Journal of Convex Analysis
Volume 26 (2019), No. 4, 1077–1088

Symmetries of Convex Sets in the Hyperbolic Plane

Jesús Jerónimo-Castro
Facultad de Ingenieria, Universidad Autónoma de Querétaro,
Cerro de las Campanas s/n, C.P. 76010, Querétaro, México
jesusjero@hotmail.com

Francisco G. Jimenez-Lopez
Facultad de Ingenieria, Universidad Autónoma de Querétaro,
Cerro de las Campanas s/n, C.P. 76010, Querétaro, México
fjimenez@uaq.mx

Received: May 09, 2018
Revised manuscript received: December 17, 2018
Accepted: December 20, 2018

We establish some results characterizing central or axial symmetry of convex sets in the hyperbolic plane. The characterizations follow the spirit of a Chakerian-Klamkin's characterization of central symmetry for Euclidean sets: if for any three point-subset M of a compact set K there is a symmetric image of M that is also contained in K, then K has a center of hyperbolic symmetry. We also study axial symmetry when the axis is either a geodesic, a horocycle, or a hypercycle. Finally, in the last section we give a characterization of the hyperbolic disc.

Keywords: Hyperbolic disc, orthogonal symmetry, hyperbolic symmetry.

2010 Mathematics Subject Classification: 52A10, 52A55.

1. Introduction

There are many interesting and useful characterizations of centrally symmetric sets in the Euclidean space. That is, there are criteria to know whether for a set $K \subset \mathbb{R}^n$ there exists a point $p \in \mathbb{R}^n$ such that $K - p = p - K$. We are especially interested in the following criterion given by G. D. Chakerian and M. S. Klamkin in [1]: *a compact set $K \subset \mathbb{R}^n$ has a center of symmetry if for any three point set $S = \{a, b, c\} \subset K$ there exists a point x_S such that $\{x_S - a, x_S - b, x_S - c\} \subset K$.* In [4], J. Jerónimo-Castro and L. Montejano proved the *colored* version of the Chakerian-Klamkin's theorem and also proved an analogue theorem for the case of axial symmetry:

Let $K \subset \mathbb{R}^n$ be a compact, convex set and let $v \in \mathbb{S}^{n-1}$ be a fixed direction. Assume that for any two of points $a, b \in K$ there is a hyperplane H, orthogonal to v, with the property that the orthogonal reflections of a and b with respect to H are also contained in K. Then K has an orthogonal hyperplane of symmetry orthogonal to v.

ISSN 0944-6532 / \$ 2.50

Using a very simple argument, we can extend this last result in order to avoid the hyphotesis of convexity over K. That argument is used, indeed, in the proof of Theorem 2.8. The main purpose of this article is to give some Chakerian-Klamkin type theorems in the hyperbolic plane and also to give a characterization of the hyperbolic disc in terms of the convexity of the union of K with some suitable reflections of it.

2. Characterizing central and axial symmetry

Let $(\mathbb{D}, d_h)$ denote the Poincaré disc model for the hyperbolic plane, i.e, $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and d_h denotes the hyperbolic metric. The unit circle is said to be the ideal boundary of $\mathbb{D}$. The geodesics in this space are given by diameters of the unit circle and by arcs of circles which are orthogonal to the ideal boundary. Given two points z and w in $\mathbb{D}$, there exists a unique geodesic L passing through them. We denote the geodesic segment joining these two points by $[z, w]$.

A *horocycle* in $\mathbb{D}$ is a circle tangent to the ideal boundary. The tangency point is called the *center of the horocycle*. For any two points z and w in $\mathbb{D}$, there exist exactly two horocycles passing through them. Moreover, one is the reflection of the other along the geodesic passing through the points z and w. The domain bounded by a horocycle is called a *horocyclic region*, while the intersection of two horocyclic regions is called a *horocycle lentil*. We will denote horocycles in $\mathbb{D}$ with the letter Υ.

A hypercycle in $\mathbb{D}$ is the set of points that are equidistant from a fixed geodesic called its axis. A hypercycle consists of an arc of a circle joining the endpoints of the axis together with its reflection along the axis. If a hypercycle does not coincide with the axis, it encloses a domain called a hypercyclic region. We will call the arcs of circle corresponding to half the boundary of hypercycles semi-hypercycles and we denote them with the letter Γ. For any two points z, w that lie in a semi-hypercycle Γ, we use the notation $[z, w]_\Gamma$ to represent the segment of the hypercycle Γ joining z and w.

A subset K of $\mathbb{D}$ is called *convex* if for any two points in K the geodesic segment joining them is also contained in K.

There is another notion of convexity in $\mathbb{D}$: A subset K of $\mathbb{D}$ is called *h-convex* if for any two points z and w in K the two segments of horocycles joining z and w belong to K.

It is easy to see that if K is h-convex, then for any z and w, the horocyclic lentil determined by z and w is contained in K. Since the segment $[z, w]$ is contained in the horocycle lentil, the set K must be convex. Moreover, since every horocycle lentil is convex in the Euclidean sense, we conclude that K is also convex in the Euclidean sense considered as a subset of the complex plane. Notice that the boundary of any h-convex set does not contain Euclidean nor hyperbolic geodesics.

In this paper we study different types of symmetry for compact sets in the hyperbolic plane. In order to do so we need the following definition.

Definition 2.1. Let K be a subset of $\mathbb{D}$.

(1) For a point $p \in \mathbb{D}$, let R_p denote the mapping corresponding to the hyperbolic rotation of degree two around p. The set K is said to have center of symmetry p if $R_p(K) = K$.

(2) For a geodesic L, let R_L denote the mapping corresponding to the reflection along L. The set K is said to be *symmetric with respect to* L if $R_L(K) = K$. In this case we call L an *axis of symmetry* for K.

(3) For a horocycle Υ tangent to the ideal boundary of $\mathbb{D}$ at a point α, let R_Υ denote the mapping sending a point z to the point w in the geodesic L passing through α and z, such that the only point in $\Upsilon \cap L$ is the middle point of the segment $[z, w]$. The set K is said to be *symmetric with respect to* Υ if $R_\Upsilon(K) = K$.

(4) For a semi-hypercycle Γ with axis L, let R_Γ denote the mapping sending a point z to the point w in the geodesic L' orthogonal to L and passing through z, such that the only point in $\Gamma \cap L'$ is the middle point of the segment $[z, w]$. The set K is said to be *symmetric with respect to* Γ if $R_\Gamma(K) = K$.

Our first result is the following.

Proposition 2.2. *Let K be a compact subset of $\mathbb{D}$. Suppose that for any three points w_1, w_2 and w_3 in K there exists a point p in $\mathbb{D}$ such that $R_p(w_1)$, $R_p(w_2)$, $R_p(w_3) \in K$. Then there exists a unique point $p_0 \in \mathbb{D}$ such that K is symmetric with respect to p_0. In particular if K is convex, then p_0 is in K.*

Proof. Let $w_0, w_0' \in K$ be such that $d_h(w_0, w_0') = \sup_{z,z' \in K} d_h(z, z')$. The only point $p_0 \in \mathbb{D}$ such that $R_{p_0}(w_0), R_{p_0}(w_0') \in K$ is the middle point of the segment $[w_0, w_0']$. For if $p_1 \neq p_0$ also satisfies $R_{p_1}(w_0), R_{p_1}(w_0') \in K$, then

$$\max\left(d_h(w_0, R_{p_1}(w_0)), d_h(w_0', R_{p_1}(w_0'))\right) > d_h(w_0, p_0).$$

This implies

$$\sup_{z,z' \in K} d_h(z, z') \geq 2\max(d_h(w_0, p_1), d_h(w_0', p_1)) > d_h(w_0, w_0'),$$

which is a contradiction. Then, for any point $w \in K$, the only point p that satisfies $R_p(w), R_p(w_0), R_p(w_0') \in K$ is p_0. It follows that $R_{p_0}(w) \in K$ for every $w \in K$.

If K is convex, then it must contain the segment $[w_0, w_0']$. In particular it must contain p_0. $\square$

The following proposition gives a criterion to know when a compact set has an axis of symmetry.

Proposition 2.3. *Let K be a compact subset of $\mathbb{D}$ containing more than one point. Then K is symmetric with respect to a geodesic L if and only if for any finite set $\{z_1, ..., z_n\}$ of elements of K, there exists a geodesic L_0 such that $R_{L_0}(z_i) \in K$ for all $i = 1, ..., n$.*

Proof. Since necessity is trivial we only prove sufficiency. Let $A = \{z_1, z_2, ...\}$ be a countable dense subset of K. For each n, let L_n be the geodesic in $\mathbb{D}$ such that $R_{L_n}(z_i) \in K$ for every $i = 1, ..., n$. Let w_n and w'_n be the endpoints of L_n in the ideal boundary of $\mathbb{D}$. Since the sequences (w_n) and (w'_n) are bounded in the Euclidean metric, we may assume by passing to a subsequence that they converge to points w and w' in the ideal boundary of $\mathbb{D}$. Moreover, the compactness of K implies that $w \neq w'$. Let L be the geodesic determined by w and w'. For any point z_j in A we have $R_{L_n}(z_j) \to R_L(z_j)$ as $n \to \infty$. Since $R_{L_n}(z_j) \in K$ for n sufficiently large and K is closed, we conclude that $R_L(z_j) \in K$. Thus $R_L(A) \subseteq K$.

Let z be any point in K and choose a sequence $(\tilde{z}_n)$ in A that converges to z. It follows that $R_L(\tilde{z}_n) \to R_L(z)$ as $n \to \infty$. Using again the fact that K is closed and that $R_L(\tilde{z}_n) \in K$ for every n we obtain $R_L(z) \in K$. This completes the proof. □

One may think that the sufficiency part of the previous proposition is asking too much. In order to prove the theorem, the condition of being symmetric with respect to a line is required for too many points. Nonetheless, it is not possible to weaken it in that sense. In order to show this we construct an example.

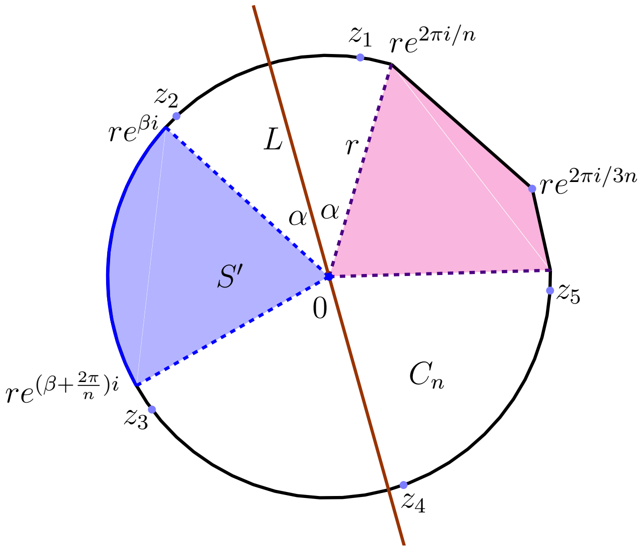

Figure 1. Example for $n = 5$.

Example 2.4. Let n be a natural number greater than 1. Define the set C_n to be the convex hull of $\{z \in \mathbb{D} : |z| = r \text{ and } 2\pi/n \leq \arg z \leq 2\pi\} \cup \{re^{2\pi i/3n}\}$ where $0 < r < 1$. It is easy to show that there is no geodesic L with respect to which the set C_n is symmetric. Let $z_1, ..., z_n$ be n points in C_n arranged in such a way that $0 \leq \arg z_1 \leq \arg z_2 \leq .... \leq \arg z_n < 2\pi$. Let S_1 be the sector determined by z_1 and z_n with central angle $\arg z_1 - \arg z_n + 2\pi$ and for $i = 2, ..., n$ let S_i be the sector in $\mathbb{D}$ determined by the points z_i and z_{i-1} with central angle $\arg z_i - \arg z_{i-1}$. The sum of the central angles of each sector S_i must equal 2π. It follows that that there is at least one sector S_j with central angle greater than or equal to $2\pi/n$. Let S' be a sector contained in S_j with central angle $2\pi/n$ and bounded by the rays starting at 0 and passing through the points $re^{\beta i}$ and $re^{(\beta+\frac{2\pi}{n})i}$. Choose the geodesic L passing through 0 which is the angle bisector of the two rays starting at 0 and passing through the points $re^{\frac{2\pi}{n}i}$ and $re^{\beta i}$. We have that $R_L(S') = \{z : |z| \leq r, 0 \leq \arg z \leq 2\pi/n\}$. Since all the z_i are in $\overline{(S')^c}$, it is readily seen that for each $i = 1, ..., n$ we have $R_L(z_i) \in \{z \in \mathbb{D} : |z| \leq r \text{ and } 2\pi/n \leq \arg z \leq 2\pi\} \subseteq C_n$.

The previous example shows that no matter how large an integer n is, we can always construct a convex set C_n that has no geodesic of symmetry through a given point (in fact, it has no geodesic of symmetry at all) but such that the reflection of any n points in it along a geodesic lies in C_n.

There is another interesting example: there exists a convex set K that satisfies that for any two points $w_1, w_2 \in K$ there exists a geodesic L orthogonal to a fixed geodesic L' such that $R_L(w_1), R_L(w_2) \in K$, and however, the set K has no geodesic as an axis of symmetry. It is known that a compact Euclidean convex set satisfying an analogue property must have an axis of symmetry (see [4]).

Example 2.5. Let L' be the geodesic determined by the diameter of $\mathbb{D}$ whose endpoints are $-i$ and i. Let $0 < r < 1$ and denote the geodesics orthogonal to the geodesic L' passing through ri and $-ri$ by L_r and L_{-r} respectively. Choose any point w_0 in L_r with positive real and imaginary parts and let Γ_0 be the semi-hypercycle passing through w_0 whose axis is L'. Then, Γ_0 must intersect the real axis at a point z_0. Let L_0 be the geodesic joining $\overline{w}_0$ with z_0 and define K to be the compact convex set bounded by L', L_r, $L_{r'}$, Γ_0 and L_0 (see Figure 2), which clearly has no geodesic as an axis of symmetry.

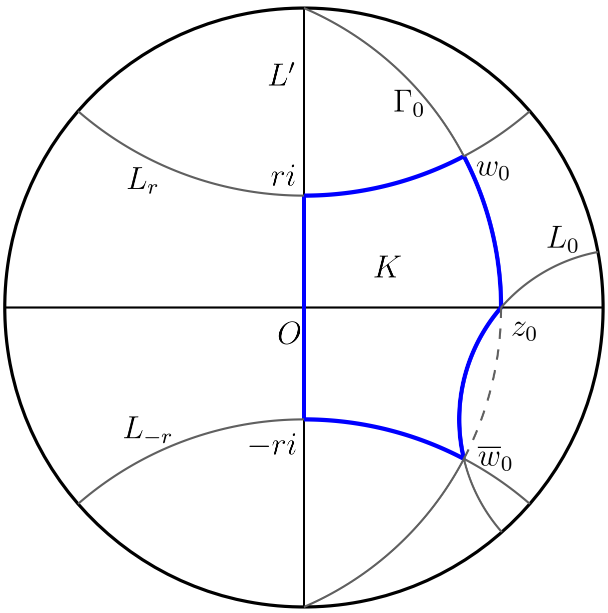

Figure 2. Counterexample.

In order to prove that for any two points $w_1, w_2 \in K$ there exists a geodesic L orthogonal to L' such that $R_L(w_1), R_L(w_2) \in K$, let Γ be any semi-hypercycle with axis L' that intersects K. Then, Γ intersects the boundary of K at points p and $\overline{p}$, where p has positive imaginary part. We will prove first that for any two points w and w' in $[p,\overline{p}]_\Gamma$ there exists a geodesic L orthogonal to L' such that $R_L(w), R_L(w') \in K$. Let w and w' be two such points. We may assume that w' is not in the hypercycle segment $[w,p]_\Gamma$. Let λ be the length of the hypercycle segment $[p,\overline{p}]_\Gamma$. Suppose first that the length of $[w,w']_\Gamma$ is at most $\lambda/2$. Choose L to be the geodesic orthogonal to L' such that $R_L(w') = p$. Since hypercycles with axis L' are invariant under R_L we must have that $R_L(w)$ is in $[p,\overline{p}]_\Gamma$. Moreover, the length of $[R_L(w'), R_L(w)]_\Gamma = [p, R_L(w)]_\Gamma$ is at most $\lambda/2$ and then $R_L(w)$ has a nonnegative imaginary part and must be in K.

Suppose now that the length of $[w,w']_\Gamma$ is greater than $\lambda/2$. Choose L to be the geodesic orthogonal to L' such that $R_L(w) = \overline{p}$. In consequence the length of $[R_L(w'), R_L(w)]_\Gamma = [R_L(w'), \overline{p}]_\Gamma$ is greater than $\lambda/2$. Since $R_L(w')$ is in $[p,\overline{p}]_\Gamma$, it follows that $R_L(w')$ has a positive imaginary part and must be in K.

Now let w_1 and w_2 be any two points in K and let Γ be the semi-hypercycle passing through w_1 whose axis is L'. Let L_2 be the geodesic passing through w_2 that is orthogonal to L'. Then, L_2 crosses Γ at some point w_1'. By applying the previous argument to the points w_1 and w_1' we know there exists a geodesic L orthogonal to L' such that $R_L(w_1), R_L(w_1') \in K$. From this it is easily seen that $R_L(w_2)$ is also in K. □

If we impose stronger conditions this situation changes. More precisely, we have the following result.

Theorem 2.6. *Let L' be a fixed geodesic and let K be a compact convex subset of $\mathbb{D}$ such that for any two points $w_1, w_2 \in K$ there exists a geodesic L orthogonal to L' satisfying $R_L(w_1), R_L(w_2) \in K$. Suppose that K satisfies one of the following:*

(1) *The geodesic L' intersects L in at most one point.*

(2) *The intersection of K with every semi-hypercycle with axis L' is either empty or a connected set.*

Then there exists a geodesic L_0 orthogonal to L' such that K is symmetric with respect to L_0.

Proof. Assume first that K and L' are disjoint. Since K is compact, there exists a point w_0 in K such that $0 < d_h(w_0, L') = d_h(K, L')$. We claim that w_0 is the unique point realizing this distance. In order to prove it, let E be the hypercyclic region whose boundary passes through w_0 and has axis L'. Since the boundary of E is the set of points whose distance from L' is $d_h(w_0, L')$, the set K must be contained in the complement of E (see Figure 3). The closure of E is a convex set. Thus, if Γ denotes the semi-hypercycle passing through w_0 determined by E, we have that for any point w_1 in Γ such that $w_1 \neq w_0$, the segment $[w_0, w_1]$ intersects E. It follows that Γ intersects K only at w_0.

On the other hand, Γ is invariant under all reflections along geodesics that are orthogonal to L'. In particular, if L is a geodesic orthogonal to L' not passing through w_0, the point $R_L(w_0)$ is in Γ. The uniqueness of w_0 implies that for any such geodesic L, $R_L(w_0) \in K$ if and only if L is the geodesic orthogonal to L' passing through w_0. Let L_0 denote this geodesic and choose any point $w \in K$. By hypothesis, there exists a geodesic L such that $R_L(w_0), R_L(w) \in K$. The previous argument shows that $L = L_0$ and we conclude that $R_L(w) \in K$ for every $w \in K$.

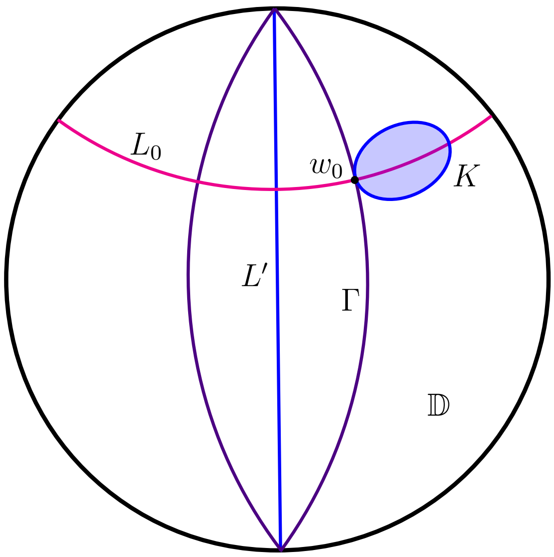

Figure 3. The set K is contained in the complement of E.

If L' crosses K at only one point w_0, it is clear that $R_L(w_0) \in K$ if and only if L is the orthogonal geodesic to L' passing through w_0. We then proceed as in the previous case.

Suppose now that the intersection of K with every hypercycle with axis L' is either empty or a connected set. For any w in K, denote by K_w the collection of geodesics orthogonal to L' along which the reflection of w is in K. Let K'_w be the set of points where the geodesics in K_w intersect L'. Since all points w in K lie in a semi-hypercycle Γ with axis L' and $\Gamma \cap K$ is connected, we must have that K'_w is connected. Let w_1 and w_2 be any two points in K. Then, by hypothesis there exists a geodesic L orthogonal to L' along which the reflection of w_1 and w_2 are in K. It follows that the intersection $K'_{w_1} \cap K'_{w_2}$ must be nonempty. Applying Helly's Theorem (see [2]) for arcs over a circle we conclude that there exists a point $z_0 \in \cap_{w \in K} K'_w$. Thus, if L_0 is the geodesic orthogonal to L' that passes through z_0 we have $R_{L_0}(w) \in K$ for every $w \in K$. $\square$

Proposition 2.7. *Let K be a compact convex subset of $\mathbb{D}$ and let α be a fixed point in the ideal boundary of $\mathbb{D}$. If for any two points $w_1, w_2 \in K$ there exists a horocycle Υ with center α such that $R_\Upsilon(w_1), R_\Upsilon(w_2) \in K$. Then there exists a unique horocycle Υ_0 with center α such that K is symmetric with respect to Υ_0.*

Proof. Let L' be a fixed geodesic with α as one of its endpoints. For each $w \in K$, let K_w be the collection of horocycles Υ centered at α such that $R_\Upsilon(w) \in K$. Each horocycle in K_w intersects L' at exactly one point. Denote by K'_w the collection of such points. Notice that K'_w must be a closed segment of L'. Since for any two points there exists a horocycle centered at α along which the reflection of those points remains in K, we must have $K'_{w_1} \cap K'_{w_2} \neq \emptyset$ for every $w_1, w_2 \in K$. It is a consequence of Helly's Theorem (see for instance [2]), for arcs over a circle, that there exists a point $z_0 \in K'_w$ for every $w \in K$. In other words, there exists an horocycle Υ_0 passing through z_0 such that $R_{\Upsilon_0}(w) \in K$ for every $w \in K$.

In order to prove uniqueness of Υ_0, let L be any geodesic with one endpoint at α that crosses K in at least two points. Let w_0 and w'_0 be the points in K such that $[w_0, w'_0] = L \cap K$. Then there is only one horocycle centered at α along which the reflection of w_0 and w'_0 remains in K, namely the horocycle passing through the middle point of $[w_0, w'_0]$. This arguments shows that Υ_0 is unique. □

Theorem 2.8. *Let K be a compact subset of $\mathbb{D}$ and let L be a fixed geodesic. If for any two points $w_1, w_2 \in K$ there exists a half-hypercycle Γ with axis L' such that $R_\Gamma(w_1), R_\Gamma(w_2) \in K$, then there exists a unique half-hypercycle Γ_0 with axis L such that K is symmetric with respect to Γ_0*

Proof. Let L' be a fixed geodesic that is orthogonal to L. Each semi-hypercycle with axis L must intersect L' at exactly one point. For each $w \in K$ let K_w denote the collection of half-hypercycles Γ with axis L such that $R_\Gamma(w) \in K$. Then there is a one to one correspondence between the points in K_w and a subset K'_w of L'. The argument used in the previous theorem shows the existence of a half-hypercycle Γ_0 such that K is symmetric with respecto to Γ_0

Uniqueness of Γ_0 is also proved in a similar way: Let L'' be a geodesic orthogonal to L such that $L'' \cap K = [w_0, w'_0]$, with $w_0 \neq w'_0$. Then the only half-hypercycle along which the reflection of w_0 and w_1 remains in K is that which crosses the middle point of $[w_0, w'_0]$. □

3. A characterization of the hyperbolic disc

In this section we will use a notion of width of convex sets introduced by Santaló in [6]. Let K be a closed convex set in $\mathbb{D}$ with differentiable boundary and let z be a point in ∂K. Let L_z be the geodesic supporting K at the point z and let L' be the orthogonal geodesic to L_z passing through z. Let L_w be another geodesic supporting K at a point w, that is also orthogonal to L'. Suppose L_w intersects L' at the point w'. The *width* with respect to the point z is the length of the segment $[z, w']$. In this paper, we will say that the supporting geodesics L_z and L_w are parallel with respect to K and we denote the bisector of $[z, w']$ by L_{zw}.

A closed convex set K is called of *constant width* λ if the width with respect to every point in its boundary is λ. It is known now, due to a result by Dekster [3] that a closed convex set K is of constant width if and only if for any parallel supporting geodesics of K, L_z and L_w, the common orthogonal to them, intersects them at z and w, respectively. In other words, a closed convex set is of constant width if and only if every normal is a double normal. It is an immediate consequence of this fact, that if K is of constant width λ, then $\mathrm{diam}K = \lambda$, where $\mathrm{diam}K$ denotes the diameter of K.

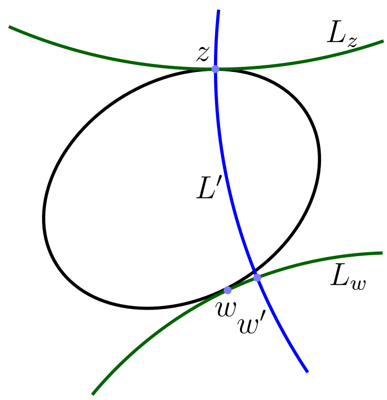

Figure 4. Width in the sense of Santaló.

The aim in this section is to give a proof of the following theorem.

Theorem 3.1. *Let K be a compact h-convex set in $\mathbb{D}$ with nonempty interior and whose boundary is differentiable. Suppose that for any pair of parallel supporting lines of K, L_z and L_w, the set $R_{L_{zw}}(K) \cup K$ is convex. Then K is a hyperbolic disc.*

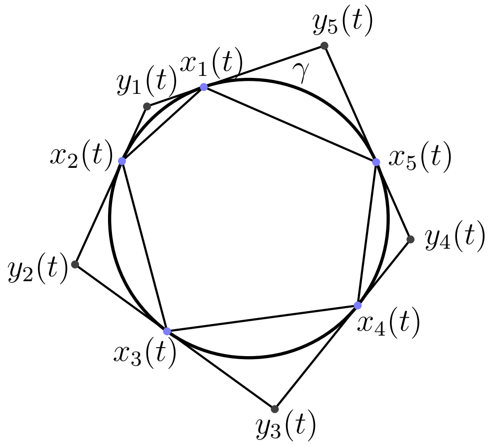

Figure 5. The circle is an n-fine curve for every n.

Before giving the proof of Theorem 3.1 we need to prove a lemma similar to that proved in [5] (as Proposition 6.3). In order to do this, we introduce the following definition.

Definition 3.2. We say that an oriented, simple, differentiable, closed, convex curve γ in the Euclidean plane, is an *n-fine curve* if the following conditions hold:

(1) There is a one parameter family of inscribed n-gons with vertices $x_1(t)$, $x_2(t)$, ..., $x_n(t)$, for $t \in \mathbb{R}$, such that γ shares tangent lines with a circle at the points $x_1(t)$, $x_2(t)$, ..., $x_n(t)$.

(2) The velocity $x_i'(t)$ has the positive direction for all values of t and i (i.e., all vertices move along γ with positive speeds),

(3) The vertices of the polygon are cyclically permuted over a period: $x_i(t+1) = x_{i+1}(t)$ for all i and t.

The definition of n-fine curves given in [5] is slightly different. It was required for convγ to be inscribed in the convex n-gon generated by the lines tangent to γ at the points $x_i(t)$, $i = 1, 2, \ldots, n$.

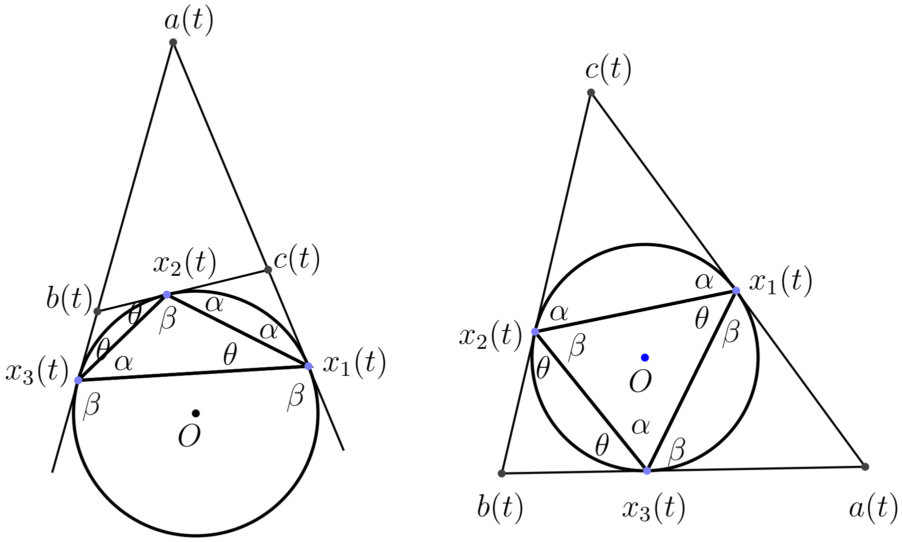

Figure 6.

However, in our definition we permit the two cases shown in Figure 6. Here is the mentioned lemma.

Lemma 3.3. *Given a natural number $n \geq 3$, the only n-fine curve γ is the Euclidean circle.*

Proof. First note that it is only necessary to prove the theorem for $n = 3$. Suppose the origin 0 is contained in the interior of convγ and for every $t \in [0, 2\pi]$ let $u(t) = (\cos t, \sin t)$ be the unitary vector which makes an angle t with respect to the x-axis. Let γ be parameterized as $\gamma(t)$, where $\gamma(t)$ is the point in γ such that $\gamma'(t)$ is orthogonal to $u(t)$ and $u(t)$ is the exterior normal vector at $\gamma(t)$. Suppose the sequence of points x_1, x_2, and x_3 is given by $\gamma(t)$, $\gamma(t + \alpha(t))$, and $\gamma(t+\alpha(t)+\beta(t))$, respectively, for two differentiable functions α and β. Denote by $c(t)$ and $r(t)$ the center and radius, respectively, of the circle passing through $\gamma(t)$, $\gamma(t+\alpha(t))$, and $\gamma(t+\alpha(t)+\beta(t))$ (see Figure 7). We have that $\gamma(t) = c(t)+r(t)u(t)$, and since $\langle \gamma'(t), u(t)\rangle = 0$, we obtain

$$\langle c'(t) + r'(t)u(t) + r(t)u'(t), u(t)\rangle = \langle c'(t), u(t)\rangle + r'(t) = 0. \tag{1}$$

Note that
$$r(t + \alpha(t) + \beta(t)) = r(t + \alpha(t)) = r(t),$$
and
$$c(t + \alpha(t) + \beta(t)) = c(t + \alpha(t)) = c(t),$$

for every $t \in [0, 2\pi]$. In an analogous way we prove that

$$\langle c'(t), u(t + \alpha(t))\rangle + r'(t) = 0, \tag{2}$$

and
$$\langle c'(t), u(t + \alpha(t) + \beta(t))\rangle + r'(t) = 0. \tag{3}$$

From (1), (2), and (3) we have

$$\langle c'(t), u(t) - u(t+\alpha(t))\rangle = 0$$

and

$$\langle c'(t), u(t) - u(t+\alpha(t)+\beta(t))\rangle = 0.$$

Now, since the vectors $u(t) - u(t+\alpha(t))$ and $u(t) - u(t+\alpha(t)+\beta(t))$ are linearly independent we have $c'(t) = 0$, i.e., $c(t) = c$ for a constant vector c. Finally, using that $c'(t) = 0$ for every $t \in [0, 2\pi]$, from equation (1) we obtain that $r'(t) = 0$. Since the centers $c(t)$ and the radii $r(t)$ are constant, we conclude that γ is a circle. □

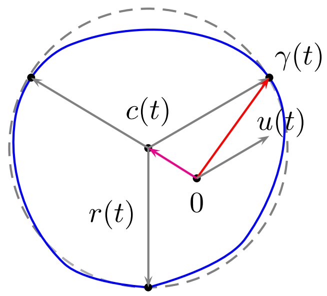

Figure 7. A circle sharing tangent lines with γ.

Proof of Theorem 3.1. Let z be any point in the boundary of K and let L_w be the corresponding geodesic parallel (in the sense explained before) to L_z. Let L_{zw} be the line defined before the statement of Theorem 3.1. Since the set $R_{L_{zw}}(K) \cup K$ is convex and $R_{L_{zw}}(z) = w'$, where the point $w' \in L_w$ is such that $[z, w']$ is orthogonal to both L_z and L_w, the segment $[w', w]$ must be contained in $R_{L_{zw'}}(K) \cup K$. Then $w' = w$, for if this is not the case, then for any point $\zeta \in (w', w)$ we have $R_{L_{zw}}(\zeta) \in \partial K$ or $\zeta \in \partial K$ contradicting the fact that ∂K has no geodesic segments (since K is an h-convex set). Then every normal is a double normal and it follows that K must be a set of constant width λ for some $\lambda > 0$.

Let p and q be the points where L_{zw} intersect ∂K. Since the geodesic passing through p and q is an axis of symmetry for $R_{L_{zw}}(K) \cup K$ it follows that $[p, q]$ is a double normal for $R_{L_{zw}}(K) \cup K$ and in particular for K. Let L_{pq} be the perpendicular bisector of $[p, q]$. We claim that L_{pq} is the geodesic passing through z and w. Suppose this is not the case and let $\tilde{z}$ and $\tilde{w}$ be the points of intersection between L_{pq} and ∂K. By repeating the argument above for the points p and q, we get that $[\tilde{z}, \tilde{w}]$ is a double normal and $\lambda = d_h(z, w) = d_h(\tilde{z}, \tilde{w})$. We may assume,without loss of generality that $[\tilde{z}, w]$ and $[\tilde{w}, z]$ intersect at a point u. Then

$$d_h(w, z) < d_h(w, u) + d_h(u, z) \text{ and } d_h(\tilde{w}, \tilde{z}) < d_h(\tilde{w}, u) + d_h(u, \tilde{w}).$$

Adding up these inequalities we obtain

$$2\lambda < d_h(\tilde{w}, u) + d_h(u, \tilde{z}) + d_h(w, u) + d_h(u, z) = d_h(w, \tilde{z}) + d_h(\tilde{w}, z).$$

Thus $\lambda < \max(d_h(w, \tilde{z}), d_h(\tilde{w}, z))$, and since the width of a set of constant width is also equal to the diameter of the set, this contradicts the fact that K is of constant width λ. We have proved that the segments $[z, w]$ and $[p, q]$ bisect each other, and since both of them have the same length, we have that z, p, w, and q are contained in a hyperbolic circle (and consequently Euclidean) with radius $\lambda/2$. A well known property of hyperbolic circles is that every tangent line is orthogonal to the radius drawn at the point of tangency, and so the circle through the points z, p, w, and q shares tangent lines with ∂K. Moreover, since for every double normal $[z, w]$ there is a unique orthogonal double normal $[p, q]$, we have that the hypotheses of Lemma 3.3 hold and therefore, we conclude that γ is a hyperbolic (Euclidean) circle. $\square$

Remark 3.4. The Euclidean analogue of this theorem is also true and the proof follows the same ideas.

References

[1] G. D. Chakerian, M. S. Klamkin: *A three point characterization of central symmetry*, Amer. Math. Montly 111 (2004) 903–905.

[2] L. Danzer, B. Grünbaum, V. Klee: *Helly's theorem and its relatives*, in: *Convexity*, V. L. Klee (ed.), Proc. Symp. Pure Math. 7, American Mathematical Society, Providence (1963) 101–180.

[3] B. V. Dekster: *Double normals characterize bodies of constant width in Riemannian manifolds*, in: *Geometric Analysis and Nonlinear Partial Differential Equations*, I. J. Bakelman (ed.), Lecture Notes Pure Appl. Math. 144, Marcel Dekker, New York (1993) 187–201.

[4] J. Jerónimo-Castro, L. Montejano: *Chakerian-Klamkin type theorems*, J. Convex Analysis 17 (2010) 643–649.

[5] J. Jerónimo-Castro, S. Tabachnikov: *Configuration spaces of plane polygons and a sub-Riemannian aproach to the equitangent problem*, J. Dyn. Control Syst. 22 (2016) 227–250.

[6] L. Santaló: *Note on convex curves on the hyperbolic plane*, Bull. Amer. Math. Soc. 51 (1957) 405–412.

[7] L. Santaló: *Convexity in the hyperbolic plane*, Univ. Nac. Tucumán Rev. Ser. A 19 (1969) 173–183.