(A) = \emptyset.$$

Theorem 2.26. *Let a proper cone K be strictly invariant for a nonzero matrix A. Then the following facts hold:*

(i) *the spectral radius $\rho(A)$ is a simple positive eigenvalue of A and $|\lambda| < \rho(A)$ for any other eigenvalue λ of A;*

(ii) $\mathrm{int}(K)$ *contains the unique leading eigenvector v (corresponding to $\rho(A)$);*

(iii) $K \cap H_A = \{0\}$.

Corollary 2.27. *In the assumptions of the previous theorem we also have*

$$K \cap \ker(A) = \{0\}.$$

In this paper, our attention is mainly directed to matrices of the following type.

Definition 2.28. A matrix A is said to be *asymptotically rank-one* if the following conditions hold:

(i) $\rho(A) > 0$;

(ii) exactly one between $\rho(A)$ and $-\rho(A)$ is an eigenvalue of A and, moreover, it is a simple eigenvalue;

(iii) $|\lambda| < \rho(A)$ for any other eigenvalue λ of A.

The unique leading eigenvalue of A will be denoted by λ_A. □

Remark 2.29. A matrix A is asymptotically rank-one if and only if A^T is so.

The term "asymptotically rank-one" is inspired by the following well-known fact, easy to prove.

Proposition 2.30. *If A is an asymptotically rank-one matrix, then there exists*

$$\hat{A}^{\infty} := \lim_{k\to\infty} A^k/\lambda_A^k$$

and such a limit is the rank-one matrix

$$\hat{A}^{\infty} = (v_A^T h_A)^{-1} v_A h_A^T,$$

where v_A and h_A are the (unique) leading eigenvectors of A and A^T, respectively.

The following characterization rephrases Theorem 4.4 in Vandergraft [22].

Theorem 2.31. *A matrix A is asymptotically rank-one if and only if A or $-A$ admits a strictly invariant proper cone.*

In other words, A is asymptotically rank-one if and only if it admits a strictly invariant proper symmetric cone (see the forthcoming Definition 6.4).

Note that, if A is an asymptotically rank-one matrix, then λ_A is nondefective (as it is simple) and, consequently, $V_{\lambda_A} = W_{\lambda_A}$ has dimension one. Briefly, we set $V_A := V_{\lambda_A}$.

Definition 2.32. If A is an asymptotically rank-one matrix, then the eigenspace V_A will be called the *leading invariant line* of A, whereas the hyperplane H_A will be called *secondary invariant hyperplane* of A and we shall write

$$H_A = \{h_A\}^{\perp}$$

for a suitable vector h_A. □

If A is an asymptotically rank-one matrix, by Remark 2.4, clearly

$$h_A^T v_A \neq 0. \tag{5}$$

Moreover, it is easy to see that

$$V_{A^T} = H_A^{\perp} \quad \text{or, equivalently,} \quad H_{A^T} = V_A^{\perp}. \tag{6}$$

3. Symmetric cones and multicones

We begin with a very natural generalization of the concept of cone, already considered by Edwards, McDonald and Tsatsomeros [7] under the name of *double cone* and also reminding the concept of *cone of rank* 1 (see Krasnosel'skiĭ and Sobolev [13]).

Definition 3.1. Any subset of $\mathbb{R}^d$ of the form $K_{sym} = K \cup -K$, where K is a cone, is called a *symmetric cone* of $\mathbb{R}^d$.

We also use to say that K and $-K$ are the *positive* and the *negative* part of K_{sym} and denote them by K_+ and K_-, respectively. Thus we can write

$$K_{sym} = K_+ \cup K_-.$$ □

It would seem natural to extend the definition to the more general notion of *symmetric quasi-cone*. However, it is not difficult to realize that this would generate an ambiguity when K_+ is a semi-subspace of dimension ≥ 2. In fact, $K_+ \cup K_-$ would be a whole subspace of dimension ≥ 2, which could be thought as the union of infinitely many pairs of semi-subspaces.

Remark 3.2. If K_{sym} is a symmetric cone, by Theorem 2.20, there exists a hyperplane H such that $H \cap K_{sym} = \{0\}$.

Finally, let us give a key definition.

Definition 3.3. Consider a finite collection of symmetric cones $K^{(1)}_{sym}, \ldots, K^{(r)}_{sym}$ such that

(m1) $$K^{(i)}_{sym} \cap K^{(j)}_{sym} = \{0\} \qquad \text{whenever } i \neq j;$$

(m2) there exists a hyperplane H of $\mathbb{R}^d$ such that

$$H \cap K^{(i)}_{sym} = \{0\} \qquad \text{for all } i = 1, \ldots, r.$$

Then the set
$$K_{mul} := \bigcup_{i=1}^{r} K^{(i)}_{sym}$$
is called a *multicone* of $\mathbb{R}^d$, the $K^{(i)}_{sym}$'s are its *symmetric components* and the $K^{(i)}_{\pm}$'s are its *(conic) components* (also denoted simply by $K^{(i)}$). The number r of symmetric components is called the *fragmentation index* of K_{mul}. Finally, H in (m2) is called a *splitting hyperplane* for K_{mul}. □

Note that the hyperplane H in $(m2)$ is such that $H \cap K_{mul} = \{0\}$.

As a particular case, a symmetric cone is a multicone with fragmentation index $r = 1$.

Remark 3.4. Unless the fragmentation index is $r \leq 2$ or the dimension is $d = 2$, condition (m1) does not automatically imply condition (m2). A nice counterexample in dimension 3 can be obtained by getting inspiration from a classical soccer ball.

Example 3.5. As is well-known, the geometric figure which the soccer ball is based on is the *truncated icosahedron* (see, e.g., Wikipedia [23]). By varying the radius from 0 to ∞, such a solid gives rise to six symmetric cones with regular pentagonal section and other ten symmetric cones with regular hexagonal section, whose boundaries are all suitably shared by one another. If we reduce a little bit the section of each cone by the same proportional ratio keeping the regular pentagonal and hexagonal shape, all the boundaries get separated from one another and we obtain a union of sixteen symmetric cones which fulfil condition (m1) of Definition 3.3. However, it is immediate to realize that no splitting plane exists.

Definition 3.6. Given a multicone K_{mul}, we can choose a specific splitting hyperplane $H = \{h\}^{\perp}$ and rearrange the signs of the conic components in order to label as positive those contained in the positive semi-space S^h_+, i.e., if

$K_{mul} = \bigcup_{i=1}^{r} K_{sym}^{(i)}$, we set

$$K_{+}^{(1)} \cup \cdots \cup K_{+}^{(r)} \setminus \{0\} \subseteq \operatorname{int}\left(S_{+}^{h}\right) \quad \text{and} \quad K_{-}^{(1)} \cup \cdots \cup K_{-}^{(r)} \setminus \{0\} \subseteq \operatorname{int}\left(S_{-}^{h}\right). \quad (7)$$

Such splitting hyperplane, chosen once for ever, will be called *labelling hyperplane* for K_{mul} in $\mathbb{R}^d$. □

Remark 3.7. Note that the hyperplane H in $(m2)$ is such that $H = \{h\}^{\perp}$, where the vector h belongs to the set $\bigcap_{i=1}^{r} \operatorname{int}\left(\left(K_{+}^{(i)}\right)^{\star}\right)$ (see Lemma 2.16). Since this set is open and nonempty, it contains infinitely many directions, corresponding to infinitely many splitting hyperplanes, all of which labelling the multicone in the same way.

It is clear that $K_{+}^{(1)} \cup \cdots \cup K_{+}^{(r)}$ is strictly supported by the chosen labelling hyperplane H and, thus, any union $K_{+}^{(i_1)} \cup \ldots \cup K_{+}^{(i_s)}$, $2 \leq s \leq r$, is strictly H-supported as well. Hence, by Proposition 2.21, all of these unions generate a cone. Of course, the same is true for the opposite unions $K_{-}^{(i_1)} \cup \ldots \cup K_{-}^{(i_s)}$.

Remark 3.8. For any pair of conic components $K_{+}^{(i)}$ and $K_{+}^{(j)}$, condition (m1) of Definition 3.3 and Theorem 2.20 guarantee the existence of a separating hyperplane (not unique), from now on denoted by H_{ij}. Obviously, the same hyperplane H_{ij} also separates $K_{-}^{(i)}$ from $K_{-}^{(j)}$.

In dimension 2 a separating hyperplane H_{ij} is necessarily also splitting. However, in dimension 3, it is easy to find examples of multicones for which this is not true. For instance, consider the soccer ball in Example 3.5.

Definition 3.9. Let K_{mul} be a multicone and let K' and K'' be two separated cones. We say that K' and K'' are *well separated with respect to K_{mul}* if there exists a hyperplane H, splitting for K_{mul}, which separates K' and K''. Otherwise, they are said to be *badly separated with respect to K_{mul}*.

In particular, if K' and K'' are themselves conic components of K_{mul}, we simply say that they are *well (respectively, badly) separated.* □

Remark 3.10. Note that, by (7), each positive component $K_{+}^{(i)}$ is always well separated from any negative component $K_{-}^{(j)}$ by the labelling hyperplane of K_{mul}.

In the sequel we shall often consider sets of conic components $K^{(i)}$ without taking care of their sign.

Definition 3.11. Let $K_{mul} = \bigcup_{i=1}^{r} K_{sym}^{(i)}$ be a multicone. If all pairs of conic components $K^{(i)}$ and $K^{(j)}$ are well separated, then we say that the multicone is *reduced.* Otherwise, it is said to be *nonreduced.* □

Clearly, the property of being reduced is intrinsic of a multicone. Therefore, it is independent of the choice of the labelling hyperplane.

In view of some applications, like those in the forthcoming Section 5, it is convenient to define a standard procedure to canonically embed a nonreduced multicone into another one which is reduced. In order to do this, we need some preliminary results.

Proposition 3.12. *Let $K_{mul} = \bigcup_{i=1}^{r} K_{sym}^{(i)}$ be a multicone and consider s of its conic components $K^{(i_1)}, \ldots, K^{(i_s)}$, where $2 \leq s \leq r-1$. Assume that $K^{(i_1)}$ is badly separated from $K^{(i_2)}, \ldots, K^{(i_s)}$, respectively, and well separated from a further component $K^{(j)}$. Then the quasi-cone generated by $K^{(i_1)} \cup \ldots \cup K^{(i_s)}$ is a cone and is well separated from $K^{(j)}$ with respect to K_{mul}.*

Proof. By assumption and by Definition 3.9, for any hyperplane $\tilde{H} = \{\tilde{h}\}^{\perp}$ splitting for K_{mul}, it is not restrictive to assume

$$K^{(i_k)} \setminus \{0\} \subseteq \operatorname{int}\left(S_{+}^{\tilde{h}}\right), \qquad k = 1, \ldots, s.$$

Therefore, by Proposition 2.21, qcone $\left(K^{(i_1)} \cup \ldots \cup K^{(i_s)}\right)$ is a cone and

$$\operatorname{cone}\left(K^{(i_1)} \cup \ldots \cup K^{(i_s)}\right) \setminus \{0\} \subseteq \operatorname{int}\left(S_{+}^{\tilde{h}}\right). \tag{8}$$

Furthermore, still by assumption, there exists a splitting hyperplane $\bar{H} = \{\bar{h}\}^{\perp}$ for K_{mul} such that

$$K^{(i_1)} \setminus \{0\} \subseteq \operatorname{int}\left(S_{+}^{\bar{h}}\right), \quad K^{(j)} \setminus \{0\} \subseteq \operatorname{int}\left(S_{-}^{\bar{h}}\right).$$

Therefore, because of (8), $\bar{H}$ separates cone $\left(K^{(i_1)} \cup \ldots \cup K^{(i_s)}\right)$ from $K^{(j)}$. □

Proposition 3.13. (Transitivity) *Let $K_{mul} = \bigcup_{i=1}^{r} K_{sym}^{(i)}$ be a multicone and $K^{(i)}$, $K^{(j)}$, $K^{(k)}$ be conic components of it. If $K^{(i)}$ is badly separated from $K^{(j)}$ and $K^{(j)}$ is badly separated from $K^{(k)}$, then $K^{(i)}$ is badly separated also from $K^{(k)}$.*

Proof. Assume, by contradiction, that $K^{(i)}$ is well separated from $K^{(k)}$. Then, by Proposition 3.12, cone $\left(K^{(i)} \cup K^{(j)}\right)$ is well separated from $K^{(k)}$ with respect to K_{mul}. In particular, $K^{(j)}$ is well separated from $K^{(k)}$ and this contradicts the assumption. □

If we conventionally say that a conic component $K^{(i)}$ is badly separated from itself, the foregoing statement tells us that the property of being badly separated is an *equivalence relation* in the set of all the conic components.

Remark 3.14. Denote by $[K^{(j)}]$ the *equivalence class* of $K^{(j)}$, $j = 1, \ldots, r$. Then Remark 3.10 implies that

$$[K_{+}^{(j)}] \subseteq \{K_{+}^{(1)}, \ldots, K_{+}^{(r)}\} \qquad \text{and} \qquad [K_{-}^{(j)}] \subseteq \{K_{-}^{(1)}, \ldots, K_{-}^{(r)}\}.$$

Therefore, if $H = \{h\}^{\perp}$ is the labelling hyperplane, we have that

$$\bigcup_{K \in [K_{+}^{(j)}]} K \subseteq \operatorname{int}\left(S_{+}^{h}\right)$$

and, consequently, Proposition 2.21 assures that the quasi-cone generated by such a set is a cone, namely,

$$\tilde{K}_+^{(j)} := \mathrm{cone}\left(\bigcup_{K \in [K_+^{(j)}]} K \right). \tag{9}$$

Up to renaming the indices, we can assume that the quotient set consists of the elements $[K_+^{(1)}], \ldots, [K_+^{(\tilde{r})}], [K_-^{(1)}], \ldots, [K_-^{(\tilde{r})}]$, where clearly $\tilde{r} \leq r$.

Definition 3.15. Let $K_{mul} = \bigcup_{i=1}^r K_{sym}^{(i)}$ be a multicone and let

$$\tilde{K}_{mul} := \bigcup_{j=1}^{\tilde{r}} \tilde{K}_{sym}^{(j)}$$

be the (reduced) multicone whose symmetric components are given by

$$\tilde{K}_{sym}^{(j)} := \tilde{K}_+^{(j)} \cup \tilde{K}_-^{(j)}$$

(see (9)). The multicone $\tilde{K}_{mul}$ is called the *reduction* of K_{mul} and its fragmentation index $\tilde{r}$ is called the *reduced fragmentation index* of K_{mul}. □

Such a construction starts from a nonreduced multicone and ends with a reduced one, obviously uniquely determined.

Remark 3.16. It is clear that $K_{mul} \subseteq \tilde{K}_{mul}$ and $\tilde{r} \leq r$. Furthermore, K_{mul} is reduced if and only if the above inclusion is an equality.

In analogy to the definition given for cones, we introduce the following notion.

Definition 3.17. A multicone K_{mul} is said to be *proper* if each of its conic components is a proper cone. If we only require that $\mathrm{span}(K_{mul}) = \mathbb{R}^d$, we say that K_{mul} is *weakly proper.* □

In particular, if $r = 1$, we have a *proper symmetric cone* and, in this case, the notions of proper and weakly proper obviously coincide.

We point out that we cannot restrict ourselves to consider proper multicones only. In fact, although in the sequel we will be mainly interested in the duality of proper multicones, their dual sets will possibly have some degenerate components.

To this aim, it is useful to give the following notion.

Definition 3.18. The largest proper multicone, if any, contained in K_{mul} is said its *proper core* and is denoted by $\mathrm{pr}(K_{mul})$.

Note that $\mathrm{pr}(K_{mul})$ equals the union of its proper conic components.

Remark 3.19. Note that $\mathrm{pr}(K_{mul}) = \mathrm{cl}(\mathrm{int}(K_{mul}))$ and $\mathrm{int}(\mathrm{pr}(K_{mul})) = \mathrm{int}(K_{mul})$. Observe also that the "pr" operator preserves the inclusion, i.e.,

$$K_{mul}^{(1)} \subseteq K_{mul}^{(2)} \quad \Longrightarrow \quad \mathrm{pr}(K_{mul}^{(1)}) \subseteq \mathrm{pr}(K_{mul}^{(2)}). \tag{10}$$

4. Duality of symmetric cones and multicones

The notion of duality cannot be directly extended to symmetric cones. Namely, if $K_{sym} = K_+ \cup K_-$ is a solid symmetric cone, then Proposition 2.13(iii) implies $K^\star_{sym} = (K_+ \cup K_-)^\star = K^\star_+ \cap K^\star_- = \{0\}$.

So we need to introduce a suitable notion of duality, regarding symmetric cones, which is advisable to denote by a different symbol.

Definition 4.1. Let $K_{sym} = K_+ \cup K_-$ be a symmetric cone of $\mathbb{R}^d$. Then

$$\begin{aligned} K^\dagger_{sym} : &= K^\star_+ \cup K^\star_- = K^\star_+ \cup -K^\star_+ \\ &= \left\{ y \in \mathbb{R}^d \;\middle|\; y^T x \geq 0 \; \forall x \in K_+ \quad \text{or} \quad y^T x \geq 0 \; \forall x \in K_- \right\} \end{aligned}$$

is called the *dual set* of K_{sym}. □

Observe that, in general, $K^\dagger_{sym}$ is not a symmetric cone itself, but the union of two quasi-cones only (though still symmetric with respect to the origin).

Indeed, by Corollary 2.15, $K^\dagger_{sym}$ is a symmetric cone if and only if K_{sym} is proper and, in this case, $K^\dagger_{sym}$ is proper, too.

Example 4.2. Consider the symmetric cone $K_{sym} = K_+ \cup K_- \subset \mathbb{R}^3$, where

$$K_+ := \{(x_1, x_2, x_3) \in \mathbb{R}^3 \;\Big|\; |x_1| \leq x_2/2, \; x_3 = 0\}.$$

Then
$$K^\star_+ = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \;\Big|\; |x_1| \leq 2x_2\}.$$

In particular, $L(K^\star_+)$ is the x_3-axis. So $K^\star_+$ is a nonpointed quasi-cone and, hence, $K^\dagger_{sym}$ is not a symmetric cone. □

It is straightforward to verify that most of the properties of the duality of cones (for instance, reflexivity and those collected in Proposition 2.17) extend to proper dual symmetric cones. In particular, the following result guarantees that also † is a "geometric duality".

Proposition 4.3. *If K_{sym} is a proper symmetric cone, then*

$$\mathrm{int}(K^\dagger_{sym}) = \left\{ y \in \mathbb{R}^d \;\middle|\; y^T x \neq 0 \quad \forall x \in K_{sym} \setminus \{0\} \right\}, \tag{11}$$

$$K^\dagger_{sym} \setminus \{0\} = \left\{ y \in \mathbb{R}^d \;\middle|\; y^T x \neq 0 \quad \forall x \in \mathrm{int}(K_{sym}) \right\} \tag{12}$$

and
$$K^{\dagger\dagger}_{sym} = K_{sym}. \tag{13}$$

Furthermore, if $K^{(1)}_{sym}$, $K^{(2)}_{sym}$ are symmetric cones, then

$$K^{(1)}_{sym} \subseteq K^{(2)}_{sym} \quad \Longleftrightarrow \quad \left(K^{(1)}_{sym}\right)^\dagger \supseteq \left(K^{(2)}_{sym}\right)^\dagger. \tag{14}$$

Proof. First note that $\mathrm{int}(K^\dagger_{sym}) = \mathrm{int}(K^\star_+) \cup \mathrm{int}(K^\star_-)$.

Therefore, Proposition 2.17 implies that $\mathrm{int}(K^\dagger_{sym})$ equals the set

$$\left\{y \in \mathbb{R}^d \;\middle|\; y^T x > 0 \;\forall x \in K_+ \setminus \{0\} \;\text{ or }\; y^T x > 0 \;\forall x \in K_- \setminus \{0\}\right\}$$

and that $K^\dagger_{sym} \setminus \{0\}$ equals the set

$$\left\{y \in \mathbb{R}^d \;\middle|\; y^T x > 0 \;\forall x \in \mathrm{int}(K_+) \;\text{ or }\; y^T x > 0 \;\forall x \in \mathrm{int}(K_-)\right\}.$$

Hence, the equalities (11) and (12) follow from the continuity of the scalar product $y^T x$ with respect to x and from the fact that $K_+ \setminus \{0\}$ and $K_- \setminus \{0\}$ are connected sets. Finally, (13) follows immediately from (3) and (14) from (4). □

In order to give a convenient extension of the notion of duality from symmetric cones to multicones, it is reasonable to require that the latter be compatible with the union operator, i.e., that it fulfils (iii) of Proposition 2.13.

Thus, the most natural definition of the dual of a multicone is the intersection of the dual sets of its symmetric components. More precisely, if $K_{mul} = \bigcup_{i=1}^r K^{(i)}_{sym}$, its "dual" should be $\bigcap_{i=1}^r (K^{(i)}_{sym})^\dagger$, even if this set is not necessarily a multicone (see Example 4.2, where $r = 1$).

For this reason, in the general case, we can only give the following notion.

Definition 4.4. Let $K_{mul} = \bigcup_{i=1}^r K^{(i)}_{sym}$ be a multicone. Then

$$K^\dagger_{mul} := \bigcap_{i=1}^r (K^{(i)}_{sym})^\dagger$$

is called the *dual set* of K_{mul}.

From the above definition and (14), it is straightforward to prove the following result.

Proposition 4.5. *Let $K^{(1)}_{mul}$ and $K^{(2)}_{mul}$ be two multicones. Then*

$$K^{(1)}_{mul} \subseteq K^{(2)}_{mul} \quad \Longrightarrow \quad \left(K^{(1)}_{mul}\right)^\dagger \supseteq \left(K^{(2)}_{mul}\right)^\dagger. \tag{15}$$

In order to study the structure of the dual set $K^\dagger_{mul}$, we introduce the auxiliary notion of *associated quasi-cone*.

Given a multicone $K_{mul} = \bigcup_{i=1}^r K^{(i)}_{sym}$, consider the generic union of r conic components

$$K_{\sigma_1 \dots \sigma_r} := K^{(1)}_{\sigma_1} \cup \cdots \cup K^{(r)}_{\sigma_r},$$

where $\sigma_i \in \{+, -\}$, and let $\qquad \tilde{K} := \mathrm{qcone}(K_{\sigma_1 \dots \sigma_r})$.

Clearly, the total number of the $K_{\sigma_1 \dots \sigma_r}$'s is 2^r and, for all of them,

$$K_{mul} = K_{\sigma_1 \cdots \sigma_r} \cup K_{-\sigma_1 \cdots -\sigma_r}.$$

We can summarize all of the possible occurencies as follows, taking Proposition 2.21 into account.

Proposition 4.6. *Let $K_{\sigma_1 \dots \sigma_r}$ and $\tilde{K}$ be as before. Then one of the following three cases necessarily occurs:*

(i) $l(\tilde{K}) = 0$, *which is equivalent to each of the following conditions:*
  (i$_1$) $\tilde{K}$ *is a cone;*
  (i$_2$) $K_{\sigma_1 \dots \sigma_r}$ *(and hence $\tilde{K}$) is strictly supported by some hyperplane $\bar{H}$.*
  In particular, $\bar{H}$ is a splitting hyperplane for K_{mul} which separates each of the $K^{(i)}_{\sigma_i}$'s from all of the opposite conic components $K^{(1)}_{-\sigma_1}, \dots, K^{(r)}_{-\sigma_r}$.

(ii) $0 < l(\tilde{K}) < d$, *which is equivalent to each of the following conditions:*
  (ii$_1$) $\tilde{K} \neq \mathbb{R}^d$ *is a quasi-cone, but not a cone;*
  (ii$_1$) $K_{\sigma_1 \dots \sigma_r}$ *(and hence $\tilde{K}$) is supported by some hyperplane $\bar{H}$, but is not strictly supported by any H.*

(iii) $l(\tilde{K}) = d$ *or, equivalently, $\tilde{K} = \mathbb{R}^d$.*

Now we explicitly provide an example of a proper multicone $K_{mul} = \bigcup_{i=1}^{3} K^{(i)}_{sym}$ whose associated quasi-cones are not all cones. In other words, we give an example of case (ii) in Proposition 4.6.

Example 4.7. Consider the proper symmetric cone of $\mathbb{R}^3$

$$K^{(1)}_{sym}: \quad 9x_1^2 + 8x_3^2 - 6x_2x_3 \leq 0.$$

Its boundary is the (circular) conic surface whose axis is the line $(x_1, x_2, x_3) = (0, 3t, t)$, $t \in \mathbb{R}$, with the x_2-axis as a generatrix. Moreover, such a surface is tangent to the plane H_3 of equation $x_3 = 0$ along the x_2-axis. Furthermore, let

$$A = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

be the anticlockwise $(2\pi/3)$-rotation matrix around the x_3-axis and set

$$K^{(2)}_{sym} := A(K^{(1)}_{sym}) \qquad \text{and} \qquad K^{(3)}_{sym} := A^2(K^{(1)}_{sym}).$$

We can write each of the above symmetric cones as the union of its conic components, i.e.,

$$K^{(i)}_{sym} = K^{(i)} \cup -K^{(i)}, \qquad i = 1, 2, 3,$$

where $K^{(i)}$ is included in the semi-space S_3^+ defined by $x_3 \geq 0$.

Now consider the set

$$K_{mul} := K^{(1)}_{sym} \cup K^{(2)}_{sym} \cup K^{(3)}_{sym}.$$

It turns out that the $K^{(i)}_{sym}$'s satisfy condition (m1) of Definition 3.3 and that the plane H_2 of equation $x_2 = 0$ is splitting for it. Therefore, K_{mul} is a proper multicone. Moreover, it is reduced since the planes H_2, $A(H_2)$ and $A^2(H_2)$ separate all the possible pairs of conic components and each of them is splitting for K_{mul}.

Nevertheless, the associated quasi-cone given by

$$\mathrm{qcone}(K^{(1)} \cup K^{(2)} \cup K^{(3)})$$

is not a cone, namely it is the entire semi-space S_3^+.

Accordingly, the union $K^{(1)} \cup K^{(2)} \cup K^{(3)}$ cannot be "separated" from

$$-K^{(1)} \cup -K^{(2)} \cup -K^{(3)}.$$

In fact, the only plane candidate to do so would be H_3, whereas, for each pair (i,j), it does not separate $K^{(i)}$ from $-K^{(j)}$, since it is tangent to both of them.

Consequently, H_3 is not splitting and, so, it cannot be labelling for K_{mul}. $\square$

Now set $\Sigma := \{(\sigma_1,\ldots,\sigma_r) \in \{+,-\}^r \mid K_{\sigma_1\ldots\sigma_r} \text{ is supported by some } \bar{H}\}$

and $\Sigma_+ := \{(+,\sigma_2,\ldots,\sigma_r) \in \Sigma\}, \quad \Sigma_- := \{(-,\sigma_2,\ldots,\sigma_r) \in \Sigma\}.$

Clearly, $\Sigma = \Sigma_+ \cup \Sigma_-, \quad \Sigma_+ \cap \Sigma_- = \emptyset, \quad \Sigma_- = -\Sigma_+.$

Definition 4.8. Let $r^\dagger$ be the cardinality of Σ_+ and, for $k = 1,\ldots,r^\dagger$, set

$$\hat{K}_+^{(k)} := \mathrm{qcone}\left(K_{+\,\sigma_2(k)\cdots\sigma_r(k)}\right), \quad (+,\sigma_2(k),\ldots,\sigma_r(k)) \in \Sigma_+$$

and $\hat{K}_-^{(k)} := -\hat{K}_+^{(k)}$. We say that $\hat{K}_\pm^{(1)},\ldots,\hat{K}_\pm^{(r^\dagger)}$ are the *quasi-cones associated* to K_{mul} and $r^\dagger$ is called the *dual fragmentation index* of K_{mul}. $\square$

Note that, since $K_{mul} \subseteq \hat{K}_+^{(k)} \cup \hat{K}_-^{(k)}$ for all $k = 1,\ldots,r^\dagger$, we obviously have

$$K_{mul} \subseteq \bigcap_{k=1}^{r^\dagger} (\hat{K}_+^{(k)} \cup \hat{K}_-^{(k)}). \tag{16}$$

Now we are able to express the dual set of K_{mul} in terms of its associated quasi-cones.

Theorem 4.9. *Let $K_{mul} = \bigcup_{i=1}^r K_{sym}^{(i)}$ be a multicone and let $\hat{K}_\pm^{(1)},\ldots,\hat{K}_\pm^{(r^\dagger)}$ be its associated quasi-cones. Then the following facts hold:*

(i) *the dual set of K_{mul} can be written as*

$$K_{mul}^\dagger = \bigcup_{k=1}^{r^\dagger} \left(\hat{K}_+^{(k)}\right)^\star \cup \left(\hat{K}_-^{(k)}\right)^\star; \tag{17}$$

(ii) *if K_{mul} is weakly proper, then the $\left(\hat{K}_\pm^{(k)}\right)^\star$'s are cones and at least one of them is solid;*

(iii) *if, in addition, K_{mul} is proper, then $K_{mul}^\dagger$ is a multicone (clearly weakly proper).*

Proof. (i) Lemma 2.18 yields

$$\left(\hat{K}_+^{(k)}\right)^\star = \left(K_{+\,\sigma_2(k)\ldots\sigma_r(k)}\right)^\star = \left(K_+^{(1)}\right)^\star \cap \cdots \cap \left(K_{\sigma_r}^{(r)}\right)^\star$$

where $(+, \sigma_2(k), \ldots, \sigma_r(k)) \in \Sigma_+$. We then obtain

$$\bigcup_{k=1}^{r^\dagger} \left(\hat{K}_+^{(k)}\right)^\star \cup \left(\hat{K}_-^{(k)}\right)^\star = \bigcup_{(\sigma_1,\ldots,\sigma_r)\in\Sigma} \left(K_{\sigma_1}^{(1)}\right)^\star \cap \cdots \cap \left(K_{\sigma_r}^{(r)}\right)^\star . \tag{18}$$

On the other hand,

$$K_{mul}^\dagger = \bigcap_{i=1}^{r} \left(K_{sym}^{(i)}\right)^\dagger = \bigcup_{(\sigma_1,\ldots,\sigma_r)\in\{+,-\}^r} \left(K_{\sigma_1}^{(1)}\right)^\star \cap \cdots \cap \left(K_{\sigma_r}^{(r)}\right)^\star ,$$

where the first equality comes from Definition 4.4 and the second one by standard set computations.

We compare the above union with (18) and take into account that, by Lemma 2.18,

$$\left(K_{\sigma_1}^{(1)}\right)^\star \cap \cdots \cap \left(K_{\sigma_r}^{(r)}\right)^\star = (K_{\sigma_1\ldots\sigma_r})^\star = \{0\}$$

whenever $(\sigma_1, \ldots, \sigma_r) \notin \Sigma$. Thus we obtain (17).

(ii) Now assume that K_{mul} is weakly proper. Then each $\hat{K}_\pm^{(k)}$ is a solid quasi-cone and, hence, $\left(\hat{K}_\pm^{(k)}\right)^\star$ is a cone (see Corollary 2.15).

Moreover, $\operatorname{cone}(K_+^{(1)} \cup \cdots \cup K_+^{(r)})$ is strictly supported by the labelling hyperplane H. Hence, its dual is solid, and this proves the second claim.

(iii) Finally, assume that K_{mul} is proper. In order to show that the set $K_{mul}^\dagger$ given by (17) is a multicone, we are left to verify conditions (m1) and (m2) of Definition 3.3.

Concerning condition (m1), we have to prove that

$$\left(\left(\hat{K}_+^{(k)}\right)^\star \cup \left(\hat{K}_-^{(k)}\right)^\star\right) \cap \left(\left(\hat{K}_+^{(h)}\right)^\star \cup \left(\hat{K}_-^{(h)}\right)^\star\right) = \{0\}$$

if $k \neq h$. By elementary set theoretical properties and Proposition 2.13(iii), it turns out that the above intersection is the union of four components of the type

$$\left(K_{\sigma_1(k)\ldots\sigma_r(k)} \cup K_{\sigma_1(h)\ldots\sigma_r(h)}\right)^\star .$$

Since $k \neq h$, the set $K_{\sigma_1(k)\ldots\sigma_r(k)} \cup K_{\sigma_1(h)\ldots\sigma_r(h)}$ entirely contains at least one symmetric component of K_{mul}, whose dual set is just $\{0\}$, since K_{mul} is proper by assumption. Hence, (m1) is fulfilled.

Concerning condition (m2), it is enough to show that there exists a hyperplane $\bar{H}$ separating all the positive conic components $(\hat{K}_+^{(k)})^\star$ of $K_{mul}^\dagger$ from all the negative ones $(\hat{K}_-^{(k)})^\star$.

Indeed, from Definition 4.8 and Lemma 2.18, we have that $(\hat{K}_+^{(k)})^\star \subseteq (K_+^{(1)})^\star$ for all k and, analogously, $(\hat{K}_-^{(k)})^\star \subseteq (K_-^{(1)})^\star$. Thus we conclude by observing that $(K_+^{(1)})^\star$ and $(K_-^{(1)})^\star$ are separated by a suitable hyperplane $\bar{H}$, since $K_{sym}^{(1)}$ is a proper symmetric cone. $\square$

Note that, even if K_{mul} is proper, its dual $K^{\dagger}_{mul}$ is a weakly proper multicone (see Theorem 4.9), but not necessarily proper. For instance, the x_3-axis is a symmetric degenerate component of the dual $K^{\dagger}_{mul}$ of the multicone considered in Example 4.7. Namely, the semi-spaces S_3^+ and S_3^- are associated quasi-cones of K_{mul}.

Observe that, if the multicone K_{mul} is a symmetric cone K_{sym} (i.e., if $r = 1$), then the set (17) is nothing but the dual set $K^{\dagger}_{sym}$.

Some useful observations can be done if K_{mul} is proper.

We start by writing the form (17) of the dual multicone in a more compact way by defining the symmetric cones

$$K^{\dagger\,(k)}_{sym} := \left(\hat{K}^{(k)}_{+}\right)^{\star} \cup \left(\hat{K}^{(k)}_{-}\right)^{\star},$$

so that

$$K^{\dagger}_{mul} = \bigcup_{k=1}^{r^{\dagger}} K^{\dagger\,(k)}_{sym}. \tag{19}$$

Remark 4.10. As we saw in the proof of Theorem 4.9 when K_{mul} is proper, the dual $K^{\dagger}_{mul}$ admits a splitting hyperplane which separates all the conic components related to Σ_+ from those related to Σ_-. In other words, there exists a canonical labelling for $K^{\dagger}_{mul}$ which is induced by the labelling of K_{mul}.

The properties of the duality of proper symmetric cones stated by Proposition 4.3 are inherited by the duality of proper multicones, possibly weakened somehow.

Proposition 4.11. *If K_{mul} is a proper multicone of $\mathbb{R}^d$, then*

$$\operatorname{int}(K^{\dagger}_{mul}) = \left\{ y \in \mathbb{R}^d \;\middle|\; y^T x \neq 0 \quad \forall x \in K_{mul} \setminus \{0\} \right\}, \tag{20}$$

$$K^{\dagger}_{mul} \setminus \{0\} = \left\{ y \in \mathbb{R}^d \;\middle|\; y^T x \neq 0 \quad \forall x \in \operatorname{int}(K_{mul}) \right\}. \tag{21}$$

Furthermore, if $K^{(1)}_{mul}$ and $K^{(2)}_{mul}$ are proper multicones, then

$$K^{(1)}_{mul} \setminus \{0\} \subseteq \operatorname{int}(K^{(2)}_{mul}) \quad \Longrightarrow \quad \operatorname{int}\left((K^{(1)}_{mul})^{\dagger}\right) \supseteq \left(K^{(2)}_{mul}\right)^{\dagger} \setminus \{0\}. \tag{22}$$

Proof. First recall that the "interior operator" distributes on finite intersections of sets and, thanks to (m1) of Definition 3.3, also on unions of conic components of a multicone. Thus, taking into account that

$$K^{\dagger}_{mul} = \bigcap_{i=1}^{r} (K^{(i)}_{sym})^{\dagger}$$

and using elementary set theoretical properties, (11) yields (20) and (12) yields (21). Finally, the implication (22) immediately follows from (20) and (21). □

In general, the *bidual set* $K^{\dagger\dagger}_{mul}$ is not a multicone (see Theorem 4.9 and Example 4.7). In any case, the following property holds.

Proposition 4.12. *If K_{mul} is a proper multicone, then*

$$K_{mul} \subseteq K_{mul}^{\dagger\dagger}. \tag{23}$$

Proof. Note that $K_{mul}^{\dagger\dagger} = \bigcap_{k=1}^{r^\dagger} \left(K_{sym}^{\dagger\,(k)}\right)^\dagger = \bigcap_{k=1}^{r^\dagger} (\hat{K}_+^{(k)} \cup \hat{K}_-^{(k)})$,

where the first equality follows from Definition 4.4 and (19) and the second one from the reflexivity of quasi-cones. The inclusion (16) concludes the proof. □

As observed in Theorem 4.9(ii), $K_{mul}^\dagger$ may possess degenerate conic components, but not all. Thus, if we remove them, we obtain a proper multicone which is nonempty (its proper core, see Definition 3.18).

Definition 4.13. Let K_{mul} be a proper multicone. Then the set consisting of all the proper conic components of $K_{mul}^\dagger$ is called the *proper dual* multicone of K_{mul} and is denoted by $K_{mul}^\times$, i.e., $K_{mul}^\times = \mathrm{pr}(K_{mul}^\dagger)$. □

From the description of $K_{mul}^\dagger$ given in (19), it is clear that $K_{mul}^\times$ is the union of the symmetric cones $K_{sym}^{\dagger\,(k)}$, where $\left(\hat{K}_+^{(k)}\right)^\star$ is a proper cone or, equivalently (by Corollary 2.15), where $\hat{K}_+^{(k)}$ is a proper cone. In particular, it is clear that

$$\mathrm{int}(K_{mul}^\times) = \mathrm{int}(K_{mul}^\dagger). \tag{24}$$

By taking Proposition 4.6 and the subsequent discussion into account and by setting

$$\Sigma^\times := \{(\sigma_1, \ldots, \sigma_r) \in \{+,-\}^r \mid K_{\sigma_1\ldots\sigma_r} \text{ is strictly supported by some } \bar{H}\},$$

whose cardinality is denoted by $2r^\times$, we obtain immediately the following fact.

Proposition 4.14. *With the above notation, we have*

$$K_{mul}^\times = \bigcup_{(\sigma_1,\ldots,\sigma_r)\in\Sigma^\times} (K_{\sigma_1\ldots\sigma_r})^\star .$$

Equivalently, if we assume, without loss of generality, that the first $r^\times$ elements in the set $\{K_{sym}^{\dagger\,(1)}, \ldots, K_{sym}^{\dagger\,(r^\dagger)}\}$ are symmetric proper cones, we have

$$K_{mul}^\times = \bigcup_{k=1}^{r^\times} K_{sym}^{\dagger\,(k)}. \tag{25}$$

Note that, in the case of the proper dual, (25) is the analog of the expression (19) of the dual multicone. Hence, it is natural to introduce the following further numerical invariant for a multicone.

Definition 4.15. Let K_{mul} be a proper multicone. Then the number $r^\times$ of the symmetric components of $K_{mul}^\times$ is called its *proper dual fragmentation index.*

Remark 4.16. The previous construction reveals that the number of possible labellings of K_{mul} is exactly $r^\times$.

Thanks to Definition 4.13, the inclusion (10) and Proposition 4.5, we get the following fact.

Proposition 4.17. *Let $K_{mul}^{(1)}$ and $K_{mul}^{(2)}$ be two proper multicones. Then*

$$K_{mul}^{(1)} \subseteq K_{mul}^{(2)} \quad \Longrightarrow \quad \left(K_{mul}^{(1)}\right)^{\times} \supseteq \left(K_{mul}^{(2)}\right)^{\times}. \tag{26}$$

The notion of proper dual multicone shows the low importance of nonreduced multicones.

Keeping the notation introduced in the previous section (see (9)), if K is a conic component of K_{mul}, by $[K]$ we denote the equivalence class of the conic components badly separated from K.

Lemma 4.18. *If $K_{\sigma_1\dots\sigma_r}$ is strictly supported by some hyperplane $\bar{H}$, then*

$$[K_{\sigma_1}^{(1)}] \cup \cdots \cup [K_{\sigma_r}^{(r)}] = \{K_{\sigma_1}^{(1)}, \dots, K_{\sigma_r}^{(r)}\}.$$

Proof. Let us first show that, if $K_+^{(i)}$ and $K_+^{(j)}$ are two badly separated conic components, then $K_+^{(i)} \subseteq K_{\sigma_1\dots\sigma_r}$ if and only if $K_+^{(j)} \subseteq K_{\sigma_1\dots\sigma_r}$ (i.e., $\sigma_i = +$ if and only if $\sigma_j = +$).

To this aim, we assume by contradiction that $K_+^{(i)} \subseteq K_{\sigma_1\dots\sigma_r}$ and $K_+^{(j)} \nsubseteq K_{\sigma_1\dots\sigma_r}$. So $K_-^{(j)} \subseteq K_{\sigma_1\dots\sigma_r}$ and, thus, setting as usual $\bar{H} = \{\bar{h}\}^{\perp}$, without loss of generality we can suppose that $K_{\sigma_1\dots\sigma_r} \setminus \{0\} \subset \mathrm{int}(S_+^{\bar{h}})$. Therefore, also $K_+^{(i)} \setminus \{0\}$ and $K_-^{(j)} \setminus \{0\}$ are contained in $\mathrm{int}(S_+^{\bar{h}})$. This implies that $K_+^{(j)} \setminus \{0\}$ is contained in $\mathrm{int}(S_-^{\bar{h}})$ and, hence, that $K_+^{(i)}$ and $K_+^{(j)}$ are well separated by the splitting hyperplane $\bar{H}$, against the assumption.

Therefore, we have shown that $[K_{\sigma_i}^{(i)}] \subseteq \{K_{\sigma_1}^{(1)}, \dots, K_{\sigma_r}^{(r)}\}$ for all $i = 1, \dots, r$ and, hence, that $[K_{\sigma_1}^{(1)}] \cup \cdots \cup [K_{\sigma_r}^{(r)}] \subseteq \{K_{\sigma_1}^{(1)}, \dots, K_{\sigma_r}^{(r)}\}$. The opposite inclusion is obvious. □

Note that the above result generalizes Remark 3.14.

Theorem 4.19. *Let $\tilde{K}_{mul}$ be the reduction of a proper multicone K_{mul}. Then*

$$K_{mul}^{\times} = \left(\tilde{K}_{mul}\right)^{\times} \tag{27}$$

and, consequently, their proper dual fragmentation indices coincide.

Proof. By Lemma 4.18, it is easy to see that K_{mul} and $\tilde{K}_{mul}$ have the same associated quasi-cones which are strictly supported by a suitable hyperplane $\bar{H}$. Therefore, by the expression of the proper dual given in (25), we get the result. □

The analogous property of Proposition 4.12 holds also for the proper dual.

Proposition 4.20. *If K_{mul} is a proper multicone, then*

$$K_{mul} \subseteq K_{mul}^{\times\times}. \tag{28}$$

Proof. Since $K^{\times}_{mul} \subseteq K^{\dagger}_{mul}$, we have that

$$K_{mul} \subseteq K^{\dagger\dagger}_{mul} \subseteq K^{\times\dagger}_{mul},$$

where the first inclusion follows from (23) and the second from (15).

Moreover, since $K^{\times\times}_{mul} = \mathrm{pr}(K^{\times\dagger}_{mul})$, we then obtain $K_{mul} \subseteq K^{\times\times}_{mul}$, since K_{mul} is proper by assumption. □

In dimension 2 any proper multicone K_{mul} satisfies the equality $K^{\times}_{mul} = K^{\dagger}_{mul}$ and, moreover, the inclusions (23) and (28) are always equalities.

This is no longer true in dimension $d \geq 3$, as we shall see soon. Therefore, we are led to give the following definitions, according to the terminology used in the abstract geometric duality context.

Definition 4.21. We say that a proper multicone K_{mul} is *reflexive* if we have $K^{\dagger\dagger}_{mul} = K_{mul}$ and *properly reflexive* if $K^{\times\times}_{mul} = K_{mul}$ holds. □

We conclude this section with some results regarding properly reflexive multicones.

Proposition 4.22. *If a proper multicone K_{mul} is properly reflexive, then it is also reduced.*

Proof. We always have that $K_{mul} \subseteq \tilde{K}_{mul}$ without any assumption (see Remark 3.16). Conversely, note that

$$\tilde{K}_{mul} \subseteq \left(\tilde{K}_{mul}\right)^{\times\times} = K^{\times\times}_{mul},$$

where the inclusion follows from (28) applied to $\tilde{K}_{mul}$ and the equality from (27). Since, by assumption, $K_{mul} = K^{\times\times}_{mul}$, we get $\tilde{K}_{mul} \subseteq K_{mul}$. □

The foregoing proposition says that, if K_{mul} is not reduced, then it is not properly reflexive. As far as the existence of nonreduced multicones is concerned (clearly, for $d \geq 3$), an example can easily be constructed by slightly modifying the "soccer ball" in Example 3.5.

The next result comes immediately from Propositions 4.5 and 4.17.

Proposition 4.23. *Let $K^{(1)}_{mul}$ and $K^{(2)}_{mul}$ be two proper multicones. If they are reflexive, then*

$$K^{(1)}_{mul} \subseteq K^{(2)}_{mul} \qquad \Longleftrightarrow \qquad \left(K^{(1)}_{mul}\right)^{\dagger} \supseteq \left(K^{(2)}_{mul}\right)^{\dagger}$$

whereas, if they are properly reflexive, then

$$K^{(1)}_{mul} \subseteq K^{(2)}_{mul} \qquad \Longleftrightarrow \qquad \left(K^{(1)}_{mul}\right)^{\times} \supseteq \left(K^{(2)}_{mul}\right)^{\times}.$$

Proposition 4.24. *Let K_{mul} be a proper multicone. Then $K^{\times}_{mul}$ is properly reflexive (i.e., $K^{\times\times\times}_{mul} = K^{\times}_{mul}$) and, hence, is reduced.*

Proof. On one hand, we can apply (28) to the proper multicone $K_{mul}^{\times}$, obtaining $K_{mul}^{\times} \subseteq K_{mul}^{\times\times\times}$. On the other hand, Proposition 4.17 applied to (28) itself, yields $K_{mul}^{\times} \supseteq K_{mul}^{\times\times\times}$.

The last claim follows from Proposition 4.22. $\square$

5. Matrices with an invariant multicone

Not all the properties of matrices having an invariant cone transfer directly to matrices having invariant symmetric cones and/or multicones.

First of all, already in dimension 2, it is easy to find pairs (A, K_{mul}) such that Corollary 2.25 does not hold, i.e., $\rho(A) > 0$ but $\operatorname{int}(K_{mul}) \cap \ker(A) \neq \emptyset$, as the following example shows.

Example 5.1. Consider the matrix $A := \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix}$ and let $K_{sym} := \mathbb{R}^2_+ \cup -\mathbb{R}^2_+$. Obviously, K_{sym} is invariant for A and $\rho(A) = 1$. Nevertheless, since $\ker(A)$ is the line of equation $x_1 = x_2$, it holds that

$$\operatorname{int}(K_{sym}) \cap \ker(A) \neq \emptyset. \qquad \square$$

Moreover, it is also easy to find pairs (A, K_{mul}) for which the dual $K_{mul}^{\dagger}$ is not invariant for the transpose matrix A^T.

Example 5.2. Consider the pair (A, K_{sym}) of the previous Example 5.1. We have that $\begin{pmatrix} 1 \\ 1 \end{pmatrix} \in K_{sym}^{\dagger} (= K_{sym})$ and $A^T = \begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix}$ but, clearly,

$$A^T \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \notin K_{sym}^{\dagger}. \qquad \square$$

Finally, the analogous result of the Perron-Frobenius theorem does not hold either.

Example 5.3. Consider the proper multicone $K_{mul} = K_{sym}^{(1)} \cup K_{sym}^{(2)}$ of $\mathbb{R}^2$, where

$$K_+^{(1)} := \{(x_1, x_2)^T \mid x_2 \geq 3|x_1|\} \qquad \text{and} \qquad K_+^{(2)} := \{(x_1, x_2)^T \mid x_1 \geq 3|x_2|\}$$

and the $(\pi/2)$-rotation matrix $A := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

It is immediate to see that K_{mul} is invariant for A, but no eigenvector of A lies in K_{mul}. Namely, in this case, they all are not real. $\square$

Therefore, we want to look for sufficient conditions to guarantee that the properties of the invariant cones keep on holding in the more general context of multicones. We begin with the following *localization* result.

Proposition 5.4. *Let a multicone $K_{mul} = \bigcup_{i=1}^{r} K_{sym}^{(i)}$ be invariant for a matrix A. Then, for each symmetric component $K_{sym}^{(i)}$, there exists a symmetric component $K_{sym}^{(j)}$, possibly different, such that*

$$A(K_{sym}^{(i)}) \subseteq K_{sym}^{(j)}. \tag{29}$$

Moreover, if

$$\mathrm{int}(K_{mul}) \cap \ker(A) = \emptyset, \tag{30}$$

then, for each proper conic component $K^{(i)}$, if any, there exists a conic component $K^{(j)}$, possibly different, such that

$$A(K^{(i)}) \subseteq K^{(j)}. \tag{31}$$

Proof. Suppose there exist two distinct vectors $x, y \in K^{(i)}$ such that $Ax \in K^{(j)}$ and $Ay \in K^{(k)}$. Since $K^{(i)}$ is convex, it contains the segment $\overline{x\,y}$ and, thus, $\overline{Ax\,Ay} = A(\overline{x\,y}) \subseteq K_{mul}$ by the invariance of K_{mul}.

Since $\overline{Ax\,Ay}$ is a segment, condition (m1) of Definition 3.3 necessarily implies that either $K^{(j)} = K^{(k)}$ or $K^{(j)} = -K^{(k)}$ and that, consequently, (29) is proved.

Now let $K^{(i)}$ be a proper conic component and assume that (30) holds. Then consider $x, y \in \mathrm{int}(K^{(i)})$ with $x \neq y$ and assume $Ax \in K^{(j)}$ and $Ay \in -K^{(j)}$. With an analogous argument as before we obtain the existence of $z \in \overline{x\,y} \subset \mathrm{int}(K^{(i)})$ such that $Az = 0$. But this fact contradicts assumption (30) and, therefore, $A(\mathrm{int}(K^{(i)}) \subseteq K^{(j)}$ and so, by continuity, (31) is proved. □

Note that condition (30) is not redundant, as shown in the following example.

Example 5.5. Consider again the pair (A, K_{sym}) of Example 5.1.

In this case, (30) does not hold and (31) is not satisfied either. In fact,

$$\begin{pmatrix}1\\2\end{pmatrix}, \begin{pmatrix}2\\1\end{pmatrix} \in \mathbb{R}^2_+, \text{ but } A\begin{pmatrix}1\\2\end{pmatrix} \in \mathbb{R}^2_+ \text{ and } A\begin{pmatrix}2\\1\end{pmatrix} \in \mathbb{R}^2_-. \qquad \square$$

The following corollary is immediate, the symmetric components of a multicone being finitely many.

Corollary 5.6. *Let a multicone $K_{mul} = \bigcup_{i=1}^{r} K_{sym}^{(i)}$ be invariant for a matrix A. Then there exist a positive integer $s \leq r$ and a symmetric component $K_{sym}^{(k)}$ with*

$$A^s(K_{sym}^{(k)}) \subseteq K_{sym}^{(k)}.$$

Moreover, if K_{mul} is proper and (30) holds, then there exist a positive integer $p \leq 2r$ and a conic component $K^{(k)}$ such that

$$A^p(K^{(k)}) \subseteq K^{(k)}.$$

Example 5.7. Consider the pair (A, K_{mul}) of the previous Example 5.3. Keeping the notation of the above corollary, we have that the fragmentation index of K_{mul} is $r = 2$ and it is immediate to see that $2r = 4$ is the minimum exponent p such that each conic component of K_{mul} is invariant for A^p (in fact, $A^4 = I$). Note that, in this case, (30) holds since $\ker(A) = \{0\}$. □

The above result allows us to suitably extend Theorem 2.24 to symmetric cones and multicones.

Theorem 5.8. *Let a proper symmetric cone $K_{sym} = K_+ \cup K_-$ be invariant for a matrix A. Then*

(i) $\pm\rho(A)$ *is an eigenvalue of A;*

(ii) *the symmetric cone K_{sym} contains a leading eigenvector v corresponding to such eigenvalue.*

Proof. If the cone K_+ is invariant for A or $-A$, then by Theorem 2.24 the claims (i) and (ii) are proved.

We are left to assume that $A(K_+) \not\subseteq K_+$ and $A(K_+) \not\subseteq K_-$.

This implies that, by Proposition 5.4, condition (30) cannot be satisfied and, hence,

$$K_+ \cap \ker(A) \neq \{0\}. \tag{32}$$

Then, by defining the sets

$$C_+ := f_A^{-1}(K_+) \cap K_+ \qquad \text{and} \qquad C_- := f_A^{-1}(K_-) \cap K_+,$$

we clearly have that

$$C_+ \cup C_- = K_+ \tag{33}$$

and that $C_+ \setminus \ker(A) \neq \emptyset$ and $C_- \setminus \ker(A) \neq \emptyset$. Therefore, for any choice of $x \in C_+ \setminus \ker(A)$ and $y \in C_- \setminus \ker(A)$, the convexity of K_+ implies $\overline{xy} \subset K_+$. Furthermore, K_{sym} being invariant for A, we get

$$\overline{AxAy} = A(\overline{xy}) \subset K_{sym}.$$

Therefore, since $Ax \in K_+ \setminus \{0\}$ and $Ay \in K_- \setminus \{0\}$, it necessarily holds that $0 \in \overline{AxAy}$, i.e., $Ax = \alpha Ay$ for some $\alpha \neq 0$. So, given the equality (33) and the arbitrariness of x and y, it follows that the whole set $A(K_+)$ is contained in one line, say $\mathrm{span}(\{v\})$ for some $v \neq 0$. Finally, K_{sym} being solid, homogeneous and invariant for A, we obtain

$$A(K_{sym}) = \mathrm{range}(A) = \mathrm{span}(\{v\}) \subset K_{sym} \tag{34}$$

which, in turn, implies $\dim(\ker(A)) = d - 1$.

This tells us that A is a rank-one matrix and, consequently, that it has the unique leading eigenvalue $\lambda_A = \pm\rho(A)$. In this way, (i) is proved.

In order to prove also (ii), we first assume that $\rho(A) = 0$. In this case, any nonzero element of $\ker(A)$ is a leading eigenvector and, indeed, such elements do exist because of (32).

Eventually, if $\rho(A) > 0$, then (34) reveals that v is the (unique) leading eigenvector of A and that $v \in K_{sym}$, as required. $\square$

The above result extends easily to multicones.

Theorem 5.9. *Let a proper multicone $K_{mul} = \bigcup_{i=1}^{r} K_{sym}^{(i)}$ be invariant for a matrix A. Then there exists a positive integer $s \leq r$ such that:*

(i) $\pm\rho(A)^s$ *is an eigenvalue of* A^s*;*

(ii) *the multicone K_{mul} contains a leading eigenvector v of A^s corresponding to such eigenvalue.*

Proof. By Corollary 5.6, there exist $k \in \{1, \ldots, r\}$ and $s \leq r$ such that $K_{sym}^{(k)}$ is invariant for A^s. Theorem 5.8 concludes the proof. $\square$

Remark 5.10. Note that Corollary 5.6 and the proof of Theorem 5.9 refer to the same integers s and k. More precisely, they show that there exist $s \leq r$ and $k \in \{1, \ldots, r\}$ such that $K_{sym}^{(k)}$ is invariant for A^s and contains a leading eigenvector of it. $\square$

As shown in Example 5.2, it is not true that K_{mul} invariant for a matrix A implies $K_{mul}^{\dagger}$ invariant for the transpose A^T. Nevertheless, this holds under suitable assumptions, as the forthcoming Theorem 5.12 shows.

Proposition 5.11. *Let a proper multicone K_{mul} be invariant for a matrix $A \neq 0$. Then the dual version of (30) holds, i.e.,*

$$\mathrm{int}(K_{mul}^{\dagger}) \cap \ker(A^T) = \emptyset.$$

Proof. Since K_{mul} is proper and $A \neq 0$, there exists $x \in K_{mul} \setminus \ker(A)$. Hence, the invariance of K_{mul} implies that $Ax \in K_{mul} \setminus \{0\}$.

Assume by contradiction that there exists $y \in \mathrm{int}(K_{mul}^{\dagger}) \cap \ker(A^T)$. Then, since $y \in \mathrm{int}(K_{mul}^{\dagger})$, by (20) we have $y^T(Ax) \neq 0$. On the other hand, $y \in \ker(A^T)$ implies $A^T y = 0$ and, so, $(A^T y)^T x = 0$. The equality $(A^T y)^T x = y^T(Ax)$ gives the absurde. $\square$

Theorem 5.12. *Let a proper multicone K_{mul} be invariant for a matrix $A \neq 0$. If (30) holds, then $K_{mul}^{\dagger}$ is invariant for A^T.*

Proof. Since
$$K_{mul}^{\dagger} = \bigcap_{j=1}^{r} (K_{sym}^{(j)})^{\dagger},$$
if $y \in K_{mul}^{\dagger}$, then $y \in (K_{sym}^{(j)})^{\dagger}$ for all $j = 1, \ldots, r$, i.e., by Definition 4.1,

$$y^T z \geq 0 \quad \forall z \in K_{+}^{(j)} \qquad \text{or} \qquad y^T z \geq 0 \quad \forall z \in K_{-}^{(j)} \tag{35}$$

for all $j = 1, \ldots, r$. On the other hand, since K_{mul} is proper, under the assumption (30) Proposition 5.4 assures that, for each $i = 1, \ldots, r$, there exists $j_i \in \{1, \ldots, r\}$ such that

$$A(K_{+}^{(i)}) \subseteq K_{+}^{(j_i)} \qquad \text{or} \qquad A(K_{-}^{(i)}) \subseteq K_{+}^{(j_i)}$$

(see (31)) so that, by using (35), we obtain

$$(A^T y)^T x = y^T A x \geq 0 \quad \forall x \in K_+^{(i)} \qquad \text{or} \qquad (A^T y)^T x = y^T A x \geq 0 \quad \forall x \in K_-^{(i)}$$

for all $i = 1, \ldots, r$. This clearly implies that $A^T y \in (K_{sym}^{(i)})^\dagger$ for all i and, hence, $A^T(K_{mul}^\dagger) \subseteq K_{mul}^\dagger$. □

Theorem 5.13. *Let a proper multicone K_{mul} be invariant for a matrix $A \neq 0$ and assume that*

$$K_{mul} \cap \ker(A) = \{0\}. \tag{36}$$

Then both $K_{mul}^\dagger$ and $K_{mul}^\times$ are invariant for A^T.

Proof. Since (36) is stronger than (30), $K_{mul}^\dagger$ is invariant for A^T by Theorem 5.12. To prove the invariance of $K_{mul}^\times$, observe first that, for all $x \in K_{mul} \setminus \{0\}$, the invariance of K_{mul} and (36) give that $Ax \in K_{mul} \setminus \{0\}$. Therefore, since $\mathrm{int}(K_{mul}^\times) = \mathrm{int}(K_{mul}^\dagger)$, if $y \in \mathrm{int}(K_{mul}^\times)$, equality (20) of Proposition 4.11 yields

$$(A^T y)^T x = y^T A x \neq 0 \quad \forall x \in K_{mul} \setminus \{0\},$$

that is, $A^T y \in \mathrm{int}(K_{mul}^\times)$. Hence, $A^T(\mathrm{int}(K_{mul}^\times)) \subseteq \mathrm{int}(K_{mul}^\times)$ and standard continuity arguments complete the proof. □

Theorem 5.14. *Let K_{mul} be a properly reflexive multicone and $A \in \mathbb{R}^{d \times d}$ and assume that*

$$K_{mul} \cap \ker(A) = \{0\} \qquad \text{and} \qquad K_{mul}^\times \cap \ker(A^T) = \{0\}.$$

Then the following statements are equivalent:

(i) *K_{mul} is invariant for A;*

(ii) *$K_{mul}^\times$ is invariant for A^T.*

Proof. The implication (i) ⇒ (ii) follows from Theorem 5.13.

For the reverse implication (ii) ⇒ (i), apply Theorem 5.13 to the pair $(A^T, K_{mul}^\times)$, taking into account that $(A^T)^T = A$ and $K_{mul}^{\times\times} = K_{mul}$, since K_{mul} is properly reflexive. □

6. Asymptotically rank-one matrices and strict invariance

In this section we consider asymptotically rank-one matrices only (see Definition 2.28).

We preliminarily observe that an asymptotically rank-one matrix is necessarily nonzero and, more generally, not nilpotent.

Theorem 6.1. *Let a weakly proper multicone K_{mul} be invariant for an asymptotically rank-one matrix A. Then $v_A \in K_{mul}$.*

Proof. Since K_{mul} is homogeneous and $\mathrm{span}(K_{mul}) = \mathbb{R}^d$, there exist $x \in K_{mul}$ and $u \in H_A$ with $x = v_A + u$. Thus Proposition 2.30 yields $\lim_{k\to\infty} A^k x/\lambda_A^k = v_A$ and, consequently, K_{mul} being also invariant for A and closed, the proof is complete. □

Another benefit for a matrix of being asymptotically rank-one concerns the bounds $s \leq r$ and $p \leq 2r$ in Corollary 5.6: they can be improved down to $s = 1$ (if K_{mul} is weakly proper) and $p \leq 2$.

Proposition 6.2. *Let a weakly proper multicone $K_{mul} = \bigcup_{i=1}^r K_{sym}^{(i)}$ be invariant for an asymptotically rank-one matrix A. Then there exists a symmetric component $K_{sym}^{(k)}$ such that*

(i) $A(K_{sym}^{(k)}) \subseteq K_{sym}^{(k)}$.

Moreover, if K_{mul} is proper and (30) *holds, then*

(ii)
$$A(K_+^{(k)}) \subseteq \begin{cases} K_+^{(k)} & \text{if } \lambda_A > 0, \\ \\ K_-^{(k)} & \text{if } \lambda_A < 0, \end{cases}$$

and, in particular, $K_+^{(k)}$ is invariant for A^2;

(iii) $\mathrm{int}(K_{sym}^{(k)}) \cap H_A = \emptyset$.

Proof. (i) By Theorem 6.1, there exists a symmetric component $K_{sym}^{(k)}$ such that $v_A \in K_{sym}^{(k)}$. Since $Av_A = \lambda_A v_A$, clearly $Av_A \in K_{sym}^{(k)}$. Thus Proposition 5.4 completes the proof.

(ii) is again a direct consequence of Proposition 5.4.

(iii) follows from (ii), Theorem 2.24(iii) and the equality $H_A = H_{-A}$. □

Being $\ker(A^T) \subseteq H_{A^T}$, the following result is a stronger form of Proposition 5.11 for asymptotically rank-one matrices.

Theorem 6.3. *Let K_{mul} be a proper multicone and A be an asymptotically rank-one matrix. Then we have:*

(i) *if K_{mul} is invariant for A, then*

$$\mathrm{int}(K_{mul}^{\times}) \cap H_{A^T} = \emptyset; \tag{37}$$

(ii) *if $K_{mul}^{\times}$ is invariant for A^T, then*

$$\mathrm{int}(K_{mul}) \cap H_A = \mathrm{int}(K_{mul}^{\times\times}) \cap H_A = \emptyset. \tag{38}$$

Proof. (i) By Theorem 6.1, the leading eigenvector v_A belongs to K_{mul}. So (20) immediately implies $\mathrm{int}(K_{mul}^{\dagger}) \cap \{v_A\}^{\perp} = \emptyset$. Hence, by (24) and because of $\{v_A\}^{\perp} = H_{A^T}$ (see (6)), we obtain (37).

(ii) We can apply (i) to $K_{mul}^{\times}$, which is proper, and to the matrix A^T, which is asymptotically rank-one as well. So we obtain $\mathrm{int}(K_{mul}^{\times\times}) \cap H_A = \emptyset$. Since $K_{mul} \subseteq K_{mul}^{\times\times}$ (see (28)), we get (38). □

The notion of strict invariance also extends to multicones, taking into account that multicones are unions of cones. Here we confine ourselves to the case of weakly proper multicones, so that $\mathrm{span}(K_{mul}) = \mathbb{R}^d$.

Definition 6.4. We say that a weakly proper multicone K_{mul} is *strictly invariant under the action of the matrix* A on $\mathbb{R}^d$ (in short, *strictly invariant for* A) if

$$A(K_{mul} \setminus \{0\}) \subseteq \mathrm{int}(K_{mul}).$$ □

The reason to treat the strict invariance in the present section is due to the forthcoming Theorem 6.7, which proves that this property forces the matrix to be asymptotically rank-one.

Proposition 6.5. *If the weakly proper multicone K_{mul} is strictly invariant for a matrix A, then its proper core* $\mathrm{pr}(K_{mul})$ *is nonempty and strictly invariant as well, and condition* (36) *holds.*

Proof. First note that $\mathrm{pr}(K_{mul}) \neq \emptyset$ since

$$\mathrm{int}(\mathrm{pr}(K_{mul})) = \mathrm{int}(K_{mul}) \supseteq A(K_{mul} \setminus \{0\}) \neq \emptyset.$$

This also obviously implies the strict invariance of $\mathrm{pr}(K_{mul})$.
Moreover, by assumption, we have $A(K_{mul} \setminus \{0\}) \subseteq \mathrm{int}(K_{mul})$ and thus, since $0 \notin \mathrm{int}(K_{mul})$, condition (36) is proved as well. □

Next properties are the analog of Proposition 5.4 in the strictly invariant case and give a tighter localization-type result.

Proposition 6.6. *Let a weakly proper multicone $K_{mul} = \bigcup_{i=1}^{r} K_{sym}^{(i)}$ be strictly invariant for a matrix A. Then we have:*

(i) *for each proper conic component $K^{(i)}$, there exists a proper conic component $K^{(j)}$, possibly different, such that*

$$A(K^{(i)} \setminus \{0\}) \subseteq \mathrm{int}(K^{(j)}); \tag{39}$$

(ii) *there exists a proper conic component $K^{(k)}$ which is strictly A^2-invariant and, more precisely, such that*

$$A(K^{(k)} \setminus \{0\}) \subseteq \begin{cases} \mathrm{int}(K^{(k)}) & \text{if } \lambda_A > 0, \\ \mathrm{int}(-K^{(k)}) & \text{if } \lambda_A < 0 \end{cases}$$

and $v_A \in \mathrm{int}(K^{(k)})$.

Proof. (i) By Proposition 6.5, condition (36) holds; hence, (30) is verified, too.

So we can apply Proposition 5.4 and obtain that, for each proper conic component $K^{(i)}$, there exists a conic component $K^{(j)}$ such that $A(K^{(i)}) \subseteq K^{(j)}$. The strict invariance implies the stronger form (39). Consequently, $K^{(j)}$ is necessarily proper.

(ii) Since also $\mathrm{pr}(K_{mul})$ is strictly invariant (by Proposition 6.5), we can apply Proposition 6.2 to it. Therefore, taking into account the strict invariance, we obtain the required inclusions. Moreover, $K^{(k)}$ is strictly invariant for A^2 and, hence, Theorem 2.26(ii) yields $v_A = v_{A^2} \in \mathrm{int}(K^{(k)})$. □

Theorem 6.7. *Let a weakly proper multicone $K_{mul} = \bigcup_{i=1}^r K_{sym}^{(i)}$ be strictly invariant for a matrix A. Then A is an asymptotically rank-one matrix.*

Proof. By Proposition 6.6(ii), we can apply Theorem 2.26(i) to the matrix A^2 in relation to its strictly invariant cone $K^{(k)}$. Thus, A^2 has a unique leading eigenvalue $\rho(A^2) > 0$, which is simple.

Clearly, $\lambda_A^2 = \rho(A)^2 = \rho(A^2) > 0$, so that $\lambda_A = \pm\rho(A) \neq 0$ is the unique, and simple, leading eigenvalue of A. □

Now we can state the main result of this section.

Theorem 6.8. *Let a proper multicone K_{mul} be strictly invariant for a matrix A. Then:*

(i) *both $K_{mul}^{\dagger}$ and $K_{mul}^{\times}$ are strictly invariant for the transpose matrix A^T;*

(ii) *both $K_{mul}^{\times\dagger}$ and $K_{mul}^{\times\times}$ are strictly invariant for A;*

(iii) $K_{mul}^{\dagger} \cap H_{A^T} = K_{mul}^{\times} \cap H_{A^T} = \{0\}$*;*

(iv) $K_{mul} \cap H_A = K_{mul}^{\dagger\dagger} \cap H_A = K_{mul}^{\times\times} \cap H_A = \{0\}$*.*

Proof. (i) First note that $K_{mul}^{\dagger}$ is a weakly proper multicone (see Theorem 4.9 (ii)–(iii)). Then, if $y \in K_{mul}^{\dagger} \setminus \{0\}$, equalities (20) and (21) of Proposition 4.11 yield

$$(A^T y)^T x = y^T A x \neq 0 \quad \forall x \in K_{mul} \setminus \{0\},$$

that is, $A^T y \in \mathrm{int}(K_{mul}^{\dagger})$, as required.

Moreover, by Proposition 6.5 also $K_{mul}^{\times}$ is strictly invariant for A^T.

(ii) It directly follows from (i) applied to the pair $(A^T, K_{mul}^{\times})$.

(iii) Assume there exists $y \in (K_{mul}^{\dagger} \cap H_{A^T}) \setminus \{0\}$. By (i), $A^T y \in \mathrm{int}(K_{mul}^{\dagger})$. On the other hand, A^T is asymptotically rank-one (see Remark 2.29) and, hence, H_{A^T} is invariant for A^T by Remark 2.4. Therefore, $A^T y \in \mathrm{int}(K_{mul}^{\dagger}) \cap H_{A^T}$, against (37). The other equality trivially follows.

(iv) By (i) and the second equality in (iii) applied to the pair $(A^T, K_{mul}^{\times})$, we have $K_{mul}^{\times\times} \cap H_A = \{0\}$. Since $K_{mul} \subseteq K_{mul}^{\times\times}$ (see (28)), we also have $K_{mul} \cap H_A = \{0\}$.

Finally, applying the first equality in (iii) to the same pair, we obtain $K_{mul}^{\times\dagger} \cap H_A = \{0\}$. Since $K_{mul}^{\dagger\dagger} \subseteq K_{mul}^{\times\dagger}$ (being $K_{mul}^{\times} \subseteq K_{mul}^{\dagger}$), we get $K_{mul}^{\dagger\dagger} \cap H_A = \{0\}$. □

We end our discussion with the following straightforward result.

Corollary 6.9. *Let K_{mul} be a properly reflexive (proper) multicone. Then K_{mul} is strictly invariant for A if and only if the proper dual $K_{mul}^{\times}$ is strictly invariant for the transpose matrix A^T.*

7. Conclusions

In this paper we have analyzed the geometric structure and properties of a certain class of subsets of $\mathbb{R}^d$, here called *multicones*, which are quite natural generalizations of the classical cones. In particular, we have introduced and investigated a suitable extension of the concept of duality, which has allowed us to treat in a convenient way many issues related to the invariance and strict invariance of multicones for real matrices.

We have seen that, in general, there exist real matrices which admit an invariant multicone but not an invariant cone, nor even an invariant symmetric cone.

Concerning asymptotically rank-one matrices, although Proposition 6.2 tells us that those having an invariant multicone are no more than those having an invariant symmetric cone, our analysis reveals to be particularly useful in order to treat *families of matrices*. In fact, in this case it is known that invariant and strictly invariant multicones may exist even if any (symmetric) cone with such properties does not (see, e.g., Bochi and Morris [4] for families of nonsingular matrices). This subject is faced in [6].

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