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Asymptotic Behavior of Solutions to a Second-Order Gradient Equation of Pseudo-Convex Type

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Dedicated to Professor Viorel Barbu on the occasion of his 75th birthday.

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Consider in a real Hilbert space H the second order gradient equation

$$u''(t) = \nabla\phi(u(t)), \quad t \geq 0.$$

We state and prove several results on the weak or strong convergence of bounded solutions of this equation to minimizers of ϕ, where $\phi\colon H \to \mathbb{R}$ is a continuously differentiable, pseudo-convex function with $\operatorname{Argmin}\phi \neq \varnothing$. Our results extend previous results in the literature that are related to the case when ϕ is convex.

Keywords: Convex function, pseudo-convex function, minimum point, critical point, second order gradient system, asymptotic behavior.

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1. Introduction

Let H be a real Hilbert space with inner product $(\cdot,\cdot)$ and norm $|\cdot|$. We denote weak convergence in H by $\rightharpoonup$ and strong convergence by $\to$.

A function $f\colon H \to \mathbb{R}$ is called

(i) *convex* iff
$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y), \quad \forall x, y \in H$ and $\forall\ 0 \leq \lambda \leq 1$;

(ii) *α-strongly convex for some* $\alpha > 0$ iff, $\forall x, y \in H$ and $\forall\ 0 \leq \lambda \leq 1$,
$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y) - \alpha\lambda(1-\lambda)|x-y|^2$;

(iii) *quasi-convex* iff, $\forall x, y \in X$ and $\forall\ 0 \leq \lambda \leq 1$,
$f(\lambda x + (1-\lambda)y) \leq \max\{f(x), f(y)\}$, or, equivalently, for all $r \in \mathbb{R}$, the sublevel set $L_r^f := \{x \in H : f(x) \leq r\}$ is a convex subset of H;

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(iv) *α-strongly quasi-convex for some* $\alpha > 0$ iff $\forall x, y \in H$, and $\forall\ 0 < \lambda < 1$, $f(\lambda x + (1-\lambda)y) \le \max\{f(x), f(y)\} - \alpha\lambda(1-\lambda)|x-y|^2$.

(v) *pseudo-convex* iff $f(y) > f(x)$ implies that there exist $\beta(x,y) > 0$ and $0 < \delta(x,y) \le 1$ such that $f(y) - f(tx + (1-t)y) \ge t\beta(x,y)$, $\forall t \in (0, \delta(x,y))$.

A differentiable function $\phi\colon H \to \mathbb{R}$ is

(vi) *quasi-convex* iff $\phi(x) \ge \phi(y) \Rightarrow (\nabla\phi(x), y - x) \le 0$;

(vii) *pseudo-convex* iff $\phi(x) > \phi(y) \Rightarrow (\nabla\phi(x), y - x) < 0$.

Obviously convexity implies pseudo-convexity and pseudo-convexity implies quasi-convexity. We refer the reader to the interesting book by Cambini and Martein [9] for information on convexity and its extensions, including their definitions, properties and examples.

Given any function $\phi\colon H \to]-\infty, +\infty]$ (not necessarily convex), its *subdifferential* is defined as a multivalued operator $\partial\phi\colon D(\partial\phi) \subset H \to H$,

$$\partial\phi(x) = \{w \in H \mid \phi(x) - \phi(y) \le (w, x - y), \quad \forall y \in H\}.$$

ϕ is called *proper* if its effective domain $D(\phi) = \{x \in H \mid \phi(x) < +\infty\}$ is a nonempty set. Obviously $D(\partial\phi) \subset D(\phi)$. It is well known that if ϕ is convex and Gâteaux differentiable at a point x, then $\partial\phi(x) = \{\nabla\phi(x)\}$. We refer the reader to Brezis [5] or Moroşanu [19] for information on subdifferentials of convex functions in Hilbert spaces.

For a proper, convex and lower semicontinuous (lsc) function $\phi\colon H \to]-\infty, +\infty]$ with $\operatorname{Argmin}\phi \ne \varnothing$, Bruck [8] proved that any solution of

$$\begin{cases} -u'(t) \in \partial\phi(u(t)), \quad t > 0\,, \\ u(0) = x \in \overline{D(\phi)} \end{cases} \tag{1}$$

converges weakly as $t \to \infty$ to a minimum point of ϕ (see also Chapter 8, Remark 8.3.10 of [13]). He also established strong convergence of solutions when ϕ is even. Equation (1) which is called a steepest descent evolution equation gives a dynamical approach to the approximation of minimizers of convex functions. In [12] Bruck's weak convergence result was extended to a quasi-convex function $\phi\colon H \to \mathbb{R}$ which possesses at least one minimum point.

Recall also that Barbu [2, 3, 4] (see also [1]) proved the existence of solutions to the following second order evolution equation

$$\begin{cases} u''(t) \in \partial\phi(u(t)), \quad t > 0\,, \\ u(0) = x, \quad \sup_{t \ge 0} |u(t)| < +\infty\,, \end{cases} \tag{2}$$

where $\phi\colon H \to]-\infty, +\infty]$ is a proper, convex, lsc function. In fact Barbu proved the existence and uniqueness of solutions in the more general case when (2) is associated with a maximal monotone operator A (instead of $\partial\phi$). Moreover, Poffald and Reich [21, 22] established the existence and uniqueness of solutions to problem (1.2) in a Banach space for an m-accretive operator A (instead of $\partial\phi$). It was shown in [18] that any solution $u(t)$ of (2) converges weakly to a minimum point of ϕ provided $\operatorname{Argmin}\phi \neq \varnothing$ (see also [19, p. 83]) and for more investigations on the weak and strong convergence of solutions to (2) the reader is refered to [10, 16, 17]). Therefore (2) may be useful for optimization purposes. Moreover the first author [14] showed that $\phi(u(t))$ has a rate of convergence to the minimum value of ϕ which is better than the corresponding rate in the case of the first order gradient equation (steepest descent equation). Also the first author and Shokri [15] compared by numerical methods the rate of convergence and stability of solutions to equations (1) and (2) and showed the advantage of equation (2) with respect to equation (1).

It is worth pointing out that the asymptotic behavior of solutions to (2) has found some applications in economics (see for example [6, 7]).

In this paper we consider the following second order gradient system

$$\begin{cases} u''(t) = \nabla\phi(u(t)), \quad t > 0\,, \\ u(0) = x, \qquad \sup_{t\geq 0}|u(t)| < +\infty\,, \end{cases} \tag{3}$$

where $\phi\colon H \to]-\infty, +\infty]$ is a C^1, pseudo-convex function which possesses at least one minimum point. The existence of solutions to (3) is an open problem. However there are particular cases when (3) has solutions. Such a case is presented in what follows.

Example 1.1. Let $\phi\colon H \to \mathbb{R}$ be defined by

$$\phi(v) = \frac{|v|^2}{1+|v|^2}\,, \quad v \in H,$$

which is a C^∞, pseudo-convex function, and

$$\nabla\phi(v) = \frac{2}{\left(1+|v|^2\right)^2}\,v\,, \quad v \in H\,.$$

Consider in H the Cauchy problem

$$u'(t) = -\frac{\sqrt{2}}{\sqrt{1+|u(t)|^2}}\,u(t)\,, \quad t \geq 0\,, \tag{4}$$

$$u(0) = u_0\,. \tag{5}$$

It is easily seen that the operator $P\colon H \to H$ defined by

$$Pv = -\frac{\sqrt{2}}{\sqrt{1+|v|^2}}\,v\,,$$

is Lipschitz. Then for all $u_0 \in H$ there exists a unique solution of (4), (5), $u \in C^\infty([0,\infty); H)$ (by Banach's Fixed Point Theorem in $X = C([0,T]; H)$, $T > 0$, equipped with a Bielecki norm). Moreover,

$$(u'(t), u(t)) = -\sqrt{2}\,\frac{|u(t)|^2}{\sqrt{1+|u(t)|^2}} \le 0\,, \quad t \ge 0\,,$$

so
$$\frac{d}{dt}|u(t)|^2 \le 0, \quad t \ge 0 \;\Rightarrow\; |u(t)| \le |u_0|\,, \quad t \ge 0\,.$$

Now, by (4) we have
$$\sqrt{1+|u|^2}u' + \sqrt{2}u = 0\,, \quad t \ge 0\,. \tag{6}$$

If we differentiate (6) and then substitute u' given by (4) in the resulting equation we infer that u satisfies $(3)_1$. Therefore u is a solution of (3) with $x = u_0$. □

2. Weak convergence

In this section we investigate the weak convergence of solutions to (3) to minimum points of ϕ. Let us first recall the following elementary lemma.

Lemma 2.1. [14] *Let $f, g\colon \mathbb{R}^+ \to \mathbb{R}^+$, $\mathbb{R}^+ = [0,\infty)$, be such that f is decreasing, $\lim_{t\to+\infty} f(t) = 0$, and $\int_0^{+\infty} f(s)g(s)ds < +\infty$. Then*

$$\lim_{t\to+\infty} f(t)(\int_0^t g(s)ds) = 0.$$

Lemma 2.2. *Assume that u is a solution of $(3)_1$. If $\operatorname{Argmin}\phi \neq \varnothing$, then $t \mapsto |u(t)-p|^2$ is a convex function on $\mathbb{R}^+$ for all $p \in \operatorname{Argmin}\phi$. If in addition u is bounded on $\mathbb{R}^+$, then $t \mapsto |u(t)-p|$ is also convex and decreasing.*

Proof. Let $p \in \operatorname{Argmin}\phi$. Since $\phi(u(t)) \ge \phi(p)$ it follows by the pseudo-convexity of ϕ that

$$(\nabla\phi(u(t)), u(t)-p) \ge 0 \quad \forall t \ge 0\,.$$

Thus we obtain from equation $(3)_1$

$$\frac{d^2}{dt^2}|u(t)-p|^2 \ge 2|u'(t)|^2 \quad \forall t \ge 0\,, \tag{7}$$

which shows in particular that $t \mapsto |u(t)-p|^2$ is a convex function. Now assume further that u is bounded on $\mathbb{R}^+$. Then a result of convex analysis implies that the function $t \mapsto |u(t)-p|^2$ is decreasing on $\mathbb{R}^+$, hence so is $t \mapsto |u(t)-p|$. On the other hand, noting that the function $t \mapsto |u(t)-p| = (u(t)-p, u(t)-p)^{1/2}$ is twice differentiable at all $t \in \mathbb{R}^+$ with $|u(t)-p| \neq 0$, we have for such t's (see (7))

$$\frac{d}{dt}\big(|u(t)-p|\frac{d}{dt}|u(t)-p|\big) \ge |u'(t)|^2\,,$$

and so
$$|u(t)-p|\frac{d^2}{dt^2}|u(t)-p| + \big(\frac{d}{dt}|u(t)-p|\big)^2 \ge |u'(t)|^2\,,$$

which implies
$$|u(t)-p|\frac{d^2}{dt^2}|u(t)-p| \geq 0\,, \tag{8}$$
since
$$\frac{d}{dt}|u(t)-p| = \frac{(u'(t),u(t)-p)}{|u(t)-p|} \leq |u'(t)|\,.$$

Now if $|u(t_0)-p|=0$ for some $t_0 \geq 0$ then $|u(t)-p|=0$ for all $t \geq t_0$. Let t_0 be the smallest number with this property. If $t_0=0$ then $u(t)=p \;\; \forall t \geq 0$. If $t_0>0$, then $|u(t)-p|>0$ for all $t \in [0,t_0)$ so we deduce from (8) that $t \mapsto |u(t)-p|$ is convex in $[0,t_0)$ hence in $\mathbb{R}^+$. If $|u(t)-p|>0$ for all $t \geq 0$ then (8) also implies the convexity of $t \mapsto |u(t)-p|$ in $\mathbb{R}^+$. □

Remark 2.3. If $u(t)$ is a solution of
$$u''(t) = \nabla\phi(u(t)), \quad t \geq 0\,, \tag{9}$$
which is unbounded on $\mathbb{R}^+$ then there exists an $s>0$ such that $|u(s)-p| > |u(0)-p|$. Denote $f(t) := |u(t)-p|^2$. Since f is convex (cf. Lemma 2.2) we have for each $t>s>0$
$$f(s) \leq \frac{s}{t}f(t) + (1-\frac{s}{t})f(0)$$
and therefore
$$f(t) \geq \frac{t}{s}(f(s)-f(0)) + f(0)\,,$$
which shows that $f(t) \to +\infty$ as $t \to +\infty$. So for every unbounded solution $u(t)$ of (9) we have $|u(t)| \to +\infty$.

Lemma 2.4. *If u is a solution of* (3)*, then either $u(t) \to p \in \operatorname{Argmin}\phi$ or for all $t>0$ there exists $\epsilon>0$ such that for all $h \in (0,\epsilon)$*
$$\phi(u(t)) < \max\{\phi(u(t+h)), \phi(u(t-h))\}.$$

Proof. Assume that there is a $t_0>0$ such that $\forall\epsilon>0\ \exists h \in (0,\epsilon)$ such that
$$\phi(u(t_0)) \geq \max\{\phi(u(t_0+h)), \phi(u(t_0-h)\}.$$
Then there is a sequence $h_n \to 0^+$ such that
$$\phi(u(t_0)) \geq \max\{\phi(u(t_0+h_n)), \phi(u(t_0-h_n))\}.$$
Thus by the pseudo-convexity of ϕ we have
$$(\nabla\phi(u(t_0)), u(t_0+h_n)-u(t_0)) \leq 0 \;\text{ and }\; (\nabla\phi(u(t_0)), u(t_0-h_n)-u(t_0)) \leq 0\,,$$
hence
$$(\nabla\phi(u(t_0)), u(t_0+h_n)-2u(t_0)+u(t_0-h_n)) \leq 0\,.$$
Dividing this inequality by h_n^2 and letting $n \to +\infty$, we obtain
$$(\nabla\phi(u(t_0)), u''(t_0)) \leq 0 \;\Rightarrow\; |\nabla\phi(u(t_0))|^2 \leq 0\,.$$
Therefore $\nabla\phi(u(t_0))=0$, i.e., $u(t_0)$ is a critical point of ϕ. As ϕ is pseudo-convex $u(t_0)$ is a minimum point of ϕ. So, by Lemma 2.2, $t \mapsto |u(t)-u(t_0)|$ is decreasing on $[t_0,+\infty)$. Hence u is a constant function, $u(t)=u(t_0) \in \operatorname{Argmin}\phi$ for all $t \geq t_0$. □

Lemma 2.5. *If $u(t)$ is a solution of* (3)*, then either $u(t) \to p \in \operatorname{Argmin} \phi$ or $\phi(u(t))$ is decreasing and $t|u'(t)| \to 0$ as $t \to +\infty$.*

Proof. Suppose to the contrary $u(t)$ is not strongly convergent to a minimum point of ϕ and $\frac{d}{dt}\phi(u(t)) > 0$ for some $t = t_0 > 0$. First we prove that $\phi(u(t))$ is eventually increasing. Since ϕ is a C^1 function, there is a $\delta > 0$ such that $h \mapsto \phi(u(t_0 + h))$ is increasing in $(0, \delta)$. Define $\epsilon = \sup\{\delta > 0\colon\ h \mapsto \phi(u(t_0 + h))$ is increasing on $(0, \delta)\}$. If $\epsilon = +\infty$ then $\phi(u(t))$ is eventually increasing. Otherwise $\epsilon < +\infty$. In this case, by Lemma 2.4, there exists $0 < \epsilon_1 < \epsilon$ such that for all $h \in (0, \epsilon_1)$

$$\phi(u(t_0 + \epsilon)) < \max\{\phi(u(t_0 + \epsilon + h)), \phi(u(t_0 + \epsilon - h))\}$$

($\epsilon_1 < \epsilon$ can be chosen arbitrarily). If for a given $h \in (0, \epsilon_1)$,

$$\max\{\phi(u(t_0 + \epsilon + h)), \phi(u(t_0 + \epsilon - h))\} = \phi(u(t_0 + \epsilon - h)),$$

then

$$\phi(u(t_0 + \epsilon)) < \phi(u(t_0 + \epsilon - h)),$$

which contradicts the fact that $t \mapsto \phi(u(t))$ is increasing in $(t_0, t_0 + \epsilon)$. This contradiction shows that

$$\max\{\phi(u(t_0 + \epsilon + h)), \phi(u(t_0 + \epsilon - h))\} = \phi(u(t_0 + \epsilon + h)),$$

and therefore

$$\phi(u(t_0 + \epsilon)) < \phi(u(t_0 + \epsilon + h)), \quad \forall h \in (0, \epsilon_1)$$

which contracdicts the choice of ϵ. Hence again $\phi(u(t))$ is eventually increasing. Now, from $(3)_1$ we derive

$$\frac{d}{dt}\phi(u(t)) = (\nabla\phi(u(t)), u'(t)) = (u''(t), u'(t)) = \frac{1}{2}\frac{d}{dt}|u'(t)|^2, \tag{10}$$

for all $t \geq 0$. Therefore $|u'(t)|$ is eventually increasing. Let $p \in \operatorname{Argmin} \phi$. Since $\phi(u(t)) \geq \phi(p)$ we deduce from the pseudo-convexity of ϕ that

$$(\nabla\phi(u(t)), u(t) - p) \geq 0\,,$$

so, in view of $(3)_1$,

$$(u''(t), u(t) - p) \geq 0. \tag{11}$$

Therefore

$$|u'(t)|^2 \leq \frac{d^2}{dt^2}|u(t) - p|^2\,.$$

Multiplying this inequality by t and then integrating over $[0, T]$ we get (see also Lemma 2.2)

$$\int_0^T t|u'(t)|^2 dt \leq T\frac{d}{dT}|u(T) - p|^2 - \int_0^T \frac{d}{dt}|u(t) - p|^2 dt$$

$$= T\frac{d}{dT}|u(T) - p|^2 - |u(T) - p|^2 + |u(0) - p|^2 \leq |u(0) - p|^2\,.$$

Letting $T \to +\infty$ we get
$$\int_0^\infty t|u'(t)|^2 dt < +\infty\,, \tag{12}$$
which contradicts the fact that $|u'(t)|$ is eventually increasing. Therefore either $u(t)$ converges strongly to an element of $\operatorname{Argmin}\phi$ or $\phi(u(t))$ is decreasing and consequently, by (10), $|u'(t)|$ is decreasing as well. If $\phi(u(t))$, (and consequently $|u'(t)|$) is decreasing, then by (12) $|u'(t)| \to 0$ as $t \to +\infty$. Finally, by Lemma 2.1 and (12) we derive $t|u'(t)| \to 0$ as $t \to +\infty$. □

Next we state a theorem on the weak convergence of solutions to (3) to minimum points of ϕ. To prove the theorem we follow an argument from the proof of Theorem 2 in [12] (which is related to first order gradient equations of a quasi-convex type). For the properties of the nearest point projection (metric projection) P, which are used in the proof of Theorem 2.5 below, see Section 3 of the interesting book by Goebel and Reich [11].

Theorem 2.6. *Suppose* $\operatorname{Argmin}\phi \neq \varnothing$. *Then every solution* $u(t)$ *of* (3) *converges weakly to a minimizer of* ϕ. *Moreover* $Pu(t)$ *converges strongly to* $p \in \operatorname{Argmin}\phi$, *where* Px *is the nearest point projection of* x *on* $\operatorname{Argmin}\phi$.

Proof. If $u(t)$ converges strongly to a minimizer of ϕ, then the theorem is proved. Otherwise, by Lemma 2.5 $\phi(u(t))$ is decreasing. Therefore $\lim_{t\to+\infty}\phi(u(t))$ exists. To prove the theorem we distinguish two cases:

(1) $\lim_{t\to+\infty}\phi(u(t)) = \phi(p)$, where $p \in \operatorname{Argmin}\phi$. By assumption $u(t)$ is bounded so there are sequences $t_n \to \infty$ such that $(u(t_n))$ is weakly convergent. If (t_n) is such a sequence, i.e., $u(t_n) \rightharpoonup q$, then
$$\phi(q) \leq \liminf \phi(u(t_n)) = \lim_{t\to+\infty}\phi(u(t)) = \phi(p)\,,$$
which implies that $q \in \operatorname{Argmin}\phi$. The theorem in this case follows from the Opial lemma (see [20]) and Lemma 2.2.

(2) $\lim_{t\to+\infty}\phi(u(t)) > \phi(p)$. Then there exist an $r > 0$ and a $t_0 > 0$ such that for all y in the ball $B_r(p)$ $\phi(u(t)) > \phi(y)$ for all $t \geq t_0$. So by the pseudo-convexity of ϕ, we have $(\nabla\phi(u(t)), u(t) - y) \geq 0$ for all $t \geq t_0$. Thus for $y = p + r\frac{\nabla\phi(u(t))}{|\nabla\phi(u(t))|}$ we have
$$(\nabla\phi(u(t)), u(t) - p - r\frac{\nabla\phi(u(t))}{|\nabla\phi(u(t))|}) \geq 0 \quad \forall t \geq t_0\,.$$
This implies that
$$\begin{aligned} r|\nabla\phi(u(t))| &\leq (\nabla\phi(u(t)), u(t) - p) = (u''(t), u(t) - p) \\ &= \frac{1}{2}\frac{d^2}{dt^2}|u(t) - p|^2 - |u'(t)|^2 \quad \forall t \geq t_0\,. \end{aligned} \tag{13}$$
In fact we can assume that $t_0 = 0$ since the equation $(3)_1$ is autonomous. Multiplying both sides by t and then integrating over $[0, T]$ we deduce (note that by Lemma 2.2 $|u(t) - p|$ is decreasing)

$$r\int_0^T t|\nabla\phi(u(t))|dt = r\int_0^T t|u''(t)|dt$$
$$\leq \frac{1}{2}T\frac{d}{dT}|u(T)-p|^2 - \frac{1}{2}|u(T)-p|^2 + \frac{1}{2}|u(0)-p|^2 \leq \frac{1}{2}|u(0)-p|^2\,.$$

Letting $T\to+\infty$ we obtain

$$\int_0^{+\infty} t|u''(t)|dt < +\infty. \tag{14}$$

Now, if $|u'(t)|\neq 0$ for all $t\in[0,+\infty)$, we get by integration by parts

$$\int_0^T |u'(t)|dt = T|u'(T)| - \int_0^T t\frac{(u'(t),u''(t))}{|u'(t)|}\,dt \leq T|u'(T)| + \int_0^T t|u''(t)|\,dt\,.$$

By letting $T\to+\infty$ we get (see Lemma 2.5 and (14))

$$\int_0^{+\infty} |u'(t)|\,dt < +\infty. \tag{15}$$

(15) implies that $u(t)$ converges strongly to a point q of H. By (14) there is a sequence $t_n\to+\infty$ such that $\lim_{n\to\infty}|u''(t_n)|=0$. So $(3)_1$ and $\phi\in C^1$ imply $\nabla\phi(q)=0$. This implies $q\in \text{Argmin}\phi$ (as ϕ is pseudo-convex).

Now let us prove the second part of the theorem. By Lemma 2.2 and the projection property (see [11]), we have

$$|u(t+h)-Pu(t+h)| \leq |u(t+h)-Pu(t)| \leq |u(t)-Pu(t)|\,.$$

Therefore $t\mapsto |u(t)-Pu(t)|$ is decreasing. Using again the projection property we get

$$\begin{aligned}|Pu(t+h)-Pu(t)|^2 &= (Pu(t+h)-u(t+h),Pu(t+h)-Pu(t))\\ &\qquad + (u(t+h)-Pu(t),Pu(t+h)-Pu(t))\\ &\leq \frac{1}{2}(|u(t+h)-Pu(t)|^2 + |Pu(t+h)-Pu(t)|^2\\ &\qquad - |u(t+h)-Pu(t+h)|^2).\end{aligned}$$

It follows that

$$\begin{aligned}|Pu(t+h)-Pu(t)|^2 &\leq |u(t+h)-Pu(t)|^2 - |u(t+h)-Pu(t+h)|^2\\ &\leq |u(t)-Pu(t)|^2 - |u(t+h)-Pu(t+h)|^2\end{aligned}$$

and therefore $\lim_{t\to+\infty}|Pu(t+h)-Pu(t)|=0$. Hence $Pu(t)$ converges strongly to a point p in $\text{Argmin}\,\phi$. Assume that q is the weak limit point of $u(t)$. Since P is a projection operator and $q\in\text{Argmin}\,\phi$ and by projection property, we have

$$(u(t)-Pu(t),q-Pu(t)) \leq 0\,.$$

Now by letting $t\to+\infty$ we get $q=p$. □

Open problem: We are wondering whether Theorem 2.6 is still valid if ϕ is a quasi-convex function.

Remark 2.7. Consider the following second order initial value problem

$$\begin{cases} u''(t) = \nabla\phi(u(t)), & t \geq 0\,, \\ u(0) = x, \qquad u'(0) = y\,, \end{cases} \tag{16}$$

where $\phi\colon H \to \mathbb{R}$ is a C^1, pseudo-convex function with $\operatorname{Argmin}\phi \neq \varnothing$ and $\nabla\phi$ is Lipschitz on H. Denoting $v(t) := u'(t)$ the above problem can be expressed as the first order initial value problem

$$\begin{cases} U'(t) = AU(t), & t \geq 0\,, \\ U(0) = (x, y) \end{cases} \tag{17}$$

where $U(t) = (u(t), v(t))$ and $A(u, v) = (v, \nabla\phi(u))$. Clearly A is Lipschitz on $H \times H$ so (17) has a solution. Therefore (16) has a solution $u(t)$ that either converges weakly to a minimum point of ϕ if it is bounded (by Theorem 2.6) or $|u(t)| \to +\infty$ if it is unbounded (cf. Remark 2.3).

3. Strong convergence

In this section we establish some strong convergence results for the solutions of (3).

Theorem 3.1. *Suppose ϕ is strongly pseudo-convex and* $\operatorname{Argmin}\phi \neq \varnothing$*. Then every solution $u(t)$ of* (3) *converges strongly to a minimum point of ϕ.*

Proof. Let $p \in \operatorname{Argmin}\phi$. Therefore $\phi(p) \leq \phi(u(t))$. By assumption there is an $\alpha > 0$ such that

$$(\nabla\phi(u(t)), p - u(t)) \leq -\alpha|u(t) - p|^2 \quad \forall t \geq 0\,.$$

Thus we get by (3)

$$\alpha|u(t) - p|^2 \leq (u''(t), u(t) - p), \quad t \geq 0\,.$$

Integrating this over $[0, T]$ we obtain

$$\begin{aligned} \alpha \int_0^T |u(t) - p|^2 &\leq \frac{d}{dT}|u(T) - p|^2 - (u'(0), u(0) - p) \\ &\leq -(u'(0), u(0) - p)\,. \end{aligned}$$

Therefore $\liminf_{t\to+\infty} |u(t) - p| = 0$. Thus we conclude by Lemma 2.2 that $u(t) \to p$ as $t \to +\infty$. □

Theorem 3.2. *Suppose that ϕ is pseudo-convex and $u(t)$ is a solution of* (3)*. If in addition the level set $\{x : \phi(x) \leq \lambda\}$ is compact for each $\lambda > 0$, then* $\operatorname{Argmin}\phi \neq \varnothing$*, and $u(t)$ converges strongly to a minimum point of ϕ.*

Proof. Obviously $\operatorname{Argmin}\phi \neq \varnothing$. Assume that $u(t)$ does not converge strongly to an element of $\operatorname{Argmin}\phi$. Then, according to Lemma 2.5, $\phi(u(t))$ is decreasing. On the other hand, $u(t) \in \{x : \phi(x) \leq \phi(u(t_0))\}$ for all $t \geq t_0$. By assumption there is a sequence $t_n \to +\infty$ such that $u(t_n) \to q$. By Theorem 2.6, $q = p \in \operatorname{Argmin}\phi$, where p is the weak limit point of $u(t)$. Since $|u(t) - p|$ is decreasing (cf. Lemma 2.2), we conclude that $u(t) \to p \in \operatorname{Argmin}\phi$ as $t \to +\infty$. $\square$

Theorem 3.3. *Suppose that ϕ is pseudo-convex and even. If* $\operatorname{Argmin}\phi \neq \varnothing$*, then every solution $u(t)$ of* (3) *converges strongly to a minimum point of ϕ.*

Proof. By assumption and Lemma 2.5 we have

$$\phi(-u(t)) = \phi(u(t)) \leq \phi(u(s)) \quad \forall s < t\,.$$

By the pseudo-convexity of ϕ and $(3)_1$, we have

$$(u''(s), u(s) + u(t)) \geq 0 \tag{18}$$

and

$$(u''(s), u(s) - u(t)) \geq 0\,. \tag{19}$$

Summing (18) and (19), we get $(u''(s), u(s)) \geq 0$ for all $s \geq 0$. As in the proof of Lemma 2.2, we deduce that $s \mapsto |u(s)|$ is decreasing. Now define

$$g(s) := 2(|u(s)|^2 - |u(t)|^2) - |u(s) - u(t)|^2, \quad s \in [0, t]\,.$$

By (18) $g''(s) \geq 0$ for all $s \in [0, t]$ so g is a convex function on $[0, t]$ and $g(t) = 0$. We can prove that g in decreasing on $[0, t]$. Otherwise there is $0 < s_0 < t$ such that g is nondecreasing on $[s_0, t]$ but since $|u(s)|$ is decreasing we have

$$g'(t) = 4(u'(t), u(t)) \leq 0\,,$$

a contradiction. Therefore $g(s) \geq 0$ for all $s \in [0, t]$. Then

$$|u(s) - u(t)|^2 \leq 2(|u(s)|^2 - |u(t)|^2).$$

This shows that $\{u(t)\}$ is a Cauchy net, and hence converges to a point in H, which is a minimum point of ϕ (cf. Theorem 2.6). $\square$

Theorem 3.4. *Suppose that ϕ is pseudo-convex with* $\operatorname{int}\operatorname{Argmin}\phi \neq \varnothing$*. Then every solution $u(t)$ of* (3) *converges strongly to a minimum point of ϕ.*

Proof. Let $p \in \operatorname{int}\operatorname{Argmin}\phi$. Then by assumption there is $r > 0$ such that $B_r(p) \subset \operatorname{Argmin}\phi$. Set $y = p + r\frac{\nabla\phi(u(t))}{|\nabla\phi(u(t))|} \in B_r(p)$. Since $\phi(u(t)) \geq \phi(y)$ and ϕ is pseudo-convex we can derive (13). The rest of the proof is similar to the proof of Case 2 of Theorem 2.6. $\square$

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