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A Geometric Characterization of Polygonal Radon Planes*

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We study unit circles of polygonal Radon planes from a geometric point of view. In particular, we prove that a two-dimensional real polygonal Banach space $\mathbb{X}$ cannot be a Radon plane if the number of vertices of its unit circle is $4n$, for some $n \in \mathbb{N}$. Also we obtain a complete characterization of polygonal Radon planes in terms of a tractable geometric concept introduced in this article. It follows from our characterization that every regular polygon with $4n+2$ vertices, where $n \in \mathbb{N}$, is the unit circle of a Radon plane. Furthermore, we describe types of Radon planes for which the unit circles are hexagons, but not regular ones.

Keywords: Radon plane, Birkhoff-James orthogonality, polygonal Banach space.

2010 Mathematics Subject Classification: Primary 46B20; secondary 52A21.

1. Introduction

The purpose of this short note is to study Radon planes, i.e., two-dimensional real Banach spaces where Birkhoff-James orthogonality is symmetric. We restrict ourselves to unit circles of the plane, which are polygons. Indeed, we obtain a complete characterization of polygonal Radon planes in terms of a novel geometric concept which is rather straightforward to verify. Let us first establish the relevant notation and the terminologies to be used in the paper.

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Let $(\mathbb{X}, \|.\|)$ be a *two-dimensional real Banach space*, i.e., a normed plane. Given any two elements $u, v \in \mathbb{X}$, let $\overline{uv} = \{(1-t)u + tv \ : t \in [0,1]\}$ denote the closed straight line segment joining u and v. Let $B_{\mathbb{X}} = \{x \in \mathbb{X} \colon \|x\| \leq 1\}$ and $S_{\mathbb{X}} = \{x \in \mathbb{X} \colon \|x\| = 1\}$ be the unit disk and the unit circle of $\mathbb{X}$, respectively. It is perhaps obvious that, without loss of generality, we may assume $\mathbb{X}$ to be $\mathbb{R}^2$, equipped with some appropriate norm. It is in this sense that we identify the zero vector with the origin $(0,0)$ in $\mathbb{R}^2$. For any two elements $x, y \in \mathbb{X}$, x is said to be *Birkhoff-James orthogonal* to y [3], written as $x \perp_B y$, if $\|x + \lambda y\| \geq \|x\|$ for all $\lambda \in \mathbb{R}$. We say that the Birkhoff-James orthogonality is *right unique* [7] in $\mathbb{X}$, if for any two elements $x, y \in \mathbb{X}$, there exists a unique number $a \in \mathbb{R}$ such that $x \perp_B ax + y$. We refer the reader to [1] for a recent and comprehensive survey of the various applications of Birkhoff-James orthogonality in studying the geometry of normed linear spaces. It is easy to observe that Birkhoff-James orthogonality is homogeneous, i.e., given any $x, y \in \mathbb{X}$ and any two scalars $\alpha, \beta \in \mathbb{R}$, we have that $x \perp_B y$ implies $\alpha x \perp_B \beta y$. However, we would like to note that Birkhoff-James orthogonality is not symmetric in $\mathbb{X}$, i.e., given any two elements x, y in $\mathbb{X}$, $x \perp_B y$ does not imply that $y \perp_B x$. For an element $x \in \mathbb{X}$, we say that x is left symmetric [11] if $x \perp_B y$ implies $y \perp_B x$ for any $y \in \mathbb{X}$. $\mathbb{X}$ is said to be a Radon plane [10] if the Birkhoff-James orthogonality is symmetric in $\mathbb{X}$. We recall from [2] that a centrally symmetric closed convex curve γ is said to be a Radon curve if and only if a suitable translate of it can be used as unit circle of a normed plane in which the Birkhoff-James orthogonality is symmetric. We would like to refer the reader to some of the excellent recent studies [2, 8, 9] for more information on various aspects of Radon curves. Let us emphasize here that the background of Minkowski geometry (i.e., the geometry of finite-dimensional real Banach spaces) is affine in nature. In particular, the image of a Radon curve under an affine transformation is again a Radon curve.

It is well-known [6] that for a Banach space $\mathbb{Y}$ of dimension greater than or equal to 3, Birkhoff-James orthogonality is symmetric in $\mathbb{Y}$ if and only if the norm is induced by an inner product. However, it follows from [4, 6] that counterexamples exist in the two-dimensional case. $\mathbb{X}$ is said to be polygonal if $B_{\mathbb{X}}$ contains only finitely many extreme points, or, equivalently, if $S_{\mathbb{X}}$ is a polygon. We say that $S_{\mathbb{X}}$ is a regular polygon if all the edges of $S_{\mathbb{X}}$ have the same length with respect to the usual metric on $\mathbb{R}^2$ and all interior angles are equal in measure. In order to obtain a complete geometric characterization of Radon planes, we introduce the following two definitions.

Definition 1.1. (*Translated vertex property of an edge*) Let $\mathbb{X}$ be a two-dimensional real polygonal Banach space. An edge L of $S_{\mathbb{X}}$ is said to satisfy the *translated vertex property* (TVP) if $0 \in x + L$ for some $x \in \mathbb{X}$ and the line determined by $x + L$ intersects $S_{\mathbb{X}}$ in two vertices $\pm v$. In that case, $\pm v$ are called the translated vertices of $S_{\mathbb{X}}$ corresponding to the edge L. Equivalently, an edge L satisfies TVP if and only if the kernel of the extreme supporting functional f, denoted by ker f, corresponding to L (see Definition 1.3 of [12]) contains a pair of vertices of $S_{\mathbb{X}}$. The space $\mathbb{X}$ is said to satisfy TVP if every edge of $S_{\mathbb{X}}$ satisfies TVP.

Definition 1.2. (*Translated edge property of two adjacent edges*) Let $\mathbb{X}$ be a two-dimensional real polygonal Banach space and let L, M be any two adjacent edges of $S_{\mathbb{X}}$ meeting at $x \in S_{\mathbb{X}}$. We say that the consecutive edges L and M satisfy the *translated edge property* (TEP) if and only if

(i) both L and M satisfy TVP with translated vertices $\pm v$ and $\pm w$, respectively.

(ii) either $\overline{vw}$ or $\overline{v(-w)}$ is an edge with the translated vertices $\pm x$.

The space $\mathbb{X}$ is said to satisfy TEP if every pair of adjacent edges of $S_{\mathbb{X}}$ satisfies the same.

The importance of these two geometric concepts, introduced in this article, becomes evident in course of our study. Indeed, the following result is the major highlight of our present study: *a two-dimensional real polygonal Banach space* $\mathbb{X}$ *is a Radon plane if and only if* $\mathbb{X}$ *satisfies TEP.* Moreover, it is easy to deduce from this geometric characterization of polygonal Radon planes that every regular polygon with $4n+2$ vertices, where $n \in \mathbb{N}$, is the unit circle of a Radon plane. On the other hand, we prove that if $\mathbb{X}$ is a two-dimensional real Banach space such that $S_{\mathbb{X}}$ is a polygon with $4n$ vertices, where $n \in \mathbb{N}$, then $\mathbb{X}$ is never a Radon plane. We further study the geometry of Radon planes whose unit circles are polygons but not necessarily regular ones. Indeed, we give a concrete example of a family of Radon planes whose unit circles are non-regular hexagons.

The notion of smoothness in a Banach space is intimately connected with the present study. An element $x \in S_{\mathbb{X}}$ is said to be a smooth point if there is a unique line H supporting $B_{\mathbb{X}}$ at x. The space $\mathbb{X}$ is said to be smooth if every point of $S_{\mathbb{X}}$ is a smooth point. It is immediate that the unit circle of any two-dimensional polygonal real Banach space consists of extreme points and smooth points only, the extreme points being the vertices of the polygon. At every smooth point on the unit circle, the only supporting line is the edge of the polygon that contains the corresponding point. However, at every vertex of the unit circle, there are infinitely many supporting lines that lie inside a cone generated by the two edges of the polygon meeting at that particular vertex of the polygon. Indeed, this simple geometric idea is at the heart of our approach towards completely characterizing Radon planes whose unit circles are polygons.

2. Main results

Let us begin with two simple observations that relate smooth points and extreme points of the unit disk of a two-dimensional real polygonal Banach space and describe the Birkhoff-James orthogonality set of any unit vector. We omit the proofs; that of the first proposition is rather obvious, and that of the second one follows from [13, Th. 2.1].

Proposition 2.1. *Let* $\mathbb{X}$ *be a two-dimensional real polygonal Banach space. Let* $x \in S_{\mathbb{X}}$. *Then* x *is an extreme point of* $B_{\mathbb{X}}$ *if and only if* x *is not a smooth point.*

Proposition 2.2. *Let* $\mathbb{X}$ *be a two-dimensional real Banach space and* $x \in S_{\mathbb{X}}$. *Then the set* $x^{\perp} = \{y \in \mathbb{X} : x \perp_B y\} = K \cup -K$, *where* K *is a normal cone in* $\mathbb{X}$.

Note that a subset K of $\mathbb{X}$ is said to be a *normal cone* in $\mathbb{X}$ if
(i) $K + K \subset K$, (ii) $\alpha K \subset K$ for all $\alpha \geq 0$, and (iii) $K \cap (-K) = \{\emptyset\}$.

Our next proposition is useful in studying polygonal Radon planes.

Proposition 2.3. *Let $\mathbb{X}$ be a two-dimensional real polygonal Banach space such that $\mathbb{X}$ is a Radon plane. Let L be an edge of $S_{\mathbb{X}}$ and v be an extreme point of $S_{\mathbb{X}}$ such that $L \perp_B v$. Then $v \perp_B u$ if and only if $u \in \pm L$.*

Proof. Let L be an edge of $S_{\mathbb{X}}$ such that $L \perp_B v$, where v be an extreme point of $S_{\mathbb{X}}$. Since $\mathbb{X}$ is a Radon plane, then $u \in \pm L$ implies that $v \perp_B u$.

Conversely, let $L \perp_B v$ and $v \perp_B u$. We want to show that $u \in \pm L$. Since $L \perp_B v$, there exists a linear functional $f \in S_{\mathbb{X}^*}$ such that $f(x) = 1$ for all $x \in L$ and $f(v) = 0$. Again, since $\mathbb{X}$ is a Radon plane, we have $u \perp_B v$. Then there exists a linear functional $g \in S_{\mathbb{X}^*}$ such that $g(u) = 1$ and $g(v) = 0$. Therefore, $v \in \ker g$. We also have $v \in \ker f$. Since the dimension of $\mathbb{X}$ is two, we have $\ker f = \ker g$. This shows that $f = \lambda g$. As $f, g \in S_{\mathbb{X}^*}$, we have $f = \pm g$. Therefore, $f(u) = \pm 1$, and so $u \in \pm L$. $\square$

Our next result illustrates the importance of the extreme points of the unit disk in studying Radon planes.

Theorem 2.4. *Let $\mathbb{X}$ be a two-dimensional real polygonal Banach space. Then $\mathbb{X}$ is a Radon plane if and only if Birkhoff-James orthogonality is left symmetric at each extreme point of $B_{\mathbb{X}}$.*

Proof. Let $\mathbb{X}$ be a Radon plane. Then Birkhoff-James orthogonality is symmetric at each point of $S_{\mathbb{X}}$. Therefore, Birkhoff-James orthogonality is left symmetric at each extreme point of $B_{\mathbb{X}}$.

Conversely, suppose Birkhoff-James orthogonality is left symmetric at each extreme point of $B_{\mathbb{X}}$. Let $x, y \in S_{\mathbb{X}}$ and $x \perp_B y$. We show that $y \perp_B x$. If x is an extreme point of $B_{\mathbb{X}}$ then we are done. In a real two-dimensional polygonal space, we know that each point of $S_{\mathbb{X}}$ is either an extreme point of $B_{\mathbb{X}}$ or a smooth point. So we consider x to be a smooth point. Clearly x can be written as $x = (1-t)v_1 + tv_2$ for some extreme points v_1, v_2 of $B_{\mathbb{X}}$. As $x \perp_B y$, there exists a linear functional $f \in S_{\mathbb{X}^*}$ such that $f(x) = 1$ and $f(y) = 0$. Then we have, $(1-t)f(v_1) + tf(v_2) = f(x) = 1 \Rightarrow f(v_1) = f(v_2) = 1$. This shows that $v_1 \perp_B y$ and $v_2 \perp_B y$. Again, by the given condition, we have $y \perp_B v_1$ and $y \perp_B v_2$. Now, by Proposition 2.2, the set $x^{\perp} = \{v \in \mathbb{X} : y \perp_B v\} = K \cup (-K)$, where K is a normal cone in $\mathbb{X}$, so it follows that $y \perp_B (1-t)v_1 + tv_2 = x$. Thus $x \perp_B y$ implies that $y \perp_B x$ so that Birkhoff-James orthogonality is symmetric. This completes the proof. $\square$

If $\mathbb{X}$ is a two-dimensional real polygonal Banach space, then the number of vertices of $S_{\mathbb{X}}$ is either $4n$ or $4n+2$, for some $n \in \mathbb{N}$. The major aim of the present paper is to show that in the first case $\mathbb{X}$ is *never* a Radon plane, while in the second case, $\mathbb{X}$ is a Radon plane provided some additional conditions are satisfied. We begin by proving the following result that addresses our first query.

Theorem 2.5. *Let $\mathbb{X}$ be a two-dimensional real Banach space such that $S_{\mathbb{X}}$ is a polygon with $4n$ vertices, where $n \in \mathbb{N}$. Then $\mathbb{X}$ is not a Radon plane.*

Proof. If possible, suppose that $\mathbb{X}$ is a Radon plane. Without loss of generality, we take an edge L of $S_{\mathbb{X}}$. Let $f \in S_{\mathbb{X}^*}$ be the extreme supporting linear functional corresponding to the edge L. Then $f(L) = 1$ and $f(v) = 0$ for some $v \in S_{\mathbb{X}}$. Clearly, $L \perp_B v$. Since $\mathbb{X}$ is a Radon plane, we have $v \perp_B L$. Since L is an edge, Birkhoff-James orthogonality at v is not right unique, and so it follows from [7, Th. 4.1] that v is not a smooth point of $\mathbb{X}$. Then, using Proposition 2.1, we have that v is an extreme point of $B_{\mathbb{X}}$, and so v is a vertex of $S_{\mathbb{X}}$. Now, let u be the midpoint of L. Since $v \perp_B u$, there exists a supporting functional $g \in S_{\mathbb{X}^*}$ at v such that $g(v) = 1$ and $g(u) = 0$. The situation is illustrated in Figure 2.1.

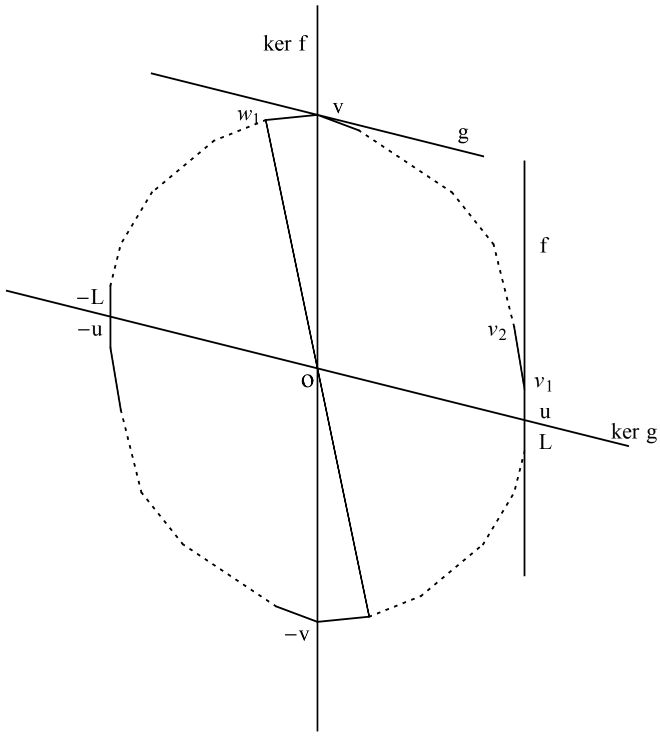

Figure 2.1:

Then the plane $\mathbb{X}$ is divided into the four closed sets:

$$S_1 = \{x \in \mathbb{X} : f(x) \geq 0,\ g(x) \geq 0\}, \quad S_2 = \{x \in \mathbb{X} : f(x) \leq 0,\ g(x) \geq 0\},$$

$$S_3 = \{x \in \mathbb{X} : f(x) \leq 0,\ g(x) \leq 0\}, \quad S_4 = \{x \in \mathbb{X} : f(x) \geq 0,\ g(x) \leq 0\}.$$

We note that ker f contains two vertices of $S_{\mathbb{X}}$, namely $\pm v$, and ker g contains two smooth points of $S_{\mathbb{X}}$, namely $\pm u$. We claim that each closed set $S_i, i = 1, 2, 3, 4$ (excluding ker f), contains exactly the same number of vertices of $S_{\mathbb{X}}$. Let k_0 denote the number of vertices of $S_{\mathbb{X}}$ in S_1 (excluding $\pm v$). To establish our claim, let us consider two adjacent vertices v_1 and v_2 (which may not be equal to v) of $S_{\mathbb{X}}$ in S_1 such that $v_1 \in S_{\mathbb{X}} \cap L$. Then, because of convexity of $B_{\mathbb{X}}$, v_2 can be written as $v_2 = (1-t)v_1 + t(\alpha v)$ for some $\alpha > 0$ and $t \in (0, 1)$. This shows that $f(v_2) < f(v_1)$.

Similarly, v_2 can be written as $v_2 = (1-s)v + s(\beta u)$ for some $\beta > 0$ and $s \in (0,1)$. This shows that $g(v_2) < g(v)$. Proceeding in the same way, we can show that $g(v_1) < g(v_2)$. Now, $\overline{v_1 v_2}$ is an edge of $S_{\mathbb{X}}$ lying in S_1 and $\overline{v_1 v_2} \perp_B v_2 - v_1$.

Let $w_1 = (v_2 - v_1)/\|v_2 - v_1\|$. Since $\mathbb{X}$ is a Radon plane, we have that $w_1 \perp_B \overline{v_1 v_2}$. As before, the Birkhoff-James orthogonality is not right unique at w_1, and so w_1 is a vertex of $S_{\mathbb{X}}$. Since $f(w_1) < 0$ and $g(w_1) > 0$, we have that w_1 lies in S_2 (excluding ker f and ker g). Continuing this process for every edge in S_1, we may and do conclude that, corresponding to every edge of $S_{\mathbb{X}}$ lying entirely in S_1, there is a vertex of $S_{\mathbb{X}}$ in S_2 (excluding ker f). Therefore, the number of vertices of $S_{\mathbb{X}}$ in S_2 (excluding ker f) is greater than or equal to k_0.

Similarly, corresponding to every edge of $S_{\mathbb{X}}$ lying entirely in S_2, there is a vertex of $S_{\mathbb{X}}$ in S_1 (excluding ker f). So the number of vertices of $S_{\mathbb{X}}$ in S_1 (excluding ker f) is greater than or equal to the number of vertices of $S_{\mathbb{X}}$ in S_2 (excluding ker f). This proves that $S_{\mathbb{X}}$ has the same number of vertices in S_1 and S_2 (excluding ker f). Moreover, by symmetry of $S_{\mathbb{X}}$ about the origin, the number of vertices of $S_{\mathbb{X}}$ in S_1 and S_3 (S_2 and S_4) must be equal. Therefore, the number of vertices of $S_{\mathbb{X}}$ in each $S_i, i = 1,2,3,4$ (excluding ker f), is k_0.

This completes the proof of our claim. Now, we count the total number of vertices of $S_{\mathbb{X}}$. Clearly, this turns out to be $4k_0 + 2$ (k_0 in each closed set $S_i, i = 1,2,3,4$ except ker f and 2 in ker f). However, this clearly contradicts the fact that $S_{\mathbb{X}}$ is a polygon with $4n$ vertices. This completes the proof. $\square$

We next obtain a complete characterization of polygonal Radon planes in terms of the translated edge property (TEP).

Theorem 2.6. *Let $\mathbb{X}$ be a two-dimensional real polygonal Banach space. Then $\mathbb{X}$ is a Radon plane if and only if $\mathbb{X}$ satisfies TEP.*

Proof. We first prove the necessary part. Let L be an edge of $S_{\mathbb{X}}$. Let $f \in S_{\mathbb{X}^*}$ be the extreme supporting linear functional corresponding to the edge L. Then $f(L) = 1$ and $f(v) = 0$ for some $v \in S_{\mathbb{X}}$. Clearly, $L \perp_B v$. Since $\mathbb{X}$ is a Radon plane, we have $v \perp_B L$. Since L is an edge, the Birkhoff-James orthogonality is not right unique, and so it follows from [7, Th. 4.1] that v is not a smooth point of $\mathbb{X}$. Again, it follows from Proposition 2.1 that v is an extreme point of $B_{\mathbb{X}}$, and so v is a vertex of $S_{\mathbb{X}}$. This shows that every edge satisfies the TVP. Let L and M be any two adjacent edges of $S_{\mathbb{X}}$ with a common vertex u. Also, let f and g be the extreme supporting functionals corresponding to the edges L and M, respectively so that $f(u) = g(u) = 1$. Also by the TVP, we get ker $f \cap S_{\mathbb{X}} = \{\pm v\}$ and ker $g \cap S_{\mathbb{X}} = \{\pm w\}$ for some extreme points v and w of $B_{\mathbb{X}}$. Figure 2.2, given below, illustrates the situation.

We show that $\overline{vw}$ or $\overline{v(-w)}$ is an edge of $S_{\mathbb{X}}$. Clearly $L \perp_B v, M \perp_B w$ and because of the symmetry of Birkhoff-James orthogonality we get $v \perp_B L$, $w \perp_B M$. Now u, v, w are vertices with the property that $u \perp_B v$ and $u \perp_B w$. Let $y_s = (1-s)v + sw$ and $z_s = (1-s)v - sw$, where $s \in (0,1)$. We claim that $u \perp_B y_s$ for all $s \in (0,1)$ or $u \perp_B z_s$ for all $s \in (0,1)$. It follows directly from Proposition 2.2, that the set

$u^{\perp} = \{x \in \mathbb{X} : u \perp_B x\} = K \cup (-K)$, where K is a normal cone in $\mathbb{X}$, and so the claim is established. We assume that $u \perp_B y_s$ for all $s \in (0,1)$. Then $y_s \perp_B u$ for all $s \in (0,1)$, i.e., $\overline{vw} \perp_B u$. Clearly, $\|y_s\| \leq 1$ for all $s \in (0,1)$. If $\|y_s\| = 1$ for all $s \in (0,1)$ then $\overline{vw}$ is an edge and we are done. If it is possible that $\overline{vw}$ is not an edge, then we can find vertices v_1, w_1, not necessarily unequal, so that $\overline{vv_1}$ and $\overline{ww_1}$ are edges, and proceeding as before we can conclude that $\overline{vv_1} \perp_B u$, $\overline{ww_1} \perp_B u$. This is not possible, and so $\overline{vw}$ is an edge. This completes the proof of the necessary part.

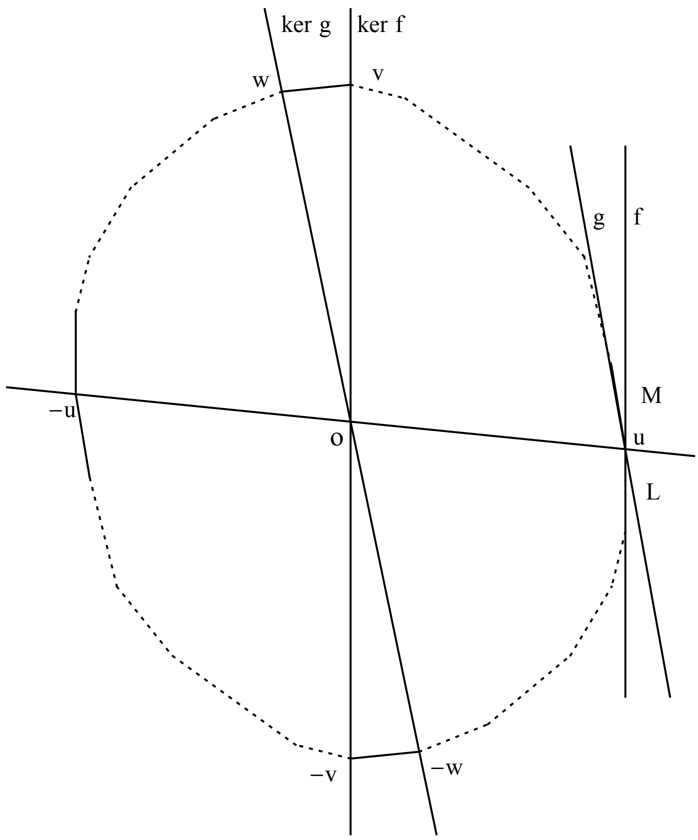

Figure 2.2:

We next prove the sufficient part using Theorem 2.4. So it is sufficient to show that $x \perp_B y$ implies $y \perp_B x$, whenever x is an extreme point of $B_{\mathbb{X}}$ and $y \in S_{\mathbb{X}}$. Let L and M be the two edges of $S_{\mathbb{X}}$ containing x. Let $f, g \in S_{\mathbb{X}^*}$ be the extreme supporting linear functionals corresponding to edges L and M such that $\ker f \cap S_{\mathbb{X}} = \{\pm v\}$ and $\ker g \cap S_{\mathbb{X}} = \{\pm w\}$. Then $\overline{vw}$ or $\overline{v(-w)}$ is an edge of $S_{\mathbb{X}}$.

Without loss of generality we may assume that $\overline{vw}$ is an edge of $S_{\mathbb{X}}$. Then, by the TVP, we get $\overline{vw} \perp_B x$. We want to show that $y \in \overline{vw}$. Following the proof of [12, Th. 2.1] we note that every supporting functional at x is a convex combination of f and g. As $x \perp_B y$, there is some $t \in [0,1]$ such that $[(1-t)f + tg](x) = 1$ and $[(1-t)f + tg](y) = 0$. This implies that $f(y)g(y) < 0$, and so either $f(y) > 0$, $g(y) < 0$ or $f(y) < 0$, $g(y) > 0$. As $\|y\| = 1$, we can conclude that either $y \in \overline{vw}$ or $-y \in \overline{vw}$. Thus we get $y \perp_B x$. This completes the proof. $\square$

As an application of the above theorem, it is easy to observe that regular polygons with $4n+2$ number of vertices, where $n \in \mathbb{N}$, are the unit circles of Radon planes.

Theorem 2.7. *Let $\mathbb{X}$ be a two-dimensional real Banach space such that $S_{\mathbb{X}}$ is a regular polygon with $4n+2$ vertices, where $n \in \mathbb{N}$. Then $\mathbb{X}$ is a Radon plane.*

Proof. We show that $\mathbb{X}$ satisfies the TEP. Let L and M be two adjacent edges of $S_{\mathbb{X}}$ with common vertex v_1. Then $-L$ is also an edge of $S_{\mathbb{X}}$, parallel to L. We choose the X-axis to be the line through the origin that bisects both L and $-L$. Clearly, there is no vertex of $S_{\mathbb{X}}$ on the X-axis. We know that for a regular polygon with $2n$ vertices, the lines of symmetries are the following straight lines:

(i) The straight line passing through the midpoints of any two parallel edges of the polygon.

(ii) The straight line passing through any two opposite vertices v_i and $-v_i$ of the polygon.

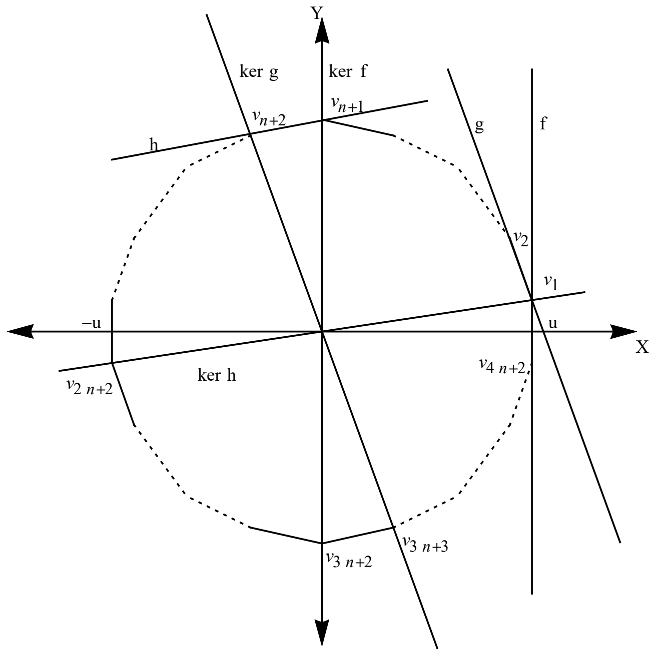

Figure 2.3:

We choose the Y-axis to be the straight line through the origin, which is parallel to L and $-L$. Because of regularity of $S_{\mathbb{X}}$ it is easy to see that both the X and Y-axes are lines of symmetry. Let the Y-axis meets $S_{\mathbb{X}}$ at two points $\pm w = \pm(0, \beta)$, where $\beta \in \mathbb{R}$. The two axes divide the plane into four quadrants. By symmetry of $S_{\mathbb{X}}$ about the two axes, each quadrant (excluding the two axes) contains exactly the same number of vertices. Since there is no vertex of $S_{\mathbb{X}}$ on the X-axis and the number of vertices of $S_{\mathbb{X}}$ is $4n+2$, we conclude that $\pm w$ are two vertices of $S_{\mathbb{X}}$. Let the vertex v_1 be in the first quadrant. The other vertices of $S_{\mathbb{X}}$ are chosen anticlockwise and are denoted by $v_2, v_3, ..., v_{4n+2}$, where $v_j = (\cos\frac{(2j-1)\pi}{4n+2},\ \sin\frac{(2j-1)\pi}{4n+2})$, $j = 1, 2, ..., 4n+2$. We observe the following:

(1) The other vertex on L is v_{4n+2}.
(2) v_i and v_{i+1} are adjacent, $i = 1, 2, ..., 4n + 1$.
(3) $w = v_{n+1}$ and $-w = v_{3n+2}$.
(4) The vertices of $S_{\mathbb{X}}$ on $-L$ are v_{2n+1} and v_{2n+2} in the third and fourth quadrant, respectively.

Without loss of generality we take the two adjacent edges $L = \overline{v_1 v_{4n+2}}$ and $M = \overline{v_1 v_2}$ of $S_{\mathbb{X}}$. Let $f, g \in S_{\mathbb{X}^*}$ be the extreme supporting linear functionals at v_1 corresponding to edges L and M respectively such that $f(v_1) = g(v_1) = 1$. It is easy to see that ker $f \cap S_{\mathbb{X}} = \{v_{n+1}, v_{3n+2}\}$ and ker $g \cap S_{\mathbb{X}} = \{v_{n+2}, v_{3n+3}\}$. Then $\overline{v_{n+1} v_{n+2}}$ and $\overline{v_{3n+2} v_{3n+3}}$ are the edges of $S_{\mathbb{X}}$ and $\overline{v_{n+1} v_{n+2}} = -\overline{v_{3n+2} v_{3n+3}}$. Now, let $h \in S_{\mathbb{X}^*}$ be the extreme supporting linear functional corresponding to edge $\overline{v_{n+1} v_{n+2}}$ such that $h(x) = 1$, where $x \in \overline{v_{n+1} v_{n+2}}$. It is easy to verify that ker $h \cap S_{\mathbb{X}} = \{v_1, v_{2n+2}\}$ and $v_1 = -v_{2n+2}$. Thus the edges L and M satisfy the TEP. So using Theorem 2.6, we conclude that $\mathbb{X}$ is a Radon plane. $\square$

Combining Theorem 2.5 and Theorem 2.7, we have the following complete characterization of Radon planes having regular polygons as unit circles.

Theorem 2.8. *Let $\mathbb{X}$ be a two-dimensional real Banach space such that $S_{\mathbb{X}}$ is a regular polygon. Then $\mathbb{X}$ is a Radon plane if and only if the number of vertices of $S_{\mathbb{X}}$ is $4n + 2$, where $n \in \mathbb{N}$. In particular, the affine image of any regular polygon with $4n + 2$ vertices is always a Radon curve.*

Remark 2.9. We would like to mention that Heil [5] was the first to observe that if $\mathbb{X}$ is a real two-dimensional polygonal Banach space such $S_{\mathbb{X}}$ is a regular polygon then $\mathbb{X}$ is a Radon plane if and only if the number of vertices of $S_{\mathbb{X}}$ is $4n + 2$ for some natural number n. In this paper, we have provided an alternative geometric approach to prove the same. Furthermore, our results in this direction generalize the relevant observations made by Heil in [5] by taking into consideration all possible convex polygons, and not just regular ones. $\square$

Continuing in the spirit of Remark 2.9, we next present a family of Radon planes whose unit circles are polygons, but not *regular* ones. We would like to mention here that the sufficient part of the following result is a consequence of the more general fact that the property of being a Radon curve is invariant under affine transformations.

Theorem 2.10. *Let $\mathbb{X} = \mathbb{R}^2$ be endowed with a norm so that $S_{\mathbb{X}}$ is a hexagon with four vertices $\pm(1, \alpha), \pm(1, -\alpha)$, where $\alpha > 0$. Then $\mathbb{X}$ is a Radon plane if and only if the other two vertices of $S_{\mathbb{X}}$ are $\pm(0, 2\alpha)$ or $\pm(2, 0)$.*

Proof. Let us first prove the easier sufficient part of the theorem. It follows from Theorem 2.8 that a regular hexagon with vertices $\pm(1, 0), \pm(\frac{1}{2}, \frac{\sqrt{3}}{2}), \pm(-\frac{1}{2}, \frac{\sqrt{3}}{2})$ is a Radon curve. Therefore, an affine transformation of this Radon curve under the linear map T given by : $T(1, 0) = (1, -\alpha), T(\frac{1}{2}, \frac{\sqrt{3}}{2}) = (1, \alpha)$ is again a Radon curve. It is easy to see that the vertices of the transformed Radon curve are given by $\pm(1, \alpha), (1, -\alpha), \pm(0, 2\alpha)$. This is shown in Figure 2.4. Similarly considering the transformation $T(-\frac{1}{2}, \frac{\sqrt{3}}{2}) = (1, \alpha), T(1, 0) = (1, -\alpha)$ we can prove that the hexagon with the vertices $\pm(1, \alpha)$, $\pm(1, -\alpha), \pm(2, 0)$ is also a Radon curve.

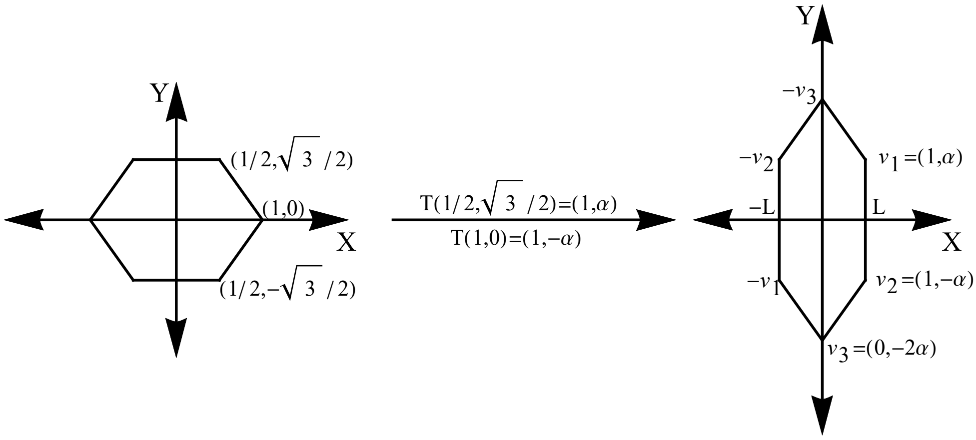

Figure 2.4:

Let us now prove the necessary part of the theorem. Let $\mathbb{X}$ be a two-dimensional real Banach space such that $S_{\mathbb{X}}$ is a hexagon with two vertices $v_1 = (1, \alpha), v_2 = (1, -\alpha)$. Then $-v_1 = (-1, -\alpha), -v_2 = (-1, \alpha)$ are also vertices of $S_{\mathbb{X}}$. Suppose $\mathbb{X}$ is a Radon plane. Now, one of the following two cases must hold.

(1) The X-axis contains no vertices of $S_{\mathbb{X}}$.

(2) The X-axis contains two vertices of $S_{\mathbb{X}}$.

First we assume that there are no vertices of $S_{\mathbb{X}}$ on the X-axis. Let us denote the edge between the vertices v_1 and v_2 by L. Then $-L$ is the edge between the verices $-v_1$ and $-v_2$. Now, it is clear that the X-axis meets L at the point $(1, 0)$ and $-L$ at the point $(-1, 0)$, which are smooth points of $S_{\mathbb{X}}$. The Y-axis meets $S_{\mathbb{X}}$ at two points $\pm v = \pm(0, \beta)$, where $\beta \in \mathbb{R}$. It is easy to observe that given any $y \in L$, we have $y \perp_B v$. Since Birkhoff-James orthogonality is symmetric in $\mathbb{X}$, we must have that $v \perp_B y$ for any $y \in L$. This shows that Birkhoff-James orthogonality is not right unique at v, and so it follows from [7, Th. 4.1] that v must be a non-smooth point of $\mathbb{X}$. Therefore, the other two vertices of $S_{\mathbb{X}}$ are of the form $\pm(0, \beta)$.

We denote the vertex $(0, -\beta)$ by v_3. Then the vertex $(0, \beta)$ is $-v_3$. Again, by the symmetry of Birkhoff-James orthogonality in $\mathbb{X}$, we must have that the edges $\overline{v_2 v_3}$ and $\overline{v_3(-v_1)}$ are parallel to $\overline{v_1(-v_1)}$ and $\overline{v_2(-v_2)}$, respectively. It is easy to see that this is true only when $\beta = 2\alpha$. The situation is illustrated in the right hand side of Figure 2.4.

Now, we assume that the X-axis contains two vertices of $S_{\mathbb{X}}$. Then the other two vertices of $S_{\mathbb{X}}$ are given by $\pm w = \pm(\eta, 0)$, where $\eta \in \mathbb{X}$. The rest of the proof of the fact that $\eta = 2$ in this case can be completed similarly as above.

This completes the proof of the necessary part of the theorem and establishes the theorem completely. $\square$

Remark 2.11. Using the fact that affine transformation of a Radon curve is a Radon curve it follows from Theorem 2.8 that there exist non-regular $4n+2$-gons which are Radon curves for each natural number n.

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