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Positive Solutions for Nonlinear Robin Problems with Concave Terms

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We consider a parametric Robin problem driven by the p-Laplacian plus a potential. In the reaction we have the combined effects of a parametric concave term and of a $(p-1)$-linear perturbation. We consider the case of uniform nonresonance with respect to the principal eigenvalue $\widehat{\lambda}_1 > 0$ and the case of nonuniform nonresonance with respect to $\widehat{\lambda}_1 > 0$. For both cases we prove a bifurcation-type theorem describing the dependence on the parameter $\lambda > 0$ of the set of positive solutions. We also establish the existence of a smallest positive solution $\widetilde{u}_\lambda^*$ for every admissible parameter $\lambda > 0$ and determine the monotonicity and continuity properties of the map $\lambda \longmapsto \widetilde{u}_\lambda^*$.

Keywords: p-Laplacian, concave nonlinearity, uniform nonresonance, nonuniform nonresonance, bifurcation-type theorem, minimal positive solution.

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1. Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$. In this paper we study the following nonlinear parametric Robin problem:

$$(P_\lambda) \qquad \begin{cases} -\Delta_p u(z) + \xi(z)u(z)^{p-1} = \lambda u(z)^{q-1} + f(z,u(z)) & \text{in } \Omega, \\ \frac{\partial u}{\partial n_p} + \beta(z)u^{p-1} = 0 \quad \text{on } \partial\Omega, \\ u > 0 \quad \text{in } \Omega \end{cases}$$

where $1 < p < +\infty$ and $\lambda > 0$. In this problem Δ_p denotes the p-Laplace differential operator defined by

$$\Delta_p u = \operatorname{div}(|Du|^{p-2}Du) \quad \forall u \in W_0^{1,p}(\Omega).$$

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The potential function $\xi \in L^\infty(\Omega)$ and $\xi \geqslant 0$. In the right hand side of (P_λ) (the reaction of the problem), we have two terms. One is the parametric term $u \longmapsto \lambda u^{q-1}$ with $\lambda > 0$ being the parameter and $1 < q < p$. Therefore this term is a concave (that is, strictly $(p-1)$-sublinear) contribution to the reaction of (P_λ). The perturbation $f(z,x)$ is a Carathéodory function (that is, for all $x \in \mathbb{R}$, $z \longmapsto f(z,x)$ is measurable and for a.a. $z \in \Omega$, $x \longmapsto f(z,x)$ is continuous). We assume that $f(z,\cdot)$ exhibits $(p-1)$-linear growth near $+\infty$ and asymptotically as $x \to +\infty$, the quotient $f(z,x)/x^{p-1}$ stays above the principal eigenvalue $\widehat{\lambda}_1$ of the operator $u \longmapsto -\Delta_p u + \xi(z)|u|^{p-2}u$ with the Robin boundary condition. We consider two cases. In the first we have uniform nonresonance with respect to $\widehat{\lambda}_1$ and in the second we have nonuniform nonresonance with respect to $\widehat{\lambda}_1$. In the boundary condition $\frac{\partial u}{\partial n_p}$ denotes the conormal derivative of u defined by extension of the map

$$C^1(\overline{\Omega}) \ni u \longmapsto |Du|^{p-2}(Du,n)_{\mathbb{R}^N} = |Du|^{p-2}\frac{\partial u}{\partial n},$$

with n being the outward unit normal on $\partial\Omega$. The boundary coefficient β is nonnegative. If $\beta \equiv 0$, then we have the Neumann problem.

We are looking for positive solutions. We establish the precise dependence of the set of positive solutions on the parameter $\lambda > 0$. More precisely, we prove a bifurcation-type result, producing a critical parameter value $\lambda^* > 0$ such that

(i) for all $\lambda \in (0,\lambda^*)$ problem (P_λ) has at least two positive smooth solutions;

(ii) for $\lambda = \lambda^*$ problem (P_λ) has at least one positive smooth solution;

(iii) for $\lambda > \lambda^*$ problem (P_λ) has no positive solutions.

Such results for parametric equations with competing nonlinearities, were proved primarily for Dirichlet problems with concave and convex (that is, $(p-1)$-superlinear) nonlinearities. We mention the works of Ambrosetti-Brezis-Cerami [3], García Azorero-Manfredi-Peral Alonso [6], Gasiński-Papageorgiou [11], Guo and Zhang [15], Marano-Papageorgiou [19], Papageorgiou-Winkert [26]. For Neumann and Robin problems, we have the works of Papageorgiou-Smyrlis [25], Papageorgiou-Rădulescu [21]. We are not aware of such a result when the perturbation $f(z,x)$ is $(p-1)$-linear in $x \in \mathbb{R}$.

Moreover, here we show that for every admissible parameter $\lambda > 0$ (that is, $\lambda \in (0,\lambda^*]$), problem (P_λ) has a minimal positive solution $\widetilde{u}_\lambda^*$ and we establish the monotonicity and continuity properties of the map $\lambda \longmapsto \widetilde{u}_\lambda^*$. Finally for other Robin boundary value problems we refer to Bai-Gasiński-Papageorgiou [5], Gasiński-O'Regan-Papageorgiou [8] and Gasiński-Papageorgiou [14].

2. Mathematical background – hypotheses

Let X be a Banach space and let X^* be its topological dual. By $\langle\cdot,\cdot\rangle$ we denote the duality brackets for the pair (X^*,X). If $\varphi \in C^1(X;\mathbb{R})$, then we say that φ satisfies the *Cerami condition* (CC), if the following property holds:

(CC) Every sequence $\{u_n\}_{n\geqslant 1} \subseteq X$ such that $\{\varphi(u_n)\}_{n\geqslant 1} \subseteq \mathbb{R}$ is bounded and $(1+\|u_n\|)\varphi'(u_n) \longrightarrow 0$ in X^* as $n \to +\infty$, admits a strongly convergent subsequence.

This compactness-type condition on the functional φ, leads to a deformation theorem from which one can derive the minimax theory of the critical values of φ. Prominent in that theory is the so called "mountain pass theorem" due to Ambrosetti-Rabinowitz [4].

Theorem 2.1. *If $\varphi \in C^1(X;\mathbb{R})$ satisfies the Cerami condition* (CC)*, $u_0, u_1 \in X$,*

$$0 < \varrho < \|u_1 - u_0\|, \quad \max\{\varphi(u_0), \varphi(u_1)\} < \inf\{\varphi(u) : \ \|u - u_0\| = \varrho\} = m_\varrho$$

and $c = \inf\limits_{\gamma\in\Gamma} \max\limits_{0\leqslant t\leqslant 1} \varphi(\gamma(t))$ with

$$\Gamma = \{\gamma \in C([0,1];X) : \ \gamma(0) = u_0, \ \gamma(1) = u_1\},$$

then $c \geqslant m_\varrho$ and c is a critical value of φ (that is, there exists $\widehat{u} \in X$ such that $\varphi(\widehat{u}) = c$ and $\varphi'(\widehat{u}) = 0$).

In the analysis of the problem (P_λ) we will use the Sobolev space $W^{1,p}(\Omega)$, the Banach space $C^1(\overline{\Omega})$ and the "boundary" Lebesgue space $L^p(\partial\Omega)$. By $\|\cdot\|$ we denote the norm of $W^{1,p}(\Omega)$ defined by

$$\|u\| = (\|u\|_p^p + \|Du\|_p^p)^{\frac{1}{p}} \quad \forall u \in W^{1,p}(\Omega).$$

The Banach space $C^1(\overline{\Omega})$ is an ordered Banach space with positive (order) cone given by

$$C_+ = \{u \in C^1(\overline{\Omega}) : \ u(z) \geqslant 0 \text{ for all } z \in \overline{\Omega}\}.$$

We will also use two open cones

$$\widehat{C}_+ = \{u \in C_+ : \ u(z) > 0 \text{ for all } z \in \Omega, \ \frac{\partial u}{\partial n}\Big|_{\partial\Omega \cap u^{-1}(0)} < 0\}.$$

and the interior of C_+, namely

$$D_+ = \{u \in C_+ : \ u(z) > 0 \text{ for all } z \in \overline{\Omega}\}.$$

In fact D_+ is also the interior of C_+ when on $C^1(\overline{\Omega})$ we consider the relative $C(\overline{\Omega})$-norm topology.

On $\partial\Omega$ we consider the $(N-1)$-dimensional Hausdorff (surface) measure $\sigma(\cdot)$. Using this measure we can define in the usual way the boundary Lebesgue spaces $L^r(\partial\Omega)$, $1 \leqslant r \leqslant +\infty$. We know that there exists a unique continuous linear map $\gamma_0 \colon W^{1,p}(\Omega) \longrightarrow L^p(\partial\Omega)$ known as the "trace map" such that $\gamma_0(u) = u|_{\partial\Omega}$ for all $u \in W^{1,p}(\Omega) \cap C(\overline{\Omega})$. So, the trace map assigns "boundary values" to all Sobolev functions. We know that

$$\operatorname{im}\gamma_0 = W^{\frac{1}{p'},p}(\partial\Omega) \quad \text{and} \quad \ker\gamma_0 = W_0^{1,p}(\Omega)$$

(with $\frac{1}{p} + \frac{1}{p'} = 1$). Also γ_0 is compact from $W^{1,p}(\Omega)$ into $L^r(\partial\Omega)$ for all r with $1 \leqslant r < \frac{(N-1)p}{N-p}$ if $p < N$ and into $L^r(\partial\Omega)$ for all $1 \leqslant r < +\infty$ if $p \geqslant N$. In the sequel for the sake of notational simplicity, we drop the use of the trace map. All restrictions of Sobolev functionals on $\partial\Omega$ are understood in the sense of traces.

Let $A\colon W^{1,p}(\Omega) \longrightarrow W^{1,p}(\Omega)^*$ be the nonlinear map defined by

$$\langle A(u), h\rangle = \int_\Omega |Du|^{p-2}(Du, Dh)_{\mathbb{R}^N}\, dz \quad \forall u, h \in W^{1,p}(\Omega).$$

The next proposition is a special case of a more general result of Gasiński-Papageorgiou [10].

Proposition 2.2. *The map $A\colon W^{1,p}(\Omega) \longrightarrow W^{1,p}(\Omega)^*$ is bounded (that is, maps bounded sets to bounded sets), continuous, monotone (hence maximal monotone too) and of type $(S)_+$, that is, if $u_n \xrightarrow{w} u$ in $W^{1,p}(\Omega)$ and*

$$\limsup_{n\to+\infty}\langle A(u_n), u_n - u\rangle \leqslant 0,$$

then $u_n \longrightarrow u$ in $W^{1,p}(\Omega)$.

Consider a Carathéodory function $f_0\colon \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ such that

$$|f_0(z,x)| \leqslant a_0(z)(1 + |x|^{r-1}) \quad \text{for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R},$$

with $a_0 \in L^\infty(\Omega)$ and $1 < r \leqslant p^*$, where

$$p^* = \begin{cases} \frac{Np}{N-p} & \text{if} \quad p < N, \\ +\infty & \text{if} \quad N \leqslant p \end{cases}$$

(the critical Sobolev exponent). Let

$$F_0(z,x) = \int_0^x f_0(z,s)\, ds$$

and $\beta \in C^{0,\alpha}(\partial\Omega)$ and consider the C^1-functional $\varphi_0\colon W^{1,p}(\Omega) \longrightarrow \mathbb{R}$ defined by

$$\varphi_0(u) = \frac{1}{p}\|Du\|_p^p + \frac{1}{p}\int_{\partial\Omega} \beta(z)|u|^p\, d\sigma - \int_\Omega F_0(z,u)\, dz \quad \forall u \in W^{1,p}(\Omega).$$

The next result can be found in a more general form in Papageorgiou-Rădulescu [23].

Proposition 2.3. *If $u_0 \in W^{1,p}(\Omega)$ is a local $C^1(\overline{\Omega})$-minimizer of φ_0, that is, there exists $r_1 > 0$ such that*

$$\varphi_0(u_0) \leqslant \varphi_0(u_0 + h) \quad \forall h \in C^1(\overline{\Omega}),\ \|h\|_{C^1(\overline{\Omega})} \leqslant r_1,$$

then $u_0 \in C^{1,\eta}(\overline{\Omega})$ for some $\eta \in (0,1)$ and u_0 is a local $W^{1,p}(\Omega)$-minimizer of φ_0, that is, there exists $r_2 > 0$ such that

$$\varphi_0(u_0) \leqslant \varphi_0(u_0 + h) \quad \forall h \in W^{1,p}(\Omega),\ \|h\| \leqslant r_2.$$

The next two lemmata will be useful in our estimates. The first is a particular case of Lemma 4.11 in Mugnai-Papageorgiou [20].

Lemma 2.4. *If $\xi \in L^\infty(\Omega)$, $\xi(z) \geqslant 0$ for a.a. $z \in \Omega$ and the inequality is strict on a set of positive measure, then there exists $c_0 > 0$ such that*

$$c_0\|u\|^p \leqslant \|Du\|_p^p + \int_\Omega \xi(z)|u|^p\,dz \quad \forall u \in W^{1,p}(\Omega).$$

The next lemma can be found in Gasiński-Papageorgiou [12, Proposition 2.4].

Lemma 2.5. *If $\beta \in L^\infty(\partial\Omega)$, $\beta(z) \geqslant 0$ σ-a.e. and the inequality is strict on a set of positive σ-measure, then*

$$u \longmapsto |u| = (\|Du\|_p^p + \int_{\partial\Omega} \beta(z)|u|^p\,d\sigma)^{\frac{1}{p}}$$

is an equivalent norm on Sobolev space $W^{1,p}(\Omega)$.

Now we introduce our hypotheses on the potential function ξ and on the boundary coefficient β.

$H(\xi)$: $\xi \in L^\infty(\Omega)$, $\xi(z) \geqslant 0$ for a.a. $z \in \Omega$.

$H(\beta)$: $\beta \in C^{0,\alpha}(\partial\Omega)$ with $\alpha \in (0,1)$ and $\beta(z) \geqslant 0$ for all $z \in \partial\Omega$.

H_0: $\xi \not\equiv 0$ or $\beta \not\equiv 0$.

We consider the following nonlinear eigenvalue problem

$$\begin{cases} -\Delta_p u(z) + \xi(z)|u(z)|^{p-2}u(z) = \widehat{\lambda}|u(z)|^{p-2}u(z) & \text{in } \Omega, \\ \frac{\partial u}{\partial n_p} + \beta(z)u^{p-1} = 0 & \text{on } \partial\Omega. \end{cases} \tag{1}$$

From Mugnai-Papageorgiou [20] (Neumann case) and Papageorgiou-Rădulescu [22] (Robin case), we know that (1) admits a smallest eigenvalue $\widehat{\lambda}_1$. Moreover, on account of Lemmata 2.4 and 2.5 we have that $\widehat{\lambda}_1 > 0$. This first eigenvalue has the following properties:

(i) $\widehat{\lambda}_1 > 0$ is isolated (that is, if $\widehat{\sigma}(p)$ denotes the spectrum of (1), then there exists $\varepsilon > 0$ such that $(\widehat{\lambda}_1, \widehat{\lambda}_1 + \varepsilon) \cap \widehat{\sigma}(p) = \emptyset$).

(ii) $\widehat{\lambda}_1$ is simple (that is, if $\widehat{u}, \widehat{v}$ are eigenfunctions corresponding to $\widehat{\lambda}_1$, then $\widehat{u} = \vartheta\widehat{v}$ for some $\vartheta \in \mathbb{R} \setminus \{0\}$).

(iii) we have

$$\widehat{\lambda}_1 = \inf_{u \in W^{1,p}(\Omega)\setminus\{0\}} \frac{\mu(u)}{\|u\|_p^p}, \tag{2}$$

where $\mu\colon W^{1,p}(\Omega) \longrightarrow \mathbb{R}$ is the C^1-functional defined by

$$\mu(u) = \|Du\|_p^p + \int_\Omega \xi(z)|u|^p\,dz + \int_{\partial\Omega} \beta(z)|u|^p\,d\sigma \quad \forall u \in W^{1,p}(\Omega).$$

The infimum in (2) is realized on the one-dimensional eigenspace corresponding to $\widehat{\lambda}_1 > 0$. The above properties imply that the elements of this eigenspace have constant sign. By $\widehat{u}_1$ we denote the positive, L^p-normalized (that is $\|\widehat{u}_1\|_p = 1$) eigenfunction corresponding to $\widehat{\lambda}_1$. The nonlinear regularity theory (see Lieberman [17, Theorem 2]) and the nonlinear maximum principle (see e.g., Gasiński-Papageorgiou [9, p. 738]) imply that $\widehat{u}_1 \in D_+$. The Ljusternik-Schnirelmann minimax scheme, gives a whole sequence of distinct elements $\{\widehat{\lambda}_n\}_{n\geqslant 1} \subseteq \widehat{\sigma}(p)$ such that $\widehat{\lambda}_n \longrightarrow +\infty$ as $n \to +\infty$. These elements are known as "variational eigenvalues" of (1) and we do not know if they exhaust $\widehat{\sigma}(p)$. Every eigenfunction corresponding to an eigenvalue $\widehat{\lambda} \neq \widehat{\lambda}_1$ is necessary nodal (that is, sign changing).

The same results are also valid for a weighted version of problem (1). So, let $\vartheta \in L^\infty(\Omega)$, $\vartheta(z) \geqslant 0$ for a.a. $z \in \Omega$, $\vartheta \not\equiv 0$. We consider the following nonlinear eigenvalue problem

$$\begin{cases} -\Delta_p u(z) + \xi(z)|u(z)|^{p-2}u(z) = \widetilde{\lambda}\vartheta(z)|u(z)|^{p-2}u(z) & \text{in } \Omega, \\ \frac{\partial u}{\partial n_p} + \beta(z)u^{p-1} = 0 & \text{on } \partial\Omega. \end{cases}$$

Again we have a smallest eigenvalue $\widetilde{\lambda}_1(\vartheta) > 0$ which has the same properties as $\widehat{\lambda}_1$. In this case the variational characterization of $\widetilde{\lambda}_1(\vartheta)$ has the following form:

$$\widetilde{\lambda}_1(\vartheta) = \inf_{u \in W^{1,p}(\Omega)\setminus\{0\}} \frac{\mu(u)}{\int_\Omega \vartheta(z)|u|^p dz}, \tag{3}$$

As before the infimum in (3) is realized on the corresponding one-dimensional eigenspace, the elements of which have constant sign. By $\widetilde{u}_1(\vartheta)$ we denote the positive, L^p-normalized eigenfunction corresponding to $\widetilde{\lambda}_1(\vartheta)$. We have that $\widetilde{u}_1(\vartheta) \in D_+$. From (3) and the aforementioned properties, we deduce easily the following monotonicity property for the map $\vartheta \longmapsto \widetilde{\lambda}_1(\vartheta)$.

Lemma 2.6. *If $\vartheta_1, \vartheta_2 \in L^\infty(\Omega)$, $0 \leqslant \vartheta_1(z) \leqslant \vartheta_2(z)$ for a.a. $z \in \Omega$ and the two inequalities are strict on sets of positive measure, then $\widetilde{\lambda}_1(\vartheta_2) < \widetilde{\lambda}_1(\vartheta_1)$.*

The next auxiliary result is a strong comparison principle which complements recent similar results proved by Gasiński-Papageorgiou [12].

Proposition 2.7. *If $\widetilde{\xi} \in L^\infty(\Omega)$, $\widetilde{\xi}(z) \geqslant 0$ for a.a. $z \in \Omega$, $h_1, h_2 \in L^\infty(\Omega)$ with*

$$0 < c_1 \leqslant h_1(z) - h_2(z) \quad \textit{for a.a. } z \in \Omega \tag{4}$$

and $u, v \in C^{1,\alpha}(\overline{\Omega})$ with $\alpha \in (0,1]$, $0 \leqslant v \leqslant u$ and

$$\begin{aligned} -\Delta_p u(z) + \widetilde{\xi}(z)|u(z)|^{p-2}u(z) &= h_1(z) \quad \textit{for a.a. } z \in \Omega, \\ -\Delta_p v(z) + \widetilde{\xi}(z)|v(z)|^{p-2}v(z) &= h_2(z) \quad \textit{for a.a. } z \in \Omega, \end{aligned}$$

then $u - v \in \widehat{C}_+$.

Proof. In what follows to shorten our notation we use the map

$$a(y) = |y|^{p-2}y \quad \forall y \in \mathbb{R}^N.$$

By hypothesis, we have

$$-\mathrm{div}\,(a(Du) - a(Dv)) = h_1 - h_2 - \widetilde{\xi}(z)(u^{p-1} - v^{p-1}) \quad \text{for a.a. } z \in \Omega. \tag{5}$$

We write $a = (a_k)_{k=1}^N$ and consider $y = (y_i)_{i=1}^N \in \mathbb{R}^N$ and $y' = (y_i')_{i=1}^N \in \mathbb{R}^N$. Then

$$a(y) - a(y') = \int_0^1 \frac{d}{dt} a(y' + t(y - y'))\,dt = \int_0^1 \nabla a(y' + t(y - y')).(y - y')\,dt$$

(by the chain rule), so

$$a_k(y) - a_k(y') = \sum_{i=1}^N \int_0^1 \frac{\partial a_k}{\partial y_i}(y' + t(y - y'))(y_i - y_i')\,dt \quad \forall k \in \{1, \ldots, N\}. \tag{6}$$

We set

$$e_{k,i}(z) = \int_0^1 \frac{\partial a_k}{\partial y_i}(Dv(z) + t(Du(z) - Dv(z)))(D_i u(z) - D_i v(z))\,dt, \tag{7}$$

with $D_i = \frac{\partial}{\partial z_i}$ for all $i \in \{1, \ldots, N\}$. We consider the following second order differential operator in divergence form

$$L(w) = -\mathrm{div}\left(\sum_{i=1}^N e_{k,i}(z) D_i w\right) = -\sum_{k,i=1}^N D_k(e_{k,i}(z) D_i w).$$

We set $\vartheta = u - v \in C^{1,\alpha}(\overline{\Omega})$. By hypothesis $\vartheta(z) \geqslant 0$ for all $z \in \overline{\Omega}$, $\vartheta \not\equiv 0$.

Suppose that for some $z_0 \in \Omega$ we have $\vartheta(z_0) = 0$, that is $u(z_0) = v(z_0)$.

Recall that $x \longmapsto x^{p-1}$ is a uniformly continuous homeomorphism on $\mathbb{R}_+$ (more precisely for $1 < p < 2$ this function is Hölder continuous, while for $p \geqslant 2$ it is locally Lipschitz; see Garcia Azorero-Manfredi-Peral Alonso [6, (3.1), p. 391]).

Since $\widetilde{\xi} \in L^\infty(\Omega)$, using (4), we see that we can find $\varrho \in (0,1)$ small such that

$$h_1 - h_2 - \widetilde{\xi}(z)(u^{p-1} - v^{p-1}) \geqslant c_2 > 0 \quad \text{for a.a. } z \in \overline{B}_\varrho(z_0). \tag{8}$$

Also using Theorem 1.1 of Lucia-Prashanth [18], we see that $e_{k,i} \in W^{1,\infty}(B_\varrho(z_0))$ (see (7)). From (5), (6) and (8) we have

$$L(\vartheta)(z) \geqslant c_2 > 0 \quad \text{for a.a. } z \in \overline{B}_\varrho(z_0).$$

Invoking the Harnack inequality (see Pucci-Serrin [27, p. 163]) or alternatively using Theorem 4 of Vázquez [28], we have $\vartheta(z) > 0$ for all $z \in B_\varrho(z_0)$, a contradiction to the fact that $\vartheta(z_0) = 0$. Therefore it follows that $\vartheta(z) > 0$ for all $z \in \Omega$.

Now let $\Sigma_0 = \{z \in \partial\Omega : \ \vartheta(z)(z) = 0\}$. If $\Sigma_0 = \emptyset$, then $\vartheta(z) > 0$ for all $z \in \overline{\Omega}$, so $\vartheta \in D_+ \subseteq \operatorname{int} C_+$ and we are done.

So, we assume that $\Sigma_0 \neq \emptyset$. Let $z_0 \in \Sigma_0$. Since $\partial\Omega$ is a C^2-manifold, we can find $r \in (0,1)$ small and an open r-ball B_r such that

$$B_r \subseteq \Omega, \qquad z_0 \in \partial\Omega \cap \partial B_r.$$

From (3), (4) and by taking $r > 0$ even smaller if necessary, we can have that $L|_{B_r}$ is strictly elliptic and $e_{k,i} \in W^{1,\infty}(B_r)$. So, invoking the Hopf boundary point lemma, we have

$$\frac{\partial\vartheta}{\partial n}(z_0) < 0,$$

hence $\vartheta = u - v \in \widehat{C}_+$. □

Remark 2.8. With the same proof the result remains valid if the differential operator has the form $-\operatorname{div} a(Du)$ with $a\colon \mathbb{R}^N \longrightarrow \mathbb{R}^N$ satisfying hypotheses $H(a)$ of Gasiński-O'Regan-Papageorgiou [7] (see also Papageorgiou-Winkert [26]). Such operators need not be homogeneous and incorporate as a special case the (p,q)-Laplacian (that is, the sum of a p-Laplacian and a q-Laplacian). □

If $x \in \mathbb{R}$, we define $x^{\pm} = \max\{\pm x, 0\}$ and for $u \in W^{1,p}(\Omega)$ we set $u^{\pm}(\cdot) = u(\cdot)^{\pm}$. We know that

$$u^{\pm} \in W^{1,p}(\Omega), \quad |u| = u^+ + u^-, \quad u = u^+ - u^-.$$

Given a measurable function $g\colon \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ (for example, a Carathéodory function), by N_g we denote the Nemitsky map corresponding to g, that is

$$N_g(u)(\cdot) = g(\cdot, u(\cdot)) \quad \forall u \in W^{1,p}(\Omega).$$

If $u, v \in W^{1,p}(\Omega)$ and $u \leqslant v$, then we define

$$[u,v] = \{y \in W^{1,p}(\Omega) : \ u(z) \leqslant y(z) \leqslant v(z) \text{ for a.a. } z \in \Omega\}.$$

Also by $\operatorname{int}_{C^1(\overline{\Omega})}[u,v]$ we denote the interior in $C^1(\overline{\Omega})$ of $[u,v] \cap C^1(\overline{\Omega})$. Finally if $\varphi \in C^1(X;\mathbb{R})$, then $K_\varphi = \{u \in X : \ \varphi'(u) = 0\}$ (the critical set of φ).

Now we will introduce the hypotheses on the perturbation term $f(z,x)$. First the case of uniform nonresonance with respect to $\widehat{\lambda}_1 > 0$.

$H(f)$: $f\colon \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ is a Carathéodory function such that $f(z,0) = 0$ for a.a. $z \in \Omega$ and

(i) for every $\varrho > 0$ there exists $a_\varrho \in L^\infty(\Omega)$ such that

$$|f(z,x)| \leqslant a_\varrho(z) \quad \text{for a.a. } z \in \Omega, \text{ all } 0 \leqslant x \leqslant \varrho;$$

(ii) there exist $\widehat{\eta} > \eta > \widehat{\lambda}_1$ such that

$$\eta \leqslant \liminf_{x \to +\infty} \frac{f(z,x)}{x^{p-1}} \leqslant \limsup_{x \to +\infty} \frac{f(z,x)}{x^{p-1}} \leqslant \widehat{\eta},$$

uniformly for a.a. $z \in \Omega$;

(iii) there exists a function $w \in C^{1,\alpha}(\overline{\Omega}) \cap D_+$ $(0 < \alpha \leqslant 1)$ such that

$$\Delta_p w \in L^{p'}(\Omega), \quad -\Delta_p w(z) + \xi(z) w(z)^{p-1} \geqslant 0 \quad \text{for a.a. } z \in \Omega,$$

$$w(z)^{p-1} + f(z, w(z)) \leqslant -\widehat{c} < 0 \quad \text{for a.a. } z \in \Omega;$$

(iv) if $\widehat{m} = \min_{\overline{\Omega}} w > 0$ (recall that $w \in D_+$), then there exists $\delta_0 \in (0, \widehat{m})$ such that for all $m \in (0, \delta_0)$ there exists $\widehat{c}_m > 0$ such that

$$f(z, x) \geqslant \widehat{c}_m \quad \text{for a.a. } z \in \Omega, \text{ all } x \in [m, \delta_0];$$

(v) for every $\varrho > 0$, there exists $\widehat{\xi}_\varrho > 0$ such that for a.a. $z \in \Omega$, the function $x \longmapsto f(z, x) + \widehat{\xi}_\varrho x^{p-1}$ is nondecreasing on $[0, \varrho]$.

Remark 2.9. Since we are interested on positive solutions and the above hypotheses concern the positive semiaxis $\mathbb{R}_+ = [0, +\infty)$, without any loss of generality we may assume that

$$f(z, x) = 0 \quad \text{for a.a. } z \in \Omega, \text{ all } x \leqslant 0. \tag{9}$$

Since $\eta > \widehat{\lambda}_1$, hypothesis $H(f)$(ii) implies that asymptotically at $+\infty$, we have uniform nonresonance with respect to $\widehat{\lambda}_1 > 0$. Hypothesis $H(f)$(iii) is satisfied, if there exists $\vartheta > 0$ such that

$$\vartheta^{p-1} + f(z, \vartheta) \leqslant -\widetilde{c} < 0 \quad \text{for a.a. } z \in \Omega.$$

Hypotheses $H(f)$(iii),(iv) dictate an oscillatory behaviour for $f(z, \cdot)$ near 0^+. Hypothesis $H(f)$(v) is satisfied if for example for a.a. $z \in \Omega$, $f(z, \cdot)$ is differentiable and for every $\varrho > 0$ there exists $\widehat{a}_\varrho > 0$ such that

$$f'_x(z, x)x \geqslant -\widehat{a}_\varrho x^{p-1} \quad \text{for a.a. } z \in \Omega, \text{ all } 0 \leqslant x \leqslant \varrho. \qquad \square$$

Example 2.10. The following function satisfies hypotheses $H(f)$. For the sake of simplicity we drop the z-dependence:

$$f(x) = \begin{cases} x^{\tau-1} - 3x^{r-1} & \text{if } 0 \leqslant x \leqslant 1, \\ \eta x^{p-1} - (\eta + 2)x^{\vartheta-1} & \text{if } 1 < x \end{cases}$$

(see (9)), with $1 < \tau < r$, $\vartheta < p$ and $\eta > \widehat{\lambda}_1$. $\square$

For the nonuniform nonresonance case we need to strengthen the hypotheses on the perturbation term $f(z, x)$. The new conditions on f are the following:

$H(f)'$: $f\colon \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ is a Carathéodory function such that $f(z, 0) = 0$ for a.a. $z \in \Omega$ and

(i) for every $\varrho > 0$ there exists $a_\varrho \in L^\infty(\Omega)$ such that

$$|f(z, x)| \leqslant a_\varrho(z) \quad \text{for a.a. } z \in \Omega, \text{ all } 0 \leqslant x \leqslant \varrho;$$

(ii) there exists a function $\vartheta \in L^\infty(\Omega)$ such that

$$\widehat{\lambda}_1 \leqslant \vartheta(z) \quad \text{for a.a. } z \in \Omega, \quad \vartheta \not\equiv \widehat{\lambda}_1$$

and
$$\vartheta(z) \leqslant \liminf_{x \to +\infty} \frac{f(z,x)}{x^{p-1}} \leqslant \limsup_{x \to +\infty} \frac{f(z,x)}{x^{p-1}} \leqslant \widehat{\eta},$$

uniformly for a.a. $z \in \Omega$;

(iii) there exists a function $w \in C^{1,\alpha}(\overline{\Omega}) \cap D_+$ $(0 < \alpha \leqslant 1)$ such that

$$\Delta_p w \in L^{p'}(\Omega), \quad -\Delta_p w(z) + \xi(z) w(z)^{p-1} \geqslant 0 \quad \text{for a.a. } z \in \Omega,$$
$$w(z)^{p-1} + f(z, w(z)) \leqslant -\widehat{c} < 0 \quad \text{for a.a. } z \in \Omega;$$

(iv) $f(z,x) \geqslant \widehat{\lambda}_1 x^{p-1} - c^* x^{q-1}$ for a.a. $z \in \Omega$, all $x \geqslant 0$, and if we have $\widehat{m} = \min\limits_{\overline{\Omega}} w > 0$ (recall that $w \in D_+$), then there exists $\delta_0 \in (0, \widehat{m})$ such that for all $m \in (0, \delta_0)$ there exists $\widehat{c}_m > 0$ such that

$$f(z,x) \geqslant \widehat{c}_m \quad \text{for a.a. } z \in \Omega, \text{ all } x \in [m, \delta_0];$$

(v) for every $\varrho > 0$, there exists $\widehat{\xi}_\varrho > 0$ such that for a.a. $z \in \Omega$, the function $x \longmapsto f(z,x) + \widehat{\xi}_\varrho x^{p-1}$ is nondecreasing on $[0, \varrho]$.

Remark 2.11. Again without any loss of generality, we assume that (9) holds. The hypotheses $H(f)'$(i), (iii), (v) are the same as the corresponding hypotheses $H(f)$(i), (iii), (v). In hypothesis $H(f)'$(ii) asymptotically we allow the quotient $f(z,x)/x^{p-1}$ to hit $\widehat{\lambda}_1$ (resonance), but this cannot happen for all $z \in \Omega$. Finally in hypothesis $H(f)'$(iv), we added an additional unilateral growth condition on $f(z,\cdot)$. □

Example 2.12. The following function satisfies hypotheses $H(f)'$. For the sake of simplicity we drop the z-dependence:

$$f(x) = \begin{cases} \widehat{\lambda}_1 x^{p-1} - c^* x^{r-1} & \text{if } \ 0 \leqslant x \leqslant 1, \\ \eta x^{p-1} - (c^* + \eta - \widehat{\lambda}_1) x^{\tau-1} & \text{if } \ 1 < x, \end{cases}$$

with $c^* - \widehat{\lambda}_1 > 1$, $\eta > \widehat{\lambda}_1$ and $1 < \tau < p < r$.

3. Bifurcation-Type Theorem

In this section we prove bifurcation-type theorem describing in a precise way the dependence of the set of positive solutions of problem (P_λ) on the parameter $\lambda > 0$. We start by considering the following auxiliary nonlinear parametric Robin problem:

$$(Au_\lambda) \qquad \begin{cases} -\Delta_p u(z) + \xi(z) u(z)^{p-1} = \lambda u(z)^{q-1} & \text{in } \Omega, \\ \frac{\partial u}{\partial n_p} + \beta(z) u^{p-1} = 0 & \text{on } \partial\Omega, \\ u > 0 & \text{in } \Omega\,. \end{cases}$$

Proposition 3.1. *If the hypotheses $H(\xi)$, $H(\beta)$, H_0 hold, then for every $\lambda > 0$ problem (Au_λ) admits a unique positive solution*

$$\overline{u}_\lambda \in D_+ \quad \textit{and} \quad \|\overline{u}_\lambda\|_{C^1(\overline{\Omega})} \longrightarrow 0 \quad \textit{as } \lambda \to 0^+.$$

Proof. Recall that $\mu\colon W^{1,p}(\Omega) \longrightarrow \mathbb{R}$ is the C^1-functional defined by

$$\mu(u) = \|Du\|_p^p + \int_\Omega \xi(z)|u|^p\, dz + \int_{\partial\Omega} \beta(z)|u|^p\, d\sigma \quad \forall u \in W^{1,p}(\Omega).$$

Lemmata 2.4 and 2.5 imply that

$$c_3\|u\|^p \leqslant \mu(u) \quad \forall u \in W^{1,p}(\Omega) \tag{10}$$

for some $c_3 > 0$. We consider the C^1-functional $\psi_\lambda\colon W^{1,p}(\Omega) \longrightarrow \mathbb{R}$ defined by

$$\psi_\lambda(u) = \frac{1}{p}\mu(u) - \frac{\lambda}{q}\|u^+\|_q^q.$$

From (10) and since $q < p$, it follows that ψ is coercive. Also from the Sobolev embedding theorem and the compactness of the trace map, we infer that ψ_λ is sequentially weakly lower semicontinuous. So, invoking the Weierstrass-Tonelli theorem, we can find $\overline{u}_\lambda \in W^{1,p}(\Omega)$ such that

$$\psi_\lambda(\overline{u}_\lambda) = \inf_{u \in W^{1,p}(\Omega)} \psi_\lambda(u). \tag{11}$$

Since $q < p$, for $\vartheta \in (0,1)$ small, we have $\psi_\lambda(\vartheta) < 0$, hence $\psi_\lambda(\overline{u}_\lambda) < 0 = \psi_\lambda(0)$ (see (11)) and thus $\overline{u}_\lambda \neq 0$.

From (10) we have $\psi_\lambda'(\overline{u}_\lambda) = 0$, hence

$$\begin{aligned}\langle A(\overline{u}_\lambda), h\rangle &+ \int_\Omega \xi(z)|\overline{u}_\lambda|^{p-2}\overline{u}_\lambda h\, dz + \int_{\partial\Omega} \beta(z)|\overline{u}_\lambda|^{p-2}\overline{u}_\lambda h\, d\sigma \\ &= \lambda \int_\Omega (\overline{u}_\lambda^+)^{q-1} h\, dz\end{aligned} \tag{12}$$

for all $h \in W^{1,p}(\Omega)$. We choose $h = -\overline{u}_\lambda^- \in W^{1,p}(\Omega)$. Then $\mu(\overline{u}_\lambda^-) = 0$, so $\overline{u}_\lambda \geqslant 0$ and $\overline{u}_\lambda \neq 0$ (see (10)). Therefore (12) becomes

$$\langle A(\overline{u}_\lambda), h\rangle + \int_\Omega \xi(z)\overline{u}_\lambda^{p-1} h\, dz + \int_{\partial\Omega} \beta(z)\overline{u}_\lambda^{p-1} h\, d\sigma = \lambda \int_\Omega \overline{u}_\lambda^{q-1} h\, dz \tag{13}$$

for all $h \in W^{1,p}(\Omega)$, so

$$\left\{\begin{array}{l} -\Delta_p \overline{u}_\lambda(z) + \xi(z)\overline{u}_\lambda(z)^{p-1} = \lambda \overline{u}_\lambda(z)^{q-1} \quad \text{a.e. in } \Omega, \\ \frac{\partial \overline{u}_\lambda}{\partial n_p} + \beta(z)\overline{u}_\lambda^{p-1} = 0 \quad \text{on } \partial\Omega \end{array}\right. \tag{14}$$

(see Papageorgiou-Rădulescu [22]).

From (14) and Papageorgiou-Rădulescu [23], we have $\overline{u}_\lambda \in L^\infty(\Omega)$. Then Theorem 2 of Lieberman [17] implies that $\overline{u}_\lambda \in C_+ \setminus \{0\}$. From (14) we have

$$\Delta_p \overline{u}_\lambda(z) \leqslant \|\xi\|_\infty \overline{u}_\lambda(z)^{p-1} \quad \text{for a.a. } z \in \Omega$$

(see hypothesis $H(\xi)$), thus $\overline{u}_\lambda \in D_+$ (from the nonlinear strong maximum principle; see Vázquez [28] and Gasiński-Papageorgiou [9, p. 738]).

We show that this positive solution $\overline{u}_\lambda \in D_+$ is unique.

Suppose that $\overline{v}_\lambda$ is another solution of (Au_λ). As for $\overline{u}_\lambda$, we show that $\overline{v}_\lambda \in D_+$. Let $t > 0$ be the biggest positive real such that $t\overline{v}_\lambda \leqslant \overline{u}_\lambda$. Suppose that $t \in (0,1)$. We have

$$\begin{aligned}-\Delta_p(t\overline{v}_\lambda) + \xi(z)(t\overline{v}_\lambda)^{p-1} &= t^{p-1}(-\Delta_p \overline{v}_\lambda + \xi(z)\overline{v}_\lambda^{p-1}) = t^{p-1}\lambda \overline{v}_\lambda^{q-1} \\ &< \lambda(t\overline{v}_\lambda)^{q-1} \leqslant \lambda \overline{u}_\lambda^{q-1} = -\Delta_p \overline{u}_\lambda + \xi(z)\overline{u}_\lambda^{p-1}\end{aligned}$$

(since $t < 1$ and $q < p$), so $\overline{u}_\lambda - t\overline{v}_\lambda \in \widehat{C}_+$ (see Proposition 2.7), which contradicts the maximality of $t > 0$. Hence $t \geqslant 1$ and so $\overline{v}_\lambda \leqslant \overline{u}_\lambda$.

Interchanging the roles of $\overline{u}_\lambda$ and $\overline{v}_\lambda$ in the above argument, we obtain $\overline{u}_\lambda \leqslant \overline{v}_\lambda$, thus $\overline{u}_\lambda = \overline{v}_\lambda$.

In (13) we choose $h = \overline{u}_\lambda \in D_+$ and obtain $\mu(\overline{u}_\lambda) = \lambda\|\overline{u}_\lambda\|_q^q$, so

$$c_3\|\overline{u}_\lambda\|^p \leqslant \lambda c_4 \|\overline{u}_\lambda\|^q \quad \forall \lambda > 0$$

for some $c_4 > 0$ (see (9)), thus

$$\{\overline{u}_\lambda\}_{\lambda \in (0,1]} \subseteq W^{1,p}(\Omega) \text{ is bounded and } \|\overline{u}_\lambda\| \longrightarrow 0 \quad \text{as } \lambda \to 0^+. \tag{15}$$

On account of (15), from Papageorgiou-Rădulescu [23], we know that there exists $c_5 > 0$ such that

$$\|\overline{u}_\lambda\|_\infty \leqslant c_5 \quad \forall \lambda \in (0,1].$$

Then Theorem 2 of Lieberman [17] implies that there exist $\alpha \in (0,1)$ and $c_6 > 0$ such that

$$\overline{u}_\lambda \in C^{1,\alpha}(\overline{\Omega}) \quad \text{and} \quad \|\overline{u}_\lambda\|_{C^{1,\alpha}(\overline{\Omega})} \leqslant c_6 \quad \forall \lambda \in (0,1]. \tag{16}$$

Then (15), (16) and the compact embedding of $C^{1,\alpha}(\overline{\Omega})$ into $C^1(\overline{\Omega})$ imply that $\overline{u}_\lambda \longrightarrow 0$ in $C^1(\overline{\Omega})$ as $\lambda \to 0^+$. □

We introduce the sets

$$\begin{aligned}\mathcal{L} &= \{\lambda > 0 : \text{ problem } (P_\lambda) \text{ admits a positive solution}\},\\ S_\lambda &= \{u : u \text{ is a positive solution of } (P_\lambda)\}.\end{aligned}$$

Proposition 3.2. *If the hypotheses $H(\xi)$, $H(\beta)$, H_0, $H(f)$ or $H(f)'$ hold, then $\mathcal{L} \neq \emptyset$.*

Proof. On account of Proposition 3.1, we can find $\lambda_0 \in (0,1)$ such that

$$\overline{u}_\lambda(z) \in (0, \delta_0] \quad \forall z \in \overline{\Omega},\ \lambda \in (0, \lambda_0]. \tag{17}$$

Here $\delta_0 > 0$ is as in hypothesis $H(f)$(iv) (or $H(f)'$(iv)). We introduce the following Carathéodory function

$$e_\lambda(z,x) = \begin{cases} \lambda \overline{u}_\lambda(z)^{q-1} + f(z, \overline{u}_\lambda(z)) & \text{if} \quad x < \overline{u}_\lambda(z), \\ \lambda x^{q-1} + f(z,x) & \text{if} \quad \overline{u}_\lambda(z) \leqslant x \leqslant w(z), \\ \lambda w(z)^{q-1} + f(z, w(z)) & \text{if} \quad w(z) < x \end{cases} \tag{18}$$

(recall that $\delta_0 < \widehat{m} = \min\limits_{\overline{\Omega}} w$). We set

$$E_\lambda(z,x) = \int_0^x e_\lambda(z,s)\,ds$$

and consider the C^1-functional $\widehat{\varphi}_\lambda\colon W^{1,p}(\Omega) \longrightarrow \mathbb{R}$ defined by

$$\widehat{\varphi}_\lambda(u) = \frac{1}{p}\mu(u) - \int_\Omega E_\lambda(z,u)\,dz \quad \forall u \in W^{1,p}(\Omega).$$

From (10) and (18) we see that $\widehat{\varphi}_\lambda$ is coercive. Also, it is sequentially weakly lower semicontinuous. Therefore we can find $u_\lambda^0 \in W^{1,p}(\Omega)$ such that

$$\widehat{\varphi}_\lambda(u_\lambda^0) = \inf_{u \in W^{1,p}(\Omega)} \widehat{\varphi}_\lambda(u),$$

so $\widehat{\varphi}_\lambda'(u_\lambda^0) = 0$ and thus

$$\langle A(u_\lambda^0), h\rangle + \int_\Omega \xi(z)|u_\lambda^0|^{p-2}u_\lambda^0 h\,dz + \int_{\partial\Omega} \beta(z)|u_\lambda^0|^{p-2}u_\lambda^0 h\,d\sigma = \int_\Omega e_\lambda(z, u_\lambda^0)h\,dz \tag{19}$$

for all $h \in W^{1,p}(\Omega)$. In (19) first we choose $h = (\overline{u}_\lambda - u_\lambda^0)^+ \in W_0^{1,p}(\Omega)$. Then for $\lambda \in (0, \lambda_0]$, we have

$$\begin{aligned} &\langle A(u_\lambda^0), (\overline{u}_\lambda - u_\lambda^0)^+\rangle + \int_\Omega \xi(z)|u_\lambda^0|^{p-2}u_\lambda^0(\overline{u}_\lambda - u_\lambda^0)^+ dz + \int_{\partial\Omega} \beta(z)|u_\lambda^0|^{p-2}u_\lambda^0(\overline{u}_\lambda - u_\lambda^0)^+ d\sigma \\ &= \int_\Omega (\lambda \overline{u}_\lambda^{q-1} + f(z, \overline{u}_\lambda))(\overline{u}_\lambda - u_\lambda^0)^+\,dz \geqslant \int_\Omega \lambda \overline{u}_\lambda^{q-1}(\overline{u}_\lambda - u_\lambda^0)^+\,dz \\ &= \langle A(\overline{u}_\lambda), (\overline{u}_\lambda - u_\lambda^0)^+\rangle + \int_\Omega \xi(z)\overline{u}_\lambda^{p-1}(\overline{u}_\lambda - u_\lambda^0)^+\,dz + \int_{\partial\Omega} \beta(z)\overline{u}_\lambda^{p-1}(\overline{u}_\lambda - u_\lambda^0)^+\,d\sigma \end{aligned}$$

see (18), (17), hypotheses $H(f)$(v) (or $H(f)'$(v)) and Proposition 3.1. Therefore $\overline{u}_\lambda \leqslant u_\lambda^0$.

Next in (19) we choose $(u_\lambda^0 - w)^+ \in W^{1,p}(\Omega)$. Then

$$\begin{aligned}
&\langle A(u_\lambda^0),(u_\lambda^0-w)^+\rangle+\int_\Omega \xi(z)(u_\lambda^0)^{p-1}(u_\lambda^0-w)^+dz+\int_{\partial\Omega}\beta(z)(u_\lambda^0)^{p-1}(u_\lambda^0-w)^+d\sigma\\
&\qquad=\int_\Omega(\lambda w^{q-1}+f(z,w))(u_\lambda^0-w)^+\,dz\\
&\qquad\leqslant\langle A(w),(u_\lambda^0-w)^+\rangle+\int_\Omega\xi(z)w^{p-1}(u_\lambda^0-w)^+\,dz
\end{aligned}$$

(see hypotheses $H(f)$(iv) (or $H(f)'$(iv)) and recall $\lambda<1$), so $u_\lambda^0\leqslant w$ (see hypothesis $H(\beta)$). So, we have proved that

$$u_\lambda^0\in[\overline{u}_\lambda,w]\quad\forall\lambda\in(0,\lambda_0].\tag{20}$$

From (18), (19) and (20) it follows that u_λ is a positive solution of (P_λ) and Theorem 2 of Lieberman [17] implies that $u_\lambda\in D_+$. Hence $(0,\lambda_0]\subseteq\mathcal{L}$. □

Corollary 3.3. *If hypotheses $H(\xi)$, $H(\beta)$, H_0, $H(f)$ or $H(f)'$ hold and $\lambda\in\mathcal{L}$, then $S_\lambda\subseteq D_+$.*

Next we show that $\mathcal{L}$ is actually an interval.

Proposition 3.4. *If hypotheses $H(\xi)$, $H(\beta)$, H_0, $H(f)$ or $H(f)'$ hold, $\lambda\in\mathcal{L}$ and $\vartheta\in(0,\lambda)$, then $\vartheta\in\mathcal{L}$.*

Proof. Since $\lambda\in\mathcal{L}$, we can find $u_\lambda\in S_\lambda\subseteq D_+$. We introduce the Carathéodory function τ_θ defined by (see (9))

$$\tau_\theta(z,x)=\begin{cases}\vartheta(x^+)^{q-1}+f(z,x) & \text{if}\quad x\leqslant u_\lambda(z),\\ \vartheta u_\lambda(z)^{q-1}+f(z,u_\lambda(z)) & \text{if}\quad u_\lambda(z)<x.\end{cases}\tag{21}$$

We set $T_\theta(z,x)=\int_0^x\tau_\theta(z,s)\,ds$ and consider the C^1-functional $\chi_\vartheta\colon W^{1,p}(\Omega)\longrightarrow\mathbb{R}$ defined by

$$\chi_\vartheta(u)=\frac{1}{p}\mu(u)-\int_\Omega T_\vartheta(z,u)\,dz\quad\forall u\in W^{1,p}(\Omega).$$

Evidently (10) and (21) imply the coercivity of χ_ϑ. Also, χ_ϑ is sequentially weakly lower semicontinuous. So, we can find $u_\vartheta\in W^{1,p}(\Omega)$ such that

$$\chi_\vartheta(u_\vartheta)=\inf_{u\in W^{1,p}(\Omega)}\chi_\vartheta(u).\tag{22}$$

Let $\widehat{\delta}_0=\min\{\delta_0,\min_{\overline{\Omega}}u_\lambda\}>0$ (recall that $u_\lambda\in D_+$) and pick $\eta\in(0,\widehat{\delta}_0)$.

Then, since $q<p$ and $f(z,\eta)>0$ for a.a. $z\in\Omega$ (see hypotheses $H(f)$(iv) (or $H(f)'$(iv))), by choosing $\eta>0$ even smaller, we see that $\chi_\vartheta(\eta)<0$, so $\chi_\vartheta(u_\vartheta)<0=\chi_\vartheta(0)$ (see (22)) and thus $u_\vartheta\neq0$. From (22) we have $\chi_\vartheta'(u_\vartheta)=0$, so for all $h\in W^{1,p}(\Omega)$

$$\langle A(u_\vartheta),h\rangle+\int_\Omega\xi(z)|u_\vartheta|^{p-2}u_\vartheta h\,dz+\int_{\partial\Omega}\beta(z)|u_\vartheta|^{p-2}u_\vartheta h\,d\sigma=\int_\Omega\tau_\vartheta(z,u_\vartheta)h\,dz.\tag{23}$$

In (23) we choose $h = -u_\vartheta^- \in W^{1,p}(\Omega)$. Then

$$\mu(u_\vartheta^-) = 0,$$

so $u_\vartheta \geqslant 0$, $u_\vartheta \neq 0$ (see (10)).

Next in (23) we choose $h = (u_\vartheta - u_\lambda)^+ \in W^{1,p}(\Omega)$. We have

$$\begin{aligned}
&\langle A(u_\vartheta), (u_\vartheta - u_\lambda)^+\rangle + \int_\Omega \xi(z) u_\vartheta^{p-1}(u_\vartheta - u_\lambda)^+ \, dx + \int_{\partial\Omega} \beta(z) u_\vartheta^{p-1}(u_\vartheta - u_\lambda)^+ \, d\sigma \\
&= \int_\Omega (\vartheta u_\lambda^{q-1} + f(z, u_\lambda))(u_\vartheta - u_\lambda)^+ \, dz < \int_\Omega (\lambda u_\lambda^{q-1} + f(z, u_\lambda))(u_\vartheta - u_\lambda)^+ \, dz \\
&= \langle A(u_\lambda), (u_\vartheta - u_\lambda)^+\rangle + \int_\Omega \xi(z) u_\lambda^{p-1}(u_\vartheta - u_\lambda)^+ \, dz + \int_{\partial\Omega} \beta(z) u_\lambda^{p-1}(u_\vartheta - u_\lambda)^+ \, d\sigma
\end{aligned}$$

(see (21) and use the fact that $\vartheta < \lambda$ and $u_\lambda \in S_\lambda$), so $u_\vartheta \leqslant u_\lambda$. Therefore,

$$u_\vartheta \in [0, u_\lambda] \setminus \{0\},$$

so $u_\vartheta \in S_\vartheta \subseteq D_+$ (see (21)) and hence $\vartheta \in \mathcal{L}$. $\square$

The above proposition implies that $\mathcal{L}$ is an interval. A byproduct of the above proof is the following corollary.

Corollary 3.5. *If the hypotheses $H(\xi)$, $H(\beta)$, H_0, $H(f)$ or $H(f)'$ hold, $\lambda \in \mathcal{L}$, $u_\lambda \in S_\lambda$ and $\vartheta \in (0, \lambda)$, then $\vartheta \in \mathcal{L}$ and we can find $u_\vartheta \in S_\vartheta$ such that $u_\vartheta \leqslant u_\lambda$, $u_\vartheta \neq u_\lambda$.*

In fact using Proposition 2.7, we can improve the conclusion of the above corollary.

Corollary 3.6. *If the hypotheses $H(\xi)$, $H(\beta)$, H_0, $H(f)$ or $H(f)'$ hold, $\lambda \in \mathcal{L}$, $u_\lambda \in S_\lambda$ and $\vartheta \in (0, \lambda)$, then $\vartheta \in \mathcal{L}$ and we can find $u_\vartheta \in S_\vartheta$ such that $u_\lambda - u_\vartheta \in \operatorname{int} C_+$.*

Proof. From Corollary 3.5, we have that $\vartheta \in \mathcal{L}$ and there exists $u_\vartheta \in S_\vartheta \subseteq D_+$ such that

$$u_\vartheta \leqslant u_\lambda. \tag{24}$$

Let $\varrho = \|u_\lambda\|_\infty$ and let $\widehat{\xi}_\varrho > 0$ be as postulated by hypothesis $H(f)$(v) (or $H(f)'$(v)). We have

$$\begin{aligned}
-\Delta_p u_\vartheta + (\xi(z) + \widehat{\xi}_\varrho) u_\vartheta^{p-1} &= \vartheta u_\vartheta^{q-1} + f(z, u_\vartheta) + \widehat{\xi}_\vartheta u_\vartheta^{p-1} \leqslant \vartheta u_\lambda^{q-1} + f(z, u_\lambda) + \widehat{\xi}_\varrho u_\lambda^{p-1} \\
&< \lambda u_\lambda^{q-1} + f(z, u_\lambda) + \widehat{\xi}_\varrho u_\lambda^{p-1} = -\Delta_p u_\lambda + (\xi(z) + \widehat{\xi}_\varrho) u_\lambda^{p-1}
\end{aligned} \tag{25}$$

for a.a. $z \in \Omega$ (see (24) and hypothesis $H(f)$(v) (or $H(f)'$(v))). Consider the functions

$$\begin{aligned}
h_1(z) &= \lambda u_\lambda(z)^{q-1} + f(z, u_\lambda(z)) + \widehat{\xi}_\varrho u_\lambda(z)^{p-1}, \\
\widehat{h}(z) &= \vartheta u_\lambda(z)^{q-1} + f(z, u_\lambda(z)) + \widehat{\xi}_\varrho u_\lambda(z)^{p-1}, \\
h_2(z) &= \vartheta u_\vartheta(z)^{q-1} + f(z, u_\vartheta(z)) + \widehat{\xi}_\varrho u_\vartheta(z)^{p-1}.
\end{aligned}$$

Evidently $h_1, \widehat{h}, h_2 \in L^\infty(\Omega)$ (see hypothesis $H(f)$(i) (or $H(f)'$(i))) and

$$h_2 \leqslant \widehat{h} \leqslant h_1.$$

Moreover, $h_1 - \widehat{h} \geqslant (\lambda - \vartheta) \min_{\overline{\Omega}} u_\lambda = c_0^* > 0$ (recall $u_\lambda \in D_+$, $\vartheta < \lambda$), so

$$h_1 - h_2 \geqslant c_0^* > 0 \quad \text{for a.a. } z \in \Omega.$$

Then from (25) and Proposition 2.7 it follows that $u_\lambda - u_\vartheta \in \operatorname{int} C_+$. □

Let $\lambda^* = \sup \mathcal{L}$.

Proposition 3.7. *If the hypotheses $H(\xi)$, $H(\beta)$, H_0, $H(f)$ hold, then $\lambda^* < \infty$.*

Proof. The hypotheses $H(f)$(i), (ii) and (iii) imply that for $\varepsilon > 0$ small we can find $\widetilde{\lambda} > 0$ big such that

$$\widetilde{\lambda} x^{q-1} + f(z,x) \geqslant (\widehat{\lambda}_1 + \varepsilon) x^{p-1} \quad \text{for a.a. } z \in \Omega, \text{ all } x \geqslant 0. \tag{26}$$

Let $\lambda > \widetilde{\lambda}$ and suppose that $\lambda \in \mathcal{L}$. So, we can find $u_\lambda \in S_\lambda \subseteq D_+$. We have

$$\begin{aligned} -\Delta_p u_\lambda + \xi(z) u_\lambda^{p-1} &= \lambda u_\lambda^{q-1} + f(z,u_\lambda) > \widetilde{\lambda} u_\lambda^{q-1} + f(z,u_\lambda) \\ &\geqslant (\widehat{\lambda}_1 + \varepsilon) u_\lambda^{p-1} \quad \text{for a.a. } z \in \Omega \end{aligned} \tag{27}$$

(see (26)). Since $\widehat{u}_1, u_\lambda \in D_+$, we can find $t \in (0,1)$ small such that

$$\widehat{v}_1 = t\widehat{u}_1 \leqslant u_\lambda. \tag{28}$$

Also, we have

$$-\Delta_p \widehat{v}_1 + \xi(z) \widehat{v}_1^{p-1} = \widehat{\lambda}_1 \widehat{v}_1^{p-1} < (\widehat{\lambda}_1 + \varepsilon) \widehat{v}_1^{p-1} \quad \text{for a.a. } z \in \Omega. \tag{29}$$

We consider the following auxiliary Robin problem

$$\begin{cases} -\Delta_p u(z) + \xi(z) u(z)^{p-1} = (\widehat{\lambda}_1 + \varepsilon) u(z)^{p-1} & \text{in } \Omega, \\ \frac{\partial u}{\partial n_p} + \beta(z) u^{p-1} = 0 & \text{on } \partial\Omega, \\ u > 0 & \text{in } \Omega. \end{cases} \tag{30}$$

Exploiting (28), we introduce the following Carathéodory function

$$\widetilde{\eta}(z,x) = \begin{cases} (\widehat{\lambda}_1 + \varepsilon) v_1(z)^{p-1} & \text{if } x < v_1(z), \\ (\widehat{\lambda}_1 + \varepsilon) x^{p-1} & \text{if } v_1(z) \leqslant x \leqslant u_\lambda(z), \\ (\widehat{\lambda}_1 + \varepsilon) u_\lambda(z)^{p-1} & \text{if } u_\lambda(z) < x. \end{cases} \tag{31}$$

Let
$$\widetilde{H}(z,x) = \int_0^x \widetilde{\eta}(z,s)\, ds$$

and consider the C^1-functional $j \colon W^{1,p}(\Omega) \longrightarrow \mathbb{R}$ defined by

$$j(u)=\frac{1}{p}\,\mu(u)-\int_\Omega \widetilde{H}(z,u)\,dz \quad \forall u\in W^{1,p}(\Omega).$$

Evidently j is coercive (see (10) and (31)) and it is sequentially weakly lower semicontinuous. So, we can find $\widetilde{u}\in W^{1,p}(\Omega)$ such that

$$j(\widetilde{u})=\inf_{u\in W^{1,p}(\Omega)} j(u),$$

so $j'(\widetilde{u})=0$ and thus

$$\langle A(\widetilde{u}),h\rangle+\int_\Omega \xi(z)|\widetilde{u}|^{p-2}\widetilde{u}h\,dz+\int_{\partial\Omega}\beta(z)|\widetilde{u}|^{p-2}\widetilde{u}h\,d\sigma=\int_\Omega \widetilde{\eta}(z,\widetilde{u})\,dz \tag{32}$$

for all $h\in W^{1,p}(\Omega)$. In (32) we choose $h=(\widehat{v}_1-\widetilde{u})^+\in W^{1,p}(\Omega)$. Then

$$\begin{aligned}
&\langle A(\widetilde{u}),(\widehat{v}_1-\widetilde{u})^+\rangle+\int_\Omega \xi(z)|\widetilde{u}|^{p-2}\widetilde{u}(\widehat{v}_1-\widetilde{u})^+dz+\int_{\partial\Omega}\beta(z)|\widetilde{u}|^{p-2}\widetilde{u}(\widehat{v}_1-\widetilde{u})^+d\sigma\\
&=(\widehat{\lambda}_1+\varepsilon)\int_\Omega \widehat{v}_1^{p-1}(\widehat{v}_1-\widetilde{u})^+\,dz\\
&\geqslant \langle A(\widehat{v}_1),(\widehat{v}_1-\widetilde{u})^+\rangle+\int_\Omega \xi(z)\widehat{v}_1^{p-1}(\widehat{v}_1-\widetilde{u})^+dz+\int_{\partial\Omega}\beta(z)\widehat{v}_1^{p-1}(\widehat{v}_1-\widetilde{u})^+d\sigma
\end{aligned}$$

(see (31) and (29)), so $\widehat{v}_1\leqslant\widetilde{u}$.

Similarly, if in (32) we choose $h=(\widetilde{u}-u_\lambda)^+\in W^{1,p}(\Omega)$, then using (27), we obtain $\widetilde{u}\leqslant u_\lambda$. Therefore we have $\widetilde{u}\in[\widehat{v}_1,u_\lambda]$, so $\widetilde{u}\in D_+$ is a solution of (30) (see (31)). But then $\widetilde{u}$ must be nodal, a contradiction.

So, $\lambda\notin\mathcal{L}$ and we have $\lambda^*\leqslant\widetilde{\lambda}<+\infty$. □

We get the same result for the nonuniform nonresonant case (hypothesis $H(f)'$).

Proposition 3.8. *If hypotheses $H(\xi)$, $H(\beta)$, H_0, $H(f)'$ hold, then $\lambda^*<+\infty$.*

Proof. Let $\lambda>c^*$ with $c^*>0$ as in hypothesis $H(f)'$(iv). Suppose that $\lambda\in\mathcal{L}$. Then we can find $u_\lambda\in S_\lambda\subseteq D_+$. We have (see hypothesis $H(f)'$(iv))

$$\begin{aligned}
-\Delta_p u_\lambda+\xi(z)u_\lambda^{p-1} &= \lambda u_\lambda^{q-1}+f(z,u_\lambda)\\
&\geqslant (\lambda-c^*)u_\lambda^{q-1}+\widehat{\lambda}_1 u_\lambda^{p-1} \quad \text{for a.a. } z\in\Omega\,.
\end{aligned} \tag{33}$$

We consider the following auxiliary nonlinear Robin problem

$$\begin{cases}
-\Delta_p u(z)+\xi(z)u(z)^{p-1}=(\lambda-c^*)u(z)^{q-1}+\widehat{\lambda}_1 u(z)^{p-1} \quad \text{in } \Omega,\\
\frac{\partial u}{\partial n_p}+\beta(z)u^{p-1}=0 \quad \text{on } \partial\Omega,\\
u>0 \quad \text{in } \Omega.
\end{cases} \tag{34}$$

We introduce the following Carathéodory function

$$\widehat{d}(z,x)=\begin{cases}
0 & \text{if } \ x<0,\\
(\lambda-c^*)x^{q-1}+\widehat{\lambda}_1 x^{p-1} & \text{if } \ 0\leqslant x\leqslant u_\lambda(z),\\
(\lambda-c^*)u_\lambda(z)^{q-1}+\widehat{\lambda}_1 u_\lambda(z)^{p-1} & \text{if } \ u_\lambda(z)<x.
\end{cases} \tag{35}$$

We set
$$\widehat{D}(z,x)=\int_0^x \widehat{d}(z,s)\,ds$$
and consider the C^1-functional $\widehat{\zeta}\colon W^{1,p}(\Omega)\longrightarrow\mathbb{R}$ defined by
$$\zeta(u)=\frac{1}{p}\mu(u)-\int_\Omega \widehat{D}(z,u)\,dz \quad \forall u\in W^{1,p}(\Omega).$$
As before, via the direct method of the calculus of variations, we find $\widetilde{u}\in W^{1,p}(\Omega)$ such that
$$\zeta(\widetilde{u})=\inf_{u\in W^{1,p}(\Omega)}\zeta(u). \tag{36}$$
Since $q<p$, we have $\zeta(\widetilde{u})<0=\zeta(0)$, so $\widetilde{u}\neq 0$. From (36) we have $\zeta'(\widetilde{u})=0$, so
$$\langle A(\widetilde{u}),h\rangle+\int_\Omega \xi(z)|\widetilde{u}|^{p-2}\widetilde{u}h\,dz+\int_{\partial\Omega}\beta(z)|\widetilde{u}|^{p-2}\widetilde{u}h\,d\sigma=\int_\Omega \widehat{d}(z,\widetilde{u})h\,dz \tag{37}$$
for all $h\in W^{1,p}(\Omega)$. In (37), first we choose $h=-\widetilde{u}^-\in W^{1,p}(\Omega)$ and obtain $\mu(\widetilde{u}^-)=0$ (see (35)), so $\widetilde{u}\geqslant 0$, $\widetilde{u}\neq 0$ (see (10)).

Also in (37) we choose $h=(\widetilde{u}-u_\lambda)^+\in W^{1,p}(\Omega)$. Then
$$\begin{aligned}
&\langle A(\widetilde{u}),(\widetilde{u}-u_\lambda)^+\rangle+\int_\Omega \xi(z)\widetilde{u}^{p-1}(\widetilde{u}-u_\lambda)^+\,dz+\int_{\partial\Omega}\beta(z)\widetilde{u}^{p-1}(\widetilde{u}-u_\lambda)^+\,d\sigma\\
&=\int_\Omega((\lambda-c^*)u_\lambda^{q-1}+\widehat{\lambda}_1u_\lambda^{p-1})(\widetilde{u}-u_\lambda)^+dz<\int_\Omega(\lambda u_\lambda^{q-1}+f(z,u_\lambda))(\widetilde{u}-u_\lambda)^+dz\\
&=\langle A(u_\lambda),(\widetilde{u}-u_\lambda)^+\rangle+\int_\Omega \xi(z)u_\lambda^{p-1}(\widetilde{u}-u_\lambda)^+dz+\int_{\partial\Omega}\beta(z)u_\lambda^{p-1}(\widetilde{u}-u_\lambda)^+d\sigma
\end{aligned}$$
(see (35), (33) and recall $u_\lambda\in S_\lambda$), so $\widetilde{u}\leqslant u_\lambda$. Thus, we have $\widetilde{u}\in[0,u_\lambda]\cap D_+$ (by the nonlinear regularity theory). Then from (35) it follows that $\widetilde{u}\in D_+$ is a positive smooth solution of (34). Let
$$R(\widehat{u},\widetilde{u})(z)=|D\widehat{u}_1(z)|^p-|D\widetilde{u}(z)|^{p-2}\left(D\widetilde{u}(z),D\left(\frac{\widehat{u}_1^p}{\widetilde{u}^{p-1}}\right)(z)\right)_{\mathbb{R}^N}.$$
From the nonlinear Picone identity of Allegretto-Huang [2], we have
$$\begin{aligned}
0&\leqslant\int_\Omega R(\widehat{u},\widetilde{u})\,dz=\|D\widehat{u}_1\|_p^p-\int_\Omega |D\widetilde{u}|^{p-2}\left(D\widetilde{u},D\left(\frac{\widehat{u}_1^p}{\widetilde{u}^{p-1}}\right)\right)_{\mathbb{R}^N}\\
&=\|D\widehat{u}_1\|_p^p-\int_\Omega(-\Delta_p\widetilde{u})\frac{\widehat{u}_1^p}{\widetilde{u}^{p-1}}\,dz+\int_{\partial\Omega}\beta(z)\widehat{u}_1^p\,d\sigma\\
&<\|D\widehat{u}_1\|_p^p+\int_\Omega \xi(z)\widehat{u}_1^p+\int_{\partial\Omega}\beta(z)\widehat{u}_1^p\,d\sigma-\widehat{\lambda}_1=\widehat{\lambda}_1-\widehat{\lambda}_1=0
\end{aligned}$$
(using the nonlinear Green's identity (see Gasiński-Papageorgiou [9, p. 211]) and since $\lambda>c^*$, $\widetilde{u}\in D_+$ and $\|\widehat{u}_1\|_p=1$), a contradiction.

Therefore $\lambda\notin\mathcal{L}$ and so $\lambda^*\leqslant c^*<+\infty$. $\square$

Next we show that the critical parameter value $\lambda^* > 0$ is admissible.

Proposition 3.9. *If the hypotheses $H(\xi)$, $H(\beta)$, H_0, $H(f)$ or $H(f)'$ hold, then $\lambda^* \in \mathcal{L}$.*

Proof. We show the proof for the nonuniform nonresonance case (that is, hypothesis $H(f)'$). The proof for the uniform nonresonance case (that is, hypothesis $H(f)$), is similar and simpler.

Let $\{\lambda_n\}_{n\geqslant 1} \subseteq (0, \lambda^*)$ and assume that $\lambda_n \nearrow \lambda^*$. Let $u_n \in S_{\lambda_n}$, $n \geqslant 1$. We have

$$\begin{aligned} &\langle A(u_n), h\rangle + \int_\Omega \xi(z) u_n^{p-1} h\,dz + \int_{\partial\Omega} \beta(z) u_n^{p-1} h\,d\sigma \\ = \ & \lambda_n \int_\Omega u_n^{q-1} h\,dz + \int_\Omega f(z, u_n) h\,dz \quad \forall h \in W^{1,p}(\Omega),\ n \geqslant 1. \end{aligned} \tag{38}$$

Suppose that $\|u_n\| \longrightarrow +\infty$. We set $y_n = \frac{u_n}{\|u_n\|}$, $n \geqslant 1$. Then $\|y_n\| = 1$, $y_n \geqslant 1$ for all $n \geqslant 1$. From (38) we have

$$\begin{aligned} &\langle A(y_n), h\rangle + \int_\Omega \xi(z) y_n^{p-1} h\,dz + \int_{\partial\Omega} \beta(z) y_n^{p-1} h\,d\sigma \\ = \ & \frac{\lambda_n}{\|u_n\|^{p-q}} \int_\Omega y_n^{q-1} h\,dz + \int_\Omega \frac{N_f(u_n)}{\|u_n\|^{p-1}} h\,dz \end{aligned} \tag{39}$$

for all $h \in W^{1,p}(\Omega)$, $n \geqslant 1$. Hypotheses $H(f)'$(i), (ii) imply that

$$|f(z,x)| \leqslant c_7(1 + x^{p-1}) \quad \text{for a.a. } z \in \Omega, \text{ all } x \geqslant 0,$$

for some $c_7 > 0$ and

$$\text{the sequence } \left\{\tfrac{N_f(u_n)}{\|u_n\|^{p-1}}\right\}_{n\geqslant 1} \subseteq L^{p'}(\Omega) \text{ is bounded.} \tag{40}$$

So, by passing to a subsequence if necessary and using hypothesis $H(f)'$(ii), we have

$$\frac{N_f(u_n)}{\|u_n\|^{p-1}} \xrightarrow{w} \vartheta_0(z) y^{p-1} \quad \text{in } L^{p'}(\Omega) \quad \text{as } n \to +\infty, \tag{41}$$

with $\vartheta(z) \leqslant \vartheta_0(z) \leqslant \widehat{\eta}$ for a.a. $z \in \Omega$ (see Aizicovici-Papageorgiou-Staicu [1, proof of Proposition 29]). Also, since $\|y_n\| = 1$, $n \geqslant 1$, passing to a subsequence if necessary, we may assume that

$$y_n \xrightarrow{w} y \quad \text{in } W^{1,p}(\Omega) \quad \text{and} \quad y_n \longrightarrow y \quad \text{in } L^p(\Omega) \text{ and in } L^p(\partial\Omega). \tag{42}$$

If in (39) we choose $h = y_n - y \in W^{1,p}(\Omega)$, pass to the limit as $n \to +\infty$ and use (40), (42), then $\lim\limits_{n\to+\infty} \langle A(y_n), y_n - y\rangle = 0$, so

$$y_n \longrightarrow y \quad \text{in } W^{1,p}(\Omega) \tag{43}$$

and thus $\|y\| = 1$, $y \geqslant 0$ (see Proposition 2.2).

Hence, if in (39) we pass to the limit as $n \to +\infty$ and we use (41), (43) and the fact that $q < p$ and $\|u_n\| \longrightarrow +\infty$, then

$$\langle A(y), h\rangle + \int_\Omega \xi(z) y^{p-1} h\, dz + \int_{\partial\Omega} \beta(z) y^{p-1} h\, d\sigma = \int_\Omega \vartheta_0(z) y^{p-1} h\, dz \tag{44}$$

for all $h \in W^{1,p}(\Omega)$. From (39) with $h = y_n \in W^{1,p}(\Omega)$ we infer that the sequence $\{y_n\}_{n\geqslant 1} \subseteq W^{1,p}(\Omega)$ is bounded (see (43)). Therefore from Papageorgiou-Rădulescu [23], we know that there exists $c_8 > 0$ such that

$$\|y_n\|_\infty \leqslant c_8 \quad \forall n \geqslant 1.$$

Then Theorem 2 of Lieberman [17] implies that there exist $\alpha \in (0,1)$ and $c_9 > 0$ such that

$$y_n \in C^{1,\alpha}(\overline{\Omega}) \quad \text{and} \quad \|y_n\|_{C^{1,\alpha}(\overline{\Omega})} \leqslant c_9 \quad \forall n \geqslant 1. \tag{45}$$

Exploiting the compact embedding of $C^{1,\alpha}(\overline{\Omega})$ into $C^1(\overline{\Omega})$ and using (43), (45), we infer that

$$y_n \longrightarrow y \quad \text{in } C^1(\overline{\Omega}), \tag{46}$$

hence $y \in C_+ \setminus \{0\}$. From (44) we have

$$\begin{cases} -\Delta_p y(z) + \xi(z) y(z)^{p-1} = \vartheta_0(z) y(z)^{p-1} & \text{in } \Omega, \\ \frac{\partial y}{\partial n_p} + \beta(z) u^{p-1} = 0 & \text{on } \partial\Omega \end{cases} \tag{47}$$

(see Papageorgiou-Rădulescu [22]). Hence $y \in D_+$ and from Lemma 2.6 and (41), we have

$$\widetilde{\lambda}_1(\vartheta_0) \leqslant \widetilde{\lambda}_1(\vartheta) < \widetilde{\lambda}_1(\widehat{\lambda}_1) = 1. \tag{48}$$

From (47), (48) it follows that y must be nodal, a contradiction.

This proves that the sequence $\{u_n\}_{n\geqslant 1} \subseteq W^{1,p}(\Omega)$ is bounded. So, passing to a subsequence if necessary, we may assume that

$$u_n \xrightarrow{w} u^* \quad \text{in } W^{1,p}(\Omega) \quad \text{and} \quad u_n \longrightarrow u^* \quad \text{in } L^p(\Omega) \text{ and in } L^p(\partial\Omega). \tag{49}$$

In (38) we choose $h = u_n - u^* \in W^{1,p}(\Omega)$, pass to the limit as $n \to +\infty$ and use (49). Then $\lim\limits_{n\to+\infty} \langle A(u_n), u_n - u^*\rangle = 0$, so

$$u_n \longrightarrow u^* \quad \text{in } W^{1,p}(\Omega) \tag{50}$$

(see Proposition 2.2). On account of Corollary 3.5, we may assume that the sequence $\{u_n\}_{n\geqslant 1}$ is nondecreasing. Therefore $u^* \neq 0$. Passing to the limit as $n \to +\infty$ in (38) and using (50), we obtain

$$\begin{aligned} &\langle A(u^*), h\rangle + \int_\Omega \xi(z)(u^*)^{p-1} h\, dz + \int_{\partial\Omega} \beta(z)(u^*)^{p-1} h\, d\sigma \\ = {}& \lambda^* \int_\Omega (u^*)^{q-1} h\, dz + \int_\Omega f(z, u^*) h\, dz \quad \forall h \in W^{1,p}(\Omega), \end{aligned}$$

so $u^* \in S_{\lambda^*}$ and $\lambda^* \in \mathcal{L}$. $\square$

Proposition 3.10. *If the hypotheses $H(\xi)$, $H(\beta)$, H_0, $H(f)$ or $H(f)'$ hold and $\lambda \in (0, \lambda^*)$, then problem (P_λ) admits at least two positive solutions*

$$u_\lambda, \widehat{u}_\lambda \in D_+, \quad \widehat{u}_\lambda - u_\lambda \in C_+ \setminus \{0\}.$$

Proof. Again we do the proof for $H(f)'$, the proof for $H(f)$ being similar and simpler.

We have $\lambda \in \mathcal{L}$ and so we can find $u_\lambda \in S_\lambda \subseteq D_+$. Moreover, on account of Corollary 3.6, we can have

$$u^* - u_\lambda \in \operatorname{int} C_+, \tag{51}$$

with $u^* \in S_{\lambda^*}$ (recall $\lambda^* \in \mathcal{L}$; see Proposition 3.9). We consider the Carathéodory function γ_λ defined by

$$\gamma_\lambda(z, x) = \begin{cases} \lambda u_\lambda(z)^{q-1} + f(z, u_\lambda(z)) & \text{if} \quad x \leqslant u_\lambda(z), \\ \lambda x^{q-1} + f(z, x) & \text{if} \quad u_\lambda(z) < x. \end{cases} \tag{52}$$

We set
$$\Gamma_\lambda(z, x) = \int_0^x \gamma_\lambda(z, s)\, ds$$

and consider the C^1-functional $\varphi_\lambda^*\colon W^{1,p}(\Omega) \longrightarrow \mathbb{R}$ defined by

$$\varphi_\lambda^*(u) = \frac{1}{p}\mu(u) - \int_\Omega \Gamma_\lambda(z, u)\, dz \quad \forall u \in W^{1,p}(\Omega).$$

Let
$$[u_\lambda) = \{y \in W^{1,p}(\Omega) : \ u_\lambda(z) \leqslant y(z) \text{ for a.a. } z \in \Omega\}$$

Claim 1. $K_{\varphi_\lambda^*} \subseteq [u_\lambda) \cap D_+$.

Let $u \in K_{\varphi_\lambda^*}$. Then

$$\langle A(u), h\rangle + \int_\Omega \xi(z)|u|^{p-2}uh\, dz + \int_{\partial\Omega} \beta(z)|u|^{p-2}uh\, d\sigma = \int_\Omega \gamma_\lambda(z, u)h\, dz \tag{53}$$

for all $h \in W^{1,p}(\Omega)$. In (53) we choose $h = (u_\lambda - u)^+ \in W^{1,p}(\Omega)$. We have

$$\begin{aligned} &\langle A(u), (u_\lambda - u)^+\rangle + \int_\Omega \xi(z)|u|^{p-2}u(u_\lambda - u)^+ dz + \int_{\partial\Omega} \beta(z)|u|^{p-2}u(u_\lambda - u)^+ d\sigma \\ &= \int_\Omega (\lambda u_\lambda^{q-1} + f(z, u_\lambda))(u_\lambda - u)^+\, dz \\ &= \langle A(u_\lambda), (u_\lambda - u)^+\rangle + \int_\Omega \xi(z)u_\lambda^{p-1}(u_\lambda - u)^+ dz + \int_{\partial\Omega} \beta(z)u_\lambda^{p-1}(u_\lambda - u)^+ d\sigma \end{aligned}$$

(see (52) and use the fact that $u_\lambda \in S_\lambda$), so $u_\lambda \leqslant u$.

Also, as before, the nonlinear regularity theory implies that $u \in D_+$. Therefore $K_{\varphi_\lambda^*} \subseteq [u_\lambda) \cap D_+$ and this proves Claim 1.

Evidently $u_\lambda \in K_{\varphi_\lambda^*}$ (see (52)). So, we may assume that

$$K_{\varphi_\lambda^*} \cap [u_\lambda, u^*] = \{u_\lambda\}. \tag{54}$$

Otherwise, we already have a second positive solution of (P_λ) and so we are done (see Claim 1 and (52)).

We introduce the Carathéodory function $\widehat{\gamma}_\lambda$ defined by

$$\widehat{\gamma}_\lambda(z,x) = \begin{cases} \gamma_\lambda(z,x) & \text{if} \quad x \leqslant u^*(z), \\ \gamma_\lambda(z,u^*(z)) & \text{if} \quad u^*(z) < x. \end{cases} \tag{55}$$

We set
$$\widehat{\Gamma}_\lambda(z,x) = \int_0^x \widehat{\gamma}_\lambda(z,s)\,ds$$

and consider the C^1-functional $\widehat{\varphi}_\lambda^*\colon W^{1,p}(\Omega) \longrightarrow \mathbb{R}$ defined by

$$\widehat{\varphi}_\lambda^*(u) = \frac{1}{p}\mu(u) - \int_\Omega \widehat{\Gamma}_\lambda(z,u)\,dz \quad \forall u \in W^{1,p}(\Omega).$$

From (10) and (55) we see that $\widehat{\varphi}$ is coercive. Also, it is sequentially weakly lower semicontinuous. So, we can find $\widetilde{u}_\lambda \in W^{1,p}(\Omega)$ such that

$$\widehat{\varphi}_\lambda^*(\widetilde{u}_\lambda) = \inf_{u \in W^{1,p}(\Omega)} \widehat{\varphi}_\lambda^*(u),$$

so $(\widehat{\varphi}_\lambda^*)'(\widetilde{u}_\lambda) = 0$, and thus

$$\begin{aligned} &\langle A(\widetilde{u}_\lambda), h\rangle + \int_\Omega \xi(z)|\widetilde{u}_\lambda|^{p-2}\widetilde{u}_\lambda h\,dz + \int_{\partial\Omega} \beta(z)|\widetilde{u}_\lambda|^{p-2}\widetilde{u}_\lambda h\,d\sigma \\ = &\int_\Omega \widehat{\gamma}_\lambda(z,\widetilde{u}_\lambda)h\,dz \quad \forall h \in W^{1,p}(\Omega). \end{aligned} \tag{56}$$

As before, if in (56) first we choose $h = (u_\lambda - \widetilde{u}_\lambda)^+ \in W^{1,p}(\Omega)$ and then $h = (\widetilde{u}_\lambda - u^*)^+ \in W^{1,p}(\Omega)$, using (52) and (55) we show that

$$\widetilde{u}_\lambda \in [u_\lambda, u^*]. \tag{57}$$

From (52), (55), (57) it follows that $\widetilde{u}_\lambda \in K_{\varphi_\lambda^*}$. This combined with (53), (56) implies that

$$\widetilde{u}_\lambda = u_\lambda \in D_+. \tag{58}$$

From (52) and (55) it is clear that

$$\varphi_\lambda^*\big|_{[0,u^*]} = \widehat{\varphi}_\lambda^*\big|_{[0,u^*]}. \tag{59}$$

Then from (51), (58), (59) it follows that u_λ is a local $C^1(\overline{\Omega})$-minimizer of φ_λ^*, and hence, see Proposition 2.3,

$$u_\lambda \text{ is a local } W^{1,p}(\Omega)\text{-minimizer of } \varphi_\lambda^*. \tag{60}$$

We assume that $K_{\varphi_\lambda^*}$ is finite. Otherwise on account of Claim 1 and (52) we already have an infinity of positive solutions for problem (P_λ) and so we are done. Then (60) implies that we can find $\varrho \in (0,1)$ small such that

$$\varphi_\lambda^*(u_\lambda) < \inf_{\|u-u_\lambda\|=\varrho} \varphi_\lambda^*(u) = m_\lambda^* \tag{61}$$

(see Aizicovici-Papageorgiou-Staicu [1, proof of Proposition 29]).

The hypotheses $H(f)'$(i), (ii) and (52) imply that given $\varepsilon > 0$, we can find $c_{10} = c_{10} > 0$ such that

$$\Gamma_\lambda(z,x) \geqslant \frac{1}{p}(\vartheta(z)-\varepsilon)x^p - c_{10} \quad \text{for a.a. } z \in \Omega, \text{ all } x \geqslant 0. \tag{62}$$

Then for $t > 0$ we have

$$\begin{aligned} \varphi_\lambda^*(t\widehat{u}_1) &= \frac{t^p}{p}\mu(\widehat{u}_1) - \int_\Omega \Gamma_\lambda(z, t\widehat{u}_1)\,dz \leqslant \frac{t^p}{p}(\mu(\widehat{u}_1) - \int_\Omega \vartheta(z)\widehat{u}_1^p + \varepsilon) + c_{11} \\ &= \frac{t^p}{p}\Big(\int_\Omega (\widehat{\lambda}_1 - \vartheta(z))\widehat{u}_1^p + \varepsilon\Big) + c_{11} \end{aligned} \tag{63}$$

for some $c_{11} > 0$ (see (62)). Note that

$$\tau_0 = \int_\Omega (\vartheta(z) - \widehat{\lambda}_1)\widehat{u}_1^p\,dz > 0$$

(recall $\widehat{u}_1 \in D_+$ and see hypothesis $H(f)'$(ii)). So, choosing $\varepsilon \in (0, \tau_0)$, from (63) we see that

$$\varphi_\lambda^*(t\widehat{u}_1) \longrightarrow -\infty \quad \text{as } t \to +\infty. \tag{64}$$

Claim 2. φ_λ^* satisfies the Cerami condition (CC).

Let $\{u_n\}_{n\geqslant 1} \subseteq W^{1,p}(\Omega)$ be a sequence such that $\{\varphi_\lambda^*(u_n)\}_{n\geqslant 1} \subseteq \mathbb{R}$ is bounded and

$$(1 + \|u_n\|)(\varphi_\lambda^*)'(u_n) \longrightarrow 0 \quad \text{in } W^{1,p}(\Omega)^*.$$

We have for all $h \in W^{1,p}(\Omega)$ with $\varepsilon_n \to 0^+$

$$\begin{aligned} &\Big|\langle A(u_n), h\rangle + \int_\Omega \xi(z)|u_n|^{p-2}u_n\,dz \\ &+ \int_{\partial\Omega} \beta(z)|u_n|^{p-2}u_n h\,d\sigma - \int_\Omega \gamma_\lambda(z,u_n)h\,dz\Big| \leqslant \frac{\varepsilon_n\|h\|}{1+\|u_n\|}. \end{aligned} \tag{65}$$

Choosing $h = -u_n^- \in W^{1,p}(\Omega)$ and using (10) and (52), we infer that

$$\text{the sequence } \{u_n^-\}_{n\geqslant 1} \subseteq W^{1,p}(\Omega) \text{ is bounded.} \tag{66}$$

Suppose that $\|u_n^+\| \longrightarrow +\infty$ and set $y_n = \frac{u_n^+}{\|u_n^+\|}$, $n \geqslant 1$. Then $\|y_n\| = 1$, $y_n \geqslant 0$ for all $n \geqslant 1$ and passing to a subsequence if necessary, we may assume that

$$y_n \xrightarrow{w} y \quad \text{in } W^{1,p}(\Omega) \quad \text{and} \quad y_n \longrightarrow y \quad \text{in } L^p(\Omega) \text{ and in } L^p(\partial\Omega), \tag{67}$$

with $y \geqslant 0$. From (65), (66) and (52), we obtain

$$\langle A(y_n), h\rangle + \int_\Omega \xi(z) y_n^{p-1} h\,dz + \int_{\partial\Omega} \beta(z) y_n^{p-1} h\,d\sigma - \int_\Omega \frac{N_{\gamma_\lambda}(u_n^+)}{\|u_n^+\|^{p-1}} h\,dz \leqslant \varepsilon_n' \|h\| \quad \text{for all } h \in W^{1,p}(\Omega), \tag{68}$$

with $\varepsilon_n' \to 0^+$. Note that

$$|\gamma_\lambda(z,x)| \leqslant c_{12}(1 + x^{p-1}) \quad \text{for a.a. } z \in \Omega, \text{ all } x \geqslant 0,$$

for some $c_{12} > 0$, so

$$\text{the sequence } \left\{\tfrac{N_{\gamma_\lambda}(u_n^+)}{\|u_n^+\|^{p-1}}\right\}_{n\geqslant 1} \subseteq L^{p'}(\Omega) \text{ is bounded.} \tag{69}$$

So, if in (68) we choose $h = y_n - y \in W^{1,p}(\Omega)$ and pass to the limit as $n \to +\infty$, then using (67) and (69) we obtain $\lim\limits_{n\to+\infty} \langle A(y_n), y_n - y\rangle = 0$, so

$$y_n \longrightarrow y \quad \text{in } W^{1,p}(\Omega) \tag{70}$$

(see Proposition 2.2), thus $\|y\| = 1$, $y \geqslant 0$. Because of (69), by passing to a subsequence if necessary and using hypothesis $H(f)'$(ii) and (56), we obtain

$$\frac{N_{\gamma_\lambda}(u_n^+)}{\|u_n^+\|^{p-1}} \xrightarrow{w} \vartheta_0(z) y^{p-1} \quad \text{in } L^{p'}(\Omega), \tag{71}$$

with $\vartheta(z) \leqslant \vartheta_0(z) \leqslant \widehat{\eta}$ for a.a. $z \in \Omega$ (see Aizicovici-Papageorgiou-Staicu [1, proof of Proposition 16]). We pass to the limit as $n \to +\infty$ in (68) and use (70), (71).

Then
$$\langle A(y), h\rangle + \int_\Omega \xi(z) y^{p-1} h\,dz + \int_{\partial\Omega} \beta(z) y^{p-1} h\,d\sigma = \int_\Omega \vartheta_0(z) y^{p-1} h\,dz \quad \forall h \in W^{1,p}(\Omega),$$

so
$$\begin{cases} -\Delta_p y(z) + \xi(z) y(z)^{p-1} = \vartheta_0(z) y(z)^{p-1} & \text{a.e. in } \Omega, \\ \dfrac{\partial y}{\partial n_p} + \beta(z) y^{p-1} = 0 \quad \text{on } \partial\Omega, \ \|y\| = 1. \end{cases} \tag{72}$$

Lemma 2.6 and (71) imply that

$$\widetilde{\lambda}_1(\vartheta_0) \leqslant \widetilde{\lambda}_1(\vartheta) < \widetilde{\lambda}_1(\widehat{\lambda}_1) = 1,$$

so y must be nodal (see (72)), a contradiction to (70). Therefore the sequence $\{u_n^+\}_{n\geqslant 1} \subseteq W^{1,p}(\Omega)$ is bounded, thus the sequence $\{u_n\}_{n\geqslant 1} \subseteq W^{1,p}(\Omega)$ is bounded (see (66)).

So, passing to a subsequence if necessary, we may assume that

$$u_n \xrightarrow{w} u \quad \text{in } W^{1,p}(\Omega) \quad \text{and} \quad u_n \longrightarrow u \quad \text{in } L^p(\Omega) \text{ and in } L^p(\partial\Omega). \tag{73}$$

Choosing $h = u_n - u \in W^{1,p}(\Omega)$ in (65), passing to the limit as $n \to +\infty$ and using (73), then $\lim\limits_{n\to+\infty} \langle A(u_n), u_n - u\rangle = 0$, and so

$$u_n \longrightarrow u \quad \text{in } W^{1,p}(\Omega)$$

(see Proposition 2.2). This proves Claim 2.

Using (61), (64) and Claim 2, we see that we can apply Theorem 2.1 (the mountain pass theorem) and find $\widehat{u}_\lambda \in W^{1,p}(\Omega)$ such that

$$\widehat{u}_\lambda \in K_{\varphi^*} \subseteq [u_\lambda) \cap D_+$$

(see Claim 1), so $\varphi^*_\lambda(u_\lambda) < m^*_\lambda \leqslant \varphi^*_\lambda(\widehat{u}_\lambda)$ (see (61)) and thus $\widehat{u}_\lambda \in D_+$ solves (P_λ) (see (52)), $\widehat{u}_\lambda \neq u_\lambda$ and $u_\lambda \leqslant \widehat{u}_\lambda$. □

Summarizing the situation for problem (P_λ), we can state the following bifurcation-type theorem describing the dependence on the parameter $\lambda > 0$ of the set of positive solutions.

Theorem 3.11. *If hypotheses $H(\xi)$, $H(\beta)$, H_0, $H(f)$ or $H(f)'$ hold, then there exists $\lambda^* \in (0, +\infty)$ such that*

(a) *for all $\lambda \in (0, \lambda^*)$ problem (P_λ) has at least two positive solutions*

$$u_\lambda, \widehat{u}_\lambda \in D_+, \quad u_\lambda \leqslant \widehat{u}_\lambda, \quad u_\lambda \neq \widehat{u}_\lambda;$$

(b) *for $\lambda = \lambda^*$ problem (P_λ) has at least one positive solution $u^* \in D_+$;*

(c) *for $\lambda > \lambda^*$ problem (P_λ) has no positive solution.*

4. Minimal positive solutions

In this section we show that for every $\lambda \in \mathcal{L} = (0, \lambda^*]$, S_λ has a smallest element $\widetilde{u}^*_\lambda \in D_+$ and we establish the monotonicity and continuity properties of the map $\lambda \longmapsto \widetilde{u}^*_\lambda$.

Proposition 4.1. *If the hypotheses $H(\xi)$, $H(\beta)$, H_0, $H(f)$ or $H(f)'$ hold and $\lambda \in \mathcal{L}$, then S_λ admits a smallest positive solution $\widetilde{u}^*_\lambda \in D_+$.*

Proof. From Proposition 7 of Papageorgiou-Rădulescu-Repovš [24], we know that S_λ is downward directed (that is, if $u_1, u_2 \in S_\lambda$, then there exists $u \in S_\lambda$ such that $u \leqslant u_1$, $u \leqslant u_2$). So, invoking Lemma 3.10 of Hu-Papageorgiou [16, p. 178], we can find a decreasing sequence $\{u_n\}_{n\geqslant 1} \subseteq S_\lambda$ such that $\inf S_\lambda = \inf_{n\geqslant 1} u_n$. Then

$$\begin{aligned} &\langle A(u_n), h\rangle + \int_\Omega \xi(z) u_n^{p-1} h\,dz + \int_{\partial\Omega} \beta(z) u_n^{p-1} h\,d\sigma \\ &= \lambda \int_\Omega u_n^{q-1} h\,dz + \int_\Omega f(z, u_n) h\,dz \quad \forall h \in W^{1,p}(\Omega),\ n \geqslant 1. \end{aligned} \tag{74}$$

From (74) and since $0 \leqslant u_n \leqslant u_1$ for all $n \geqslant 1$, we infer that the sequence $\{u_n\}_{n\geqslant 1} \subseteq W^{1,p}(\Omega)$ is bounded.

So, passing to a subsequence if necessary, we may assume that

$$u_n \xrightarrow{w} \widetilde{u}_\lambda^* \quad \text{in } W^{1,p}(\Omega) \quad \text{and} \quad u_n \longrightarrow \widetilde{u}_\lambda^* \quad \text{in } L^p(\Omega) \text{ and in } L^p(\partial\Omega). \tag{75}$$

If in (74) we choose $h = u_n - \widetilde{u}_\lambda^* \in W^{1,p}(\Omega)$, pass to the limit as $n \to +\infty$ and use (75), then $\lim\limits_{n\to+\infty} \langle A(u_n), u_n - \widetilde{u}_\lambda^*\rangle = 0$, so

$$u_n \longrightarrow \widetilde{u}_\lambda^* \quad \text{in } W^{1,p}(\Omega) \tag{76}$$

(see Prop. 2.2). If in (74) we pass to the limit as $n \to +\infty$ and use (76), then

$$\begin{aligned} &\langle A(\widetilde{u}_\lambda^*), h\rangle + \int_\Omega \xi(z)(\widetilde{u}_\lambda^*)^{p-1} h\,dz + \int_{\partial\Omega} \beta(z)(\widetilde{u}_\lambda^*)^{p-1} h\,d\sigma \\ = \ &\lambda \int_\Omega (\widetilde{u}_\lambda^*)^{q-1} h\,dz + \int_\Omega f(z, \widetilde{u}_\lambda^*) h\,dz \end{aligned} \tag{77}$$

for all $h \in W^{1,p}(\Omega)$. If we show that $\widetilde{u}_\lambda^* \neq 0$, then from (77) we have $\widetilde{u}_\lambda^* \in S_\lambda \subseteq D_+$.

The hypotheses $H(f)$(i), (ii), (iv) (or $H(f)'$(i), (ii), (iv)) imply that

$$f(z,x) \geqslant -c_{13} x^{r-1} \quad \text{for a.a. } z \in \Omega, \text{ all } x \geqslant 0, \tag{78}$$

with $c_{13} > 0$, $r > p$. We consider the following auxiliary nonlinear parametric Robin problem:

$$(Au_\lambda)' \qquad \begin{cases} -\Delta_p u(z) + \xi(z) u(z)^{p-1} = \lambda u(z)^{q-1} - c_{13} u(z)^{r-1} & \text{in } \Omega, \\ \frac{\partial u}{\partial n_p} + \beta(z) u^{p-1} = 0 \quad \text{on } \partial\Omega, \\ u > 0 \quad \text{in } \Omega, \end{cases}$$

with $\lambda > 0$, $q < p < r$. From Proposition 3.3 of Gasiński-Papageorgiou [13], we know that $(Au_\lambda)'$ has a unique solution $u_\lambda^* \in D_+$.

Claim. $u_\lambda^* \leqslant u$ for all $u \in S_\lambda$.

Let $u \in S_\lambda$ and consider the Carathéodory function

$$w_\lambda(z,x) = \begin{cases} 0 & \text{if } \ x < 0, \\ \lambda x^{q-1} - c_{13} x^{r-1} & \text{if } \ 0 \leqslant x \leqslant u(z), \\ \lambda u(z)^{q-1} - c_{13} u(z)^{r-1} & \text{if } \ u(z) < x. \end{cases} \tag{79}$$

We set

$$W_\lambda(z,x) = \int_0^x w_\lambda(z,s)\,ds$$

and consider the C^1-functional $\widetilde{\chi}_\lambda \colon W^{1,p}(\Omega) \longrightarrow \mathbb{R}$ defined by

$$\widetilde{\chi}_\lambda(u) = \frac{1}{p}\mu(u) - \int_\Omega W_\lambda(z,u)\,dz \quad \forall u \in W^{1,p}(\Omega).$$

As before $\widetilde{\chi}_\lambda$ is coercive (see (79)) and sequentially weakly lower semicontinuous. So, we can find $\widehat{u}_\lambda^* \in W^{1,p}(\Omega)$ such that

$$\widetilde{\chi}_\lambda(\widehat{u}_\lambda^*) = \inf_{u \in W^{1,p}(\Omega)} \widetilde{\chi}_\lambda(u). \tag{80}$$

Since $q < p < r$, for $\vartheta \in (0,1)$ small $\widetilde{\chi}_\lambda(\vartheta) < 0$, so $\widetilde{\chi}_\lambda(\widehat{u}_\lambda^*) < 0 = \widetilde{\chi}_\lambda(0)$ (see (80)) and thus $\widehat{u}_\lambda^* \neq 0$. From (79), we have $\widetilde{\chi}_\lambda'(\widehat{u}_\lambda^*) = 0$, so

$$\begin{aligned} &\langle A(\widehat{u}_\lambda^*), h\rangle + \int_\Omega \xi(z)|\widehat{u}_\lambda^*|^{p-2}\widehat{u}_\lambda^* h\,dz + \int_{\partial\Omega} \beta(z)|\widehat{u}_\lambda^*|^{p-2}\widehat{u}_\lambda^* h\,d\sigma \\ = &\int_\Omega w_\lambda(z,\widehat{u}_\lambda^*)h\,dz \quad \forall h \in W^{1,p}(\Omega). \end{aligned} \tag{81}$$

In (81) we choose $h = -(\widehat{u}_\lambda^*)^- \in W^{1,p}(\Omega)$. Then $\mu((\widehat{u}_\lambda^*)^-) = 0$ (see (79)), so

$$c_3\|(\widehat{u}_\lambda^*)^-\|^p \leqslant 0$$

(see (10)) and thus $\widehat{u}_\lambda^* \geqslant 0$, $\widehat{u}_\lambda^* \neq 0$.

Next in (81) we choose $h = (\widehat{u}_\lambda^* - u)^+ \in W^{1,p}(\Omega)$. Then

$$\begin{aligned} &\langle A(\widehat{u}_\lambda^*), (\widehat{u}_\lambda^* - u)^+\rangle + \int_\Omega \xi(z)(\widehat{u}_\lambda^*)^{p-1}(\widehat{u}_\lambda^* - u)^+ dz + \int_{\partial\Omega} \beta(z)(\widehat{u}_\lambda^*)^{p-1}(\widehat{u}_\lambda^* - u)^+ d\sigma \\ &= \int_\Omega (\lambda u^{q-1} - c_{13}u^{r-1})(\widehat{u}_\lambda^* - u)^+\,dz \leqslant \int_\Omega (\lambda u^{q-1} + f(z,u))(\widehat{u}_\lambda^* - u)^+\,dz \\ &\leqslant \langle A(u), (\widehat{u}_\lambda^* - u)^+\rangle + \int_\Omega \xi(z)u^{p-1}(\widehat{u}_\lambda^* - u)^+\,dz + \int_{\partial\Omega} \beta(z)u^{p-1}(\widehat{u}_\lambda^* - u)^+\,d\sigma \end{aligned}$$

(see (79), (78) and use the fact that $u \in S_\lambda$), so $\widehat{u}_\lambda^* \leqslant u$.

So, we have proved that
$$\widehat{u}_\lambda^* \in [0,u] \setminus \{0\}, \tag{82}$$

hence $\widehat{u}_\lambda^*$ solves $(Au_\lambda)'$ (see (79)), which leads to $\widehat{u}_\lambda^* = u_\lambda^*$.

Therefore $u_\lambda^* \leqslant u$ for all $u \in S_\lambda$ (see (82)). This proves the Claim.

As a consequence of the Claim, we have

$$u_\lambda^* \leqslant u_n \quad \forall n \geqslant 1, \quad \text{and so} \quad u_\lambda^* \leqslant \widetilde{u}_\lambda^*$$

(see (76)) and thus $\widetilde{u}_\lambda^* \in S_\lambda \subseteq D_+$ and $\widetilde{u}_\lambda^* = \inf S_\lambda$. □

Next we establish the monotonicity and continuity properties of the map $\lambda \longmapsto \widetilde{u}_\lambda^*$.

Proposition 4.2. *If the hypotheses $H(\xi)$, $H(\beta)$, H_0, $H(f)$ or $H(f)'$ hold, then the map $\lambda \longmapsto \widetilde{u}_\lambda^*$ from $\mathcal{L} = (0, \lambda^*]$ into $C^1(\overline{\Omega})$ is strictly increasing (that is, if $\eta < \lambda$, then $\widetilde{u}_\lambda^* - \widetilde{u}_\eta^* \in \operatorname{int} C_+$) and left continuous.*

Proof. Let $0 < \eta < \lambda \leqslant \lambda^*$ and let $\widetilde{u}_\lambda^* \in D_+$ be the minimal solution of (P_λ) (see Proposition 4.1). According to Corollary 3.6, we can find $u_\eta \in S_\eta \subseteq D_+$ such that

$$\widetilde{u}_\lambda^* - u_\eta \in \operatorname{int} C_+, \quad \text{and so} \quad \widetilde{u}_\lambda^* - \widetilde{u}_\eta^* \in \operatorname{int} C_+.$$

Hence $\lambda \longmapsto \widetilde{u}_\lambda^*$ is strictly increasing.

Next let $\{\lambda_n, \lambda\}_{n\geqslant 1} \subseteq \mathcal{L}$ and assume that $\lambda_n \nearrow \lambda$. We have that

$$\text{the sequence } \{\widetilde{u}_n^* = \widetilde{u}_{\lambda_n}^*\}_{n\geqslant 1} \subseteq W^{1,p}(\Omega) \text{ is bounded.}$$

From this and Theorem 2 of Lieberman [17], we infer that

$$\text{the sequence } \{\widetilde{u}_n^*\}_{n\geqslant 1} \subseteq C^1(\overline{\Omega}) \text{ is relatively compact.}$$

So, passing to a subsequence if necessary, we may assume that

$$\widetilde{u}_n^* \longrightarrow \overline{u}_\lambda \quad \text{in } C^1(\overline{\Omega}) \quad \text{as } n \to +\infty. \tag{83}$$

Suppose that $\overline{u}_\lambda \neq \widetilde{u}_\lambda^*$. Then we can find $z_0 \in \Omega$ such that

$$\widetilde{u}_\lambda^*(z_0) < \overline{u}_\lambda(z_0), \quad \text{and so} \quad \widetilde{u}_\lambda^*(z_0) < \widetilde{u}_n^*(z_0) \quad \forall n \geqslant n_0$$

(see (83)), which is a contradiction to the strict monotonicity of $\lambda \longmapsto \widetilde{u}_\lambda^*$. Therefore $\overline{u}_\lambda = \widetilde{u}_\lambda^*$ and so $\widetilde{u}_n^* \longrightarrow \widetilde{u}_\lambda^*$ in $C^1(\overline{\Omega})$. This proves the left continuity of $\lambda \longmapsto \widetilde{u}_\lambda^*$. $\square$

We conclude by mentioning some open problems.

(1) Is it possible to have such a bifurcation-type theorem for the resonant case (that is, if $\vartheta(z) = \widehat{\lambda}_1$ for a.a. $z \in \Omega$). It seems that without additional conditions on $f(z, \cdot)$ this is not possible.

(2) Is it possible to replace the p-Laplacian by a more general not necessarily homogeneous differential operator as the one in Gasiński-O'Regan-Papageorgiou [7]? In our proofs here, in several occasions, we have exploited the $(p-1)$-homogeneity of the p-Laplacian. So, if we want to treat nonhomogeneous equations we need to modify our approach.

(3) Is it possible to extend the result to problems with an indefinite potential ξ? In this direction we mention the work of Gasiński-Papageorgiou [13].

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