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Shortest Paths along a Sequence of Line Segments in Euclidean Spaces

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We present some analytical and geometric properties of shortest ordered paths joining two given points with respect to a sequence of line segments in a Euclidean space, especially their existence, uniqueness, characteristics, and conditions for concatenation of two shortest ordered paths to be a shortest ordered path. We then focus on straightest paths lying on a sequence of adjacent convex polygons in 2 or 3 dimensional spaces.

Keywords: Shortest paths, ordered paths, shortest ordered paths, straightest paths, concatenation of paths.

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1. Introduction

Finding shortest paths is a classical geometric optimization problem and has many applications in different areas such as robotics, geographic information systems and navigation [1], [28]. To date, many algorithms have been proposed: touring polygons problems [2], [17], the shortest path problem on polyhedral surfaces [13], [22], [31], the weighted region problem [6], [23], the shortest descending path problem [3], [4], [14], and the shortest gentle descending path problem [5] (see also [11], [20], [21]). However, the problem of computing the shortest path between

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two points in three dimensions in the presence of general polyhedral obstacles is known to be computationally difficult [9], [12].

In some algorithms, exact and approximate solutions are computed based on solving the subproblem: finding the shortest path joining two given points along a triangle sequence (the adjacent triangles on a polyhedral surface). Several algorithms need to extend a shortest path from an old triangle sequence to a new one [10], [13], [15], [16], [26], [32]. Not many properties of shortest paths on polyhedral surfaces have been shown (see [9], [12], [22], [29]). They usually suppose paths as a polyline. And the properties of shortest paths along triangle sequences is even less than that ([22]). But the question is even more basic: "Is the shortest path joining two points along a triangle sequence existent and unique?" . The uniqueness of such a path is assumed even in geodesic spaces and generalized segment spaces [18]. Thus, this question is important not only in a computational sense but also in a theoretical one. In this paper, we will consider the existence and uniqueness of such shortest paths. We show how a shortest path bends at an edge, and moreover how two shortest paths can be "glued" together to form one shortest path.

The problem of finding shortest path joining two points along a sequence of adjacent convex faces (polygons) can be simplified to the following: Given two points a, b and a sequence of line segments $e_1, e_2, \ldots, e_k$ in $\mathbb{R}^3$, find a shortest path that joins a to b and passes through $e_1, e_2, \ldots, e_k$ in this order. In this paper, we consider the above problem in a general setting: the space $\mathbb{E}$ is a non-trivial Euclidian space, line segments $e_1, e_2, \ldots, e_k$ in $\mathbb{E}$ are arbitrary and not necessarily distinct, some of them may be singletons, and two consecutive line segments may not lie in a plane. This problem covers even the one of finding the shortest path lying in a convex polygon $\mathcal{P}$ in $\mathbb{R}^2$ that joints two given points $a, b \in \mathcal{P}$ and meets a given sequence of sides and vertices of $\mathcal{P}$ in a required order.

Our problem is studied under analytical and geometrical viewpoints. In this respect a path $\gamma\colon [t_0, t_1] \subset \mathbb{R} \to \mathbb{E}$ joining a to b is called an *ordered path* with respect to the sequence $e_1, e_2, \ldots, e_k$ if there exists a sequence of numbers $t_0 \leq \bar{t}_1 \leq \cdots \leq \bar{t}_k \leq t_1$ with $\gamma(\bar{t}_i) \in e_i$ for $i = 1, 2, \ldots, k$, and $\gamma(t_0) = a$, $\gamma(t_1) = b$.

We discuss the existence and uniqueness of shortest ordered path (Theorem 3.8 and Corollary 3.11). We then investigate some characteristics of angles between the shortest ordered path and line segments e_is, especially when the path goes through common points of e_is. Theorem 4.4 and Corollaries 4.7–4.8 give a step-by-step method to determine whether an ordered path is shortest. Sufficient conditions under which concatenation of two shortest ordered paths is shortest are also given (Theorem 4.9 and Corollary 4.11).

In the last section we introduce the concept of straightest path on a sequence S of adjacent convex polygons in $\mathbb{R}^2$ or $\mathbb{R}^3$, establish a relation between shortest paths and straightest paths joining two points on S, and consider a discrete initial value problem for straightest paths. Theorem 5.2 can be used to find a straightest path on a sequence of convex polygons from a given point and a given direction. Moreover, using Proposition 5.5 one can construct the shortest path joining two points and passing through a sequence of convex polygons from its finite straightest paths (see [7]).

2. Preliminaries

Let be given a metric space (X, ρ). A *path* in X is a continuous mapping γ from an interval $[t_0, t_1] \subset \mathbb{R}$ to X. We say that γ *joins* the point $\gamma(t_0)$ to the point $\gamma(t_1)$. The *length* of $\gamma\colon [t_0, t_1] \to X$ is the quantity

$$l(\gamma) = \sup_{\sigma} \sum_{i=1}^{k} \rho\big(\gamma(\tau_{i-1}), \gamma(\tau_i)\big),$$

where the supremum is taken over the set of partitions $t_0 = \tau_0 < \tau_1 < \cdots < \tau_k = t_1$ of $[t_0, t_1]$. The length of a path is additive, i.e., for any path $\gamma\colon [t_0, t_1] \to X$ and $t_* \in [t_0, t_1]$, $l(\gamma) = l\big(\gamma_{|_{[t_0,t_*]}}\big) + l\big(\gamma_{|_{[t_*,t_1]}}\big)$, where $\gamma_{|_{[t_0,t_*]}}$ and $\gamma_{|_{[t_*,t_1]}}$ are restrictions of γ on $[t_0, t_*]$ and $[t_*, t_1]$, respectively (see [25]). For instance, let $(X, \|\cdot\|)$ be a normed space. A mapping $\gamma\colon [t_0, t_1] \to X$ is said to be an *affine path* if for any $\lambda \in [0, 1]$, $\gamma((1-\lambda)t_0 + \lambda t_1) = (1-\lambda)\gamma(t_0) + \lambda\gamma(t_1)$. For $x, y \in X$, $t_0, t_1 \in \mathbb{R}$, $t_0 < t_1$, a path $\gamma\colon [t_0, t_1] \to X$ joining x to y is affine iff $\gamma(t) = (t_1 - t_0)^{-1}\big[(t_1 - t)x + (t - t_0)y\big]$. In this case, γ has length $\|x - y\|$ and the image $\gamma([t_0, t_1])$ is the *line segment* $[x, y] := \{(1-\lambda)x + \lambda y : 0 \le \lambda \le 1\}$. We assume that all paths in this paper have finite length.

Let t_0, t_1 and t_2 be real numbers satisfying $t_0 \le t_1 \le t_2$. If $\gamma_1\colon [t_0, t_1] \to X$ and $\gamma_2\colon [t_1, t_2] \to X$ are two paths satisfying $\gamma_1(t_1) = \gamma_2(t_1)$, then we can define the path $\gamma\colon [t_0, t_2] \to X$ by setting $\gamma(t) = \gamma_1(t)$ if $t_0 \le t \le t_1$, $\gamma(t) = \gamma_2(t)$ if $t_1 \le t \le t_2$. γ is called the *concatenation* of γ_1 and γ_2, denoted by $\gamma_1 * \gamma_2$, and we have $l(\gamma) = l(\gamma_1) + l(\gamma_2)$.

Let $\gamma\colon [t_0, t_1] \to X$ and $\eta\colon [\tau_0, \tau_1] \to X$ be two paths in X. We say that γ *is obtained from η by a change of parameter* if there exists a function $\psi\colon [t_0, t_1] \to [\tau_0, \tau_1]$ that is monotonic (in the weak sense), surjective and that satisfies $\gamma = \eta \circ \psi$. The function ψ is called *the change of parameter*. It is proved that the equality $l(\gamma) = l(\eta)$ always holds. We say that $\gamma\colon [t_0, t_1] \to X$ is *parametrized by arclength* if for all τ and τ' satisfying $t_0 \le \tau \le \tau' \le t_1$, we have $l\big(\gamma_{|_{[\tau,\tau']}}\big) = \tau' - \tau$. If $\gamma\colon [t_0, t_1] \to X$ is any path, then there always exists a path $\lambda\colon [0, l(\gamma)] \to X$ such that λ is parametrized by arclength and γ is obtained from λ by the change of the parameter $\psi\colon [t_0, t_1] \to [0, l(\gamma)]$ defined by $\psi(\tau) = l\big(\gamma_{|_{[t_0,\tau]}}\big)$. Most of definitions and results above can be found in [25].

In this paper, $(\mathbb{E}, \|\cdot\|)$ denotes a non-trivial Euclidian space and the induced distance is $\rho(x, y) = \|x - y\|$. For $x, y \in \mathbb{E}$, denote $]x, y] = [x, y] \setminus \{x\}$, $[x, y[= [x, y] \setminus \{y\}$, and $]x, y[= [x, y] \setminus \{x, y\}$. Note that when $x = y$, $[x, y] = \{x\}$ and $[x, y[=]x, y] =]x, y[= \emptyset$. If $x \ne y$, each point $z \in]x, y[$ is called an *interior point* of $[x, y]$. By abuse of notation, sometimes we also call the image $\gamma([t_0, t_1])$ the path $\gamma\colon [t_0, t_1] \to \mathbb{E}$. For practical purposes, $\mathbb{E}$ is usually chosen to be the finite dimensional space $\mathbb{R}^d$ with $d = 2$ or $d = 3$.

3. Shortest ordered paths

Definition 3.1. Let a, b be points in $\mathbb{E}$ and let $e_1, \ldots, e_k$ be a sequence of line segments (these line segments are not necessarily distinct and some of them may be singletons). A path $\gamma\colon [t_0, t_1] \to \mathbb{E}$ that joins a to b is called an *ordered path* with respect to the sequence $e_1, \ldots, e_k$ if there is a sequence of numbers $t_0 \le \bar{t}_1 \le \cdots \le \bar{t}_k \le t_1$ such that $\gamma(\bar{t}_i) \in e_i$ for $i = 1, \ldots, k$ (see Figure 3.1). γ is called a *shortest ordered path* if its length does not exceed any ordered path.

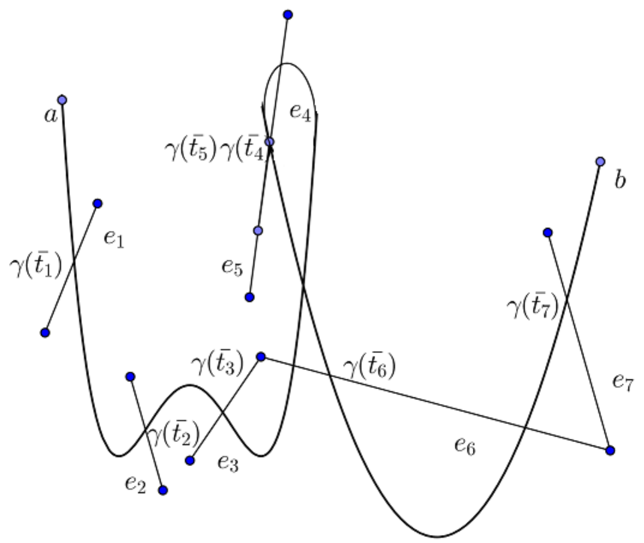

Figure 3.1: An ordered path joining a and b, passing through $e_1, e_2, e_3, e_4, e_5, e_6$ and e_7.

If $\gamma\colon [t_0, t_1] \to \mathbb{E}$ joins a to b and is an ordered path with respect to the sequence $e_1, \ldots, e_k$, we call it an $OP(a,b)_{(e_1,\ldots,e_k)}$. If we want to emphasize the intersection points $\gamma(\bar{t}_i)$ of γ and e_i, $t_0 \le \bar{t}_1 \le \cdots \le \bar{t}_k \le t_1$, we say γ is an $OP(a,b)_{(e_1,\ldots,e_k)}$ corresponding to $\gamma(\bar{t}_i) \in e_i$, $1 \le i \le k$. If, in addition, γ is a shortest path, then we say that it is an $SOP(a,b)_{(e_1,\ldots,e_k)}$.

In this paper we assume for convenience that $e_i \neq e_{i+1}$ for all $i = 1, \ldots, k-1$. We also use the notation $SOP(a,b)_\emptyset$ to denote any path that joins a to b, one-to-one, and has length $\|a - b\|$ and image $[a,b]$. Usually $SOP(a,b)_\emptyset$ is assumed to be an affine path joining a to b. The aim of this section is to prove that shortest ordered path joining two given points and with respect to a given sequence of line segments exists and in some sense is unique.

We start with an intuitive property of paths.

Lemma 3.2. *Let $t_0 < t_1$ and let $\gamma\colon [t_0, t_1] \to \mathbb{E}$ be any path.*

(a) *$l(\gamma) \ge \|\gamma(t_0) - \gamma(t_1)\|$ and equality holds only if $\gamma([t_0, t_1]) = [\gamma(t_0), \gamma(t_1)]$.*

(b) *If γ is parametrized by arclength and $l(\gamma) = \|\gamma(t_0) - \gamma(t_1)\| > 0$, then γ is affine, that is* $\gamma(t) = \dfrac{t_1 - t}{t_1 - t_0}\gamma(t_0) + \dfrac{t - t_0}{t_1 - t_0}\gamma(t_1),\ \ t \in [t_0, t_1].$

Proof. Set $a = \gamma(t_0)$ and $b = \gamma(t_1)$.

(a) Choosing the partition $\{t_0, t_1\}$ of $[t_0, t_1]$ gives $l(\gamma) \geq \|\gamma(t_0) - \gamma(t_1)\| = \|a - b\|$. Suppose that $l(\gamma) = \|a - b\|$. If there were $x = \gamma(\tau) \in \gamma([t_0, t_1])$ such that $x \notin [a, b]$, then we would have $l(\gamma) \geq \|\gamma(t_0) - \gamma(\tau)\| + \|\gamma(\tau) - \gamma(t_1)\| = \|a - x\| + \|x - b\| > \|a - b\|$, a contradiction. Thus $\gamma([t_0, t_1]) \subset [a, b]$. Moreover, since $\gamma([t_0, t_1])$ is connected, $a = \gamma(t_0)$ and $b = \gamma(t_1)$, it follows that $\gamma([t_0, t_1]) = [a, b]$.

(b) First observe that $t_1 - t_0 = l(\gamma) = \|a - b\| > 0$ and by (a), $\gamma([t_0, t_1]) = [a, b]$. Fix $t_0 < t < t_1$ and suppose $t = (1 - \lambda)t_0 + \lambda t_1$ and $\gamma(t) = (1 - \mu)a + \mu b$, $\lambda, \mu \in [0, 1]$. Since γ is parametrized by arclength, $l\big(\gamma_{|[t_0,t]}\big) = t - t_0 \geq \|\gamma(t) - a\|$ and $l\big(\gamma_{|[t,t_1]}\big) = t_1 - t \geq \|b - \gamma(t)\|$. Adding these inequalities gives $t_1 - t_0 \geq \|\gamma(t) - a\| + \|b - \gamma(t)\| \geq \|b - a\| = t_1 - t_0$. Hence $t - t_0 = \|\gamma(t) - a\|$, that is $\lambda(t_1 - t_0) = \mu\|b - a\| = \mu(t_1 - t_0)$, whence $\mu = \lambda = (t - t_0)/(t_1 - t_0)$. □

If $\gamma\big([t_0, t_1]\big)$ is a path which joins $a = \gamma(t_0)$ to $b = \gamma(t_1)$, and $c \in \mathbb{E}$, for abbreviation, we denote by $\gamma * [b, c]$ the concatenation of γ and any one-to-one shortest path $\xi\colon [t_1, t_2] \to \mathbb{E}$ joining b and c. (If $b = c$, $\gamma * [b, c] = \gamma$.) The notation $[c, a] * \gamma$ is defined similarly. Clearly $l\big(\gamma * [b, c]\big) = l(\gamma) + \|c - b\|$ and $l\big([c, a] * \gamma\big) = l(\gamma) + \|c - a\|$. Likewise, if $\gamma_1\big([t_0, t_1]\big)$, $\gamma_2\big([t_1', t_2]\big)$ are paths which join a to b and c to d, respectively, and $t_1 < t_1'$, then $\gamma_1 * [b, c] * \gamma_2$ denotes the concatenation $\gamma_1 * \xi * \gamma_2$, where ξ is defined on $[t_1, t_1']$, one-to-one, and is a shortest path joining b to c. In these notations, ξ is usually chosen to be affine.

In studying properties of shortest ordered paths, the following simple result is essential for others.

Proposition 3.3. *Let $\gamma\big([t_0, t_1]\big)$ be an $SOP(a, b)_{(e_1,\ldots,e_k)}$ corresponding to $\gamma(\bar{t}_i) \in e_i$, $i = 1, \ldots, k$.*

(a) *If $\bar{t}_{j-1} \leq \alpha < \beta \leq \bar{t}_j$, then $l\big(\gamma_{|[\alpha,\beta]}\big) = \|\gamma(\alpha) - \gamma(\beta)\|$ and $\gamma([\alpha, \beta]) = [\gamma(\alpha), \gamma(\beta)]$. If, in addition, γ is parametrized by arclength, then γ is affine on $[\alpha, \beta]$.*

(b) *$\gamma\big([t_0, t_1]\big) = \cup_{i=0}^{k}[\gamma(\bar{t}_i), \gamma(\bar{t}_{i+1})]$. Thus $\gamma\big([t_0, t_1]\big)$ is a polyline.*

Proof. Let $x := \gamma(\alpha)$, $y := \gamma(\beta)$. If $l\big(\gamma_{|[\alpha,\beta]}\big) > \|x - y\|$, then the path $\eta = \gamma_{|[t_0,\alpha]} * [x, y] * \gamma_{|[\beta,t_1]}$ would be an $OP(a, b)_{(e_1,\ldots,e_k)}$ and $l(\eta) = l\big(\gamma_{|[t_0,\alpha]}\big) + \|x - y\| + l\big(\gamma_{|[\beta,t_1]}\big) < l(\gamma)$, a contradiction. Thus $l\big(\gamma_{|[\alpha,\beta]}\big) = \|x - y\|$ and hence, by Lemma 3.2, $\gamma\big([\alpha, \beta]\big) = [x, y]$. In particular, $\gamma\big([\bar{t}_i, \bar{t}_{i+1}]\big) = [\gamma(\bar{t}_i), \gamma(\bar{t}_{i+1})]$ for $0 \leq i \leq k$. Therefore, $\gamma\big([t_0, t_1]\big) = \cup_{i=0}^{k}\gamma([\bar{t}_i, \bar{t}_{i+1}]) = \cup_{i=0}^{k}[\gamma(\bar{t}_i), \gamma(\bar{t}_{i+1})]$, which is a polyline. Finally, if γ is parametrized by arclength, Lemma 3.2 says that γ is affine on $[\alpha, \beta]$. □

Remark 3.4. Let $f_1, \ldots, f_{k+1}$ be a sequence of convex polygons (not necessarily distinct and f_i and f_{i+1} may be identical), let e_i be a common edge of f_i and f_{i+1} and let $a \in f_1$, $b \in f_{k+1}$ ($k \geq 1$). If $\gamma([t_0, t_1])$ is an $SOP(a, b)_{(e_1,\ldots,e_k)}$, then by Proposition 3.3, $\gamma([t_0, t_1]) = \cup_{i=0}^{k}[\gamma(\bar{t}_i), \gamma(\bar{t}_{i+1})] \subset \cup_{i=1}^{k+1} f_i$, i.e., this shortest ordered path lies on the polygons. Conversely, if $\gamma\colon [t_0, t_1] \to \cup_{i=1}^{k+1} f_i$ is an

$OP(a,b)_{(e_1,\dots,e_k)}$ and has the minimum length in the family of all $OP(a,b)_{(e_1,\dots,e_k)}$ that lie totally on the polygons f_i, then $\gamma([t_0,t_1])$ is an $SOP(a,b)_{(e_1,\dots,e_k)}$. $\square$

Thus determining a shortest path on the sequence of convex polygons $f_1,\dots,f_{k+1}$ in $\mathbb{R}^3$ (or in $\mathbb{R}^2$) is equivalent to determining an $SOP(a,b)_{(e_1,\dots,e_k)}$. This is a reason why we investigate in this paper shortest ordered paths with respect to a sequence of line segments. The approach allows us to study ordered paths even in cases where the unfolded images of convex polygons on a plane overlap.

The following lemma says that restriction of a shortest ordered path on each subinterval of its domain is again a shortest ordered path. It can be seen as a "local property" of shortest ordered paths.

Lemma 3.5. *Let $\gamma\big([t_0,t_1]\big)$ be an $SOP(a,b)_{(e_1,\dots,e_k)}$ and $t_0 =: \bar t_0 \le \bar t_1 \le \cdots \le \bar t_k \le \bar t_{k+1} := t_1$, $\gamma(\bar t_i) \in e_i$ for $i=1,\dots,k$. If $1 \le m \le n \le k$ and $\bar t_{m-1} \le \alpha \le \bar t_m \le \bar t_n \le \beta \le \bar t_{n+1}$, then $\gamma_{|[\alpha,\beta]}$ is an $SOP(\gamma(\alpha),\gamma(\beta))_{(e_m,\dots,e_n)}$.*

Proof. Let $x := \gamma(\alpha)$ and $y := \gamma(\beta)$. Clearly $\gamma_{|[\alpha,\beta]}$ is an $OP(x,y)_{(e_m,\dots,e_n)}$. If $\gamma_{|[\alpha,\beta]}$ is not shortest, there exists a path $\eta\colon [\alpha,\beta] \to \mathbb{E}$ that is an $OP(x,y)_{(e_m,\dots,e_n)}$ with $l(\eta) < l\big(\gamma_{|[\alpha,\beta]}\big)$. Then $\xi\colon = \gamma_{|[t_0,\alpha]} * \eta * \gamma_{|[\beta,t_1]}$ is an $OP(a,b)_{(e_1,\dots,e_k)}$ but $l(\xi) < l(\gamma)$, a contradiction. $\square$

Lemma 3.6. *Suppose $\gamma\big([t_0,t_1]\big)$ is an $SOP(a,b)_{(e_1,\dots,e_k)}$ and $t_0 =: \bar t_0 \le \bar t_1 \le \cdots \le \bar t_k \le \bar t_{k+1} := t_1$, $\gamma(\bar t_i) \in e_i$ for $i=1,\dots,k$. If $l\big(\gamma_{|[t,t']}\big) > 0$ for all $t,t' \in [t_0,t_1]$ and $t<t'$, then γ is one-to-one on each subinterval $[\bar t_i,\bar t_{i+1}]$, $0 \le i \le k$.*

Proof. Suppose conversely that γ is not one-to-one on $[\bar t_j,\bar t_{j+1}]$, i.e., there exist $\bar t_j \le \alpha < \beta \le \bar t_{j+1}$ with $\gamma(\alpha)=\gamma(\beta)$. Set $\delta := \beta-\alpha > 0$. Define $\eta\colon [t_0,t_1-\delta] \to \mathbb{E}$ by $\eta(t)=\gamma(t)$ if $t_0 \le t \le \alpha$ and $\eta(t)=\gamma(t+\delta)$ if $\alpha < t \le t_1-\delta$. Then η is an $OP(a,b)_{(e_1,\dots,e_k)}$ and since $l\big(\gamma_{|[\alpha,\beta]}\big) > 0$,

$$l(\eta) = l\big(\eta_{|[t_0,\alpha]}\big) + l\big(\eta_{|[\alpha,t_1-\delta]}\big) = l\big(\gamma_{|[t_0,\alpha]}\big) + l\big(\gamma_{|[\beta,t_1]}\big) = l(\gamma) - l\big(\gamma_{|[\alpha,\beta]}\big) < l(\gamma).$$

This contradicts the fact that γ is an $SOP(a,b)_{(e_1,\dots,e_k)}$. Thus γ is one-to-one on each $[\bar t_i,\bar t_{i+1}]$. $\square$

In this paper, sometimes we use the following assumption.

(A) *The restriction of the path on each non-singleton interval of its domain has positive length.*

Assumption (A) means that the path is not constant on any interval with positive length in its domain. It is satisfied, for instance, if the path is parametrized by arclength or if it is one-to-one.

We now turn to the problem of existence and uniqueness of a shortest ordered path.

Lemma 3.7. *Let $\mathcal{E} = (e_1,\dots,e_k)$, $\gamma([t_0,t_1])$ and $\eta([\tau_0,\tau_1])$ be $OP(a,b)_{\mathcal{E}}$ corresponding to $x_i=\gamma(\bar t_i)$ and $y_i=\eta(\bar\tau_i)$, $1 \le i \le k$, respectively.*

(a) *If $a \neq x_1$, $a \neq y_1$ and the angle between $x_1 - a$ and $y_1 - a$ is not zero, γ and η are not both shortest.*

(b) *If $b \neq x_k$, $b \neq y_k$ and the angle between $x_k - b$ and $y_k - b$ is not zero, γ and η are not both shortest.*

Proof. We prove for case (a), similar arguments apply to case (b). Suppose on the contrary that γ and η are both $SOP(a,b)_{\mathcal{E}}$ and $l(\gamma) = l(\eta) = \sigma$. Let $x_0 = y_0 := a$, $x_{k+1} = y_{k+1} := b$ and $z_i := (x_i + y_i)/2$, $0 \leq i \leq k+1$, so $z_0 = a$ and $z_{k+1} = b$. Let $\bar{\alpha}_0 := 0$, $\bar{\alpha}_1 := \|z_1 - z_0\|$, $\bar{\alpha}_2 := \bar{\alpha}_1 + \|z_2 - z_1\|$,..., $\bar{\alpha}_{k+1} := \bar{\alpha}_k + \|z_{k+1} - z_k\|$. Define $\varphi\colon [0, \bar{\alpha}_{k+1}] \to \mathbb{E}$ by $\varphi(\bar{\alpha}_i) = z_i$ for $i = 0, \dots, k+1$, and φ is affine on each subinterval $[\bar{\alpha}_i, \bar{\alpha}_{i+1}]$. Then φ is an $OP(a,b)_{\mathcal{E}}$ and

$$l(\varphi) = \sum_{i=0}^{k} l\big(\varphi_{|[\bar{\alpha}_i, \bar{\alpha}_{i+1}]}\big) = \sum_{i=0}^{k} \|z_i - z_{i+1}\|. \tag{1}$$

Likewise, according to linearity of length and Theorem 3.3,

$$l(\gamma) = \sum_{i=0}^{k} \|x_i - x_{i+1}\| \quad \text{and} \quad l(\eta) = \sum_{i=0}^{k} \|y_i - y_{i+1}\|. \tag{2}$$

Since the angle between $x_1 - a$ and $y_1 - a$ is not zero,

$$2\|a - z_1\| < \|a - x_1\| + \|a - y_1\|. \tag{3}$$

Moreover, for $1 \leq i \leq k$,

$$2\|z_i - z_{i+1}\| = \|(x_i + y_i) - (x_{i+1} + y_{i+1})\| \leq \|x_i - x_{i+1}\| + \|y_i - y_{i+1}\|. \tag{4}$$

Using (1)–(4) we get

$$\begin{aligned}
2l(\varphi) = 2\sum_{i=0}^{k} \|z_i - z_{i+1}\| &< \|a - x_1\| + \|a - y_1\| + 2\sum_{i=1}^{k} \|z_i - z_{i+1}\| \\
&\leq \|a - x_1\| + \|a - y_1\| + \sum_{i=1}^{k} \|x_i - x_{i+1}\| + \sum_{i=1}^{k} \|y_i - y_{i+1}\| \\
&= \sum_{i=0}^{k} \|x_i - x_{i+1}\| + \sum_{i=0}^{k} \|y_i - y_{i+1}\| = l(\gamma) + l(\eta) = 2\sigma,
\end{aligned}$$

implying $l(\varphi) < \sigma$. This is impossible. Therefore γ and η are not both shortest paths. $\square$

Suppose that $\gamma([t_0, t_1])$ and $\eta([\tau_0, \tau_1])$ are $OP(a,b)_{\mathcal{E}}$. We say that γ *equals* η if at least one path can be obtained from the other by a strictly increasing change of parameter. Clearly, if γ equals η, then $l(\gamma) = l(\eta)$. If $t_0 < t_1$, $\tau_0 < \tau_1$, and $\gamma([t_0, t_1])$ is an $OP(a,b)_{\mathcal{E}}$, then there always exists an $OP(a,b)_{\mathcal{E}}$ defined on

$[\tau_0, \tau_1]$ that is equal to γ, say $\eta = \gamma \circ \varphi$, where $\varphi \colon [\tau_0, \tau_1] \to [t_0, t_1]$ is defined by $\varphi(\tau) = (\tau_1 - \tau_0)^{-1}(\tau_1 - \tau)t_0 + (\tau_1 - \tau_0)^{-1}(\tau - \tau_0)t_1$. This equality relation is in fact an equivalence relation on the family of $OP(a, b)_{\mathcal{E}}$. Whenever we identify ordered paths this way, the shortest ordered path joining two points with respect to a given sequence of line segments is unique. This is stated in the following theorem which is also the main result in this section.

Theorem 3.8. *Let $a, b \in \mathbb{E}$ and let $e_1, \ldots, e_k$ be a sequence of line segments (these line segments are not necessarily distinct and some of them may be singletons). There exists a shortest ordered path joining a to b with respect to the sequence $e_1, \ldots, e_k$. Moreover, this shortest ordered path is unique in the family of ordered paths satisfying assumption* (A).

Proof. *Existence.* Denote $\mathcal{E} = (e_1, \ldots, e_k)$, $K := e_1 \times \cdots \times e_k \subset \mathbb{E}^k$, $x_0 := a$, and $x_{k+1} := b$. Consider the function $\Phi \colon \mathbb{E}^k \to \mathbb{R}$, $\Phi(x_1, \ldots, x_k) = \sum_{i=0}^{k} \|x_i - x_{i+1}\|$. As Φ is continuous and K is compact, there exists $(x_1^*, \ldots, x_k^*) \in K$ such that $\Phi(x_1^*, \ldots, x_k^*) = \sigma := \min_K \Phi$. Let

$$x_0^* := a, \quad x_{k+1}^* := b, \quad \bar{t}_0 := 0, \quad \bar{t}_{i+1} := \bar{t}_i + \|x_i^* - x_{i+1}^*\|, \quad 0 \le i \le k.$$

Clearly, $\bar{t}_{k+1} = \sum_{i=0}^{k} \|x_i^* - x_{i+1}^*\| = \sigma$. Consider the mapping $\gamma_0 \colon [0, \sigma] \to \mathbb{E}$ defined by $\gamma_0(\bar{t}_i) = x_i^*$, $i = 0, \ldots, k+1$, and γ_0 is affine on each subinterval $[\bar{t}_i, \bar{t}_{i+1}]$. Then γ_0 is an $OP(a, b)_{\mathcal{E}}$ and

$$l(\gamma_0) = \sum_{i=0}^{k} l\big(\gamma_{0|[\bar{t}_i, \bar{t}_{i+1}]}\big) = \sum_{i=0}^{k} \|x_i^* - x_{i+1}^*\| = \Phi(x_1^*, \ldots, x_k^*) = \sigma.$$

Suppose now that $\tilde{\gamma} \colon [t_0, t_1] \to \mathbb{E}$ is an $OP(a, b)_{\mathcal{E}}$ corresponding to $\tilde{x}_i = \tilde{\gamma}(\tilde{t}_i) \in e_i$, $1 \le i \le k$. Set $\tilde{x}_0 := a$, $\tilde{x}_{k+1} := b$, $\tilde{t}_0 := t_0$, and $\tilde{t}_{k+1} := t_1$. Then, by Lemma 3.2,

$$l(\tilde{\gamma}) = \sum_{i=0}^{k} l\big(\tilde{\gamma}_{|[\tilde{t}_i, \tilde{t}_{i+1}]}\big) \ge \sum_{i=0}^{k} \|\tilde{x}_i - \tilde{x}_{i+1}\| = \Phi(\tilde{x}_1, ..., \tilde{x}_k) \ge \Phi(x_1^*, ..., x_k^*) = l(\gamma_0).$$

Therefore γ_0 is an $SOP(a, b)_{\mathcal{E}}$. Observe further that γ_0 is parametrized by arclength.

Uniqueness. Suppose that $\eta([\tau_0, \tau_1])$ is an $SOP(a, b)_{\mathcal{E}}$ corresponding to $y_i = \eta(\bar{\tau}_i) \in e_i$, $1 \le i \le k$. Set $\tau_0 =: \bar{\tau}_0$, $\bar{\tau}_{k+1} := \tau_1$, $y_0 := \eta(\bar{\tau}_0) = a$, and $y_{k+1} := \eta(\bar{\tau}_{k+1}) = b$.

We first consider the case when η is parametrized by arclength and $\tau_0 = 0$. Then $\tau_1 = \tau_1 - \tau_0 = l(\eta) = \sigma$, so $[\tau_0, \tau_1] = [0, \sigma]$.

Suppose that the set $I = \{i \colon y_i \ne x_i^*\}$ is nonempty and $m := \min I$. As $y_0 = x_0^* = a$ and $y_{k+1} = x_{k+1}^* = b$ we have $1 \le m \le k$. Furthermore, by definition, $y_{m-1} = x_{m-1}^*$. If $x_m^* \notin [x_{m-1}^*, y_m]$ and $y_m \notin [x_{m-1}^*, x_m^*]$, then Lemma 3.7 says that $\gamma_{0|[\bar{t}_{m-1}, t_1]}$ and $\eta_{|[\bar{\tau}_{m-1}, \tau_1]}$ are not both $SOP(x_{m-1}^*, b)_{(e_m, \ldots, e_k)}$. This and Lemma 3.5 imply that

γ_0 and η are not both $SOP(a,b)_{\mathcal{E}}$, a contradiction. Therefore $x_m^* \in [y_{m-1}, y_m[$ or $y_m \in [x_{m-1}^*, x_m^*[$ (since $y_{m-1} = x_{m-1}^*$).

If $y_m \in [x_{m-1}^*, x_m^*[$, by Theorem 3.3, there exists $\bar{t}'_m \in [\bar{t}_{m-1}, \bar{t}_m[$ for which $\gamma_0(\bar{t}'_m) = y_m$. Since γ_0 is parametrized by arclength, Theorem 3.3 says that γ_0 is still affine on each subintervals $[\bar{t}_{m-1}, \bar{t}'_m]$ and $[\bar{t}'_m, \bar{t}_{m+1}]$, and $\|x_{m-1}^* - y_m\| = l\big(\gamma_{0|[\bar{t}_{m-1},\bar{t}'_m]}\big) = \bar{t}'_m - \bar{t}_{m-1}$, $\|y_m - x_{m+1}^*\| = \bar{t}_{m+1} - \bar{t}'_m$. Thus we can relabel $\bar{t}_m := \bar{t}'_m$ and $x_m^* := y_m$. Likewise, if $x_m^* \in [y_{m-1}, y_m[$, we can relabel $y_m := x_m^*$. If $J = I \setminus \{m\}$ is nonempty, set $n = \min J$. It follows that $x_n^* \in [y_{n-1}, y_n[$ or $y_n \in [x_{n-1}^*, x_n^*[$ and we continue to reduce the number of elements of J. After finite steps we get $y_i = x_i^*$ for all i. Thus we can assume $y_i = x_i^*$ for all $i = 0, 1, \ldots, k+1$ and will prove that $\eta = \gamma_0$ on $[0, \sigma]$.

Observe that $\bar{\tau}_{i+1} - \bar{\tau}_i = l\big(\eta_{|[\bar{\tau}_i,\bar{\tau}_{i+1}]}\big) = \|\eta(\bar{\tau}_i) - \eta(\bar{\tau}_{i+1})\| = \|x_i^* - x_{i+1}^*\| = \bar{t}_{i+1} - \bar{t}_i$ for $i = 0, \ldots, k$. Condition $\bar{\tau}_0 = \bar{t}_0 = 0$ implies that $\bar{\tau}_1 = \bar{t}_1$, $\bar{\tau}_2 = \bar{t}_2, \ldots, \bar{\tau}_{k+1} = \bar{t}_{k+1} = \sigma$. Suppose $\bar{\tau}_{j+1} > \bar{\tau}_j$. Since η is parametrized by arclength and $l\big(\eta_{|[\bar{\tau}_j,\bar{\tau}_{j+1}]}\big) = \|\eta(\bar{\tau}_j) - \eta(\bar{\tau}_{j+1})\|$, Lemma 3.2 says that η is affine on $[\bar{\tau}_j, \bar{\tau}_{j+1}]$. Hence $\eta(t) = \gamma_0(t)$ on $[\bar{\tau}_j, \bar{\tau}_{j+1}]$ and therefore $\eta(t) = \gamma_0(t)$ on $[0, \sigma]$.

For the general case when $[\tau_0, \tau_1]$ is arbitrary and η is not necessarily parametrized by arclength but satisfies assumption (A), we express $\eta = \zeta \circ \psi$, where $\psi\colon [\tau_0, \tau_1] \to [0, \sigma]$ is defined by $\psi(\tau) = l\big(\eta_{|[\tau_0,\tau]}\big)$ and $\zeta\colon [0,\sigma] \to \mathbb{E}$ is parametrized by arclength (see [25]). Since ψ is strictly increasing and onto, we have $0 = \psi(\bar{\tau}_0) \leq \psi(\bar{\tau}_1) \leq \cdots \leq \psi(\bar{\tau}_{k+1}) = \sigma$. This and condition $\zeta\big(\psi(\bar{\tau}_i)\big) = \eta(\bar{\tau}_i) = y_i$ for $i = 0, 1, \ldots, k+1$ imply that ζ is an $OP(a,b)_{\mathcal{E}}$. Moreover, ζ is parametrized by arclength, $l(\zeta) = \sigma - 0 = \sigma$, so that ζ is an $SOP(a,b)_{\mathcal{E}}$. By the proof above, $\zeta = \gamma_0$. Therefore, $\eta = \gamma_0 \circ \psi$, i.e., the two paths η and γ_0 are equal. The proof is complete. □

Remark 3.9. Notice that the function Φ in the proof above is in general not strictly convex and hence the k-tuple $(x_1^*, \ldots, x_k^*) \in K$ is not unique. For instance, if $a, b \in \mathbb{E}$, $a \neq b$, and $\Phi\colon \mathbb{E} \to \mathbb{R}$ is defined by $\Phi(x) = \|x - a\| + \|x - b\|$, then Φ is not strictly convex, $\operatorname{argmin} \Phi = [a, b]$, and for each $x \in e := [a, b]$, $[a, x] * [x, b]$ is an $SOP(a,b)_{(e)}$.

We can apply the arguments in the proof of the first part of Theorem 3.8 to prove existence of solutions of problems with variable endpoints.

Corollary 3.10. *Let A,B be nonempty compact subsets of $\mathbb{E}$ and let $\mathcal{E} = (e_1, \ldots, e_k)$.*

(a) *If $b \in \mathbb{E}$ is fixed, in the family of $OP(a,b)_{\mathcal{E}}$, where $a \in A$, there exists a shortest path.*

(b) *If $a \in \mathbb{E}$ is fixed, in the family of $OP(a,b)_{\mathcal{E}}$, where $b \in B$, there exists a shortest path.*

(c) *In the family of $OP(a,b)_{\mathcal{E}}$, where $a \in A$ and $b \in B$, there exists a shortest path.*

(d) *In the family of $OP(a,a)_{\mathcal{E}}$, where $a \in A$, there exists a shortest path.*

Proof. (a) The proof is similar to the first part of that of Theorem 3.8, where Φ is replaced with $\tilde{\Phi}\colon \mathbb{E}^{k+1} \to \mathbb{R}$, $\tilde{\Phi}(x_0, x_1, \ldots, x_k) = \big(\sum_{i=0}^{k-1} \|x_i - x_{i+1}\|\big) + \|x_k - b\|$ and K is replaced with $A \times e_1 \times \cdots \times e_k$. Proofs of parts (b)–(d) are similar. □

Observe that shortest ordered paths in Corollary 3.10 may not be unique. Moreover, by Theorem 3.3, all shortest paths in this corollary are always polylines.

Applying Corollary 3.10(d) to the problem of finding an inscribed polygon in a given convex polygon $\mathcal{P} \subset \mathbb{R}^2$ with a minimum perimeter, we find that this problem has a solution. Some properties of angles of this inscribed polygon will be derived in the next section.

Corollary 3.11. *If $\gamma([t_0, t_1])$ and $\eta([\tau_0, \tau_1])$ are $SOP(a,b)_{(e_1,\dots,e_k)}$ and parametrized by arclength, then $\eta(\tau) = \gamma(\tau - \tau_0 + t_0)$ and $\gamma(t) = \eta(t - t_0 + \tau_0)$. If, in addition, $\tau_0 = t_0$ (hence $\tau_1 = t_1$), then $\eta(t) = \gamma(t)$ for all $t \in [t_0, t_1]$.*

Proof. By Theorem 3.8, there is a strictly increasing and surjective function $\psi\colon [\tau_0, \tau_1] \to [t_0, t_1]$ such that $\eta = \gamma \circ \psi$. We have $\psi(\tau_0) = t_0$ and $\psi(\tau_1) = t_1$. As γ and η are parametrized by arclength,

$$\tau - \tau_0 = l\big(\eta_{|[\tau_0,\tau]}\big) = l\big(\gamma_{|[\psi(\tau_0),\psi(\tau)]}\big) = \psi(\tau) - \psi(\tau_0) = \psi(\tau) - t_0$$

for every $\tau \in [\tau_0, \tau_1]$. It follows that $\psi(\tau) = \tau - \tau_0 + t_0$, giving $\eta(\tau) = \gamma\big(\psi(\tau)\big) = \gamma(\tau - \tau_0 + t_0)$. Conversely, for each $t \in [t_0, t_1]$, $\tau := t - t_0 + \tau_0 \in [\tau_0, \tau_1]$ and so $\eta(t - t_0 + \tau_0) = \eta(\tau) = \gamma(\tau - \tau_0 + t_0) = \gamma(t)$. $\square$

Corollary 3.12. *Let $\mathcal{D}$ be a polytope in $\mathbb{R}^3$ whose faces are convex. Let $f_1, \dots, f_k$ be a sequence of faces of $\mathcal{D}$ such that $f_i \cap f_{i+1}$ is an edge for $i = 1, \dots, k-1$, $a \in f_1$, and $b \in f_k$. In the family of ordered paths satisfying assumption (A), there exists uniquely a shortest ordered path lying on the surface of $\mathcal{D}$, joining a to b, and going orderly through $f_1, \dots, f_k$.*

As in Remark 3.4, this shortest path is an $SOP(a,b)_{(e_1,\dots,e_{k-1})}$, where $e_i = f_i \cap f_{i+1}$, $i = 1, \dots, k-1$.

Applying Theorem 3.8 to polytopes in $\mathbb{R}^3$ and simple polygons on the plane $\mathbb{R}^2$ we get the following well-known results that are shown in [22], [29].

Corollary 3.13. (a) *Let a and b be any two points on a domain $\mathcal{S}$ in $\mathbb{R}^3$ (or in $\mathbb{R}^2$) consisting of finitely many adjacent convex polygons. In the family of paths joining a to b and lying totally on $\mathcal{S}$, there exists a shortest path. This shortest path is a polyline.*

(b) *Let a and b be any two points in a simple polygon $\mathcal{P} \subset \mathbb{R}^2$. There exists a shortest path joining a to b that lies totally in $\mathcal{P}$ and furthermore, it is a polyline.*

Proof. (a) The case both a and b lie on the same polygon is trivial, so we assume that they belong to different polygons. Each path joining a to b and lying on $\mathcal{S}$ is an ordered path with respect to some sequence $\mathcal{E}$ consisting of common edges of adjacent polygons in $\mathcal{S}$. Let $\gamma_{\mathcal{E}}$ be an $SOP(a,b)_{(\mathcal{E})}$. Since the family of such sequences $\mathcal{E}$ is finite, the shortest path γ in the family $\{\gamma_{\mathcal{E}}\}$ is the required path. By Theorem 3.3, each $\gamma_{\mathcal{E}}$ is a polyline, so is γ.

(b) Partition $\mathcal{P}$ into nonoverlapping convex polygons and apply part (a) to obtain the required result. $\square$

4. Conditions for concatenation of two shortest ordered paths to be a shortest ordered path

We have shown in Theorem 3.3 that every shortest ordered path is a polyline. In this section we consider some geometric characteristics of shortest ordered paths and represent conditions under which concatenation of two shortest ordered paths is a shortest ordered path. If $j < i$, then $SOP(x,y)_{(e_i,\dots,e_j)}$ is understood to be $SOP(x,y)_{\emptyset}$.

Lemma 4.1. *Suppose γ is an $OP(a,b)_{(e_1,\dots,e_k)}$ and $b \in e_n \cap e_{n+1} \cap \cdots \cap e_k$. γ is an $SOP(a,b)_{(e_1,\dots,e_k)}$ iff it is an $SOP(a,b)_{(e_1,\dots,e_{n-1})}$. Similarly, if $a \in e_1 \cap e_2 \cap \cdots \cap e_m$, then γ is an $SOP(a,b)_{(e_1,\dots,e_k)}$ iff it is an $SOP(a,b)_{(e_{m+1},\dots,e_k)}$.*

This is derived from the fact that under the condition $b \in e_n \cap \cdots \cap e_k$, a path is an $OP(a,b)_{(e_1,\dots,e_{n-1})}$ iff it is an $OP(a,b)_{(e_1,\dots,e_k)}$. The other case is similar.

Lemma 4.2. *Let $\gamma\big([t_0,t_1]\big)$ be an $SOP(a,b)_{(e_1,\dots,e_k)}$ corresponding to $x_i = \gamma(\bar{t}_i) \in e_i$, $i = 1,\dots,k$.*

(a) *If e_* is a line segment such that $e_* \cap [x_{j-1},x_j] \neq \emptyset$, $2 \le j \le k$, and $e_* \neq e_{j-1}, e_* \neq e_j$, then γ is an $SOP(a,b)_{(e_1,\dots,e_{j-1},e_*,e_j,\dots,e_k)}$. If $e_* \cap [a,x_1] \neq \emptyset$, and $e_* \neq e_1$, then γ is an $SOP(a,b)_{(e_*,e_1,\dots,e_k)}$. The case $e_* \cap [x_k,b] \neq \emptyset$ is similar.*

(b) *If $e_{*1},\dots,e_{*m}$ are line segments that contain the same point belonging to $[x_{j-1},x_j]$ for some $j \in \{2,\dots,k\}$, then γ is also a shortest ordered path $SOP(a,b)_{(e_1,\dots,e_{j-1},e_{*1},\dots,e_{*m},e_j,\dots,e_k)}$. The cases $e_{*1},\dots,e_{*m}$ contain the same point of $[a,x_1]$ or $[x_k,b]$ are similar.*

Proof. (a) Consider the case $2 \le j \le k$. Let $\mathcal{E}_* = (e_1,\dots,e_{j-1},e_*,e_j,\dots,e_k)$ and $\eta\big([\tau_0,\tau_1]\big)$ an $OP(a,b)_{\mathcal{E}_*}$ corresponding to $y_i = \eta(\bar{\tau}_i) \in e_i$, $y_* = \eta(\bar{\tau}_*) \in e_*$ $(\bar{\tau}_{j-1} \le \bar{\tau}_* \le \bar{\tau}_j)$. Clearly, η is also an $OP(a,b)_{(e_1,\dots,e_k)}$, so $l(\gamma) \le l(\eta)$. Now take an $x_* \in e_* \cap [x_{j-1},x_j]$. Since $\gamma([\bar{t}_{j-1},\bar{t}_j]) = [x_{j-1},x_j]$, there is $\bar{t}_* \in [\bar{t}_{j-1},\bar{t}_j]$ with $\gamma(\bar{t}_*) = x_*$. Thus γ is also an $OP(a,b)_{\mathcal{E}_*}$ and therefore γ is an $SOP(a,b)_{\mathcal{E}_*}$. Other cases are proved similarly. (b) follows directly from (a). $\square$

Next we turn to our main problem of concatenation of two shortest paths. We first consider the simplest case: the concatenation of a path and a line segment. The following lemma says that if we elongate the first or last line segment of a shortest ordered path, we get a new shortest ordered path.

Lemma 4.3. *Let $\gamma\big([t_0,t_1]\big)$ be an $SOP(a,b)_{\mathcal{E}}$ corresponding to $x_i = \gamma(\bar{t}_i) \in e_i$, $e_i \in \mathcal{E} = (e_1,\dots,e_n)$.*

(a) *If $x_n \neq b$ and $q \in \mathbb{E}$ such that $b \in]x_n,q[$, then $\gamma * [b,q]$ is an $SOP(a,q)_{\mathcal{E}}$.*

(b) *If $x_1 \neq a$ and $p \in \mathbb{E}$ such that $a \in]p,x_1[$, then $[p,a] * \gamma$ is an $SOP(p,b)_{\mathcal{E}}$.*

(c) *Set $x_0 = a$. If $x_m \neq x_{m+1} = \cdots = x_n = b$, $(0 \leq m \leq n-1)$, and $b \in]x_m, q[$ then $\gamma * [b, q]$ is an $SOP(a, q)_{\mathcal{E}}$.*

Proof. (a) Let $\eta([\tau_0, \tau_1])$ be any $OP(a, q)_{\mathcal{E}}$ corresponding to $y_i = \eta(\bar{\tau}_i) \in e_i$, $i = 1, \ldots, n$. Suppose $b = (1-\lambda)x_n + \lambda q$ for some $\lambda \in]0, 1[$. Set

$$z_i := (1-\lambda)x_i + \lambda y_i, \quad 1 \leq i \leq n,$$
$$z_0 := (1-\lambda)x_1 + \lambda a, \quad z_{n+1} := (1-\lambda)x_n + \lambda q = b.$$

Then for $i = 1, \ldots, n-1$,

$$\begin{aligned} \|z_i - z_{i+1}\| &= \|(1-\lambda)(x_i - x_{i+1}) + \lambda(y_i - y_{i+1})\| \\ &\leq (1-\lambda)\|x_i - x_{i+1}\| + \lambda\|y_i - y_{i+1}\|. \end{aligned} \tag{5}$$

Let ξ be the path going through $z_0, z_1, \ldots, z_{n+1}$ such that ξ is an affine path joining z_i and z_{i+1}, $0 \leq i \leq n$. ξ is an $OP(z_0, b)_{\mathcal{E}}$. By Theorem 3.3, there exits a $t^* \in [t_0, \bar{t}_1]$ satisfying $\gamma(t^*) = z_0$. Lemma 3.5 states that $\gamma_{|[t^*, t_1]}$ is an $SOP(z_0, b)_{\mathcal{E}}$. According to Theorem 3.3 and (5) we have

$$\begin{aligned} \|z_0 - x_1\| + \sum_{i=1}^{n-1} \|x_i - x_{i+1}\| + \|x_n - b\| = l\big(\gamma_{|[t^*, t_1]}\big) \leq l(\xi) = \sum_{i=0}^{n} \|z_i - z_{i+1}\| \\ \leq \|z_0 - z_1\| + \left[(1-\lambda)\sum_{i=1}^{n-1} \|x_i - x_{i+1}\| + \lambda \sum_{i=1}^{n-1} \|y_i - y_{i+1}\|\right] + \|z_n - b\|. \end{aligned} \tag{6}$$

Noting that

$$\|z_0 - x_1\| = \lambda\|a - x_1\|, \quad \|x_n - b\| = \lambda\|x_n - q\|,$$
$$\|z_0 - z_1\| = \lambda\|a - y_1\|, \quad \|z_n - b\| = \lambda\|y_n - q\|,$$

we deduce from (6) that

$$\lambda\|a - x_1\| + \lambda\sum_{i=1}^{n-1} \|x_i - x_{i+1}\| + \lambda\|x_n - q\| \leq \lambda\|a - y_1\| + \lambda\sum_{i=1}^{n-1} \|y_i - y_{i+1}\| + \lambda\|y_n - q\|.$$

Since $\|x_n - q\| = \|x_n - b\| + \|b - q\|$, the above inequality yields $\lambda\big(l(\gamma) + \|b - q\|\big) \leq \lambda l(\eta)$, which implies $l(\gamma * [b, q]) = l(\gamma) + \|b - q\| \leq l(\eta)$. Thus, $\gamma * [b, q]$ is an $SOP(a, q)_{\mathcal{E}}$.

Part (b) is proved similarly. To prove (c) we observe that, by Lemma 4.1, γ is an $SOP(a, b)_{(e_1, \ldots, e_m)}$. Applying part (a) we find that $\gamma * [b, q]$ is an $SOP(a, q)_{(e_1, \ldots, e_m)}$. That $\gamma * [b, q]$ is an $SOP(a, q)_{\mathcal{E}}$ follows from Lemma 4.2. □

We now study some characteristics of a shortest ordered path basing on properties of angles between the path and line segments e_is. If u and v are nonzero vectors in $\mathbb{E}$, we denote by $\angle(u, v)$ the angle between u and v, which does not exceed π.

Theorem 4.4. *Let $\mathcal{E} = (e_1, \dots, e_k)$ be a sequence of line segments and $\gamma([t_0, t_1])$ an $SOP(a, b)_{(e_1,\dots,e_{n-1})}$ corresponding to $x_i = \gamma(\bar{t}_i) \in e_i$ $(1 \le i \le n-1)$ and $n \le k$. Suppose that $b \in e_n \cap \dots \cap e_k$, each e_j is non-singleton for $j = n, \dots, k$, and that $x_{n-1} \neq b$ (if $n = 1$, $x_0 := a$). Let $q \in \mathbb{E}$, $q \neq b$. Then $\gamma * [b, q]$ is an $SOP(a, q)_{\mathcal{E}}$ iff for any $y_j \in e_j$ and $y_j \neq b$, $j = n, \dots, k$, we have $\theta \ge \pi$, where*

$$\theta := \angle(x_{n-1} - b, y_n - b) + \sum_{j=n}^{k-1} \angle(y_j - b, y_{j+1} - b) + \angle(y_k - b, q - b).$$

Proof. Suppose that $\gamma * [b, q]$ is an $SOP(a, q)_{\mathcal{E}}$ and that $y_j \in e_j$, $y_j \neq b$, $j = n, \dots, k$. We show that $\theta \ge \pi$. By Lemma 3.5, $\xi := \gamma_{|[\bar{t}_{n-1}, t_1]} * [b, q]$ is an $SOP(x_{n-1}, q)_{(e_n,\dots,e_k)}$. Setting $y_{k+1} := q$, $v_j := y_j - b$, $j = n, \dots, k+1$, we have $\theta = \angle(x_{n-1} - b, v_n) + \sum_{j=n}^{k} \angle(v_j, v_{j+1})$. Let L_j be the line containing b and y_j, $n \le j \le k+1$. Let P be the plane containing L_n and x_{n-1} (if $x_{n-1} \in L_n$, P is any fixed plane containing L_n).

Set $y'_n := y_n$. If $L_n \neq L_{n+1}$, denote by $y'_{n+1} \in P$ the point such that $\triangle x_{n-1} b y'_n$ and $\triangle y'_n b y'_{n+1}$ do not overlap (i.e., x_{n-1} and y'_{n+1} are on opposite sides of L_n) and $\triangle y'_n b y'_{n+1} = \triangle y_n b y_{n+1}$; if $x_{n-1} \in L_n$, y'_{n+1} lies on any side of L_n. If $L_n = L_{n+1}$, i.e., $y_{n+1} - b = \lambda(y_n - b)$ for some λ, then we choose y'_{n+1} satisfying $y'_{n+1} - b = \lambda(y'_n - b)$. Similarly, $y'_{n+2} \in P$ is the point such that $\triangle y'_{n+1} b y'_{n+2} = \triangle y_{n+1} b y_{n+2}$ and the triangles $\triangle y'_n b y'_{n+1}$ and $\triangle y'_{n+1} b y'_{n+2}$ do not overlap, and so on. We then obtain a sequence $y'_n, \dots, y'_{k+1}$ in P such that $\|y'_j - y'_{j+1}\| = \|y_j - y_{j+1}\|$, $\|y'_j - b\| = \|y_j - b\|$, and $\angle(v'_j, v'_{j+1}) = \angle(v_j, v_{j+1})$, where $v'_j = y'_j - b$ for $j \ge n$. Then $\theta = \angle(x_{n-1} - b, v'_n) + \sum_{j=n}^{k} \angle(v'_j, v'_{j+1})$. If $\theta < \pi$, there exist $z_{n-1} \in]x_{n-1}, b[$, $z'_n \in]y'_n, b[$,..., $z'_{k+1} \in]y'_{k+1}, b[$ that are near b and such that they are collinear. Let $z_j \in]y_j, b[$ satisfy $\|z_j - b\| = \|z'_j - b\|$, $j \ge n$. We have

$$\begin{aligned} &\|z_{n-1} - z_n\| + \dots + \|z_k - z_{k+1}\| = \|z_{n-1} - z'_n\| + \dots + \|z'_k - z'_{k+1}\| \\ &= \|z_{n-1} - z'_{k+1}\| < \|z_{n-1} - b\| + \|b - z'_{k+1}\| = \|z_{n-1} - b\| + \|b - z_{k+1}\| \end{aligned}$$

and hence $[x_{n-1}, z_{n-1}] * [z_{n-1}, z_n] * \dots * [z_k, z_{k+1}] * [z_{k+1}, q]$ is an $OP(x_{n-1}, q)_{(e_n,\dots,e_k)}$ whose length is less than that of ξ. This is impossible. Therefore $\theta \ge \pi$.

Next suppose conversely that for any $y_j \in e_j$ and $y_j \neq b$, $j = n, \dots, k$, we have $\theta \ge \pi$. We first observe that $\gamma * [b, q]$ is an $OP(a, q)_{\mathcal{E}}$. Let $\eta([\tau_0, \tau_1])$ be any $SOP(a, q)_{\mathcal{E}}$ corresponding to $y_i = \eta(\bar{\tau}_i) \in e_i$, $i = 1, \dots, k$. Let $\bar{\tau}_{k+1} := \tau_1$ and $y_{k+1} := \eta(\bar{\tau}_{k+1}) = q$. We need to show that $l(\gamma * [b, q]) \le l(\eta)$.

If $b \in [y_j, y_{j+1}]$ for some $j \ge n$, then there is $\tau_b \in [\bar{\tau}_j, \bar{\tau}_{j+1}]$ with $\eta(\tau_b) = b$. Since $\eta_{|[\tau_0, \tau_b]}$ is an $OP(a, b)_{(e_1,\dots,e_{n-1})}$,

$$l(\eta) = l\big(\eta_{|[\tau_0, \tau_b]}\big) + l\big(\eta_{|[\tau_b, \tau_1]}\big) \ge l(\gamma) + \|b - q\| = l(\gamma * [b, q]).$$

Consider the case $b \notin [y_j, y_{j+1}]$ (and so $y_j \neq b$) for all $j = n, \dots, k$. Let $y'_n, \dots, y'_{k+1}$ be points defined as in the first part of the proof. Set $\theta_{n-1} := \angle(x_{n-1} - b, y'_n - b)$ and for $n \le j \le k$,

$$\theta_j := \angle(x_{n-1} - b, y'_n - b) + \angle(y'_n - b, y'_{n+1} - b) + \dots + \angle(y'_j - b, y'_{j+1} - b).$$

We have $\theta_k = \theta \geq \pi$. Let $r = \min\{j \geq n : \theta_j \geq \pi\}$. As $b \notin [y_r, y_{r+1}]$,

$$0 < \angle(y'_r - b, y'_{r+1} - b) = \angle(y_r - b, y_{r+1} - b) < \pi$$

and there exists $y^{*\prime} \in]y'_r, y'_{r+1}]$ such that $\theta_{r-1} + \angle(y'_r - b, y^{*\prime} - b) = \pi$. Let $y^* \in]y_r, y_{r+1}]$ satisfy $\|y^* - y_r\| = \|y^{*\prime} - y'_r\|$ and $\tau^* \in]\bar{\tau}_r, \bar{\tau}_{r+1}]$ satisfy $\eta(\tau^*) = y^*$. Since $b \in]x_{n-1}, y^{*\prime}[$, by Lemma 4.3(c), $\gamma * [b, y^{*\prime}]$ is an $SOP(a, y^{*\prime})_{(e_1,\dots,e_n)}$. Thus

$$\begin{aligned} l\big(\eta_{|[\tau_0,\tau^*]}\big) &= l\big(\eta_{|[\tau_0,\bar{\tau}_n]}\big) + \|y_n - y_{n+1}\| + \cdots + \|y_{r-1} - y_r\| + \|y_r - y^*\| \\ &= l\big(\eta_{|[\tau_0,\bar{\tau}_n]}\big) + \|y'_n - y'_{n+1}\| + \cdots + \|y'_{r-1} - y'_r\| + \|y'_r - y^{*\prime}\| \\ &\geq l\big(\gamma * [b, y^{*\prime}]\big) = l(\gamma) + \|b - y^{*\prime}\| = l(\gamma) + \|b - y^*\|. \end{aligned}$$

Hence
$$\begin{aligned} l(\eta) &\geq l\big(\eta_{|[\tau_0,\tau^*]}\big) + \|y^* - q\| \geq l(\gamma) + \|b - y^*\| + \|y^* - q\| \\ &\geq l(\gamma) + \|b - q\| = l(\gamma * [b, q]). \end{aligned}$$
□

Theorem 4.4 gives a step-by-step method to check whether an ordered path is shortest: we just measure the angles between line segments of the path and e_is at points of intersection, from the starting point to the terminating point.

We now consider several consequences of Theorem 4.4. In computational geometry, edges of polygons intersect at their common endpoints. In such cases, Theorem 4.4 reduces to the following simple form.

Corollary 4.5. *Let $\mathcal{E} = (e_1, \dots, e_k)$ be a sequence of line segments and $\gamma([t_0, t_1])$ an $SOP(a, b)_{(e_1,\dots,e_{n-1})}$ corresponding to $x_i = \gamma(\bar{t}_i) \in e_i$ $(1 \leq i \leq n-1)$ and $n \leq k$. Suppose that b is the common endpoint of non-singleton line segments $e_n = [b, b_n]$,..., $e_k = [b, b_k]$, and that $x_{n-1} \neq b$ (if $n = 1$, $x_0 := a$). Let $q \in \mathbb{E}$, $q \neq b$. Then $\gamma * [b, q]$ is an $SOP(a, q)_{\mathcal{E}}$ iff $\theta \geq \pi$, where*

$$\theta := \angle(x_{n-1} - b, b_n - b) + \sum_{j=n}^{k-1} \angle(b_j - b, b_{j+1} - b) + \angle(b_k - b, q - b).$$

Applying Corollary 4.5 we can prove the following result in [29]:

> *A shortest path joining two points on the surface of a convex polyhedron $\mathcal{D} \subset \mathbb{R}^3$ cannot pass through any vertex of $\mathcal{D}$.*

Indeed, suppose a shortest path γ on the surface $\mathcal{S}$ of $\mathcal{D}$ joins a and b on $\mathcal{S}$ and goes through a vertex $v \in \mathcal{S}$. As γ can be seen as a shortest ordered path, by Corollary 4.5, both the angle to the left and the angle to the right of γ at v are not less than π. Hence the angle of $\mathcal{D}$ at v is not less than 2π, a contradiction. Notice that the proof still holds for polyhedrons $\mathcal{D}$ whose angle at every vertex is strictly less than 2π.

Let us consider an example. Given $a, q \in \mathbb{R}^3 \setminus \{\mathbf{0}\}$, we show that $\gamma = [a, \mathbf{0}] * [\mathbf{0}, q]$ is the shortest path that joins a to q and meets the x-, y-, and z-axes. Suppose η is any path joining a and q and meets x-, y-, and z-axes at u, v, and w, respectively. If one of the points u, v, w is coincident with $b := \mathbf{0} = (0, 0, 0)$, then clearly, $l(\gamma) \leq l(\eta)$. Assume that $b \notin \{u, v, w\}$ and that η is an $OP(a, q)_{([b,v],[b,u],[b,w])}$.

Since $\angle(a-b, v-b) + \angle(v-b, u-b) + \angle(u-b, w-b) + \angle(w-b, q-b) \geq \pi$, Corollary 4.5 says that, in the family of all $OP(a,q)_{([b,v],[b,u],[b,w])}$, $l(\gamma) \leq l(\eta)$.

Corollary 4.6. *Suppose $\gamma\big([t_0,t_1]\big)$ is an $SOP(a,b)_{(e_1,\dots,e_n)}$ corresponding to $x_i = \gamma(\bar{t}_i) \in e_i$, $1 \leq i \leq n$, $x_n \neq b$. Let $y \in e_n$, $y \neq x_n$, and let q be any point in $\mathbb{E}$ such that $q \neq x_n$ and $\angle(y-x_n, q-x_n) = \angle(y-x_n, b-x_n)$. Then $\gamma_{|[t_0,\bar{t}_n]} * [x_n, q]$ is an $SOP(a,q)_{(e_1,\dots,e_n)}$. In particular, if we rotate the last line segment $[x_n, b]$ about the line containing e_n, we then get a new shortest ordered path.*

Proof. Set $x_0 := a$. If $x_0 = x_1 = \cdots = x_n$, then there is nothing to prove. So we assume that $m = \max\{i : x_i \neq b\}$ exists. By Theorem 4.4, for any $y_i \in e_i$, $y_i \neq x_n$, $m+1 \leq i \leq n$,

$$\angle(x_m - x_n, y_{m+1} - x_n) + \angle(y_{m+1} - x_n, y_{m+2} - x_n) + \dots + \angle(y_n - x_n, b - x_n) \geq \pi.$$

Hence $\angle(x_m - x_n, y_{m+1} - x_n) + \cdots + \angle(y_n - x_n, q - x_n) \geq \pi$ since $\angle(y_n - x_n, q - x_n) = \angle(y_n - x_n, b - x_n)$. Applying Theorem 4.4 once more, we find that $\gamma_{|[t_0,\bar{t}_n]} * [x_n, q]$ is an $SOP(a,q)_{(e_1,\dots,e_n)}$. $\square$

Corollary 4.7. *Let $\gamma([t_0,t_1])$ be an $SOP(a,b)_{(e_1,\dots,e_n)}$ corresponding to $x_i = \gamma(\bar{t}_i) \in e_i$, $1 \leq i \leq n$. Set $x_0 := a$, $x_{n+1} := b$. Suppose also that for some j, e_j is non-singleton, $x_{j-1} \neq x_j$, and $x_{j+1} \neq x_j$.*

(a) *If $y_j \in e_j$, $y_j \neq x_j$, then $\theta := \angle(x_{j-1} - x_j, y_j - x_j) + \angle(y_j - x_j, x_{j+1} - x_j) \geq \pi$. In particular, if x_j is an interior point of e_j, then $\theta = \pi$.*

(b) *Let L be the line containing e_j. If $x_{j-1} \in L$ and $x_{j+1} \notin L$, then x_j is an end point of e_j with $\|x_{j-1} - x_j\| = \min_{x \in e_j} \|x_{j-1} - x\|$ and $\theta > \pi$. The case $x_{j+1} \in L$ and $x_{j-1} \notin L$ is similar.*

(c) *If $x_{j-1}, x_{j+1} \in L$, then either $\angle(x_{j-1} - x_j, x_{j+1} - x_j) = \pi$ or $\angle(x_{j-1} - x_j, y_j - x_j) = \angle(x_{j+1} - x_j, y_j - x_j) = \pi$, where $y_j \in e_j$, $y_j \neq x_j$. In the latter case x_j is an end point of e_j with $\|x_{j-1} - x_j\| = \min_{x \in e_j} \|x_{j-1} - x\|$.*

Proof. (a) Applying Theorem 4.4 to $\gamma_{|[t_0,\bar{t}_{j+1}]}$ ($\bar{t}_{n+1} := t_1$) and the sequence $e_1, \dots, e_j$, we get $\theta \geq \pi$. If x_j is an interior point of e_j, take $y_j' \in e_j$ such that x_j is an interior point of $[y_j, y_j']$. We then also have

$$\theta' := \angle(x_{j-1} - x_j, y_j' - x_j) + \angle(y_j' - x_j, x_{j+1} - x_j) \geq \pi.$$

Since $\theta + \theta' = 2\pi$, $\theta = \theta' = \pi$.

(b) Suppose $x_{j-1} \in L$ and $x_{j+1} \notin L$. Since $\theta \neq \pi$, $\theta > \pi$ and according to part (a), x_j must be an endpoint. If $\|x_{j-1} - x_j\| > \min_{x \in e_j} \|x_{j-1} - x\|$, there exists $\bar{x} \in e_j$ with $\|x_{j-1} - x_j\| > \|x_{j-1} - \bar{x}\|$, i.e., $\bar{x} \in [x_{j-1}, x_j[$. Since $x_{j+1} \notin L$,

$$\begin{aligned}\|x_{j-1} - x_j\| + \|x_j - x_{j+1}\| &= \|x_{j-1} - \bar{x}\| + \|\bar{x} - x_j\| + \|x_j - x_{j+1}\| \\ &> \|x_{j-1} - \bar{x}\| + \|\bar{x} - x_{j+1}\|,\end{aligned}$$

i.e, $l\big(\gamma_{|[\bar{t}_{j-1},\bar{t}_{j+1}]}\big) > l\big([x_{j-1}, \bar{x}] * [\bar{x}, x_{j+1}]\big)$, a contradiction. Thus $\|x_{j-1} - x_j\| = \min_{x \in e_j} \|x_{j-1} - x\|$.

(c) If $x_j \in]x_{j-1}, x_{j+1}[$, then $\angle(x_{j-1} - x_j, x_{j+1} - x_j) = \pi$. Otherwise, say $x_{j-1} \in]x_j, x_{j+1}]$, an analysis which is similar to that in part (b) shows that x_j is an end point of e_j with $\|x_{j-1} - x_j\| = \min_{x \in e_j} \|x_{j-1} - x\|$ and $\angle(x_{j-1} - x_j, y_j - x_j) = \angle(x_{j+1} - x_j, y_j - x_j) = \pi$, for $y_j \in e_j$, $y_j \neq x_j$. □

The following result is a converse of Corollary 4.7 and it gives sufficient conditions for an ordered path to be shortest.

Corollary 4.8. *Let $\mathcal{E} = (e_1, \ldots, e_k)$ be a sequence of line segments and $\gamma([t_0, t_1])$ an $SOP(a, b)_{(e_1,\ldots,e_{n-1})}$ corresponding to $x_i = \gamma(\bar{t}_i) \in e_i$, $1 \leq i \leq n-1$. Let $b \in e_n$ and $q \in \mathbb{E}$, $q \neq b$. Set $x_0 = a$ and suppose that $m = \max\{i : x_i \neq b\}$ exits and that there is $y \in e_n, y \neq b$.*

(a) *If $\theta := \angle(x_m - b, y - b) + \angle(y - b, q - b) = \pi$, then $\gamma * [b, q]$ is an $SOP(a, q)_{\mathcal{E}}$.*

(b) *If b is an end point of e_n and $\theta > \pi$, then $\gamma * [b, q]$ is an $SOP(a, q)_{\mathcal{E}}$.*

Proof. (a) We always have $\angle(x_m - b, y' - b) + \angle(y' - b, q - b) = \pi$ for any $y' \in e_n$, $y' \neq b$. Thus by Theorem 4.4, $\gamma * [b, q]$ is an $SOP(a, q)_{(e_1,\ldots,e_m,e_n)}$. If $m < n - 1$, it follows from Lemma 4.2(b) that $\gamma * [b, q]$ is an $SOP(a, q)_{\mathcal{E}}$.
(b) follows directly from Corollary 4.5. □

To illustrate Corollaries 4.7 and 4.8 let us consider a triangle in $\mathbb{R}^2$ having acute angles (Figure 4.1). For $u \in e_1$ fixed, Corollary 4.8 states that the triangle uvw has minimum perimeter because angles θ at v and w are π. Figure 4.1 shows that $\angle(w - u, z_1 - u) + \angle(z_1 - u, v - u) \neq \pi$. Since the angles at u do not satisfy Corollary 4.7(a), uvw is not the inscribed triangle with minimum perimeter.

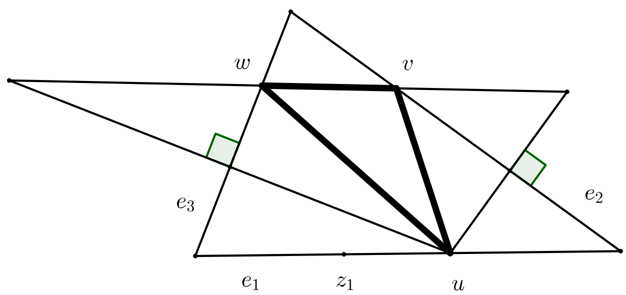

Figure 4.1: $SOP(u, u)_{(e_1,e_2,e_3)} = [u, v] * [v, w] * [w, u]$

We are now in a position to consider the general case of concatenation of two shortest ordered paths. Roughly speaking, the following theorem says that if the last line segment of a shortest ordered path and the first of the other overlap, the two paths can be joined to become a shortest ordered path.

Theorem 4.9. *Let $\mathcal{E} = (e_1, \ldots, e_k)$ be a sequence of line segments. Suppose that $\gamma_1([t_0, t_1])$ is an $SOP(a, b)_{(e_1,\ldots,e_{n-1})}$ and $\gamma_2([t^*, t_2])$ is an $SOP(c, d)_{(e_n,\ldots,e_k)}$, where $t^* < t_1 < t_2$, and $\gamma_1(t_1) = \gamma_2(t_1) = b$. Suppose also that γ_1, γ_2 satisfy assumption (A). If $t_1 \leq \bar{t}_n \leq \cdots \leq \bar{t}_k \leq t_2$, $x_i := \gamma_2(\bar{t}_i) \in e_i$ for $n \leq i \leq k$, and if there exists $\epsilon > 0$ such that $\gamma_1([t_1 - \epsilon, t_1]) \subset \gamma_2([t^*, t_1])$ then the concatenation γ of γ_1 and $\gamma_{2|[t_1,t_2]}$ is an $SOP(a, d)_{\mathcal{E}}$.*

Proof. Let $t_0 =: \bar{t}_0 \leq \bar{t}_1 \leq \cdots \leq \bar{t}_{n-1} \leq t_1$, $x_i := \gamma_1(\bar{t}_i) \in e_i$ for $i = 1, \ldots, n-1$, and $x_0 := \gamma_1(\bar{t}_0) = a$. We can also assume, by applying Lemma 4.3 if necessary, that $b = x_n \in e_n$, i.e., $\bar{t}_n = t_1$. Set $\bar{t}_{k+1} := t_2$ and note that γ is an $OP(a,d)_{\mathcal{E}}$. The proof is divided into two cases.

Case 1: All e_j, $j \geq n$, are non-singleton. Set $x_{k+1} := \gamma_2(\bar{t}_{k+1}) = d$, $m = \max\{i < n : x_i \neq b\}$, $r = \min\{j > n : x_j \neq b\}$. m and r exist due to assumption (A) and Theorem 3.3. Assume that $c \in [x_m, b[$. Then, by Lemma 3.5, $\gamma_{2|[t^*,\bar{t}_r]} = [c,b] * [b,x_r]$ is an $SOP(c,x_r)_{(e_n,\ldots,e_{r-1})}$. Applying Theorem 4.4 to $\gamma_{2|[t^*,\bar{t}_r]}$ we find that all angles θ at b are not less than π. Thus this theorem also says that $\gamma_1 * \gamma_{2|[t_1,\bar{t}_r]} = \gamma_1 * [b,x_r]$ is an $SOP(a,x_r)_{(e_1,\ldots,e_m,e_n,\ldots,e_{r-1})}$. If $m < n-1$, Lemma 4.2 again shows that $\gamma_1 * \gamma_{2|[t_1,\bar{t}_r]}$ is an $SOP(a,x_r)_{(e_1,\ldots,e_{r-1})}$. We continue in this fashion to obtain, after a finite number of times, that γ is an $SOP(a,d)_{\mathcal{E}}$.

Case 2: e_j is a singleton for some $j \geq n$. The particular case when e_n is a singleton, i.e., $e_n = \{b\}$, is derived immediately from the following lemma.

Lemma 4.10. *Let $\zeta([\alpha_0,\alpha_1])$ be an ordered path $OP(p,u)_{(e_1,\ldots,e_l)}$ and $\xi([\alpha_1,\alpha_2])$ an $OP(u,q)_{(e_{l+1},\ldots,e_m)}$. Then $\psi = \zeta * \xi$ is an $SOP(p,q)_{(e_1,\ldots,e_l,\{u\},e_{l+1},\ldots,e_m)}$ iff ζ is an $SOP(p,u)_{(e_1,\ldots,e_l)}$ and ξ an $SOP(u,q)_{(e_{l+1},\ldots,e_m)}$.*

Proof. Indeed, if ψ is a shortest ordered path, then so are ζ and ξ (Lemma 3.5). Conversely, suppose ζ and ξ are shortest ordered paths. Let $\eta([\tau_0,\tau_1])$ be any $OP(p,q)_{(e_1,\ldots,e_l,\{u\},e_{l+1},\ldots,e_m)}$, corresponding to $y_i = \eta(\bar{\tau}_i) \in e_i$, $i = 1,\ldots,m$, and $\eta(\bar{\tau}_u) = u$ ($\bar{\tau}_l \leq \bar{\tau}_u \leq \bar{\tau}_{l+1}$). Since $\eta_{|[\tau_0,\bar{\tau}_u]}$ is an $OP(p,u)_{(e_1,\ldots,e_l)}$ and $\eta_{|[\bar{\tau}_u,\tau_1]}$ is an $OP(u,q)_{(e_{l+1},\ldots,e_m)}$, $l(\eta) = l\big(\eta_{|[\tau_0,\bar{\tau}_u]}\big) + l\big(\eta_{|[\bar{\tau}_u,\tau_1]}\big) \geq l(\zeta) + l(\xi) = l(\psi)$. Therefore ψ is an $SOP(p,q)_{(e_1,\ldots,e_l,\{u\},e_{l+1},\ldots,e_m)}$.

We now prove the theorem for the case $s = \min\{j \geq n : e_j \text{ is a singleton}\} > n$, say $e_s = \{u\}$. If $t_1 = \bar{t}_{n+1} = \cdots = \bar{t}_s$, then $b = u \in e_{n+1} \cap \cdots \cap e_s$ and by Lemma 4.1, γ_1 is also an $SOP(a,b)_{(e_1,\ldots,e_{s-1})}$ and $\gamma_{2|[t_1,t_2]}$ an $SOP(b,d)_{(e_{s+1},\ldots,e_k)}$. Lemma 4.10 above states that γ is an $SOP(a,d)_{(e_1,\ldots,e_k)}$. If $\bar{t}_s > t_1$, by Case 1 we find that $\tilde{\gamma} = \gamma_1 * \gamma_{2|[t_1,\bar{t}_s]}$ is an $SOP(a,u)_{(e_1,\ldots,e_{s-1})}$. We then apply Lemma 4.10 again to $\tilde{\gamma}$ and $\gamma_{2|[\bar{t}_s,t_2]}$ to obtain the required result. The proof of the theorem is complete. □

If the last line segment of a shortest ordered path and the first of the other do not overlap, we can use the following.

Corollary 4.11. *Suppose $\gamma_1([t_0,t_1])$ is an $SOP(a,b)_{(e_1,\ldots,e_n)}$ and $\gamma_2([t_1,t_2])$ is an $SOP(b,c)_{(e_{n+1},\ldots,e_k)}$, $t_0 < t_1 < t_2$. Suppose also that γ_1,γ_2 satisfy assumption (A). Then $\gamma = \gamma_1 * \gamma_2$ is an $SOP(a,c)_{(e_1,\ldots,e_k)}$ iff there exists $\epsilon > 0$ such that $\gamma_{1|[t_1-\epsilon,t_1]} * \gamma_{2|[t_1,t_1+\epsilon]}$ is a shortest ordered path.*

Loosely speaking, if the last segment of the first shortest ordered path and the first segment of the second form a shortest ordered path, then their concatenation is also a shortest ordered path.

Proof. If $\gamma = \gamma_1 * \gamma_2$ is an $SOP(a,c)_{(e_1,\dots,e_k)}$ then $\zeta := \gamma_{1|[t_1-\epsilon,t_1]} * \gamma_{2|[t_1,t_1+\epsilon]} = \gamma_{|[t_1-\epsilon,t_1+\epsilon]}$ is a shortest ordered path for each $\epsilon \le \min\{t_1 - t_0, t_2 - t_1\}$ (Lemma 3.5). Conversely suppose that ζ is a shortest ordered path and satisfies assumption (A). Since $\gamma_{1|[t_1-\epsilon,t_1]} = \zeta_{|[t_1-\epsilon,t_1]}$, applying Theorem 4.9 to γ_1 and ζ we find that $\xi = \gamma_1 * \zeta_{|[t_1,t_1+\epsilon]}$ is a shortest ordered path. Applying Theorem 4.9 again to ξ and γ_2 we obtain that $\gamma = \xi * \gamma_{2|[t_1+\epsilon,t_2]}$ is an $SOP(a,c)_{(e_1,\dots,e_k)}$. □

In [8] and [19] we used a *direct multiple shooting method* to approximate shortest paths in a simple polygon in $\mathbb{R}^2$ and on surfaces of convex polytopes in $\mathbb{R}^3$. We divided the given region into subregions, found a shortest path in each subregion and then "glued" them. We applied one straightness condition, a version of Corollary 4.11, to adjust the paths to get better aproximations.

5. Straightest paths and a discrete initial value problem

In [27] Polthier and Schimies presented a new concept of geodesics: straightest geodesics are paths that have equal path angle on both sides at each point. In this section we consider "straightest paths" which are lightly differ from the original and in fact are particular shortest ordered paths. Let $\mathcal{S} = (f_1, f_2, \dots, f_{k+1})$ be a sequence of (not necessary distinct) adjacent convex polygons in $\mathbb{R}^2$ or $\mathbb{R}^3$, i.e., $e_i := f_i \cap f_{i+1}$ is an edge for $1 \le i \le k$.

Definition 5.1. A *straightest path* $\gamma\colon [t_0, t_1] \to \cup_{i=1}^{k+1} f_i$ on the sequence $\mathcal{S}$ is a path that satisfies assumption (A) and the following conditions:

(a) there exist integers $1 \le m \le n \le k$ and a sequence of numbers $t_0 =: \bar{t}_{m-1} < \bar{t}_m \le \dots \le \bar{t}_n < \bar{t}_{n+1} := t_1$ such that $x_i := \gamma(\bar{t}_i) \in e_i$ for $m \le i \le n$ and $x_{m-1} := \gamma(t_0) \in f_m$, $x_{n+1} := \gamma(t_1) \in f_{n+1}$;

(b) $l\big(\gamma_{|[\bar{t}_i,\bar{t}_{i+1}]}\big) = \|x_i - x_{i+1}\|$ for $m - 1 \le i \le n$;

(c) For $z_i \in e_i$, $z_i \ne x_i$, we have $\angle(x_{i-1} - x_i, z_i - x_i) + \angle(z_i - x_i, x_{i+1} - x_i) = \pi$ if $x_i \in e_i$ is not a common vertex, and

$$\angle(x_{i^*-1} - x_i, z_{i^*} - x_i) + \sum_{j=i^*}^{r-1} \angle(z_j - x_i, z_{j+1} - x_i) + \\ + \angle(z_r - x_i, x_{r+1} - x_i) = \pi$$

if $x_{i^*-1} \ne x_{i^*} = x_{i^*+1} = \dots = x_r \ne x_{r+1}$ and $i^* \le i \le r$.

It is conventional to define straightest path joining a to b on the same polygon (with respect to an empty sequence of common edges) to be any path that joins a to b, one-to-one, and has length $\|a - b\|$, i.e., an $SOP(a,b)_\emptyset$. □

Condition (a) states that a straightest path is an ordered path. Condition (b) and Lemma 3.2 imply that $\gamma([\bar{t}_i, \bar{t}_{i+1}]) = [x_i, x_{i+1}] \subset f_{i+1}$ for $m - 1 \le i \le n$. We observe also that condition (c) does not depend on the choice of $z_i \in e_i$, $m \le i \le n$. Corollary 4.8 and Theorem 4.4 show that γ is an $SOP(x_{m-1}, x_{n+1})_{(e_m,\dots,e_n)}$. Note also that assumption (A), the conditions $t_0 < \bar{t}_m$, $\bar{t}_n < t_1$, and (b) imply that $x_0 \ne x_1$ and $x_n \ne x_{n+1}$. These conditions do not restrict the definition of straightest paths since if x_0 belongs to a common edge then we choose $m = \max\{j : x_0 \in f_j\}$. Similarly, we can assume that x_{n+1} does not belong to e_n.

There are reasons why we study straightest paths: First the shortest path joining two given points on the surface of a convex polyhedron $\mathcal{D} \subset \mathbb{R}^3$ does not go through any vertex of $\mathcal{D}$ (see [29]). Thus, by Corollary 4.7, the path is straightest. Second, every shortest ordered path joining two points on a surface $\mathcal{S}$ consisting of convex polygons is composed of straightest paths joining vertexes of $\mathcal{S}$ (Proposition 5.5).

Let $\gamma([t_0, t_1])$ be a straightest path on $\mathcal{S} = (f_1, f_2, \dots, f_{k+1})$, $\gamma(t_0) \in f_m$, and v a nonzero vector that is parallel to f_m. If there exists $t^* > t_0$ such that $\gamma([t_0, t^*]) \subset f_m$ and $\gamma(t^*) - \gamma(t_0) = \lambda v$ for some $\lambda > 0$ we say that γ *starts* at $\gamma(t_0)$ in the direction of v.

As usual, we first consider the problem of existence of straightest paths.

Theorem 5.2. *Let $\mathcal{S} = (f_1, f_2, \dots, f_{k+1})$ be a sequence of adjacent convex polygons. Let $a, p \in f_m$, $a \neq p$, and $v = p - a$. Then there exists uniquely a longest straightest path $\gamma([t_0, t_1])$ on $\mathcal{S}$ starting at a and in the direction of v. Moreover, if $\eta([\tau_0, \tau_1])$ is any straightest path on $\mathcal{S}$ starting at a and in the direction of v, then η equals $\gamma_{|[t_0, t^*]}$ for some $t_0 \le t^* \le t_1$. Thus every straightest path on $\mathcal{S}$ can be extended to a longest straightest path.*

Proof. There is nothing to prove if $k = 0$, $\mathcal{S} = \{f_1\}$. Thus we assume, without loss of the generality, that $k \ge 1$, $a, p \in f_1$, and $a \notin f_2$.

First we construct γ. Denote $e_i = f_i \cap f_{i+1}$, $i = 1, \dots, k$. Set $t_0 = 0$ and $\bar{t}_1 = \max\{t : a + tv \in f_1\}$. Since $p = a + v \in f_1$, $\bar{t}_1 \ge 1$. Define $\gamma_1 \colon [0, \bar{t}_1] \to f_1$ by $\gamma_1(0) = a$, $\gamma_1(\bar{t}_1) = x_1 := a + \bar{t}_1 v \in f_1$, and γ_1 is affine on $[0, \bar{t}_1]$. If $x_1 \notin e_1$ we put $t_1 := \bar{t}_1$ and $\gamma := \gamma_1$.

Suppose $x_1 \in e_1$. Choose any $z_1 \in e_1$, $z_1 \neq x_1$. If there exists $p_1 \in f_2$ such that

$$\angle(a - x_1, z_1 - x_1) + \angle(z_1 - x_1, p_1 - x_1) = \pi, \tag{7}$$

set $v_1 := p_1 - x_1$, $\sigma_1 := \max\{t : x_1 + tv_1 \in f_2\} \ge 1$, $\bar{t}_2 := \bar{t}_1 + \sigma_1$, and $x_2 := x_1 + \sigma_1 v_1 \in f_2$. We then define $\gamma_2 \colon [\bar{t}_1, \bar{t}_2] \to f_2$ by $\gamma_2(\bar{t}_1) = x_1$, $\gamma_2(\bar{t}_2) = x_2$, and γ_2 is affine on $[\bar{t}_1, \bar{t}_2]$. Clearly $\gamma_1 * \gamma_2$ is a straightest path starting at a, in the direction of v, and joining a to x_2. Observe that such a p_1 always exists if x_1 is an interior point of e_1.

Suppose x_1 is an endpoint of e_1 and there is no $p_1 \in f_2$ satisfying (7). If x_1 is not a common vertex of e_1 and e_2, set $t_1 := \bar{t}_1$ and $\gamma := \gamma_1$. Assume that x_1 is a common vertex of e_1 and e_2. Let $r := \max\{i : x_1 \in e_j,\ 1 \le j \le i\}$. Choose $z_j \in e_j$, $z_j \neq x_1$, $1 \le j \le r$, and let z^*_{r+1} be a point on the second edge of f_{r+1} that has an endpoint x_1 and $z^*_{r+1} \neq x_1$. Let

$$\begin{aligned} \theta_1 := {} & \angle(a - x_1, z_1 - x_1) + \angle(z_1 - x_1, z_2 - x_1) + \cdots \\ & \cdots + \angle(z_{r-1} - x_1, z_r - x_1) + \angle(z_r - x_1, z^*_{r+1} - x_1). \end{aligned}$$

We call θ_1 the *angle of incidence at* x_1. If $\theta_1 < \pi$, set $t_1 := \bar{t}_1$ and $\gamma := \gamma_1$. Assume $\theta_1 \geq \pi$. Put

$$s := \max\left\{i : \angle(a-x_1, z_1-x_1)+\angle(z_1-x_1, z_2-x_1)+\cdots+\angle(z_{i-1}-x_1, z_i-x_1) < \pi\right\}.$$

Since there is no $p_1 \in f_2$ satisfying (7), $2 \leq s \leq r$ and there exists $p_s \in f_{s+1} \setminus f_s$, such that

$$\angle(a - x_1, z_1 - x_1) + \angle(z_1 - x_1, z_2 - x_1) + \cdots + \angle(z_s - x_1, p_s - x_1) = \pi.$$

Set $$\bar{t}_2 = \cdots = \bar{t}_s := \bar{t}_1, \quad x_2 = \cdots = x_s := x_1, \quad v_s := p_s - x_1,$$

$$\sigma_s := \max\{t : x_s + tv_s \in f_{s+1}\}, \quad \bar{t}_{s+1} := \bar{t}_s + \sigma_s, \text{ and } x_{s+1} := x_s + \sigma_s v_s.$$

As $\sigma_s \geq 1$, $\bar{t}_{s+1} > \bar{t}_s$. Define $\gamma_i\colon [\bar{t}_{i-1}, \bar{t}_i] \to f_i$, $\gamma_i(t) = x_1$, $2 \leq i \leq s$, $\gamma_{s+1}\colon [\bar{t}_s, \bar{t}_{s+1}] \to f_{s+1}$, $\gamma_{s+1}(\bar{t}_s) = x_s$, $\gamma_{s+1}(\bar{t}_{s+1}) = x_{s+1}$, and γ_{s+1} is affine. From the construction we find that $\gamma_1 * \gamma_2 * \cdots * \gamma_{s+1}$ is a straightest path joining a to x_{s+1} and starting at a in the direction of v.

We continue in this fashion to finally obtain a sequence of points $x_1, \ldots, x_{n+1}$ satisfying the following conditions: i) $x_n \in e_n$, $x_{n+1} \in f_{n+1} \setminus f_n$, and $1 = \max\{t > 0 : x_n + t(x_{n+1} - x_n) \in f_{n+1}\}$; ii) either $x_{n+1} \notin e_{n+1}$ (for instance, when $n = k$) or iii) $x_{n+1} \in e_{n+1}$ and the angle of incidence at x_{n+1} is less than π. We then set $t_1 := \bar{t}_{n+1} = \bar{t}_n + \sigma_n$ and define $\gamma_{n+1}\colon [\bar{t}_n, t_1] \to f_{n+1}$ by $\gamma_{n+1}(\bar{t}_n) = x_n$, $\gamma_{n+1}(t_1) = x_{n+1} = x_n + \sigma_n v_n$, and γ_{n+1} is affine. Let $\gamma = \gamma_1 * \gamma_2 * \cdots * \gamma_{n+1}$. We find that γ satisfies assumption (A) and is an $OP(a, x_{n+1})_{(e_1,\ldots,e_n)}$. Moreover, γ also satisfies conditions (b) and (c) of Definition 5.1. Thus γ is a straightest path starting at a and in the direction of v.

Suppose now that $\eta([\tau_0, \tau_1])$ is a straightest path on $\mathcal{S}$ starting at a and in the direction of v. Let $c := \eta(\tau_1)$. The step-by-step construction of γ shows that there exists $t_c \in]t_0, t_1]$ such that $c = \gamma(t_c)$ and η and $\gamma_{|[t_0,t_c]}$ are shortest ordered paths joining a to c with respect to the same sequence of line segments. Thus by the uniqueness of shortest ordered path (Theorem 3.8), η equals $\gamma_{|[t_1,t_c]}$. This implies also that η can be extended to γ and γ is the unique longest straightest path. □

Remark 5.3. Notice that shortest ordered path joining two given points on a sequence of adjacent convex polygons always exists but this may not be true for straightest path.

In the problem of computing shortest paths on a polyhedral surface, a key geometric concept is the notion of *planar unfolding* around a sequence of adjacent convex polygons $\mathcal{S} = (f_1, f_2, \ldots, f_{k+1})$ (see [22]). We unfold the sequence $\mathcal{S}$ as follow: Rotate f_1 around e_1 until its plane coincides with that of f_2, rotate f_1 and f_2 around e_2 until their plane coincides with that of f_3, continue this way until all faces $f_1, f_2, \ldots, f_k$ lie on the plane of f_{k+1}. The idea of planar unfolding was used to prove Theorem 4.4. We now investigate the image of a straightest path under a planar unfolding.

Suppose that $\gamma([t_0, t_1])$ is a straightest path joining $a \in f_1$ and $b \in f_{k+1}$, $t_0 < \bar{t}_1 \leq \cdots \leq \bar{t}_k < t_1$, $x_i := \gamma(\bar{t}_i) \in e_i$ for $1 \leq i \leq k$. For each i, choose $z_i \in e_i$, $z_i \neq x_i$.

Let a_1 be the image of a under the planar unfolding around e_1. Since $a \neq x_1$, $a_1 \neq x_1$. If $x_1 \neq x_2$, we have $\angle(a_1 - x_1, z_1 - x_1) = \angle(a - x_1, z_1 - x_1)$ so that $\angle(a_1 - x_1, z_1 - x_1) + \angle(z_1 - x_1, x_2 - x_1) = \pi$. Hence $x_1 \in]a_1, x_2[$, i.e., a_1, x_1, x_2 are collinear. Similarly, let a_2 and $x_{1,2}$ be the images of a_1 and x_1 under the planar unfolding around e_2, respectively. If $x_2 \neq x_3$, then $\angle(x_{1,2} - x_2, z_2 - x_2) = \angle(x_1 - x_2, z_2 - x_2)$ whence $\angle(x_{1,2} - x_2, z_2 - x_2) + \angle(z_2 - x_2, x_3 - x_2) = \pi$, i.e., $a_2, x_{1,2}, x_2, x_3$ are collinear. If $x_1 = x_2 = \cdots = x_r \neq x_{r+1}$, then by condition (c), a_r, x_1, x_{r+1} are collinear, where a_r is the image of a under the planar unfolding around e_1, e_2, ..., e_r. Repeating this argument we finally find that the image of γ under the planar unfolding around $e_1, \ldots, e_k$ is a line segment. Conversely, if the image of γ under the planar unfolding around $e_1, \ldots, e_k$ is a line segment, then the angles of γ at edges are π, so γ is a straightest path. We thus arrive at the following result.

Lemma 5.4. *Let γ be an $OP(a, b)_{(e_1,\ldots,e_k)}$ ($a \in f_1$, $b \in f_{k+1}$) on the sequence $\mathcal{S}$ in $\mathbb{R}^3$. Then γ is straightest iff its planar unfolding around $e_1, \ldots, e_k$ is a line segment.*

Thus straightest paths are ordered paths whose images under planar unfoldings are line segments. Therefore if there exists a straightest path joining two points $a \in f_m$ and $b \in f_{n+1}$ on the sequence $\mathcal{S}$, ($m \leq n$), then it is the shortest path joining these points that lies entirely in the polygons and passes through $e_m, \ldots, e_n$. Conversely, we have the following result presented by O'Rourke et al. (see [24]).

Proposition 5.5. *Every shortest ordered path joining two vertexes on the sequence $\mathcal{S} = (f_1, f_2, ..., f_{k+1})$ is composed of straightest paths joining vertexes of $\mathcal{S}$.*

Proof. Let $\gamma([t_0, t_1])$ be a shortest ordered path joining two vertexes $a \in f_1$ and $b \in f_{k+1}$ on $\mathcal{S}$, $t_0 < \bar{t}_1 \leq \cdots \leq \bar{t}_k < t_1$, $x_i = \gamma(\bar{t}_i) \in e_i$, $1 \leq i \leq k$. If γ does not pass through any vertexes of $\mathcal{S}$ except a and b, then each x_i is an interior point of e_i. By Corollary 4.7, for $z_i \in e_i, z_i \neq x_i$, $i = 1, \ldots, k$, $\angle(x_{i-1} - x_i, z_i - x_i) + \angle(z_i - x_i, x_{i+1} - x_i) = \pi$, where $x_0 := a$ and $x_{k+1} := b$. Thus γ is straightest.

If γ passes through vertexes $v_0 := a$, $v_1 = \gamma(t_1^*), ..., v_l = \gamma(t_l^*)$, $v_{l+1} := b$ (in this order) and there are no vertexes belonging to $\gamma(]t_0, t_1^*[)$, $\gamma(]t_1^*, t_2^*[), ..., \gamma(]t_l^*, t_1[)$, then by Lemma 3.5, restrictions of γ on $[t_0, t_1^*], ..., [t_l^*, t_1]$ are shortest ordered paths joining the vertices v_j and v_{j+1} of $\mathcal{S}$. By the proof above, each restriction is a straightest path. Thus γ is composed of straightest paths joining vertices of $\mathcal{S}$. □

Theorem 5.6. *Let $\mathcal{S} = (f_1, f_2, \ldots, f_{k+1})$ be a sequence of adjacent convex polygons and let a, q_1, q_2, q_3 be points in f_m such that $q_2 \in]q_1, q_3[$ and a, q_1, q_3 are not collinear. Let $v_i = q_i - a$, $i = 1, 2, 3$. Assume that γ_1, γ_2 and γ_3 are straightest paths starting at a and in the directions of v_1, v_2, v_3, respectively, and γ_1, γ_3 cut a line segment $e \subset f_n$ ($n \geq m$) at y_1 and y_3, respectively. If γ_2 is the longest straightest path, then it meets e at some point $y_2 \in]y_1, y_3[$.*

Proof. We prove for the case $\mathcal{S}$ is in $\mathbb{R}^3$, the other case is similar. Without restriction of generality we assume that $m = 1$ and $n = k + 1$. Let $\mathcal{U}$ be the planar unfolding around the sequence $e_1, \dots, e_k$. By Lemma 5.4, the images of $\gamma_1, \gamma_2, \gamma_3$ under $\mathcal{U}$ are line segments $[a', y_1^*]$, $[a', y_2^*]$, $[a', y_3^*]$, respectively and we have $y_1 \in [a', y_1^*]$ and $y_3 \in [a', y_3^*]$. Let q_1', q_2', q_3' be the images of q_1, q_2, q_3 under $\mathcal{U}$. Since q_2 is between q_1 and q_3, q_2' is between q_1' and q_3'. Assume $q_2' = \alpha q_1' + \beta q_3'$ where $\alpha, \beta > 0$, $\alpha + \beta = 1$. Letting $v_i' := q_i' - a'$, $i = 1, 2, 3$, we get $v_2' = \alpha v_1' + \beta v_3'$. Since q_i lies on the path γ_i and $q_i \neq a$, $q_2' \in]a', y_2^*]$, $q_1' \in]a', y_1]$, and $q_3' \in]a', y_3]$. Assume $y_1 - a' = \lambda v_1'$ and $y_3 - a' = \nu v_3'$, $\lambda, \nu > 0$. Set $\mu := \lambda\nu/(\alpha\nu + \beta\lambda) > 0$. We have $y_2 := a' + \mu v_2' = a' + \mu\alpha v_1' + \mu\beta v_3' = (\alpha\nu + \beta\lambda)^{-1}(\alpha\nu y_1 + \beta\lambda y_3) \in]y_1, y_3[$. Let $\mathcal{S}'$ be the sequence of images of $f_1, \dots, f_{k+1}$ under the planar unfolding $\mathcal{U}$. $\mathcal{S}'$ is the sequence of convex polygons with adjacent edges being images of $e_1, \dots, e_k$ under $\mathcal{U}$ and the triangle $a'y_1y_3$ is contained in the union of polygons in $\mathcal{S}'$. As the image of γ_2 is $[a', y_2^*]$, the longest line segment starting at a' in the direction of v_2', we have $[a', y_2] \subset [a', y_2^*]$. This means that the line segment $[a', y_2^*]$ meets $[y_1, y_3]$ at $y_2 \in]y_1, y_3[$, i.e., the longest straightest path γ_2 meets e at y_2. $\square$

6. Conclusions

Recently, An [7] presented a related work which is concerned with straightest paths. He proved that straightest paths solve the boundary value problem on a processed domain which is extended from a "funnel" and an alternative triangle face of that funnel. Similar to [18], we can consider Blaschke's Convergence Theorems for straightest paths.

We hope that some results in this paper could be used to study Steiner's problem (see [30]): Given a convex polygon in the plane, find an inscribed polygon of minimal perimeter.

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