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Convexity of Suns in Tangent Directions

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A direction d is called a tangent direction to the unit sphere S if the conditions $s \in S$ and $\operatorname{aff}(s+d)$ is a tangent line to the sphere S at s imply that $\operatorname{aff}(s+d)$ is a one-sided tangent to the sphere S, i.e., it is the limit of secant lines at the point s. A set M is called convex with respect to a direction d if $[x, y] \subset M$ whenever $x, y \in M$, $(y-x) \parallel d$. It is shown that in an arbitrary normed space an arbitrary sun (in particular, a boundedly compact Chebyshev set) is convex with respect to any tangent direction of the unit sphere.

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1. Introduction

We show that in arbitrary normed linear space an arbitrary sun (and in particular, a boundedly compact Chebyshev set) is convex with respect to any tangent direction to the unit sphere.

Below X is a real normed linear space, $B(x,r)$ is the closed ball with center x and radius r, and $\mathring{B}(x,r)$ and $S(x,r)$ are the corresponding open ball and sphere respectively. It will be convenient to put $S = S(0,1)$.

Definition 1.1. A set M is called a *Chebyshev set* if it is both a set of existence and a set of uniqueness; that is, for any $x \in X$ the set $P_M x$ of nearest points from M to x is a singleton. For a set $\varnothing \neq M \subset X$, a point $x \in X \setminus M$ is called a *solar point* if there exists a point $y \in P_M x \neq \varnothing$ (a *luminosity point*) such that $y \in P_M\big((1-\lambda)y + \lambda x\big)$ for all $\lambda \geqslant 0$. A set $M \subset X$ is a *sun* if any point $x \in X \setminus M$ is a solar point for M. A set $M \subset X$ is a *strict sun* if, for any point $x \in X \setminus M$, $P_M x \neq \emptyset$ and any point $y \in P_M x$ is a luminosity point.

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"Suns" are the most natural objects satisfying the Kolmogorov criterion for a nearest element. They possess some separation properties or other: a ball can be separated away from such a set by a larger ball or a support cone. Strict suns are in a sense close to the so-called sets with support condition of weak convexity [7, Definition 1.6], [8], [11], which were studied in particular in the context of weak convexity of sets. Moreover, according to [6], [12] for example, the solar and local solar properties of sets are closely related to the one important direction in geometrical optics which deals with solutions to the eikonal equation with singularities in some fixed subset of the domain. It is pretty amazing that many astonishing points of overlap between these two theories can be found even in the finite-dimensional setting. It is also worth mentioning that even though the concepts like a solar point, a ray of a sun, and a luminosity point have appeared in the geometric approximation theory much earlier than their relations to the geometric optics (and in general, to the Hamilton–Jacobi equations) were identified, these concepts were found to be in good accord with the geometrical optic concepts like a light ray, wave front, caustics, etc. In this context, the study of solar properties of sets is important from the standpoint of both the theory of approximation, geometrical optics, and convex analysis.

Below we follow the definitions given in the survey [6]. The basic definitions will be given below.

The paper is concerned with the convexity of suns with respect to tangent directions to the sphere. This statement of the problem is new and was first considered in [5], in which Chebyshev sets in the finite-dimensional setting were shown to be convex with respect to any tangent direction to the unit sphere. It is worth pointing out that the problem of convexity of suns (as opposed to the Chebyshev sets convexity problem) is not very challenging: in a space X any sun is convex if and only if X is smooth (i.e., at any point of the unit sphere S of the space X a support hyperplane to S is unique). Note that there is a discontinuous sun — the only known example of a disconnected sun was constructed by V. A. Koshcheev in an infinite-dimensional subspace of the space $C[0, 1]$ with a special norm (see [6, § 8.3]). This example does not contradict our result on the convexity of suns with respect to any tangent direction to the sphere: even on the plane one can easily construct a disconnected set which is convex with respect to any tangent direction to the sphere (see also Remark 2.12 below).

2. Results

Definition 2.1. Given $y \in S$, we let Λ_y denote the set of *one-sided tangents* to S at y (or the contingence of S at y or the *Bouligand contingent cone* to S at y). By a *one-sided tangent* to S emanating from a point $y \in S$ we mean a ray with origin at y and which is the limit of the ray passing through y and points x_n, $x_n \in S$.

Definition 2.2. A direction a is called a *(globally) tangent direction* to the sphere S if, for any point $y \in S$, the condition that a is a support direction at y implies that $a \in \Lambda_y$; i.e., a is a globally tangent direction at y.

A globally tangent direct direction can be interpreted as a line whose translations always "smoothly" travel around the sphere.

For example, in the space ℓ^∞_n, $n \geq 2$, the only tangent directions to the unit sphere are those which are parallel to edges of the unit ball (unit cube). In ℓ^1_3 there are no tangent directions. The problem of describing tangent directions to the unit sphere in finite-dimensional Banach spaces of dimension ≥ 3 remains open.

Definition 2.3. A set M is *convex with respect to a direction* d if $[x, y] \subset M$ whenever $x, y \in M$, $(y - x) \parallel d$ (see [5]).

The main result of the present paper is the following

Theorem 2.4. *In a normed linear space any sun is convex with respect to any tangent direction to the unit sphere.*

It is well known that in any finite-dimensional space any Chebyshev set is a sun. Hence Theorem 2.4 is an extension of the following result from [5].

Theorem 2.5. *In a finite-dimensional Banach space any Chebyshev set is convex with respect to any tangent direction to the unit sphere.*

In Theorems 2.5 and 2.4 it is important that the direction with respect to which the convexity of a Chebyshev set is examined is tangent for the *entire* sphere. One can easily construct an example of a three-dimensional space (for example, $X = \ell^2(2) \oplus_1 \mathbb{R}$) and a Chebyshev set (a sun) in it which fails to be convex with respect to a direction which is tangent to the sphere at *some* point, but not at *all* points at which it supports the sphere (see also Remark 2.10 below). In a two-dimensional Banach space any tangent direction at a point of the sphere is a (global) tangent direction to the sphere.

Definition 2.6. A set is called *boundedly compact* if its intersection with any closed ball is compact. It is well known that in a normed linear space a boundedly compact Chebyshev set is a sun (see [13, Theorem 4.13]).

Corollary 2.7. *In a normed linear space a boundedly compact Chebyshev set is convex with respect to any tangent direction to the unit sphere.*

A stronger version of Theorem 2.5 in the space ℓ^∞_n is given in the following Theorem 2.8 (see [6]).

Let $X = \ell^\infty_n$. Also let $k \in \mathbb{Z}_+$, $k \leq \dim X$.

We let $\mathrm{cAff}_k(X)$ denote the class of all affine coordinate subspaces of X of finite dimension k, i.e., affine subspaces of the form $\{x \in \ell^\infty_n \mid x_{i_1} = c_1, \dots, x_{i_k} = c_k\}$ for some fixed family of indexes $i_1, \dots, i_k$ and constants $c_1, \dots, c_k$. In the next theorem, $\mathrm{ri}\, A$ is the relative interior of a set A, a closed span Π is any intersection of extreme (coordinate) hyperstrips of the form $\{x \in \ell^\infty_n \mid a \leq f(x) \leq b\}$, $-\infty \leq a \leq b \leq +\infty$, $f \in \mathrm{ext}\, S^*$, generated in the original space ℓ^∞_n by extreme functionals from the dual space.

Theorem 2.8. *A Chebyshev set in $\ell^\infty(n)$ is an extremal Chebyshev set. In other words, if Π is a closed span in $\ell^\infty(n)$ and M is a Chebyshev set such that $M \cap \Pi \neq \varnothing$ and $M \cap \operatorname{ri} \Pi = \varnothing$, then the intersection $M \cap \Pi$ is a singleton.*

As a corollary, if P is a coordinate subspace and $M \cap P \neq \emptyset$, then $M \cap P$ is a Chebyshev set in P.

We note another result from [4] which partially extends Theorem 2.5. A direction d will be called a *hypertangent direction* to the unit sphere S if d is parallel to some facet of the unit ball.

Theorem 2.9. *A Chebyshev curve in ℓ^∞_n is convex with respect to any hypertangent direction to the unit sphere.*

Remark 2.10. An arbitrary Chebyshev set in ℓ^∞_n need not be convex with respect to a hypertangent direction. A corresponding example can be easily constructed in a space of any finite dimension $n \geq 3$.

Remark 2.11. A strict sun (and in particular, a sun) need not be convex with respect to a (locally) tangent direction at a point on the sphere. Consider the corresponding example in the space ℓ^∞_3. On the plane $\mathbb{R}^2$ for any $t \in \mathbb{R}$ consider the broken line

$$M_t = \Big\{ r \geq 0,\ \varphi = -\frac{1}{4} \arctan t \Big\} \cup \Big\{ r \geq 0,\ \varphi = \frac{\pi}{2} + \frac{1}{4} \arctan t \Big\}$$

in the polar coordinates (r, φ). The set $M = \bigcup \big\{ (M_t + (0,0,t)) \mid t \in \mathbb{R} \big\}$ is a strict sun in ℓ^∞_3, but its intersection with the tangent line at some points of the sphere is not convex with respect to the direction of this line.

Remark 2.12. In the three-dimensional space ℓ^∞_3, one can construct an example of a closed convex set which is convex with respect to any tangent direction to the sphere S, but which is not a sun (and, as a corollary, is not a Chebyshev set).

Remark 2.13. In the space ℓ^∞_n the convexity of suns with respect to any tangent direction was known earlier (in a different form). This result is a consequence of the monotone path-connectedness of suns in ℓ^∞_n (see, for example, [6, § 9]).

Proof of Theorem 2.4. Let M be a sun in the space X and let $u, v \in M$, $u - v$ be a tangent direction for the sphere S. Arguing on the contrary, assume

$$(u, v) \cap M = \emptyset, \tag{1}$$

and, without loss of generality, that $(u+v)/2 = 0$, $\rho(0, M) = 1$. Recall that the set

$$\mathring{K}(y, x) = \bigcup_{r>0} \mathring{B}\big(-ry + (r+1)x, (r+1)\|x - y\|\big), \tag{2}$$

which consists of the union of all homothetic copies of the ball $\mathring{B}(x, \|x - y\|)$ with respect to its boundary point y, is called the *support cone* of the ball $B(x, \|x - y\|)$ at the point y (see [6]). By the Kolmogorov criterion for a nearest element,

$y \in P_M x$ if and only if $\mathring{K}(y,x) \cap M = \varnothing$ (see [6]). Consequently, M is a sun if and only if, for any $x \in X$, there exists a point $y \in P_M x$ such that $\mathring{K}(y,x) \cap M = \varnothing$.

Let y be a luminosity point from M for x. Since $u, v \in M$ and since M is a sun,

$$u, v \notin \mathring{K}(y, 0). \tag{3}$$

Let $\mathscr{L}$ be the family of all closed linear sets containing the points y and u. By Zorn's lemma, in the family $\mathscr{L}$ there exists a maximal element H, which is a closed support hyperplane to the ball $B(0,1)$ at the point y. This element H is also a support hyperplane to any ball $B\big(-ry + (r+1)x, (r+1)\|x-y\|\big)$ from (2). Let f be the linear functional from the dual unit sphere S^* for which the hyperplane H is the level set $\{z \in X \mid f(z) = 1\}$. Then $f(B) \le 1$, $f(y) = 1$, $f(u) \ge 1$, and since $u = -v$, we have $f(v) < 0$. By a similar argument we construct a functional $g \in S^*$ such that $g(B) \le 1$, $g(v) \ge 1$, $g(y) = 1$, $g(u) < 0$.

From (3) it follows that $u - v$ is a support direction to the ball B at the point y. Since by the hypothesis the direction $u-v$ is a tangent direction to the sphere (and in particular, it is a one-sided direction to S at the point y, at which this direction supports the sphere by what has been just proved), we can find a sequence $(v_n) \subset S$ such that the chords $[y_n, y]$ tend to the half-tangent to the sphere S at the point y. This leaves us with the following two alternatives:

$$f(y_n - y) \to \alpha(f(u) - f(v)) \quad \text{and} \quad g(y_n - y) \to \beta(g(u) - g(v)) \tag{4}$$

or

$$f(y - y_n) \to \alpha(f(u) - f(v)) \quad \text{and} \quad g(y - y_n) \to \beta(g(u) - g(v)) \tag{5}$$

for some $\alpha, \beta > 0$.

In case (4), the left-hand side of the first relation is nonpositive and the right-hand side is positive, in case (5) the left-hand side of the first relation is positive and the right-hand side is negative. This contradiction shows that that assumption (1) was false. Theorem 2.4 is proved. $\square$

Remark 2.14. Theorem 2.4 shows that in an arbitrary normed linear space an arbitrary sun is convex with respect to any tangent direction to the unit sphere. Theorem 2.5, in which the same conclusion is proved for Chebyshev sets in the finite-dimensional setting, was applied in [3], [4] in the study of locally Chebyshev sets. The authors hope that Theorem 2.4 will also prove useful in the study of more general sets with given locally approximative properties (for such problems, see also [2], [9], [10]).

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