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Evolution Inclusion on a Real Hilbert Space with Quasi-Variational Structure for Inner Products

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We consider a Cauchy problem of an abstract evolution inclusion on a real Hilbert space associated with subdifferentials of time-dependent proper lower semicontinuous convex functions. The abstract evolution inclusion, which is treated in this paper, contains not only convex functions but also inner products of the Hilbert space depending on unknown functions. Especially, we call such structures for convex functions and inner products "quasi-variational structures for convex functions and inner products". The main purposes of this paper are to show the existence of global-in-time solutions to the Cauchy problem, which has a quasi-variational structure, and to apply this existence result to a phase field model with a quasi-variational boundary condition.

Keywords: Evolution inclusion, quasi-variational structures, inner products, initial-value problems, global-in-time solutions.

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1. Introduction

We consider a Cauchy problem of an abstract evolution inclusion on a real Hilbert space associated with subdifferentials of time-dependent proper lower semicontinuous (l.s.c. for short) convex functions. In order to formulate the evolution inclusion treated in this paper, we denote by H and X a real Hilbert space with an inner product $(\cdot,\cdot)_H$ and a real Banach space with a norm $\|\cdot\|_X$, respectively. A closed subset of X, denoted by A, and a family of inner products of H, denoted by $\{(\cdot,\cdot)_v\,;\, v \in A\}$, are prescribed. Moreover, for each $\tilde{u} \in C([0,T];H)$ families of time-dependent proper, l.s.c. convex functions on H, denoted by $\{\varphi(t,\tilde{u},v)\,;\, 0 \le t \le T,\ v \in A\}$, and operators from X into itself, denoted by $\{S(\tilde{u},t,s)\,;\, 0 \le s \le t \le T\}$, are also prescribed.

Using the above families we consider in this paper the following Cauchy problem of an evolution inclusion $\{(1.1)–(1.3)\}$, which is denoted by $(\mathrm{P})_{u_0,v_0}$:

$$u'(t) + \partial_{v(t)}\varphi(t,u,v(t);u(t)) \ni f(t) \quad \text{in} \quad H(v(t)), \quad \text{a.a. } t \in (0,T), \tag{1.1}$$

$$(\mathrm{P})_{u_0,v_0} \qquad v(t) = S(u,t,0)v_0 \quad \text{in} \quad X, \quad \forall\, t \ge 0, \tag{1.2}$$

$$u(0) = u_0 \quad \text{in} \quad H, \tag{1.3}$$

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where $H(v)$ is the real Hilbert space H with the inner product $(\cdot,\cdot)_v$; $\partial_v\varphi(t,u,v)$ is the subdifferential of $\varphi(t,u,v)$ on $H(v)$, that is, $z^* \in \partial_v\varphi(t,u,v;z)$ if and only if $z \in D(\varphi(t,u,v))$ and

$$(z^*, y-z)_v \le \varphi(t,u,v;y) - \varphi(t,u,v;z), \qquad \forall\, y \in H. \tag{1.4}$$

Although there are many mathematical results for abstract Cauchy problems of evolution inclusions, we give [4, 9, 10] as the papers which established the general framework for abstract evolution inclusions associated with subdifferentials of time-dependent proper l.s.c. convex functions on a real Hilbert space in this paper.

In [10] Kenmochi considered the following Cauchy problem (K) with "a variational structure for time-dependent convex functions", that is, associated to the subdifferentials of prescribed time-dependent convex functions $\{\varphi(t)\,;\, 0 \le t \le T\}$, and showed the well-posedness of its evolution inclusion:

$$\text{(K)} \qquad \begin{cases} u'(t) + \partial_H\varphi(t;u(t)) \ni f(t) \quad \text{in} \quad H, \quad \text{a.a. } t \in (0,T), \\ u(0) = u_0 \quad \text{in} \quad H, \end{cases}$$

where $\partial_H\varphi(t)$ denotes the subdifferential of a time-dependent proper l.s.c. convex function $\varphi(t)$ on H with the inner product $(\cdot,\cdot)_H$. In (K), the effective domains of $\varphi(t)$ are moving in time smoothly in general, but independently of the unknown function u itself. In addition, the inner product of H does not depend on this function u neither.

In [9] Kano, Kenmochi and Murase considered the case that time-dependent convex functions depend on the unknown function u. Roughly speaking, first of all a family of proper l.s.c. convex functions, denoted by $\{\varphi(t,\tilde{u});\ 0 \le t \le T,\ \tilde{u} \in C([0,T];H)\}$, is prescribed, and the following Cauchy problem (KKM) with "a quasi-variational structure for time-dependent convex functions" is considered:

$$\text{(KKM)} \qquad \begin{cases} u'(t) + \partial_H\varphi(t,u;u(t)) \ni f(t) \quad \text{in} \quad H, \quad \text{a.a. } t \in (0,T), \\ u(s) = u_0(s) \quad \text{in} \quad H, \quad \forall\, s \in [-\delta, 0], \end{cases}$$

where $\delta > 0$ is a prescribed constant. In [9] the authors showed the existence of global-in-time solutions to (KKM). As a typical example to which we can apply the theory established in [9], we state the following Cauchy problem $\{(1.5)–(1.15)\}$, which is denoted by (T) here. It is one of the mathematical models describing a tumor invasion phenomenon with constraints, precisely analyzed in [6, 7, 8]:

$$u_t = \nabla\cdot(\kappa(f)\nabla u) - \nabla\cdot(u\nabla f) \quad \text{in} \quad \bigcup_{0<t<T} \Omega_1(f(t)) \times \{t\}, \tag{1.5}$$

$$u_t \le \nabla\cdot(\kappa(f)\nabla u) - \nabla\cdot(u\nabla f) \quad \text{in} \quad \bigcup_{0<t<T} \Omega_2(f(t)) \times \{t\}, \tag{1.6}$$

$$u_t \ge \nabla\cdot(\kappa(f)\nabla u) - \nabla\cdot(u\nabla f) \quad \text{in} \quad \bigcup_{0<t<T} \Omega_3(f(t)) \times \{t\}, \tag{1.7}$$

$$
\text{(T)} \quad
\begin{aligned}
&f_t = -wf \quad \text{in} \quad \Omega \times (0,T), && (1.8)\\
&w_t = \Delta w + u - w \quad \text{in} \quad \Omega \times (0,T), && (1.9)\\
&0 \le u \le 1 - f,\ f \ge 0 \quad \text{in} \quad \Omega \times (0,T), && (1.10)\\
&u = 0 \quad \text{on} \quad \Gamma \times (0,T), && (1.11)\\
&\nabla w \cdot \nu = 0 \quad \text{on} \quad \Gamma \times (0,T), && (1.12)\\
&u(0) = u_0 \quad \text{in} \quad \Omega, && (1.13)\\
&f(0) = f_0 \quad \text{in} \quad \Omega, && (1.14)\\
&w(0) = w_0 \quad \text{in} \quad \Omega, && (1.15)
\end{aligned}
$$

where Ω is a bounded domain in $\mathbb{R}^N$ $(N = 1,2,3)$ with a smooth boundary $\Gamma = \partial\Omega$; κ is a strictly positive function from $\mathbb{R}$ into itself; for each measurable function f on Ω the measurable subsets $\Omega_i(f)$ $(i = 1,2,3)$ of Ω are defined by

$$
\begin{aligned}
\Omega_1(f) &= \{x \in \Omega\,;\, 0 < f(x) < 1\},\\
\Omega_2(f) &= \{x \in \Omega\,;\, f(x) = 0\},\\
\Omega_3(f) &= \{x \in \Omega\,;\, f(x) = 1\}.
\end{aligned}
$$

Here we roughly explain the approach to (T), which was used in [8]. As the function u is given in (1.9), we can define an operator S_1 which assigns a unique solution $w = S_1(u)$ of the Cauchy problem $\{(1.9), (1.12), (1.15)\}$. Moreover, by using this function $w := S_1 u$, we define an operator S_2 to assign a unique solution of the Cauchy problem $\{(1.8), (1.14)\}$. Defining an operator $S := S_2 \circ S_1$, we can regard (T) as an initial boundary problem of $(\mathrm{T})_1$, represented by $\{(1.11),(1.13)\}$:

$$
(\mathrm{T})_1
\begin{cases}
u_t = \nabla\cdot(\kappa(S(u))\nabla u) - \nabla\cdot(u\nabla S(u)) & \text{in} \quad \bigcup\limits_{0<t<T} \Omega_1((S(u))(t)) \times \{t\},\\
u_t \le \nabla\cdot(\kappa(S(u))\nabla u) - \nabla\cdot(u\nabla S(u)) & \text{in} \quad \bigcup\limits_{0<t<T} \Omega_2((S(u))(t)) \times \{t\},\\
u_t \ge \nabla\cdot(\kappa(S(u))\nabla u) - \nabla\cdot(u\nabla S(u)) & \text{in} \quad \bigcup\limits_{0<t<T} \Omega_3((S(u))(t)) \times \{t\},\\
0 \le u \le 1 - S(u),\ S(u) \ge 0 & \text{in} \quad \Omega \times (0,T).
\end{cases}
$$

Since the constraint condition $0 \le u \le 1 - S(u)$ for u in $(\mathrm{T})_1$ depends on the unknown function u itself, $(\mathrm{T})_1$ has a quasi-variational structure for the proper l.s.c. convex functions $\varphi(t,u)$ on $L^2(\Omega)$ defined by

$$
\varphi(t,u;z) :=
\begin{cases}
\dfrac{1}{2}\displaystyle\int_\Omega \kappa((S(u))(t,x))|\nabla z(x)|^2 dx - \int_\Omega u(t,x)\nabla(S(u))(t,x)\cdot\nabla z(x)dx,\\
\qquad\qquad \text{if} \ \ z \in D((S(u))(t)),\\
+\infty, \quad \text{if} \ \ z \in L^2(\Omega) \setminus D((S(u))(t)),
\end{cases}
$$

where for all $t \in [0,T]$ the effective domain $D((S(u))(t))$ of $\varphi(t,u)$ is given by

$$
D((S(u))(t)) := \{z \in L^2(\Omega)\,;\, 0 \le z(x) \le 1 - (S(u))(t,x) \text{ a.a. } x \in \Omega\}.
$$

Under the above settings, in [8] the problem (T) is precisely analyzed by using the framework of (KKM), so that a further discussion can be omitted here.

As is seen from the typical example (T), some kinds of systems have quasi-variational structures for (time-dependent) proper l.s.c. convex functions and we can treat them with the framework of (KKM). The inner product of H, however, does not change in time. From this viewpoint, (KKM) does not have a variational structure for inner products of H.

On the other hand, in [4] Damlamian considered the following Cauchy problem (D), which has a variational structure for inner products in its evolution inclusion, and showed the existence and uniqueness of solutions to (D):

$$\text{(D)} \qquad \begin{cases} u'(t) + \partial_t \varphi(t; u(t)) \ni f(t) \quad \text{in } H_t, \quad \text{a.a. } t \in (0,T), \\ u(0) = u_0 \quad \text{in } H. \end{cases}$$

In the setting of (D), a family of inner products of H, denoted by $\{(\cdot,\cdot)_t\,;\, 0 \le t \le T\}$, is also prescribed and $\partial_t \varphi(t)$ denotes the subdifferential of the time-dependent function $\varphi(t)$ with respect to the inner product $(\cdot,\cdot)_t$ (cf. (1.4)). Damlamian calls the structure in this evolution inclusion "a variational norm" in [4]. Even if (D) has a variational structure for inner products, Damlamian assumes that the effective domains of $\varphi(t)$, denoted by $D(\varphi(t))$, are independent of t, that is, $D(\varphi(t)) = D(\varphi(0))$ for all $t \in [0,T]$. This condition seems too strong for us to apply the results obtained in [4] to the mathematical models with variational structures in general. Furthermore, the inner products do not depend on the unknown function u.

Comparing $(\mathrm{P})_{u_0,v_0}$ with (K), (KKM) and (D), it is important and essential that $(\mathrm{P})_{u_0,v_0}$ has quasi-variational structures for not only time-dependent proper l.s.c. convex functions $\varphi(t,u,v(t))$ but also inner products $(\cdot,\cdot)_{v(t)}$. Here we especially emphasize that the quasi-variational structure for inner products depends on the unknown function v, which is determined by the relation (1.2), not the unknown function u directly.

The main purpose of this paper is to show the existence of solutions to $(\mathrm{P})_{(u_0,v_0)}$ as well as to give a sufficient condition for it. Furthermore, in the final section we apply the main theorem in this paper to a phase field model with a quasi-variational boundary condition.

At the end of this section, we give the definitions for solutions to $(\mathrm{P})_{u_0,v_0}$.

Definition 1.1. For each $\tau \in (0,T]$ a function u is called a *solution* to $(\mathrm{P})_{u_0,v_0}$ on $[0,\tau]$ if and only if $u \in C([0,\tau];H) \cap W^{1,2}(0,\tau;H)$ and u satisfies (1.1), (1.2) and (1.3). Moreover,

(1) A function u is called a *local-in-time solution* to $(\mathrm{P})_{u_0,v_0}$ if and only if there exists a time $\tau_0 \in (0,T]$ such that u is a solution to $(\mathrm{P})_{u_0,v_0}$ on $[0,\tau]$ for all $\tau \in (0,\tau_0)$.

(2) A function u is called a *global-in-time solution* to $(\mathrm{P})_{u_0,v_0}$ if and only if u is a solution to $(\mathrm{P})_{u_0,v_0}$ on $[0,T]$.

2. Preliminaries

2.1. Assumptions for the prescribed data

First of all, a nonempty subset A of X and a family of inner products of H, denoted by $\{(\cdot,\cdot)_v\,;\,v\in A\}$, are previously given throughout this paper. Then, we assume that the following properties are satisfied:

(A1) (*Uniform equivalence of inner products*) The family $\{(\cdot,\cdot)_v\,;\,v\in A\}$ is uniformly equivalent to the inner product $(\cdot,\cdot)_H$, that is, there exist constants $c_1>0$ and $c_2>0$ such that

$$c_1\|z\|_H \le \|z\|_v \le c_2\|z\|_H, \quad \forall\, v\in A,\ \forall\, z\in H,$$

where $\|z\|_v := \sqrt{(z,z)_v}$ and $\|z\|_H := \sqrt{(z,z)_H}$. In order to emphasize the dependence of inner products with respect to $v\in A$, we denote by $H(v)$ a Hilbert space H with the inner product $(\cdot,\cdot)_v$.

(A2) (*Continuity of norms*) There exists a constant $\gamma_0>0$ such that

$$\left|\|z\|_{v_1}^2 - \|z\|_{v_2}^2\right| \le \gamma_0\|v_1-v_2\|_X\|z\|_{v_2}^2, \quad \forall\, v_1,v_2\in A,\ \forall\, z\in H.$$

For each $\tilde{u}\in C([0,T];H)$ we consider a family $\{S(\tilde{u},t,s)\,;\,0\le s\le t\le T\}$ of operators $S(\tilde{u},t,s)$ from X into itself, which satisfies the following conditions, where if $s=t$ then $C([s,t];X)=X$ as a matter of convenience:

(S1) For each $\tilde{u}\in C([0,T];H)$ and $s,t\in[0,T]$ with $s\le t$, the operator $S(\tilde{u},t,s)$ is continuous with respect to the strong topology of X, and its domain and range are contained in A, that is, $D(S(\tilde{u},t,s))\subset A$ and $R(S(\tilde{u},t,s))\subset A$.

(S2) We fix an element $v\in A$ and $s,\ t\in[0,T]$ with $s\le t$, and assume that a sequence $\{\tilde{u}_m\}_{m\in\mathbb{N}}\subset C([0,T];H)$ and an element $\tilde{u}\in C([0,T];H)$ satisfy

$$\tilde{u}_m \longrightarrow \tilde{u} \quad \text{in} \quad C([0,T];H) \quad \text{as} \quad m\to\infty.$$

Then, we have

$$S(\tilde{u}_m,\cdot,s)v \longrightarrow S(\tilde{u},\cdot,s)v \quad \text{in} \quad C([s,t];X) \quad \text{as} \quad m\to\infty.$$

(S3) For $\tilde{u}\in C([0,T];H)$ we have $S(\tilde{u}_1,t,0)=S(\tilde{u}_2,t,0)$ on X for all $t\in[0,t_1]$ whenever for some $t_1\in[0,T]$ we have $\tilde{u}_1(t)=\tilde{u}_2(t)$ in H for all $t\in[0,t_1]$.

(S4) $S(\tilde{u},t,t)$ is the identity operator on X for all $\tilde{u}\in C([0,T];H)$ and $t\in[0,T]$.

(S5) $S(\tilde{u},t,s)=S(\tilde{u},t,\tau)\circ S(\tilde{u},\tau,s)$ on X for all $s,t,\tau\in[0,T]$ with $s\le\tau\le t$.

(S6) For $\tilde{u}\in C([0,T];H)$ and $\tau\in[-T,T]$ we define $\tilde{u}_\tau\in C([0,T];H)$ by

$$\tilde{u}_\tau(t) := \begin{cases} \tilde{u}(0) & \text{if} \quad t\in[0,\max\{-\tau,0\}), \\ \tilde{u}(t+\tau) & \text{if} \quad t\in[\max\{-\tau,0\},\min\{T-\tau,T\}], \\ \tilde{u}(T) & \text{if} \quad t\in(\min\{T-\tau,T\},T]. \end{cases}$$

Then, for all $s,t\in[\max\{-\tau,0\},\min\{T-\tau,T\}]$ with $s\le t$ the following property is satisfied: $S(\tilde{u}_\tau,t,s)=S(\tilde{u},t+\tau,s+\tau)$ on X.

Next, we consider the set $\mathcal{X}$ of families of time-dependent proper l.s.c. convex functions $\varphi(t,\tilde{u},v)$ on H, which satisfies the conditions (C1), (C2) and (C3) below:

$$\mathcal{X} := \{\{\varphi(t,\tilde{u},\tilde{v}(t))\,;\, 0 \le t \le T\}\,;\, \tilde{u} \in C([0,T];H),\ v \in A,\ \tilde{v}(t) = S(\tilde{u},t,0)v\}\,.$$

(C1) There exists a proper l.s.c. convex function φ on H such that

$$\varphi(z) \le \varphi(t,\tilde{u},\tilde{v}(t);z), \quad \forall\,\tilde{u} \in C([0,T];H),\ \forall\, t \in [0,T],\ \forall\, v \in A,\ \forall\, z \in H.$$

Moreover, for each $r > 0$ the set $\{z \in H\,;\, \|z\|_H \le r,\ |\varphi(z)| \le r\}$ is relatively compact in H.

(C2) For each $\tilde{u}_i \in C([0,T];H)$ $(i = 1,2)$ and $\tilde{v}_0 \in A$ we define a function $v_i \in W^{1,1}(0,T;X)$ by $\tilde{v}_i(t) = S(\tilde{u}_i,t,0)\tilde{v}_0$ for all $t \in [0,T]$. Then, we have $\varphi(t,\tilde{u}_1,\tilde{v}_1(t)) = \varphi(t,\tilde{u}_2,\tilde{v}_2(t))$ on H for all $t \in [0,\tau_1]$ whenever for some $\tau_1 \in [0,T]$ we have $\tilde{u}_1(t) = \tilde{u}_2(t)$ in H for all $t \in [0,\tau_1]$.

(C3) Assume sequences $\{\tilde{u}_m\}_{m\in\mathbb{N}} \subset C([0,T];H)$, $\{v_{0m}\}_{m\in\mathbb{N}} \subset A$ and elements $\tilde{u} \in C([0,T];H)$, $v_0 \in A$ such that

$$\tilde{u}_m \longrightarrow \tilde{u} \text{ in } C([0,T];H),\ v_{0m} \longrightarrow v_0 \text{ in } X \text{ as } m \to \infty,$$

and put $\tilde{v}_m(t) := S(\tilde{u}_m,t,0)v_{0m}$ and $\tilde{v}(t) := S(\tilde{u},t,0)v_0$ for all $t \in [0,T]$. Then, for each $t \in [0,T]$ we have $\varphi(t,\tilde{u}_m,\tilde{v}_m(t)) \longrightarrow \varphi(t,\tilde{u},\tilde{v}(t))$ on $H(\tilde{v}(t))$ in the sense of Mosco as $m \to \infty$, that is, the following properties are satisfied:

(1) For each $z \in D(\varphi(t,\tilde{u},\tilde{v}(t)))$ there exists a sequence $\{z_m\}_{m\in\mathbb{N}} \subset H$ with

$$z_m \longrightarrow z \text{ in } H(\tilde{v}(t)) \text{ as } m \to \infty,$$

$$\lim_{m\to\infty} \varphi(t,\tilde{u}_m,\tilde{v}_m(t);z_m) = \varphi(t,\tilde{u},\tilde{v}(t);z).$$

(2) Let $\{\tilde{u}_{m_k}\}_{k\in\mathbb{N}}$ and $\{v_{0m_k}\}_{k\in\mathbb{N}}$ be subsequences of $\{\tilde{u}_m\}_{m\in\mathbb{N}}$ and $\{v_{0m}\}_{m\in\mathbb{N}}$, respectively, and $\{z_k\}_{k\in\mathbb{N}}$ be a sequence in H such that $z_k \longrightarrow z$ weakly in $H(\tilde{v}(t))$ as $k \to \infty$ for a certain element $z \in H$. Then, we have

$$\varphi(t,\tilde{u},\tilde{v}(t);z) \le \liminf_{k\to\infty} \varphi(t,\tilde{u}_{m_k},\tilde{v}_{m_k}(t);z_k).$$

With respect to the detailed properties for the Mosco convergence of convex functions on a Hilbert space we trust [11] and omit them in this paper.

For each $(u_0,v_0) \in H \times A$ we define a subset $\mathcal{W}(u_0,v_0)$ of $C([0,T];H)$ in the following way: $\tilde{u} \in C([0,T];H)$ belongs to $\mathcal{W}(u_0,v_0)$ if and only if

$$\begin{cases} \text{(i)} & \tilde{v}(t) = S(\tilde{u},t,0)v_0 \text{ for all } t \in [0,T], \\ \text{(ii)} & \{\varphi(t,\tilde{u},\tilde{v}(t))\,;\, 0 \le t \le T\} \in \mathcal{X}, \\ \text{(iii)} & \tilde{u}(0) = u_0 \in D(\varphi(0,\tilde{u},v_0)) = D(\varphi(0,u_0,v_0)), \\ \text{(iv)} & \{\varphi(t,\tilde{u},\tilde{v}(t))\,;\, 0 \le t \le T\} \text{ satisfies Condition 1 below.} \end{cases}$$

For $\tilde{u} \in C([0,T];H)$ and $t \in [0,T]$ we define the function $[u(t)] \in C([0,T];H)$ by

$$[\tilde{u}(t)](s) = \begin{cases} \tilde{u}(s) & \text{if} \quad s \in [0,t], \\ u(t) & \text{if} \quad s \in (t,T]. \end{cases}$$

In particular we simply write $[\tilde{u}(0)]$ by $\tilde{u}(0)$ when $t = 0$.

Condition 1. (cf. [10]) For each $r > 0$ there exist nonnegative functions $\alpha_r(\tilde{u}) \in L^2(0,T)$ and $\beta_r(\tilde{u}) \in L^1(0,T)$ such that the following property is satisfied: for each $s, t \in [0,T]$ and $z_s(\tilde{u}) \in D(\varphi(s,\tilde{u},\tilde{v}(s)))$ with $\|z_s(\tilde{u})\|_{\tilde{v}(s)} \leq r$ there exists $z_t(\tilde{u},s) \in D(\varphi(t,\tilde{u},\tilde{v}(t)))$ such that

$$\|z_t(\tilde{u},s) - z_s(\tilde{u})\|_{\tilde{v}(t)} \leq \left(\sqrt{|\varphi(s,\tilde{u},\tilde{v}(s);z_s(\tilde{u}))|} + 1\right) \left| \int_s^t \alpha_r(\tilde{u};\tau)d\tau \right|, \text{ and}$$

$$|\varphi(t,\tilde{u},\tilde{v}(t);z_t(\tilde{u},s)) - \varphi(s,\tilde{u},\tilde{v}(s);z_s(\tilde{u}))| \\ \leq (|\varphi(s,\tilde{u},\tilde{v}(s);z_s(\tilde{u}))| + 1) \left| \int_s^t \beta_r(\tilde{u};\tau)d\tau \right|.$$

On the other hand, from (A2) we have the next condition, which gives a certain absolute continuity of variational norms $\{\|\cdot\|_{\tilde{v}(t)}\,;\, 0 \leq t \leq T\}$ and was originally proposed in [4] for the time-dependent *variational* norms $\{\|\cdot\|_t\,;\, 0 \leq t \leq T\}$.

Condition 2. For all $\tilde{u} \in \mathcal{W}(u_0,v_0)$, for all $s,t \in [0,T]$, and for all $z \in H$ we have

$$\left| \|z\|^2_{\tilde{v}(t)} - \|z\|^2_{\tilde{v}(s)} \right| \leq \gamma_0 \|z\|^2_{\tilde{v}(s)} \left| \int_s^t \|\tilde{v}'(\tau)\|_X d\tau \right|.$$

2.2. Continuity of time-dependent convex functions

Throughout the rest of this section we fix a pair $(u_0,v_0) \in H \times A$ and an element $\tilde{u} \in \mathcal{W}(u_0,v_0)$. The main purpose of this subsection is to construct a function $h(\tilde{u}) \in \mathcal{W}(u_0,v_0)$, which gives a certain kind of continuity of the set-valued function $t \mapsto D(\varphi(t,\tilde{u},\tilde{v}(t)))$ from $[0,T]$ into 2^H and is precisely stated in Proposition 2.5 below. In order to show Proposition 2.5, we prepare some lemmas which are used as we constract the function $h(\tilde{u})$.

Lemma 2.1. *There exists a constant $C_1 > 0$, which is independent of the choice of u_0, v_0 and $\tilde{u}$, such that for all $t \in [0,T]$ and all $z \in H$*

$$|\varphi(t,\tilde{u},\tilde{v}(t);z)| \leq \varphi(t,\tilde{u},\tilde{v}(t);z) + C_1(\|z\|_{\tilde{v}(t)} + 1), \text{ and}$$

$$|\varphi(z)| \leq \varphi(z) + C_1(\|z\|_H + 1).$$

Proof. We see from (A1) and (C1) that there exists a constant $C_{1,1} > 0$ such that

$$\varphi(t,\tilde{u},\tilde{v}(t);z) \geq \varphi(z) \geq -C_{1,1}(\|z\|_H + 1) \geq -C_{1,1} \max\left\{\frac{1}{c_1}, 1\right\} \left(\|z\|_{\tilde{v}(t)} + 1\right)$$

for all $t \in [0,T]$ and all $z \in H$, which implies the lemma. □

Lemma 2.2. *Consider sequences $\{y_n\}_{n\in\mathbb{N}}, \{z_n\}_{n\in\mathbb{N}} \subset H$, $\{v_n\}_{n\in\mathbb{N}} \subset A$ and elements $y, z \in H$, $v \in A$ satisfying the following convergences $(*)$ as $n \to \infty$:*

$$(*) \qquad \begin{cases} y_n \longrightarrow y & \text{in } H, \\ v_n \longrightarrow v & \text{in } X, \\ z_n \longrightarrow z & \text{weakly in } H(v). \end{cases}$$

Then, we have $\lim_{n\to\infty}(z_n, y_n)_{v_n} = (z, y)_v$.

Proof. We note the following elementary equality for an inner product:

$$(z_1, z_2)_v = \frac{1}{2}\left(\|z_1 + z_2\|_v^2 - \|z_1\|_v^2 - \|z_2\|_v^2\right), \quad \forall\, v \in A,\ \forall\, z_1, z_2 \in H. \tag{2.1}$$

By using (A2), we have

$$\begin{aligned} &|(z_n, y_n)_{v_n} - (z_n, y_n)_v| \\ &\le \frac{1}{2}\left(\left|\|z_n + y_n\|_{v_n}^2 - \|z_n + y_n\|_v^2\right| + \left|\|z_n\|_{v_n}^2 - \|z_n\|_v^2\right| + \left|\|y_n\|_{v_n}^2 - \|y_n\|_v^2\right|\right) \\ &\le \frac{\gamma_0}{2}\|v_n - v\|_X \left(\|z_n + y_n\|_v^2 + \|z_n\|_v^2 + \|y_n\|_v^2\right) \\ &\le \frac{3\gamma_0}{2}\|v_n - v\|_X \left(\sup_{n\in\mathbb{N}}\|z_n\|_v^2 + \sup_{n\in\mathbb{N}}\|y_n\|_v^2\right), \end{aligned}$$

which immediately implies

$$\lim_{n\to\infty}\{(z_n, y_n)_{v_n} - (z_n, y_n)_v\} = 0 \tag{2.2}$$

because of the convergences $(*)$. Using (A1), (2.2), the convergences $(*)$ again and the following inequality:

$$|(z_n, y_n)_{v_n} - (z, y)_v| \le |(z_n, y_n)_{v_n} - (z_n, y_n)_v| + |(z_n, y_n)_v - (z, y)_v|,$$

we see that this lemma holds. □

Lemma 2.3. *There exist a constant $C_2 > 0$ depending on $\tilde{u}$, $\|u_0\|_H$ and $\varphi(0, u_0, v_0; u_0)$, and a family $\{z_t(\tilde{u})\,;\, 0 \le t \le T\}$ such that*

$$\sup_{0\le t\le T}\|z_t(\tilde{u})\|_{\tilde{v}(t)} + \sup_{0\le t\le T}|\varphi(t, \tilde{u}, \tilde{v}(t); z_t(\tilde{u}))| \le C_2,$$

hence, we have $z_t(\tilde{u}) \in D(\varphi(t, \tilde{u}, \tilde{v}(t)))$ for all $t \in [0, T]$.

Proof. We put $z_0(\tilde{u}) := u_0 \in D(\varphi(0, \tilde{u}, v_0))$. Using Condition 1 as $r = r_1 \ge \|u_0\|_{v_0} + 1$, we see that there exist nonnegative functions $\alpha_{r_1}(\tilde{u}) \in L^2(0,T)$, $\beta_{r_1}(\tilde{u}) \in L^1(0,T)$ and a family $\{z_t(\tilde{u})\,;\, 0 \le t \le T\}$ such that

$$\|z_t(\tilde{u}) - u_0\|_{\tilde{v}(t)} \le \left(\sqrt{|\varphi(0, \tilde{u}, v_0; u_0)|} + 1\right)\int_0^T \alpha_{r_1}(\tilde{u}; t)dt =: C_{2,1},$$

$$|\varphi(t,\tilde{u},\tilde{v}(t);z_t(\tilde{u}))-\varphi(0,\tilde{u},v_0;u_0)|\le(|\varphi(0,\tilde{u},v_0;u_0)|+1)\int_0^T\beta_{r_1}(\tilde{u};t)dt=:C_{2,2},$$

Using (A1) and taking out $C_2>0$ such that

$$C_2\ge c_2\|u_0\|_H+|\varphi(0,u_0,v_0;u_0)|+C_{2,1}+C_{2,2},$$

we see that this lemma holds. □

Lemma 2.4. *Consider sequences* $\{z_n\}_{n\in\mathbb{N}}\subset H$, $\{t_n\}_{n\in\mathbb{N}}\subset[0,T]$ *and elements* $z\in H$, $t\in[0,T]$ *satisfying the following conditions:*

$$z_n\in D(\varphi(t_n,\tilde{u},\tilde{v}(t_n))),\quad\forall\, n\in\mathbb{N},\tag{2.3}$$

$$z_n\longrightarrow z\quad\textit{weakly in}\quad H(\tilde{v}(t))\quad\textit{as}\quad n\to\infty,\tag{2.4}$$

$$t_n\longrightarrow t\quad\textit{as}\quad n\to\infty.\tag{2.5}$$

Then, the following inequality holds:

$$\varphi(t,\tilde{u},\tilde{v}(t);z)\le\liminf_{n\to\infty}\varphi(t_n,\tilde{u},\tilde{v}(t_n);z_n).\tag{2.6}$$

Proof. It is enough to consider the case $K_1:=\liminf_{n\to\infty}\varphi(t_n,\tilde{u},\tilde{v}(t_n);z_n)<\infty$. Then, we can choose a subsequence $\{t_{n_k}\}_{k\in\mathbb{N}}$ of $\{t_n\}_{n\in\mathbb{N}}$ satisfying

$$\lim_{k\to\infty}\varphi(t_{n_k},\tilde{u},\tilde{v}(t_{n_k});z_{n_k})=K_1.\tag{2.7}$$

We see from (A1) and (2.4) that there exists a constant $K_2>0$ such that

$$\sup_{n\in\mathbb{N}}\|z_n\|_{\tilde{v}(t)}\le K_2,\quad\text{hence,}\quad\sup_{n\in\mathbb{N}}\|z_n\|_{\tilde{v}(t_n)}\le\frac{c_2K_2}{c_1}.\tag{2.8}$$

Hence we see from (2.7), (2.8) and Lemma 2.1 that there exists a constant $K_3>0$ such that

$$\sup_{k\in\mathbb{N}}|\varphi(t_{n_k},\tilde{u},\tilde{v}(t_{n_k});z_{n_k})|\le K_3.\tag{2.9}$$

Using Condition 1 as $r=r_2\ge\frac{c_2K_2}{c_1}+1$ and (2.9), we choose nonnegative functions $\alpha_{r_2}(\tilde{u})\in L^2(0,T)$, $\beta_{r_2}(\tilde{u})\in L^1(0,T)$ and a sequence $\{\tilde{z}_k\}_{k\in\mathbb{N}}\subset D(\varphi(t,\tilde{u},\tilde{v}(t)))$ such that for all $k\in\mathbb{N}$

$$\|\tilde{z}_k-z_{n_k}\|_{\tilde{v}(t)}\le\left(\sqrt{K_3}+1\right)\left|\int_{t_{n_k}}^t\alpha_{r_2}(\tilde{u};\tau)d\tau\right|,\tag{2.10}$$

$$|\varphi(t,\tilde{u},\tilde{v}(t);\tilde{z}_k)-\varphi(t_{n_k},\tilde{u},\tilde{v}(t_{n_k});z_{n_k})|\le(K_3+1)\left|\int_{t_{n_k}}^t\beta_{r_2}(\tilde{u};\tau)d\tau\right|.\tag{2.11}$$

We see from (2.10) that for all $z\in H$

$$\lim_{k\to\infty}|(\tilde{z}_k-z,y)_{\tilde{v}(t)}|\le\lim_{k\to\infty}\|\tilde{z}_k-z_{n_k}\|_{\tilde{v}(t)}\|y\|_{\tilde{v}(t)}+\lim_{k\to\infty}|(z_{n_k}-z,y)_{\tilde{v}(t)}|$$

$$\le\left(\sqrt{K_3}+1\right)\lim_{k\to\infty}\left|\int_{t_{n_k}}^t\alpha_{r_2}(\tilde{u};\tau)d\tau\right|\|y\|_{\tilde{v}(t)}+\lim_{k\to\infty}|(z_{n_k}-z,y)_{\tilde{v}(t)}|=0,$$

which implies $\tilde{z}_k \longrightarrow z$ weakly in $H(\tilde{v}(t))$ as $k \to \infty$. Since $\varphi(t, \tilde{u}, \tilde{v}(t))$ is weakly sequentially l.s.c. on $H(\tilde{v}(t))$, we have

$$\varphi(t, \tilde{u}, \tilde{v}(t); z) \leq \liminf_{k\to\infty} \varphi(t, \tilde{u}, \tilde{v}(t); \tilde{z}_k). \tag{2.12}$$

Moreover, from (2.11) we have

$$\begin{aligned} K_1 &= \lim_{k\to\infty} \varphi(t_{n_k}, \tilde{u}, \tilde{v}(t_{n_k}); z_{n_k}) \\ &= \lim_{k\to\infty} \{\varphi(t_{n_k}, \tilde{u}, \tilde{v}(t_{n_k}); z_{n_k}) - \varphi(t, \tilde{u}, \tilde{v}(t); \tilde{z}_k)\} + \liminf_{k\to\infty} \varphi(t, \tilde{u}, \tilde{v}(t); \tilde{z}_k) \\ &= \liminf_{k\to\infty} \varphi(t, \tilde{u}, \tilde{v}(t); \tilde{z}_k). \end{aligned} \tag{2.13}$$

Hence, we see from (2.12) and (2.13) that (2.6) is satisfied. □

At the end of this subsection we show Proposition 2.5 below whose original proof has already been given in [10, Lemma 1.5.3]. But we give its proof again in this paper because it contains an important information which is needed when we derive some uniform estimates of solutions to Cauchy problems of auxiliary evolution equations considered in Sections 3 and 4. Moreover, this proof is much more delicate than that of [10, Lemma 1.5.3] because the inner products themselves on H depend on a time-dependent function $\tilde{v}$.

Proposition 2.5. *There exist nonnegative functions $\alpha(\tilde{u}) \in L^2(0,T)$ and $\beta(\tilde{u}) \in L^1(0,T)$, a number $n(\tilde{u}) \in \mathbb{N}$ which is uniquely determined, a sequence $\{z_n(\tilde{u})\,;\ 0 \leq n \leq n(\tilde{u})\}$, a partition*

$$\{t_0(\tilde{u}) = 0 < t_1(\tilde{u}) < t_2(\tilde{u}) < \cdots < t_{n(\tilde{u})-1}(\tilde{u}) < t_{n(\tilde{u})}(\tilde{u}) = T\}$$

of $[0,T]$, constants $r(\tilde{u}) > 0$ and $M(\tilde{u}) > 0$, and a function $h(\tilde{u}) \in W^{1,2}(0,T;H)$ such that

$$\max_{0\leq n\leq n(\tilde{u})} \|z_n(\tilde{u})\|_H + \frac{1}{c_2} \leq r(\tilde{u}), \tag{2.14}$$

$$\max_{0\leq n\leq n(\tilde{u})} |\varphi(t_n(\tilde{u}), \tilde{u}, \tilde{v}(t_n(\tilde{u})); z_n(\tilde{u}))| + C_1\{c_2 r(\tilde{u}) + 1\} + 1 \leq M(\tilde{u}), \tag{2.15}$$

$$\left\{1 + M(\tilde{u}) \exp\left(\int_0^T \beta(\tilde{u};t)dt\right)\right\} \int_{t_{n-1}(\tilde{u})}^{t_n(\tilde{u})} \alpha(\tilde{u};t)dt = \frac{c_1}{c_2}, \quad \text{for all } n = 1, 2, \ldots, n(\tilde{u}) - 1, \tag{2.16}$$

$$\left\{1 + M(\tilde{u}) \exp\left(\int_0^T \beta(\tilde{u};t)dt\right)\right\} \int_{t_{n(\tilde{u})-1}(\tilde{u})}^{T} \alpha(\tilde{u};t)dt \leq \frac{c_1}{c_2}, \tag{2.17}$$

and for any $n \in \{0, 1, 2, \ldots, n(\tilde{u}) - 1\}$ the following properties are satisfied:

$$h(\tilde{u};t) \longrightarrow z_n(\tilde{u}) \quad \text{in } H \quad \text{as} \quad t \searrow t_n(\tilde{u}), \tag{2.18}$$

$$\limsup_{t\searrow t_n(\tilde{u})} \varphi(t, \tilde{u}, \tilde{v}(t); h(\tilde{u};t)) \leq \varphi(t_n(\tilde{u}), \tilde{u}, \tilde{v}(t_n(\tilde{u})); z_n(\tilde{u})), \tag{2.19}$$

$$\sup_{t_n(\tilde{u})<t\leq t_{n+1}(\tilde{u})} \|h(\tilde{u};t)\|_H \leq r(\tilde{u}), \tag{2.20}$$

$$\begin{aligned} &\sup_{t_n(\tilde{u})<t\leq t_{n+1}(\tilde{u})} |\varphi(t,\tilde{u},\tilde{v}(t);h(\tilde{u};t))| \\ &\qquad \leq M(\tilde{u})\left\{1+\exp\left(\int_0^T \beta(\tilde{u};t)dt\right)\int_{t_n(\tilde{u})}^t \beta(\tilde{u};\tau)d\tau\right\}, \end{aligned} \tag{2.21}$$

$$\|(h(\tilde{u}))'(t)\|_H \leq \left\{1+M(\tilde{u})\exp\left(\int_0^T \beta(\tilde{u};t)dt\right)\right\}\alpha(\tilde{u};t), \ \ a.a.\ t\in(0,T), \tag{2.22}$$

where $C_1>0$ *is the same constant that is obtained in Lemma* 2.1.

Proof. We let $C_1>0$, $C_2>0$ and $\{z_s(\tilde{u})\,;\,0\leq s\leq T\}$ the same constants and family that are obtained in Lemmas 2.1 and 2.3. We choose $r(\tilde{u})>0$ and $M(\tilde{u})>0$ such that

$$r(\tilde{u}) \geq \frac{C_2}{c_1}+\frac{1}{c_2}, \qquad M(\tilde{u})\geq C_2+c_2C_1r(\tilde{u})+1. \tag{2.23}$$

Then we see from (A1) that the following estimates are satisfied:

$$\sup_{0\leq s\leq T}\|z_s(\tilde{u})\|_H+\frac{1}{c_2}\leq \frac{1}{c_1}\sup_{0\leq s\leq T}\|z_s(\tilde{u})\|_{\tilde{v}(s)}+\frac{1}{c_2}\leq r(\tilde{u}), \tag{2.24}$$

$$\sup_{0\leq s\leq T}\|z_s(\tilde{u})\|_{\tilde{v}(s)}+1\leq c_2\left(\sup_{0\leq s\leq T}\|z_s(\tilde{u})\|_H+\frac{1}{c_2}\right)\leq c_2r(\tilde{u}), \tag{2.25}$$

$$\sup_{0\leq s\leq T}|\varphi(s,\tilde{u}(s),\tilde{v}(s);z_s(\tilde{u}))|+c_2C_1r(\tilde{u})+1\leq M(\tilde{u}). \tag{2.26}$$

Using Condition 1 as $r=c_2r(\tilde{u})$, we see that there exist nonnegative functions $\alpha(\tilde{u})\in L^2(0,T)$, $\beta(\tilde{u})\in L^1(0,T)$ and a family $\{y_s(\tilde{u};t)\,;\,0\leq s\leq T,\ 0\leq t\leq T\}$ such that

$$\|y_s(\tilde{u};t)-z_s(\tilde{u})\|_{\tilde{v}(t)}\leq\left(\sqrt{|\varphi(s,\tilde{u},\tilde{v}(s);z_s(\tilde{u}))|}+1\right)\left|\int_s^t\alpha(\tilde{u};\tau)d\tau\right|, \tag{2.27}$$

$$\begin{aligned} &|\varphi(t,\tilde{u},\tilde{v}(t);y_s(\tilde{u};t))-\varphi(s,\tilde{u},\tilde{v}(s);z_s(\tilde{u}))| \\ &\qquad \leq (|\varphi(s,\tilde{u},\tilde{v}(s);z_s(\tilde{u}))|+1)\left|\int_s^t\beta(\tilde{u};\tau)d\tau\right|. \end{aligned} \tag{2.28}$$

We define a number $n(\tilde{u})\in\mathbb{N}$ by

$$n(\tilde{u})=\begin{cases}[A]+1 & \text{if}\quad A\notin\{0\}\cup\mathbb{N},\\ [A] & \text{if}\quad A\in\{0\}\cup\mathbb{N},\end{cases} \tag{2.29}$$

where $[A]$ denotes the Gauss notation of A and

$$A:=\frac{c_2}{c_1}\left\{1+M(\tilde{u})\exp\left(\int_0^T\beta(\tilde{u};t)dt\right)\right\}\int_0^T\alpha(\tilde{u};t)dt.$$

Since the functions $\alpha(\tilde{u}) \in L^2(0,T)$ and $\beta(\tilde{u}) \in L^1(0,T)$ are nonnegative, we can choose a partition $\{0 = t_0(\tilde{u}) < t_1(\tilde{u}) < \cdots < t_{n(\tilde{u})-1}(\tilde{u}) < t_{n(\tilde{u})}(\tilde{u}) = T\}$ of $[0,T]$ such that (2.16) and (2.17) are satisfied. Moreover, we easily see that the sequence $\{z_n(\tilde{u})\,;\, 0 \le n \le n(\tilde{u})\} := \{z_{t_n(\tilde{u})}(\tilde{u})\,;\, 0 \le n \le n(\tilde{u})\}$ satisfies (2.14) and (2.15).

Next, we construct a function $h_n(\tilde{u}) \in W^{1,2}(t_{n-1}(\tilde{u}), t_n(\tilde{u}); H)$ for each $n \in \{1, 2, \ldots, n(\tilde{u})\}$ which satisfies (2.18)–(2.22). In order to do this, first of all for any $m \in \mathbb{N}$ we consider a partition $\{t_{n-1}(\tilde{u}) = t_0 < t_1 < t_2 < \cdots < t_m = t_n(\tilde{u})\}$ of $[t_{n-1}(\tilde{u}), t_n(\tilde{u})]$ and construct a sequence

$$\{z_k^m\,;\, 0 \le k \le m\} \subset \overline{B(0, r(\tilde{u}))} := \{z \in H\,;\, \|z\|_H \le r(\tilde{u})\}$$

such that the following properties are satisfied for all $k \in \{1, 2, \ldots, m\}$:

$$\|z_k^m - z_{k-1}^m\|_{\tilde{v}(t_k)} \le \left\{1 + M(\tilde{u}) \exp\left(\int_0^{t_{k-1}} \beta(\tilde{u};t)dt\right)\right\} \int_{t_{k-1}}^{t_k} \alpha(\tilde{u};t)dt, \tag{2.30}$$

$$\begin{aligned} &|\varphi(t_k, \tilde{u}, \tilde{v}(t_k); z_k^m) - \varphi(t_{k-1}, \tilde{u}, \tilde{v}(t_{k-1}; z_{k-1}^m)| \\ &\qquad \le M(\tilde{u}) \exp\left(\int_0^{t_{k-1}} \beta(\tilde{u};t)dt\right) \int_{t_{k-1}}^{t_k} \beta(\tilde{u};t)dt, \end{aligned} \tag{2.31}$$

$$1 + |\varphi(t_k, \tilde{u}, \tilde{v}(t_k); z_k^m)| \le M(\tilde{u}) \exp\left(\int_0^{t_k} \beta(\tilde{u};t)dt\right). \tag{2.32}$$

Putting $z_0^m := z_{t_n(\tilde{u})}(\tilde{u})$, we see from (2.24) that $z_0^m \in \overline{B(0, r(\tilde{u}))}$. Now, we assume that we have already constructed a sequence

$$\{z_j^m\,;\, 0 \le j \le k\} \subset \overline{B(0, r(\tilde{u}))}, \quad \forall\, k = 0, 1, 2, \ldots, m-1,$$

which satisfies (2.30)–(2.32). We note from (A1) that $\|z_k^m\|_{\tilde{v}(t_k)} \le c_2 r(\tilde{u})$ because of $z_k^m \in \overline{B(0, r(\tilde{u}))}$. Using Condition 1 and (2.32), we see that there exists an element $z_{k+1}^m \in D(\varphi(t_{k+1}, \tilde{u}, \tilde{v}(t_{k+1})))$ such that

$$\begin{aligned} \|z_{k+1}^m - z_k^m\|_{\tilde{v}(t_{k+1})} &\le \left(\sqrt{|\varphi(t_k, \tilde{u}, \tilde{v}(t_k); z_k^m)|} + 1\right) \int_{t_k}^{t_{k+1}} \alpha(\tilde{u};t)dt \\ &\le \int_{t_k}^{t_{k+1}} \alpha(\tilde{u};t)dt + (|\varphi(t_k, \tilde{u}, \tilde{v}(t_k); z_k^m)| + 1) \int_{t_k}^{t_{k+1}} \alpha(\tilde{u};t)dt \\ &\le \left\{1 + M(\tilde{u}) \exp\left(\int_0^{t_k} \beta(\tilde{u};t)dt\right)\right\} \int_{t_k}^{t_{k+1}} \alpha(\tilde{u};t)dt, \text{ and} \end{aligned} \tag{2.33}$$

$$\begin{aligned} &|\varphi(t_{k+1}, \tilde{u}, \tilde{v}(t_{k+1}); z_{k+1}^m) - \varphi(t_k, \tilde{u}, \tilde{v}(t_k); z_k^m)| \\ &\qquad \le (1 + |\varphi(t_k, \tilde{u}, \tilde{v}(t_k); z_k^m)|) \int_{t_k}^{t_{k+1}} \beta(\tilde{u};t)dt \\ &\qquad \le M(\tilde{u}) \exp\left(\int_0^{t_k} \beta(\tilde{u};t)dt\right) \int_{t_k}^{t_{k+1}} \beta(\tilde{u};t)dt, \end{aligned} \tag{2.34}$$

where $\alpha(\tilde{u})$ and $\beta(\tilde{u})$ are the same functions which are obtained in the argumentation above. Then, (2.33) and (2.34) directly imply that (2.30) and (2.31) are valid for $j = k + 1$. Moreover, we see from (2.16), (2.17), (2.30) and (2.33) that

$$
\begin{aligned}
\|z_{k+1}^m\|_H &\le \|z_0^m\|_H + \sum_{j=1}^{k+1} \|z_j^m - z_{j-1}^m\|_H \le \|z_0\|_H + \frac{1}{c_1}\sum_{j=1}^{k+1} \|z_j^m - z_{j-1}^m\|_{\tilde{v}(t_j)} \\
&\le \|z_0^m\|_H + \frac{1}{c_1}\sum_{j=1}^{k+1} \left\{1 + M(\tilde{u})\exp\left(\int_0^{t_{j-1}} \beta(\tilde{u};t)dt\right)\right\} \int_{t_{j-1}}^{t_j} \alpha(\tilde{u};t)dt \\
&\le \|z_0^m\|_H + \frac{1}{c_1}\left\{1 + M(\tilde{u})\exp\left(\int_0^{t_k} \beta(\tilde{u};t)dt\right)\right\} \int_{t_0}^{t_{k+1}} \alpha(\tilde{u};t)dt \\
&\le \|z_0^m\|_H + \frac{1}{c_1}\left\{1 + M(\tilde{u})\exp\left(\int_0^{T} \beta(\tilde{u};t)dt\right)\right\} \int_{t_{n-1}(\tilde{u})}^{t_n(\tilde{u})} \alpha(\tilde{u};t)dt \\
&\le \|z_{t_{n-1}(\tilde{u})}(\tilde{u})\|_H + \frac{1}{c_2} \le \frac{\|z_{t_{n-1}(\tilde{u})}(\tilde{u})\|_{\tilde{v}(t_{n-1}(\tilde{u}))}}{c_1} \le r(\tilde{u}),
\end{aligned}
\tag{2.35}
$$

which implies $z_{k+1}^m \in \overline{B(0, r(\tilde{u}))}$. Hence, we have $\|z_{k+1}^m\|_{\tilde{v}(t_{k+1})} \le c_2 r(\tilde{u})$. By using (2.32) and (2.34), we have

$$
\begin{aligned}
1 + |\varphi(t_{k+1}, \tilde{u}, \tilde{v}(t_{k+1}); z_{k+1}^m)| &\le (1 + |\varphi(t_k, \tilde{u}, \tilde{v}(t_k); z_k^m)|)\left(\int_{t_k}^{t_{k+1}} \beta(\tilde{u};t)dt + 1\right) \\
&\le M(\tilde{u})\left(\int_{t_k}^{t_{k+1}} \beta(\tilde{u};t)dt + 1\right)\exp\left(\int_0^{t_k} \beta(\tilde{u};t)dt\right) \\
&\le M(\tilde{u})\exp\left(\int_0^{t_{k+1}} \beta(\tilde{u};t)dt\right),
\end{aligned}
$$

which implies that (2.32) is valid for $k+1$. Hence, the set $\{z_k^m\,;\, n = 0, 1, 2, \dots, m\}$ is as desired.

Next, for any $m \in \mathbb{N}$ we consider a sequence $\{t_k^m\,;\, k = 0, 1, 2, \dots, m\}$ defined by

$$
t_k^m = t_{n-1}(\tilde{u}) + \frac{k\{t_n(\tilde{u}) - t_{n-1}(\tilde{u})\}}{m}, \quad \forall\, k = 0, 1, 2, \dots, m,
$$

and construct a set $\{z_k^m\,;\, k = 0, 1, 2, \dots, m\} \subset B(0, r(\tilde{u}))$ according to the previous argumentation. For $m \in \mathbb{N}$ we define $h_{n,m}(\tilde{u}) \in C([t_{n-1}(\tilde{u}), t_n(\tilde{u})]; H)$ by

$$
h_{n,m}(\tilde{u};t) := \begin{cases} z_0^m = z_{t_{n-1}(\tilde{u})}(\tilde{u}) & \text{if } t = t_{n-1}(\tilde{u}), \\ \dfrac{m(t - t_{k-1}^m)}{t_n(\tilde{u}) - t_{n-1}(\tilde{u})}(z_k^m - z_{k-1}^m) + z_{k-1}^m & \\ & \text{if } t \in (t_{k-1}^m, t_k^m],\ \forall\, k \in \{1, 2, \dots, m\}. \end{cases}
$$

Because of $\{z_k^m\,;\,k=0,1,2,\dots,m\}\subset B(0,r(\tilde u))$, we have

$$\max_{t_{n-1}(\tilde u)\le t\le t_n(\tilde u)} \|h_{n,m}(\tilde u;t)\|_H \le r(\tilde u). \tag{2.36}$$

Using (C1) with the convexities of φ and $\varphi(t,\tilde u,\tilde v(t))$, (2.32) and (2.26), we have

$$\begin{aligned}\sup_{t_{k-1}^m<t\le t_k^m} |\varphi(h_{n,m}(\tilde u;t))| &\le \sup_{t_{k-1}^m<t\le t_k^m} \{\varphi(h_{n,m}(\tilde u;t)) + C_1\left(\|h_{n,m}(\tilde u;t)\|_H+1\right)\} \\ &\le \sup_{t_{k-1}^m<t\le t_k^m} \{\varphi(t,\tilde u,\tilde v(t);h_{n,m}(\tilde u;t)) + C_1(\|h_{n,m}(\tilde u;t)\|_H+1)\} \\ &\le M(\tilde u)\exp\left(\int_0^T \beta(\tilde u;t)dt\right) + C_1\{r(\tilde u)+1\},\ \forall\, k\in\{1,2,\dots,m\} \end{aligned}\tag{2.37}$$

as well as

$$\begin{aligned}|\varphi(h_{n,m}(\tilde u;t_{n-1}(\tilde u)))| &\le \varphi(t,\tilde u,\tilde v(t_{n-1}(\tilde u));z_0^m) + C_1(\|z_0^m\|_H+1) \\ &\le M(\tilde u) + C_1\{r(\tilde u)+1\}. \end{aligned}\tag{2.38}$$

Using (C1) with (2.36)–(2.38) and applying Ascoli-Arzelà's theorem, we see that there exist a subsequence $\{m_k\}_{k\in\mathbb N}$ of $\{m\}_{m\in\mathbb N}$ and $h_n(\tilde u)\in C([t_{n-1}(\tilde u),t_n(\tilde u)];H)$ such that

$$h_{n,m_k}(\tilde u) \longrightarrow h_n(\tilde u) \quad \text{in} \quad C([t_{n-1}(\tilde u),t_n(\tilde u)];H) \quad \text{as} \quad k\to\infty. \tag{2.39}$$

From (2.30) we have for all $s,t\in[t_{n-1}(\tilde u),t_n(\tilde u)]$ with $s\le t$

$$\begin{aligned}\int_s^t \|(h_{n,m}(\tilde u))'(\tau)\|_H^2 d\tau &\le \sum_{j=k_s}^{k_t}\int_{t_{j-1}^m}^{t_j^m}\left(\frac{m}{t_n(\tilde u)-t_{n-1}(\tilde u)}\right)^2 \|z_j^m - z_{j-1}^m\|_H^2 d\tau \\ \le \frac{m}{t_n(\tilde u)-t_{n-1}(\tilde u)} &\sum_{j=k_s}^{k_t}\left\{1+M(\tilde u)\exp\left(\int_0^T\beta(\tilde u;t)dt\right)\right\}^2\left(\int_{t_{j-1}^m}^{t_j^m}\alpha(\tilde u;t)dt\right)^2 \\ \le \left\{1+M(\tilde u)\exp\left(\int_0^T\beta(\tilde u;t)dt\right)\right\}^2 &\int_{t_{k_s-1}^m}^{t_{k_t}^m}|\alpha(\tilde u;t)|^2dt, \end{aligned}\tag{2.40}$$

where $k_s,k_t\in\mathbb N$ are determined so that $s\in(t_{k_s-1}^m,t_{k_s}^m]$ and $t\in(t_{k_t-1}^m,t_{k_t}^m]$, respectively. By (2.40) the sequence $\{h_{n,m}(\tilde u)\}_{m\in\mathbb N}$ is bounded in $W^{1,2}(t_{n-1}(\tilde u),t_n(\tilde u);H)$, we see that there exists a subsequence of $\{m_k\}_{k\in\mathbb N}$, which is denoted by the same notation $\{m_k\}_{k\in\mathbb N}$ again, such that the following convergence holds as $k\to\infty$:

$$h_{n,m_k}(\tilde u)\longrightarrow h_n(\tilde u) \quad \text{weakly in } W^{1,2}(t_{n-1}(\tilde u),t_n(\tilde u);H). \tag{2.41}$$

It follows from (2.36)–(2.41) that (2.20), (2.21) and

$$\int_s^t\|(h_n(\tilde u))'(\tau)\|_H^2d\tau \le \left\{1+M(\tilde u)\exp\left(\int_0^T\beta(\tilde u;t)dt\right)\right\}^2\int_s^t|\alpha(\tilde u;\tau)|^2d\tau, \tag{2.42}$$

which implies that (2.22) holds for a.a. $t \in (t_{n-1}(\tilde{u}), t_n(\tilde{u}))$. Moreover, from (2.30) and (2.39) we have

$$\begin{aligned}
\|z^m_{k_t-1} - h_n(\tilde{u};t)\|_H &\le \|h_{n,m}(\tilde{u};t) - h_n(\tilde{u};t)\|_H + \frac{m(t - t^m_{k_t-1})}{t_n(\tilde{u}) - t_{n-1}(\tilde{u})}\|z^m_{k_t} - z^m_{k_t-1}\|_H \\
&\le \|h_{n,m}(\tilde{u};t) - h_n(\tilde{u};t)\|_H + \left(\frac{t_n(\tilde{u}) - t_{n-1}(\tilde{u})}{m}\right)^{\frac{1}{2}} \\
&\qquad \times \left\{1 + M(\tilde{u}) \exp\left(\int_0^T \beta(\tilde{u};t)dt\right)\right\} \left(\int_{t^m_{k_t-1}}^{t^m_{k_t}} |\alpha(\tilde{u};t)|^2 dt\right)^{\frac{1}{2}},
\end{aligned}$$

which implies

$$z^{m_k}_{k_t-1} \longrightarrow h_n(\tilde{u};t) \quad \text{in } H \quad \text{for all } t \in [t_{n-1}(\tilde{u}), t_n(\tilde{u})] \quad \text{as} \quad k \to \infty. \tag{2.43}$$

Using Lemma 2.4 and (2.31), (2.43), we have

$$\begin{aligned}
&\varphi(t, \tilde{u}, \tilde{v}(t); h_n(\tilde{u};t)) \le \liminf_{k\to\infty} \tilde{\varphi}(t^{m_k}_{k_t-1}; z^{m_k}_{k_t-1}) \\
&\le \liminf_{k\to\infty} \left\{\varphi(t_0, \tilde{u}, \tilde{v}(t_0); z_0) + M(\tilde{u}) \exp\left(\int_0^T \beta(\tilde{u};t)dt\right) \int_{t_n(\tilde{u})}^{t^{m_k}_{k_t-1}} \beta(\tilde{u};t)dt\right\} \\
&\le \varphi(t_{n-1}(\tilde{u}), \tilde{u}, \tilde{v}(t_{n-1}(\tilde{u})); z_{t_{n-1}(\tilde{u})}(\tilde{u})) + M(\tilde{u}) \exp\left(\int_0^T \beta(\tilde{u};t)dt\right) \int_{t_n(\tilde{u})}^t \beta(\tilde{u};s)ds,
\end{aligned}$$

hence, from Lemma 2.1

$$\begin{aligned}
&\sup_{t_{n-1}(\tilde{u}) < t \le t_n(\tilde{u})} |\tilde{\varphi}(t, \tilde{u}, \tilde{v}(t); h_n(\tilde{u};t))| \\
&\le \sup_{t_{n-1}(\tilde{u}) < t \le t_n(\tilde{u})} \varphi(t, \tilde{u}, \tilde{v}(t); h_n(\tilde{u};t)) + C_1(\|h_n(\tilde{u};t)\|_{\tilde{v}(t)} + 1) \\
&\le |\varphi(t_{n-1}(\tilde{u}), \tilde{u}, \tilde{v}(t_{n-1}(\tilde{u})); z_{t_{n-1}(\tilde{u})}(\tilde{u}))| + C_1\{r(\tilde{u}) + 1\} \\
&\qquad\qquad + M(\tilde{u}) \exp\left(\int_0^T \beta(\tilde{u};t)dt\right) \int_{t_{n-1}(\tilde{u})}^t \beta(\tilde{u};s)ds \\
&\le M(\tilde{u}) \left\{1 + \exp\left(\int_0^T \beta(\tilde{u};t)dt\right) \int_{t_{n-1}(\tilde{u})}^t \beta(\tilde{u};s)ds\right\}, \quad \text{and}
\end{aligned}$$

$$\limsup_{t \searrow t_{n-1}(\tilde{u})} \varphi(t, \tilde{u}, \tilde{v}(t); h_n(\tilde{u};t)) \le \varphi(t_{n-1}(\tilde{u}), \tilde{u}, \tilde{v}(t_{n-1}(\tilde{u})); z_{t_{n-1}(\tilde{u})}(\tilde{u})), \tag{2.44}$$

which imply that (2.19) holds. At last, we define the function $h(\tilde{u})$ by

$$h(\tilde{u};t) := \begin{cases} u_0 & \text{if} \quad t = 0, \\ h_n(\tilde{u};t) & \text{if} \quad t \in (t_{n-1}(\tilde{u}), t_n(\tilde{u})],\ \forall\, n = 1, 2, \ldots, n(\tilde{u}). \end{cases}$$

Then, the function $h(\tilde{u})$ is as desired and the proof is complete. □

Remark 2.6. Let $h(\tilde{u})$ be the same function that is constructed in Proposition 2.5. From (2.21) we have $h(\tilde{u};t) \in D(\varphi(t,\tilde{u},\tilde{v}(t)))$ for all $t \in [0,T]$ and

$$\sup_{0\le t\le T} |\varphi(t,\tilde{u},\tilde{v}(t);h(\tilde{u};t))| \le M(\tilde{u}) \left\{ 1 + \exp\left(\int_0^T \beta(\tilde{u};t)dt \right) \int_0^T \beta(\tilde{u};t)dt \right\},$$

$$\|(h(\tilde{u}))'\|^2_{L^2(0,T;H)} \le \left\{ 1 + M(\tilde{u}) \exp\left(\int_0^T \beta(\tilde{u};t)dt \right) \right\} \int_0^T |\alpha(\tilde{u};t)|^2 dt.$$

Hence, we see that there exists a constant $C_3 > 0$ depending on $\tilde{u}$, $\|u_0\|_H$ and $\varphi(0,u_0,v_0;u_0)$, such that

$$\begin{aligned}\langle h(\tilde{u})\rangle &:= n(\tilde{u}) + \sup_{0\le t\le T} \|h(\tilde{u};t)\|_H + \sup_{0\le t\le T} |\varphi(t,\tilde{u},\tilde{v}(t);h(\tilde{u};t))| + \|(h(\tilde{u}))'\|^2_{L^2(0,T;H)} \\ &\le C_3.\end{aligned}$$

2.3. Approximation of time-dependent convex functions

In this subsection, we consider the subdifferential $\partial_{\tilde{v}(t)}\varphi(t,\tilde{u},\tilde{v}(t))$ of a proper l.s.c. convex function $\varphi(t,\tilde{u},\tilde{v}(t))$ on $H(\tilde{v}(t))$, that is, $z^* \in \partial_{\tilde{v}(t)}\varphi(t,\tilde{u},\tilde{v}(t);z)$ if and only if $z \in D(\varphi(t,\tilde{u},\tilde{v}(t))$ and

$$(z^*, y - z)_{\tilde{v}(t)} \le \varphi(t,\tilde{u},\tilde{v}(t);y) - \varphi(t,\tilde{u},\tilde{v}(t);z), \quad \forall\, y \in H. \tag{2.45}$$

For each $\lambda \in (0,1]$ and each $t \in [0,T]$ we denote by $J_\lambda(t,\tilde{u},\tilde{v}(t))$ and $A_\lambda(t,\tilde{u},\tilde{v}(t))$ Yosida's approximation and the resolvent of $\partial_{\tilde{v}(t)}\varphi(t,\tilde{u},\tilde{v}(t))$, respectively, and by $\varphi_\lambda(t,\tilde{u},\tilde{v}(t))$ Yosida's regularization of $\varphi(t,\tilde{u},\tilde{v}(t))$ on $H(\tilde{v}(t))$, defined by

$$J_\lambda(t,\tilde{u},\tilde{v}(t)) := (I + \lambda\partial_{\tilde{v}(t)}\varphi(t,\tilde{u},\tilde{v}(t)))^{-1}, \tag{2.46}$$

$$A_\lambda(t,\tilde{u},\tilde{v}(t)) := \frac{1}{\lambda}(I - J_\lambda(t,\tilde{u},\tilde{v}(t))), \tag{2.47}$$

$$\forall\, z \in H, \quad \varphi_\lambda(t,\tilde{u},\tilde{v}(t);z) := \inf_{y\in H} \left\{ \frac{1}{2\lambda}\|z-y\|^2_{\tilde{v}(t)} + \varphi(t,\tilde{u},\tilde{v}(t);y) \right\}. \tag{2.48}$$

According to the general theory, for example, in [1], we already know that the following properties hold for all $t \in [0,T]$, all $\lambda \in (0,1]$ and all $z \in H$:

$$A_\lambda(t,\tilde{u},\tilde{v}(t);z) = \partial_{\tilde{v}(t)}\varphi_\lambda(t,\tilde{u},\tilde{v}(t);z), \tag{2.49}$$

$$A_\lambda(t,\tilde{u},\tilde{v}(t);z) \in \partial_{\tilde{v}(t)}\varphi(t,\tilde{u},\tilde{v}(t);J_\lambda(t,\tilde{u},\tilde{u}(t);z)), \tag{2.50}$$

$$\varphi_\lambda(t,\tilde{u},\tilde{v}(t);z) = \frac{\lambda}{2}\|A_\lambda(t,\tilde{u},\tilde{v}(t);z)\|^2_{\tilde{v}(t)} + \varphi(t,\tilde{u},\tilde{v}(t);J_\lambda(t,\tilde{u},\tilde{v}(t);z)). \tag{2.51}$$

Moreover, we show the next lemma, which is repeatedly used in Section 3 in order to derive the uniform estimates for approximate solutions to Cauchy problems of approximate evolution equations to auxiliary problems for $(\mathrm{P})_{u_0,v_0}$.

Lemma 2.7. *The following uniform boundedness properties are satisfied:*

(1) *There exists a constant $C_4 > 0$ depending on $\tilde{u}$, $\|u_0\|_H$ and $\varphi(0, u_0, v_0; u_0)$, such that for all $t \in [0, T]$ and all $z \in H$*

$$\sup_{0<\lambda\leq 1} \|J_\lambda(t, \tilde{u}, \tilde{v}(t); z)\|_{\tilde{v}(t)} \leq \|z\|_{\tilde{v}(t)} + C_4, \tag{2.52}$$

hence, for all $\lambda \in (0, 1]$, all $t \in [0, T]$ and all $z \in H$ we have

$$\|A_\lambda(t, \tilde{u}, \tilde{v}(t); z)\|_{\tilde{v}(t)} \leq \frac{2\|z\|_{\tilde{v}(t)} + C_4}{\lambda}, \quad \text{and} \tag{2.53}$$

$$|\varphi_\lambda(t, \tilde{u}, \tilde{v}(t); z)| \leq \varphi_\lambda(t, \tilde{u}, \tilde{v}(t); z) + 2C_1 \left(\|z\|_{\tilde{v}(t)} + C_4 + 1\right). \tag{2.54}$$

(2) *There exists a constant $C_5 > 0$ depending on $\tilde{u}$, $\|u_0\|_H$ and $\varphi(0, u_0, v_0; u_0)$, such that for all $\lambda \in (0, 1]$, all $t \in [0, T]$ and all $z \in H$*

$$|\varphi(t, \tilde{u}, \tilde{v}(t); J_\lambda(t, \tilde{u}, \tilde{v}(t); z))| \leq \frac{C_5 \left(\|z\|_{\tilde{v}(t)} + 1\right)^2}{\lambda}. \tag{2.55}$$

(3) *There exists a constant $C_6 > 0$ depending on $\tilde{u}$, $\|u_0\|_H$ and $\varphi(0, u_0, v_0; u_0)$, such that for all $\lambda \in (0, 1]$, all $t \in [0, T]$ and all $z \in H$*

$$|\varphi_\lambda(t, \tilde{u}(t), \tilde{v}(t); z)| \leq \frac{C_6 \left(\|z\|_{\tilde{v}(t)} + 1\right)^2}{\lambda}. \tag{2.56}$$

Proof. Throughout this proof we fix a parameter $\lambda \in (0, 1]$, a time $t \in [0, T]$ and an element $z \in H$.

(1) Let $\{z_s(\tilde{u})\,;\, 0 \leq s \leq T\}$ be the same family in H as obtained in Lemma 2.3. Using Lemmas 2.1 and 2.3 with (2.47), (2.48), (2.51), we have for all $s \in [0, T]$

$$\begin{aligned}
C_2 &\geq \varphi(s, \tilde{u}, \tilde{v}(t); z_s(\tilde{u})) \geq \varphi_1(s, \tilde{u}, \tilde{v}(s); z_s(\tilde{u})) \\
&= \frac{1}{2}\|z_s(\tilde{u}) - J_1(t, \tilde{u}, \tilde{v}(t); z_s(\tilde{u}))\|^2_{\tilde{v}(s)} + \varphi(s, \tilde{u}, \tilde{v}(s); J_1(s, \tilde{u}, \tilde{v}(s); z_s(\tilde{u}))) \\
&\geq \frac{1}{2}\|z_s(\tilde{u}) - J_1(s, \tilde{u}, \tilde{v}(s); z_s(\tilde{u}))\|^2_{\tilde{v}(s)} - C_1(\|J_1(s, \tilde{u}, \tilde{v}(s); z_s(\tilde{u}))\|_{\tilde{v}(s)} + 1).
\end{aligned}$$

Hence, we see that there exists a constant $C_{4,1} > 0$ depending on $\tilde{u}$, $\|u_0\|_H$ and $\varphi(0, u_0, v_0; u_0)$, such that

$$\sup_{0\leq s\leq T} \|J_1(s, \tilde{u}, \tilde{v}(s); z_s(\tilde{u}))\|_{\tilde{v}(s)} \leq C_{4,1}. \tag{2.57}$$

Since for all $s \in [0, T]$ the resolvent $J_1(s, \tilde{u}, \tilde{v}(s))$ is a contraction on $H(\tilde{v}(s))$, we have

$$\begin{aligned}
&\sup_{0\leq s\leq T} \|J_1(s, \tilde{u}, \tilde{v}(s); u_0)\|_{\tilde{v}(s)} \\
&\leq \sup_{0\leq s\leq T} \|J_1(s, \tilde{u}, \tilde{v}(s); u_0) - J_1(s, \tilde{u}, \tilde{v}(s); z_s(\tilde{u}))\|_{\tilde{v}(s)} + \sup_{0\leq s\leq T} \|J_1(s, \tilde{u}, \tilde{v}(s); z_s(\tilde{u}))\|_{\tilde{v}(s)}
\end{aligned}$$

$$\leq \sup_{0\leq s\leq T}\|u_0 - z_s(\tilde{u})\|_{\tilde{v}(s)} + \sup_{0\leq s\leq T}\|J_1(s,\tilde{u},\tilde{v}(s);z_s(\tilde{u}))\|_{\tilde{v}(s)}$$
$$\leq c_2\|u_0\|_H + C_2 + C_{4,1}.$$

Using the resolvent equation for all $\lambda \in (0,1]$, all $z \in H$ and all $s \in [0,T]$

$$J_1(s,\tilde{u},\tilde{v}(s);z) = J_\lambda(s,\tilde{u},\tilde{v}(s);\lambda z + (1-\lambda)J_1(s,\tilde{u},\tilde{v}(s);z)),$$

and the contraction property of $J_\lambda(s,\tilde{u},\tilde{v}(s))$ on $H(\tilde{v}(s))$, we have

$$\begin{aligned}
&\sup_{\substack{0<\lambda\leq 1\\0\leq s\leq T}}\|J_\lambda(s,\tilde{u},\tilde{v}(s);u_0)\|_{\tilde{v}(s)}\\
&\leq \sup_{\substack{0<\lambda\leq 1\\0\leq s\leq T}}\|J_\lambda(s,\tilde{u},\tilde{v}(s);u_0) - J_1(s,\tilde{u},\tilde{v}(s);u_0)\|_{\tilde{v}(s)} + \sup_{0\leq s\leq T}\|J_1(s,\tilde{u},\tilde{v}(s);u_0)\|_{\tilde{v}(s)}\\
&\leq \sup_{\substack{0<\lambda\leq 1\\0\leq s\leq T}}\|J_\lambda(s,\tilde{u},\tilde{v}(s);u_0) - J_\lambda(s,\tilde{u},\tilde{v}(s);\lambda u_0 + (1-\lambda)J_1(s,\tilde{u},\tilde{v}(s);u_0))\|_{\tilde{v}(s)}\\
&\qquad\qquad + \sup_{0\leq s\leq T}\|J_1(s,\tilde{u},\tilde{v}(s);u_0)\|_{\tilde{v}(s)}\\
&\leq \sup_{0<\lambda\leq 1}(1-\lambda)\left\{\sup_{0\leq s\leq T}\|u_0 - J_1(s,\tilde{u},\tilde{v}(s);u_0)\|_{\tilde{v}(s)}\right\} + \sup_{0\leq s\leq T}\|J_1(s,\tilde{u},\tilde{v}(s);u_0)\|_{\tilde{v}(s)}\\
&\leq c_2\|u_0\|_H + 2\sup_{0\leq s\leq T}\|J_1(s,\tilde{u},\tilde{v}(s);u_0)\|_{\tilde{v}(s)} \leq 3c_2\|u_0\|_H + 2(C_2 + C_{4,1}).
\end{aligned}$$

Using the contraction property of $J_\lambda(t,\tilde{u},\tilde{v}(t))$ on $H(\tilde{v}(t))$ again, for all $t \in [0,T]$ we have

$$\begin{aligned}
&\sup_{0<\lambda\leq 1}\|J_\lambda(t,\tilde{u},\tilde{v}(t);z)\|_{\tilde{v}(t)}\\
&\leq \sup_{0<\lambda\leq 1}\|J_\lambda(t,\tilde{u},\tilde{v}(t);z) - J_\lambda(t,\tilde{u},\tilde{v}(t);u_0)\|_{\tilde{v}(t)} + \sup_{0<\lambda\leq 1}\|J_\lambda(t,\tilde{u},\tilde{v}(t);u_0)\|_{\tilde{v}(t)}\\
&\leq \|z - u_0\|_{\tilde{v}(t)} + 3c_2\|u_0\|_H + 2(C_2 + C_{4,1})\\
&\leq 4c_2\|u_0\|_H + 2(C_2 + C_{4,1}) + \|z\|_{\tilde{v}(t)},
\end{aligned}$$

which implies (2.52). The estimate (2.53) is derived from (2.47) and (2.52).

(2) Using Lemma 2.3, (2.49), (2.51) and the definition of $\partial_{\tilde{v}(t)}\varphi_\lambda(t,\tilde{u},\tilde{v}(t))$ with (2.49), we have

$$\begin{aligned}
&\varphi(t;\tilde{u},\tilde{v}(t);J_\lambda(t,\tilde{u},\tilde{v}(t);z)) \leq \varphi_\lambda(t,\tilde{u},\tilde{v}(t);z)\\
\leq&\varphi_\lambda(t,\tilde{u},\tilde{v}(t);z_t(\tilde{u})) - (A_\lambda(t,\tilde{u},\tilde{v}(t);z), z_t(\tilde{u}) - z)_{\tilde{v}(t)}\\
\leq&C_2 + \|A_\lambda(t,\tilde{u},\tilde{v}(t);z)\|_{\tilde{v}(t)}\left(\|z\|_{\tilde{v}(t)} + C_2\right).
\end{aligned}$$

We see from the second estimate of (1) in this lemma that the following uniform estimate holds:

$$\varphi(t,\tilde{u},\tilde{v}(t);J_\lambda(t,\tilde{u},\tilde{v}(t);z)) \leq C_2 + \frac{\left(2\|z\|_{\tilde{v}(t)} + C_2 + C_4\right)^2}{\lambda}.$$

Using Lemma 2.1 and the first estimate of (1) in this lemma, we have

$$
\begin{aligned}
&|\varphi(t,\tilde{u},\tilde{v}(t);J_\lambda(t,\tilde{u},\tilde{v}(t);z))|\\
\leq&\varphi(t,\tilde{u},\tilde{v}(t);J_\lambda(t,\tilde{u},\tilde{v}(t);z))+C_2\left(\|J_\lambda(t,\tilde{u},\tilde{v}(t);z)\|_{\tilde{v}(t)}+1\right)\\
\leq&\frac{\left(2\|z\|_{\tilde{v}(t)}+C_2+C_4\right)^2}{\lambda}+C_2\left(\|z\|_{\tilde{v}(t)}+C_4+2\right),
\end{aligned}
$$

which implies that the required estimate holds.

(3) The required estimate is derived from (2.51), and the estimates of (1) and (2) in this lemma. □

At the end of this subsection, for each $z\in C([0,T];H)$ we show the continuity of functions $t\mapsto J_\lambda(t,\tilde{u},\tilde{v}(t);z(t))$ and $t\mapsto A_\lambda(t,\tilde{u},\tilde{v}(t);z(t))$.

Lemma 2.8. *For all $z\in C([0,T];H)$, all $\lambda\in(0,1]$ and all $t\in[0,T]$, we have*

$$\lim_{s\to t}\|J_\lambda(t,\tilde{u},\tilde{v}(t);z(s))\|_{\tilde{v}(s)}=\|J_\lambda(t,\tilde{u},\tilde{v}(t);z(t))\|_{\tilde{v}(t)},\quad \textit{hence,} \tag{2.58}$$

$$\lim_{s\to t}\|A_\lambda(t,\tilde{u},\tilde{v}(t);z(s))\|_{\tilde{v}(s)}=\|A_\lambda(t,\tilde{u},\tilde{v}(t);z(t))\|_{\tilde{v}(t)}. \tag{2.59}$$

Proof. It is enough to show (2.58) because of (2.47). We use Condition 2, (A7), the contraction property of $J_\lambda(t,\tilde{u},\tilde{v}(t))$ on $H(\tilde{v}(t))$ and the first estimate of (1) in Lemma 2.7. Then, we have

$$
\begin{aligned}
&\left|\|J_\lambda(t,\tilde{u},\tilde{v}(t);z(s))\|^2_{\tilde{v}(s)}-\|J_\lambda(t,\tilde{u},\tilde{v}(t);z(t))\|^2_{\tilde{v}(t)}\right|\\
\leq\ &\gamma_0\|J_\lambda(t,\tilde{u},\tilde{v}(t);z(s))\|^2_{\tilde{v}(t)}\left|\int_s^t\|\tilde{v}'(\tau)\|_X d\tau\right|\\
&\qquad+\left|\|J_\lambda(t,\tilde{u},\tilde{v}(t);z(s))\|^2_{\tilde{v}(t)}-\|J_\lambda(t,\tilde{u},\tilde{v}(t);z(t))\|^2_{\tilde{v}(t)}\right|\\
\leq\ &\gamma_0\left(\|z(s)\|_{\tilde{v}(t)}+C_4\right)^2\left|\int_s^t\|\tilde{v}'(\tau)\|_X d\tau\right|\\
&\qquad+\left(\|z(s)\|_{\tilde{v}(t)}+\|z(t)\|_{\tilde{v}(t)}+2C_4\right)\|z(s)-z(t)\|_{\tilde{v}(t)}\\
\leq\ &\gamma_0\left(c_2\max_{0\leq t\leq T}\|z(t)\|_H+C_4\right)^2\left|\int_s^t\|\tilde{v}'(\tau)\|_X d\tau\right|\\
&\qquad+2c_2\left(c_2\max_{0\leq t\leq T}\|z(t)\|_H+C_4\right)\|z(s)-z(t)\|_H,
\end{aligned}
$$

which implies (2.58). □

Lemma 2.9. *For each $z\in C([0,T];H)$ and each $\lambda\in(0,1]$ the function $t\mapsto J_\lambda(t,\tilde{u},\tilde{v}(t);z(t))$ is continuous from $[0,T]$ into H. Hence, the function $t\mapsto A_\lambda(t,\tilde{u},\tilde{v}(t);z(t))$ is also continuous from $[0,T]$ into H.*

Proof. Throughout this proof we fix a time $t\in[0,T]$ and consider any sequence $\{t_n\}_{n\in\mathbb{N}}\subset[0,T]$ satisfying $t_n\longrightarrow t$ as $n\to\infty$. Using Condition 2 and the first estimate of (1) in Lemma 2.8, we have

$$
\begin{aligned}
&\left| \|J_\lambda(t_n,\tilde{u},\tilde{v}(t_n);z(t_n))\|^2_{v(t_n)} - \|J_\lambda(t_n,\tilde{u},\tilde{v}(t_n);z(t_n))\|^2_{\tilde{v}(t)} \right| \\
&\le \gamma_0 \|J_\lambda(t_n,\tilde{u},\tilde{v}(t_n);z(t_n))\|^2_{\tilde{v}(t_n)} \left| \int_{t_n}^t \|\tilde{v}'(\tau)\|_X d\tau \right| \\
&\le \gamma_0 \left(c_2 \max_{0\le t\le T} \|z(t)\|_H + C_4 \right)^2 \left| \int_{t_n}^t \|\tilde{v}'(\tau)\|_X d\tau \right| ,
\end{aligned}
$$

which implies

$$
\lim_{n\to\infty} \left\{ \|J_\lambda(t_n,\tilde{u},\tilde{v}(t_n);z(t_n))\|^2_{v(t_n)} - \|J_\lambda(t_n,\tilde{u},\tilde{v}(t_n);z(t_n))\|^2_{\tilde{v}(t)} \right\} = 0. \tag{2.60}
$$

Fixing any $y_t \in D(\varphi(t,\tilde{u},\tilde{v}(t)))$ and using Condition 1 as $r = r_3 > \|y_t\|_{\tilde{v}(t)} + 1$, we see that there exist nonnegative functions $\alpha_{r_3}(\tilde{u}) \in L^2(0,T)$, $\beta_{r_3}(\tilde{u}) \in L^1(0,T)$ and a sequence $\{y_t(t_n)\}_{n\in\mathbb{N}}$ such that

$$
\|y_t(t_n) - y_t\|_{\tilde{v}(t_n)} \le \left(\sqrt{|\varphi(t,\tilde{u},\tilde{v}(t);y_t)} + 1 \right) \left| \int_t^{t_n} \alpha_{r_3}(\tilde{u};\tau) d\tau \right| , \tag{2.61}
$$

$$
\begin{aligned}
&|\varphi(t_n,\tilde{u},\tilde{v}(t_n);y_t(t_n)) - \varphi(t,\tilde{u},\tilde{v}(t);y_t)| \\
&\qquad\qquad \le (|\varphi(t,\tilde{u},\tilde{v}(t);y_t)| + 1) \left| \int_t^{t_n} \beta_{r_3}(\tilde{u};\tau) d\tau \right| ,
\end{aligned} \tag{2.62}
$$

hence, from (A7)

$$
\|y_t(t_n) - y_t\|_{\tilde{v}(t_n)} \ge c_1 \|y_t(t_n) - y_t\|_H \ge \frac{c_1}{c_2} \|y_t(t_n) - y_t\|_{\tilde{v}(t)}. \tag{2.63}
$$

Then, we see from (2.61), (2.62) and (2.63) that the following convergences hold:

$$
y_t(t_n) \longrightarrow y_t \quad \text{in} \quad H(\tilde{v}(t)) \ \text{ as } \ n \to \infty, \tag{2.64}
$$

$$
\lim_{k\to\infty} \varphi(t_n,\tilde{u},\tilde{v}(t_n);y_t(t_n)) = \varphi(t,\tilde{u},\tilde{v}(t);y_t). \tag{2.65}
$$

On the other hand, we see from (A1) and the first estimate of (1) in Lemma 2.7 that the sequence $\{J_\lambda(t_n,\tilde{u},\tilde{v}(t_n);z(t_n))\}_{n\in\mathbb{N}}$ is bounded in $H(\tilde{v}(t))$. So, we choose a subsequence $\{t_{n_k}\}_{k\in\mathbb{N}}$ of $\{t_n\}_{n\in\mathbb{N}}$ and an element $\tilde{z}_t \in H$ satisfying

$$
J_\lambda(t_{n_k},\tilde{u},\tilde{v}(t_{n_k});z(t_{n_k})) \longrightarrow \tilde{z}_t \quad \text{weakly in} \quad H(\tilde{v}(t)) \quad \text{as} \quad k \to \infty. \tag{2.66}
$$

We see from (2.60) and (2.66) that the following estimate holds:

$$
\begin{aligned}
&\liminf_{k\to\infty} \|J_\lambda(t_{n_k},\tilde{u},\tilde{v}(t_{n_k});z(t_{n_k}))\|^2_{\tilde{v}(t_{n_k})} \\
&= \lim_{k\to\infty} \left\{ \|J_\lambda(t_{n_k},\tilde{u},\tilde{v}(t_{n_k});z(t_{n_k}))\|^2_{\tilde{v}(t_{n_k})} - \|J_\lambda(t_{n_k},\tilde{u},\tilde{v}(t_{n_k});z(t_{n_k}))\|^2_{\tilde{v}(t)} \right\} \\
&\qquad + \liminf_{k\to\infty} \|J_\lambda(t_{n_k},\tilde{u},\tilde{v}(t_{n_k});z(t_{n_k}))\|^2_{\tilde{v}(t)} \ge \|\tilde{z}_t\|^2_{\tilde{v}(t)}.
\end{aligned} \tag{2.67}
$$

Using Condition 2, (A1), the inequality obtained in the proof of Lemma 2.2 and the first estimate of (1) in Lemma 2.7, we have

$$
\begin{aligned}
&\left|(z(t_{n_k}), y_t(t_{n_k}))_{\tilde v(t_{n_k})} - (z(t_{n_k}), y_t(t_{n_k}))_{\tilde v(t)}\right| \\
&\le \frac{\gamma_0}{2}\left(\|z(t_{n_k}) + y_t(t_{n_k})\|^2_{\tilde v(t)} + \|z(t_{n_k})\|^2_{\tilde v(t)} + \|y_t(t_{n_k})\|^2_{\tilde v(t)}\right)\left|\int_t^{t_{n_k}} \|\tilde v'(\tau)\|_X d\tau\right| \\
&\le \frac{3\gamma_0}{2}\left(c_2^2 \max_{0\le t\le T}\|z(t)\|^2_H + \sup_{n\in\mathbb{N}}\|y_t(t_n)\|^2_{\tilde v(t)}\right)\left|\int_t^{t_{n_k}} \|\tilde v'(\tau)\|_X d\tau\right|, \quad \text{and}
\end{aligned}
$$

$$
\begin{aligned}
&\Big|(J_\lambda(t_{n_k}, \tilde u, \tilde v(t_{n_k}); z(t_{n_k})), (z+y_t)(t_{n_k}))_{\tilde v(t_{n_k})} \\
&\qquad\qquad - (J_\lambda(t_{n_k}, \tilde u, \tilde v(t_{n_k}); (z(t_{n_k})), (z+y_t)(t_{n_k}))_{\tilde v(t)}\Big| \\
&\le \frac{5\gamma_0}{2}\left(\|J_\lambda(t_{n_k}, \tilde u, \tilde v(t_{n_k}); z(t_{n_k}))\|^2_{\tilde v(t)} + \|z(t_{n_k})\|^2_{\tilde v(t)} + \|y_t(t_{n_k})\|^2_{\tilde v(t)}\right) \\
&\qquad\qquad \times \left|\int_t^{t_{n_k}} \|v'(\tau)\|_X d\tau\right| \\
&\le \frac{5\gamma_0}{2}\left[\frac{c_2^2}{c_1^2}\left(C_4 + c_2\max_{0\le t\le T}\|z(t)\|_H\right)^2 + c_2^2\max_{0\le t\le T}\|z(t)\|^2_H + \max_{n\in\mathbb{N}}\|y_t(t_n)\|^2_{\tilde v(t)}\right] \\
&\qquad\qquad \times \left|\int_t^{t_{n_k}} \|v'(\tau)\|_X d\tau\right|,
\end{aligned}
$$

which imply that the following convergences hold:

$$
\lim_{k\to\infty}(z(t_{n_k}), y_t(t_{n_k}))_{\tilde v(t_{n_k})} = (z(t), y_t)_{\tilde v(t)}, \tag{2.68}
$$

$$
\lim_{k\to\infty}(J_\lambda(t_{n_k}, \tilde u, \tilde v(t_{n_k}); z(t_{n_k})), (z+y_t)(t_{n_k}))_{\tilde v(t_{n_k})} = (\tilde z_t, z(t) + y_t)_{\tilde v(t)}. \tag{2.69}
$$

Next, we fix a constant $r_4 > 0$ satisfying $r_4 \ge C_4 + c_2\max_{0\le t\le T}\|z(t)\|_H + 1$. From the first estimate of (1) and (2) in Lemma 2.7 we have

$$
\begin{cases}
\displaystyle\sup_{0\le s\le T}\|J_\lambda(s, \tilde u, \tilde v(s); z(s))\|_{\tilde v(s)} \le r_4, \quad \text{and} \\[2ex]
\displaystyle\sup_{0\le s\le T}|\varphi(s, \tilde u, \tilde v(s); J_\lambda(s, \tilde u, \tilde v(s); z(s)))| \le \frac{r_4^2 C_5}{\lambda}.
\end{cases} \tag{2.70}
$$

Using (A1), Condition 2 as $r = r_4$ and (2.70), we see that there exist nonnegative functions $\alpha_{r_4}(\tilde u) \in L^2(0,T)$, $\beta_{r_4}(\tilde u) \in L^1(0,T)$ and sequences $\{y_{t_n}(t)\}_{n\in\mathbb{N}}$, $\{\bar y_t(t_n)\}_{n\in\mathbb{N}}$ such that

$$
\begin{aligned}
&\|y_{t_n}(t) - J_\lambda(t_n, \tilde u, \tilde v(t_n); z(t_n))\|_{\tilde v(t)} \le \left(r_4\sqrt{\frac{C_5}{\lambda}} + 1\right)\left|\int_t^{t_n}\alpha_{r_4}(\tilde u; \tau)d\tau\right|, \\
&|\varphi(t, \tilde u, \tilde v(t); y_{t_n}(t)) - \varphi(t_n, \tilde u, \tilde v(t_n); J_\lambda(t_n, \tilde u, \tilde v(t_n); z(t_n)))| \\
&\qquad\qquad \le \left(\frac{r_4^2 C_5}{\lambda} + 1\right)\left|\int_t^{t_n}\beta_{r_4}(\tilde u; \tau)d\tau\right|,
\end{aligned}
$$

$$\|\bar{y}_t(t_n) - J_\lambda(t,\tilde{u},\tilde{v}(t);z(t))\|_{\tilde{v}(t)} \le \frac{c_2}{c_1}\left(r_4\sqrt{\frac{C_5}{\lambda}}+1\right)\left|\int_t^{t_n}\alpha_{r_4}(\tilde{u};\tau)d\tau\right|,$$

$$|\varphi(t_n,\tilde{u},\tilde{v}(t_n);\bar{y}_t(t_n)) - \varphi(t,\tilde{u},\tilde{v}(t);J_\lambda(t,\tilde{u},\tilde{v}(t);z(t)))|$$
$$\le \left(\frac{r_4^2C_5}{\lambda}+1\right)\left|\int_t^{t_n}\beta_{r_4}(\tilde{u};\tau)d\tau\right|,$$

which imply that the following convergences hold:

$$y_{t_n}(t) - J_\lambda(t_n,\tilde{u},\tilde{v}(t_n);z(t_n)) \longrightarrow 0 \quad \text{in } H(\tilde{v}(t)) \text{ as } n\to\infty, \tag{2.71}$$

$$\lim_{n\to\infty}\{\varphi(t,\tilde{u},\tilde{v}(t);y_{t_n}(t)) - \varphi(t_n,\tilde{u},\tilde{v}(t_n);J_\lambda(t_n,\tilde{u},\tilde{v}(t_n);z(t_n)))\} = 0, \tag{2.72}$$

$$\bar{y}_t(t_n) \longrightarrow J_\lambda(t,\tilde{u},\tilde{v}(t);z(t)) \quad \text{in } H(\tilde{v}(t)) \text{ as } n\to\infty, \tag{2.73}$$

$$\lim_{n\to\infty}\varphi(t_n,\tilde{u},\tilde{v}(t_n);\bar{y}_t(t_n)) = \varphi(t,\tilde{u},\tilde{v}(t);J_\lambda(t,\tilde{u},\tilde{v}(t);z(t))). \tag{2.74}$$

Repeating the argumententation similar to that for the sequence $\{y_t(t_n)\}_{n\in\mathbb{N}}$ and using (2.66), (2.73), we have

$$\lim_{k\to\infty}(z(t_{n_k}),\bar{y}_t(t_{n_k}))_{\tilde{v}(t_{n_k})} = (z(t),J_\lambda(t,\tilde{u},\tilde{v}(t);z(t)))_{\tilde{v}(t)}, \tag{2.75}$$

$$\lim_{k\to\infty}(J_\lambda(t_{n_k},\tilde{u},\tilde{v}(t_{n_k});z(t_{n_k})),(z+\bar{y}_t)(t_{n_k}))_{\tilde{v}(t_{n_k})}$$
$$= (\tilde{z}_t, z(t)+J_\lambda(t,\tilde{u},\tilde{v}(t);z(t)))_{\tilde{v}(t)}, \tag{2.76}$$

where $\{t_{n_k}\}_{k\in\mathbb{N}}$ and $\tilde{z}_t$ are the same subsequence of $\{t_n\}_{n\in\mathbb{N}}$ and element which are taken out in (2.66). From (2.66) and (2.71) for all $y\in H$ we have

$$\lim_{k\to\infty}\left|(y_{t_{n_k}}(t)-\tilde{z}_t,y)_{\tilde{v}(t)}\right|$$
$$\le \lim_{k\to\infty}\Big\{\|y\|_{\tilde{v}(t)}\|y_{t_{n_k}}(t) - J_\lambda(t_{n_k},\tilde{u},\tilde{v}(t_{n_k});z(t_{n_k}))\|_{\tilde{v}(t)}$$
$$+\left|(J_\lambda(t_{n_k},\tilde{u},\tilde{v}(t_{n_k});z(t_{n_k}))-\tilde{z}_t,y)_{\tilde{v}(t)}\right|\Big\} = 0,$$

which implies

$$y_{t_{n_k}}(t) \longrightarrow \tilde{z}_t \quad \text{weakly in} \quad H(\tilde{v}(t)) \quad \text{as} \quad k\to\infty. \tag{2.77}$$

Since $\varphi(t,\tilde{u},\tilde{v}(t))$ is weakly sequentially l.s.c. on $H(\tilde{v}(t))$, we see from (2.72) and (2.77) that

$$\liminf_{k\to\infty}\varphi(t_{n_k},\tilde{u},\tilde{v}(t_{n_k});J_\lambda(t_{n_k},\tilde{u},\tilde{v}(t_{n_k});z(t_{n_k})))$$
$$\ge \lim_{k\to\infty}\left\{\varphi(t_{n_k},\tilde{u},\tilde{v}(t_{n_k});J_\lambda(t_{n_k},\tilde{u},\tilde{v}(t_{n_k});z(t_{n_k}))) - \varphi(t,\tilde{u},\tilde{v}(t);y_{t_{n_k}}(t))\right\}$$
$$+\liminf_{k\to\infty}\varphi(t,\tilde{u},\tilde{v}(t);y_{t_{n_k}}(t)) \ge \varphi(t,\tilde{u},\tilde{v}(t);\tilde{z}_t). \tag{2.78}$$

Moreover, from (2.46) we have

$$\begin{aligned}
&\|J_\lambda(t_{n_k},\tilde{u},\tilde{v}(t_{n_k});z(t_{n_k}))\|^2_{v(t_{n_k})}+\lambda\varphi(t_{n_k},\tilde{u},\tilde{v}(t_{n_k});J_\lambda(t_{n_k},\tilde{u},\tilde{v}(t_{n_k});z(t_{n_k})))\\
&\le \lambda\varphi(t_{n_k},\tilde{u},\tilde{v}(t_{n_k});y_t(t_{n_k}))-(z(t_{n_k}),y_t(t_{n_k}))_{\tilde{v}(t_{n_k})}\\
&\qquad\qquad +(J_\lambda(t_{n_k},\tilde{u},\tilde{v}(t_{n_k});z(t_{n_k})),(z+y_t)(t_{n_k}))_{\tilde{v}(t_{n_k})},\quad\text{and} \qquad (2.79)
\end{aligned}$$

$$\begin{aligned}
&\|J_\lambda(t_{n_k},\tilde{u},\tilde{v}(t_{n_k});z(t_{n_k}))\|^2_{\tilde{v}(t_{n_k})}\\
&\le \lambda\left\{\varphi(t_{n_k},\tilde{u},\tilde{v}(t_{n_k});\bar{y}_t(t_{n_k}))-\varphi(t_{n_k},\tilde{u},\tilde{v}(t_{n_k});J_\lambda(t_{n_k},\tilde{u},\tilde{v}(t_{n_k});z(t_{n_k})))\right\}\\
&\quad +(J_\lambda(t_{n_k},\tilde{u},\tilde{v}(t_{n_k});z(t_{n_k})),(z+\bar{y}_t)(t_{n_k}))_{\tilde{v}(t_{n_k})}-(z(t_{n_k}),\bar{y}_t(t_{n_k}))_{\tilde{v}(t_{n_k})}. \qquad (2.80)
\end{aligned}$$

We take $\liminf_{n\to\infty}$ in both sides of (2.79) and use (2.64), (2.67), (2.78), (2.69), (2.78). Then, we have

$$\|\tilde{z}_t\|^2_{\tilde{v}(t)}+\lambda\varphi(t,\tilde{u},\tilde{v}(t);\tilde{z}_t)\le\lambda\varphi(t,\tilde{u},\tilde{v}(t);y_t)+(\tilde{z}_t,z(t)+y_t)_{\tilde{v}(t)}-(z(t),y_t)_{\tilde{v}(t)},$$

hence, $$\left(\frac{z(t)-\tilde{z}_t}{\lambda},y_t-\tilde{z}_t\right)_{\tilde{v}(t)}\le\varphi(t,\tilde{u},\tilde{v}(t);y_t)-\varphi(t,\tilde{u},\tilde{v}(t);\tilde{z}_t). \qquad (2.81)$$

Since y_t is any element in $D(\varphi(t,\tilde{u},\tilde{v}(t)))$, the inequality (2.81) implies

$$z(t)-\tilde{z}_t\in\lambda\partial_{\tilde{v}(t)}\varphi(t,\tilde{u},\tilde{v}(t);\tilde{z}_t),\quad\text{hence,}\quad \tilde{z}_t=J_\lambda(t,\tilde{u},\tilde{v}(t);z(t)). \qquad (2.82)$$

Moreover, taking $\limsup_{n\to\infty}$ in both sides of (2.80) and using (2.73), (2.75), (2.76), (2.82), we have

$$\begin{aligned}
&\limsup_{k\to\infty}\|J_\lambda(t_{n_k};z(t_{n_k}))\|^2_{\tilde{v}(t_{n_k})}\le\lambda\Big\{\varphi(t,\tilde{u},\tilde{v}(t);J_\lambda(t,\tilde{u},\tilde{v}(t);z(t))) \qquad (2.83)\\
&-\liminf_{k\to\infty}\varphi(t_{n_k},\tilde{u},\tilde{v}(t_{n_k});J_\lambda(t_{n_k},\tilde{u},\tilde{v}(t_{n_k});z(t_{n_k})))\Big\}+\|J_\lambda(t,\tilde{u},\tilde{v}(t);z(t))\|^2_{\tilde{v}(t)}.
\end{aligned}$$

Applying Lemma 2.4 because of (2.66) and (2.82), we have

$$\begin{aligned}
&\varphi(t,\tilde{u},\tilde{v}(t);J_\lambda(t,\tilde{u},\tilde{v}(t);z(t)))\\
&\qquad\le\liminf_{k\to\infty}\varphi(t_{n_k},\tilde{u},\tilde{v}(t_{n_k});J_\lambda(t_{n_k},\tilde{u},\tilde{v}(t_{n_k});z(t_{n_k}))), \qquad (2.84)
\end{aligned}$$

hence, from (2.83) and (2.84)

$$\limsup_{k\to\infty}\|J_\lambda(t_{n_k},\tilde{u},\tilde{v}(t_{n_k});z(t_{n_k}))\|^2_{\tilde{v}(t_{n_k})}\le\|J_\lambda(t,\tilde{u},\tilde{v}(t);z(t))\|^2_{\tilde{v}(t)}. \qquad (2.85)$$

From (2.66), (2.67), (2.82) and (2.85) we have

$$\lim_{n\to\infty}\|\tilde{J}_\lambda(t_n;z(t_n))\|_{v(t_n)}=\|\tilde{J}_\lambda(t;z(t))\|_{\tilde{v}(t)}. \qquad (2.86)$$

Using Condition 2, the first estimate of (1) in Lemma 2.7, (2.86) and the following inequality:

$$
\begin{aligned}
&\left| J_\lambda(t_{n_k}, \tilde{u}, \tilde{v}(t_{n_k}); z(t_{n_k}))\|^2_{\tilde{v}(t)} - \|J_\lambda(t, \tilde{u}, \tilde{v}(t); z)(t)\|^2_{\tilde{v}(t)} \right| \\
&\le \gamma_0 \|J_\lambda(t_{n_k}, \tilde{u}, \tilde{v}(t_{n_k}); z(t_{n_k}))\|^2_{\tilde{v}(t)} \left| \int_t^{t_{n_k}} \|v'(\tau)\|_X d\tau \right| \\
&\qquad + \left| \|J_\lambda(t_{n_k}, \tilde{u}, \tilde{v}(t_{n_k}); z(t_{n_k}))\|^2_{v(t_{n_k})} - \|J_\lambda(t, \tilde{u}, \tilde{v}(t); z(t))\|^2_{\tilde{v}(t)} \right| \\
&\le \gamma_0 \left(C_4 + c_2 \max_{0\le t\le T} \|z(t)\|_H \right)^2 \left| \int_t^{t_{n_k}} \|v'(\tau)\|_X d\tau \right| \\
&\qquad + \left| \|J_\lambda(t_{n_k}, \tilde{u}, \tilde{v}(t_{n_k}); z(t_{n_k}))\|^2_{v(t_{n_k})} - \|J_\lambda(t, \tilde{u}, \tilde{v}(t); z(t))\|^2_{\tilde{v}(t)} \right|,
\end{aligned}
$$

we have

$$
\lim_{k\to\infty} \|J_\lambda(t_{n_k}, \tilde{u}, \tilde{v}(t_{n_k}); z(t_{n_k}))\|_{\tilde{v}(t)} = \|J_\lambda(t, \tilde{u}, \tilde{v}(t); z)(t)\|_{\tilde{v}(t)}. \tag{2.87}
$$

Finally, we see from (2.66), (2.82) and (2.87) that

$$
J_\lambda(t_{n_k}, \tilde{u}, \tilde{v}(t_{n_k}); z(t_{n_k})) \longrightarrow J_\lambda(t, \tilde{u}, \tilde{v}(t); z(t)) \ \text{ in } H_{v(t)} \text{ as } k \to \infty. \tag{2.88}
$$

Because of (A1) the convergence (2.88) directly implies that the function $t \mapsto J_\lambda(t, \tilde{u}, \tilde{v}(t); z(t))$ is continuous from $[0, T]$ into H. □

2.4. Absolute continuity of norms and the Yosida regularization

In this subsection, we show Lemmas 2.10 and 2.11 below, which play important roles to obtain the uniform estimates of approximate solutions to auxiliary evolution equations considered in Section 3 and whose original proofs have been given in [4, Proposition 2.2.5 and Lemma 2.10], respectively.

Lemma 2.10. *For each $z \in W^{1,1}(0, T; H)$ the function $t \mapsto \|z(t)\|^2_{\tilde{v}(t)}$ is absolutely continuous from $[0, T]$ into $\mathbb{R}$, and we have*

$$
\left| \frac{d}{dt} \|z(t)\|^2_{\tilde{v}(t)} - 2(z'(t), z(t))_{\tilde{v}(t)} \right| \le \gamma_0 \|\tilde{v}'(t)\|_X \|z(t)\|^2_{\tilde{v}(t)}, \quad a.a.\ t \in (0, T).
$$

Proof. We put $\ell_1(t) := \|z(t)\|^2_{\tilde{v}(t)}$. Using Condition 2 with the equality

$$
\ell_1(s) - \ell_1(t) = \|z(s)\|^2_{\tilde{v}(s)} - \|z(s)\|^2_{\tilde{v}(t)} + (z(s) - z(t), z(s) + z(t))_{\tilde{v}(t)}, \tag{2.89}
$$

valid for all $s, t \in [0, T]$, we have $|\ell_1(s) - \ell_1(t)|$

$$
\begin{aligned}
&\le \gamma_0 \|z(s)\|^2_{\tilde{v}(t)} \left| \int_s^t \|\tilde{v}'(\tau)\|_X d\tau \right| + \|z(s) - z(t)\|_{\tilde{v}(t)} (\|z(s)\|_{\tilde{v}(t)} + \|z(t)\|_{\tilde{v}(t)}) \\
&\le c_2^2 \gamma_0 \max_{0\le t\le T} \|z(t)\|^2_H \left| \int_s^t \|\tilde{v}'(\tau)\|_X d\tau \right| + 2c_2^2 \max_{0\le t\le T} \|z(t)\|_H \left| \int_s^t \|z'(\tau)\|_H d\tau \right|,
\end{aligned}
$$

which implies that the function ℓ_1 is absolutely continuous from $[0,T]$ into $\mathbb{R}$. Going back to (2.89) and dividing its both sides by $|s-t|$, we obtain

$$\left|\frac{\ell_1(s)-\ell_1(t)}{s-t} - \left(\frac{z(s)-z(t)}{s-t}, z(s)+z(t)\right)_{\tilde{v}(t)}\right|$$

$$= \left|\frac{\|z(s)\|^2_{\tilde{v}(s)} - \|z(s)\|^2_{\tilde{v}(t)}}{s-t}\right| \le \gamma_0 \|z(s)\|^2_{\tilde{v}(t)} \left|\frac{1}{s-t}\int_s^t \|\tilde{v}'(\tau)\|_X d\tau\right|. \tag{2.90}$$

Taking the limit $s \to t$ on both sides of (2.90), we get the lemma. □

Lemma 2.11. *For each $z \in W^{1,1}(0,T;H)$ and each $\lambda \in (0,1]$ the function $t \mapsto \varphi_\lambda(t, \tilde{u}(t), \tilde{v}(t); z(t))$ is absolutely continuous from $[0,T]$ into $\mathbb{R}$.*
Moreover, the following inequality holds for a.a. $t \in (0,T)$:

$$\left|\frac{d}{dt}\varphi_\lambda(t,\tilde{u},\tilde{v}(t);z(t)) - (z'(t), A_\lambda(t,\tilde{u},\tilde{v}(t);z(t))_{\tilde{v}(t)}\right| \tag{2.91}$$

$$\le \|A_\lambda(t,\tilde{u},\tilde{v}(t);z(t))\|_{\tilde{v}(t)} \left(\sqrt{|\varphi_\lambda(t,\tilde{u},\tilde{v}(t);z(t))|} + \sqrt{r_4 C_1} + 1\right) \alpha_{r_4}(\tilde{u};t)$$

$$+ \left(|\varphi_\lambda(t,\tilde{u},\tilde{v}(t);z(t))| + r_4 C_1 + 1\right) \beta_{r_4}(\tilde{u};t) + \frac{\gamma_0 \lambda}{2} \|A_\lambda(t,\tilde{u},\tilde{v}(t);z(t))\|^2_{\tilde{v}(t)} \|\tilde{v}'(t)\|_X,$$

where $r_4 > 0$ and $\alpha_{r_4}(\tilde{u}) \in L^2(0,T)$, $\beta_{r_4}(\tilde{u}) \in L^1(0,T)$ are the same constant and nonnegative functions that are used in the proof of Lemma 2.9.

Proof. We see from the first estimate of (1) in Lemma 2.7 that

$$\max\left\{\|J_\lambda(t,\tilde{u},\tilde{v}(t);z(s))\|_{\tilde{v}(t)}, \|J_\lambda(s,\tilde{u},\tilde{v}(s);z(s))\|_{\tilde{v}(s)}\right\} + 1$$
$$\le C_4 + c_2 \max_{0\le\tau\le T} \|z(\tau)\|_H + 1 \le r_4$$

and from Lemma 2.1, (2.51) and the above inequality that

$$\max\{|\varphi(t,\tilde{u},\tilde{v}(t);J_\lambda(t,\tilde{u},\tilde{v}(t);z(s)))|, |\varphi(s,\tilde{u},\tilde{v}(s);J_\lambda(s,\tilde{u},\tilde{v}(s);z(s)))|\}$$
$$\le \max\{\varphi(t,\tilde{u},\tilde{v}(t);J_\lambda(t,\tilde{u},\tilde{v}(t);z(s))), \varphi(s,\tilde{u},\tilde{v}(s);J_\lambda(s,\tilde{u},\tilde{v}(s);z(s)))\} + r_4 C_1$$
$$\le \max\{|\varphi_\lambda(t,\tilde{u},\tilde{v}(t);z(s))|, |\varphi_\lambda(s,\tilde{u},\tilde{v}(s);z(s))|\} + r_4 C_1. \tag{2.92}$$

Using Condition 1, we see from (2.92) that for each $t \in [0,T]$ there exist two families $\{\tilde{z}_t(s)\,;\, 0 \le s \le T\}$ and $\{\bar{z}_s(t)\,;\, 0 \le s \le T\}$ such that for each $s \in [0,T]$ the following inequalities hold:

$$\max\{\|\tilde{z}_t(s) - J_\lambda(t,\tilde{u},\tilde{v}(t);z(s))\|_{\tilde{v}(s)}, \|\bar{z}_s(t) - J_\lambda(s,\tilde{u},\tilde{v}(s);z(s))\|_{\tilde{v}(t)}\}$$
$$\le \left(\max\left\{\sqrt{|\varphi_\lambda(s,\tilde{u},\tilde{v}(s);z(s))|}, \sqrt{|\varphi_\lambda(t,\tilde{u},\tilde{v}(t);z(s))|}\right\} + \sqrt{r_4 C_1} + 1\right)$$
$$\times \left|\int_s^t \alpha_{r_4}(\tilde{u};\tau)d\tau\right|, \quad \text{and} \tag{2.93}$$

$$
\begin{aligned}
&\max\{|\varphi(s,\tilde u,\tilde v(s);\tilde z_t(s))-\varphi(t,\tilde u,\tilde v(t);J_\lambda(t,\tilde u,\tilde v(t);z(s)))|,\\
&\qquad\qquad |\varphi(t,\tilde u,\tilde v(t);\bar z_s(t))-\varphi(s,\tilde u,\tilde v(s);J_\lambda(s,\tilde u,\tilde v(s);z(s)))|\}\\
&\le (\max\{|\varphi_\lambda(s,\tilde u,\tilde v(s);z(s))|,|\varphi_\lambda(t,\tilde u,\tilde v(t);z(s))|\}+r_4C_1+1)\left|\int_s^t \beta_{r_4}(\tilde u;\tau)d\tau\right|.
\end{aligned}
\tag{2.94}
$$

In order to show the absolute continuity of $\varphi_\lambda(t,\tilde u,\tilde v(t);z(t))$, first of all we estimate the function $\ell_2(s,t):=\varphi_\lambda(s,\tilde u,\tilde v(s);z(s))-\varphi_\lambda(t,\tilde u,\tilde v(t);z(s))$. Using Condition 2 and (2.47), (2.48), we obtain two estimations below:

$$
\begin{aligned}
\ell_2(s,t)&\le \varphi(s,\tilde u,\tilde v(s);\tilde z_t(s))-\varphi(t,\tilde u,\tilde v(t);J_\lambda(t,\tilde u,\tilde v(t);z(s)))\\
&\quad+\frac{1}{2\lambda}\|\tilde z_t(s)-z(s)\|^2_{\tilde v(s)}-\frac{\lambda}{2}\|A_\lambda(t,\tilde u,\tilde v(t);z(s))\|^2_{\tilde v(t)}\\
&\le |\varphi(s,\tilde u,\tilde v(s);\tilde z_t(s))-\varphi(t,\tilde u,\tilde v(t);J_\lambda(t,\tilde u,\tilde v(t);z(s)))|\\
&\quad+\frac{1}{2\lambda}\|\tilde z_t(s)-J_\lambda(t,\tilde u,\tilde v(t);z(s))\|^2_{\tilde v(s)}\\
&\quad-(A_\lambda(t,\tilde u,\tilde v(t);z(s)),\tilde z_t(s)-J_\lambda(t,\tilde u,\tilde v(t);z(s)))_{\tilde v(s)}\\
&\quad+\frac{\lambda}{2}\left(\|A_\lambda(t,\tilde u,\tilde v(t);z(s))\|^2_{\tilde v(s)}-\|A_\lambda(t,\tilde u,\tilde v(t);z(s))\|^2_{\tilde v(t)}\right)\\
&\le |\varphi(s,\tilde u,\tilde v(s);\tilde z_t(s))-\varphi(t,\tilde u,\tilde v(t);J_\lambda(t,\tilde u,\tilde v(t);z(s)))|\\
&\quad+\frac{1}{2\lambda}\|\tilde z_t(s)-J_\lambda(t,\tilde u,\tilde v(t);z)(s)\|^2_{v(s)}\\
&\quad+\|A_\lambda(t,\tilde u,\tilde v(t);z(s))\|_{\tilde v(s)}\|\tilde z_t(s)-J_\lambda(t,\tilde u,\tilde v(t);z(s))\|_{\tilde v(s)}\\
&\quad+\frac{\gamma_0\lambda}{2}\|A_\lambda(t,\tilde u,\tilde v(t);z(s))\|^2_{\tilde v(t)}\left|\int_s^t\|\tilde v'(\tau)\|_X d\tau\right|, \text{ and}
\end{aligned}
\tag{2.95}
$$

$$
\begin{aligned}
\ell_2(s,t)&\ge \varphi(s,\tilde u,\tilde v(s);J_\lambda(s,\tilde u,\tilde v(s);z(s)))-\varphi(t,\tilde u,\tilde v(t);\bar z_s(t))\\
&\quad+\frac{\lambda}{2}\|A_\lambda(s,\tilde u,\tilde v(s);z(s))\|^2_{\tilde v(s)}-\frac{1}{2\lambda}\|\bar z_s(t)-z(s)\|^2_{\tilde v(t)}\\
&\ge -|\varphi(t,\tilde u,\tilde v(t);\bar z_s(t))-\varphi(s,\tilde u,\tilde v(s);J_\lambda(s,\tilde u,\tilde v(s);z(s)))|\\
&\quad-\frac{1}{2\lambda}\|\bar z_s(t)-J_\lambda(s,\tilde u,\tilde v(s);z(s))\|^2_{\tilde v(t)}\\
&\quad-\|A_\lambda(s,\tilde u,\tilde v(s);z(s))\|_{\tilde v(t)}\|\bar z_s(t)-J_\lambda(s,\tilde u,\tilde v(s);z(s))\|_{\tilde v(t)}\\
&\quad-\frac{\gamma_0\lambda}{2}\|A_\lambda(s,\tilde u,\tilde v(s);z(s))\|^2_{\tilde v(t)}\left|\int_s^t\|\tilde v'(\tau)\|_X d\tau\right|.
\end{aligned}
\tag{2.96}
$$

We see from (1) and (3) of Lemma 2.7 and (2.93)–(2.96) that

$$
\begin{aligned}
|\ell_2(s,t)|\le\frac{1}{2\lambda}&\left(\max\left\{\sqrt{|\varphi_\lambda(s,\tilde u,\tilde v(s);z(s))|},\sqrt{|\varphi_\lambda(t,\tilde u,\tilde v(t);z(s))|}+\sqrt{r_4C_1}+1\right\}\right)^2\\
&\times\left|\int_s^t\alpha_{r_4}(\tilde u;\tau)d\tau\right|^2
\end{aligned}
$$

$$
\begin{aligned}
&+ \left(\max\left\{\sqrt{|\varphi_\lambda(s,\tilde{u},\tilde{v}(s);z(s))|}, \sqrt{|\varphi_\lambda(t,\tilde{u},\tilde{v}(t);z(s))|} + \sqrt{r_4 C_1} + 1\right\}\right) \\
&\times \max\{\|A_\lambda(s,\tilde{u},\tilde{v}(s);z(s))\|_{\tilde{v}(t)}, \|A_\lambda(t,\tilde{u},\tilde{v}(t);z(s))\|_{\tilde{v}(s)}\} \left|\int_s^t \alpha_{r_4}(\tilde{u};\tau)d\tau\right| \\
&+ (\max\{|\varphi_\lambda(s,\tilde{u},\tilde{v}(s);z(s))|, |\varphi_\lambda(t,\tilde{u},\tilde{v}(t);z(s))|\} + r_4 C_1 + 1) \left|\int_s^t \beta_{r_4}(\tilde{u};\tau)d\tau\right| \\
&+ \frac{\gamma_0 \lambda}{2} \max\left\{\|A_\lambda(t,\tilde{u},\tilde{v}(t);z(s))\|^2_{\tilde{v}(t)}, \|A_\lambda(s,\tilde{u},\tilde{v}(s);z(s))\|^2_{\tilde{v}(t)}\right\} \\
&\qquad\qquad \times \left|\int_s^t \|v'(\tau)\|_X d\tau\right| \qquad (2.97)
\end{aligned}
$$

$$
\begin{aligned}
&\le \frac{1}{2\lambda}\left\{\sqrt{\frac{C_6}{\lambda}}\left(1 + c_2 \max_{0\le t\le T}\|z(t)\|_H\right) + \sqrt{r_4 C_1} + 1\right\}^2 \left|\int_s^t \alpha_{r_4}(\tilde{u};\tau)d\tau\right|^2 \\
&+ \frac{1}{\lambda}\left\{\sqrt{\frac{C_6}{\lambda}}\left(1 + c_2 \max_{0\le t\le T}\|z(t)\|_H\right) + \sqrt{r_4 C_1} + 1\right\} \\
&\qquad\qquad \times \left(2c_2 \max_{0\le t\le T}\|z(t)\|_H + C_4\right) \left|\int_s^t \alpha_{r_4}(\tilde{u};\tau)d\tau\right| \\
&+ \left\{\frac{C_6}{\lambda}\left(1 + c_2 \max_{0\le t\le T}\|z(t)\|_H\right)^2 + r_4 C_1 + 1\right\} \left|\int_s^t \beta_{r_4}(\tilde{u};\tau)d\tau\right| \\
&+ \frac{\gamma_0}{2\lambda}\left(2c_2 \max_{0\le t\le T}\|z(t)\|_H + C_4\right)^2 \left|\int_s^t \|v'(\tau)\|_X d\tau\right|. \qquad (2.98)
\end{aligned}
$$

We estimate the function $\ell_3(s,t) := \varphi_\lambda(t,\tilde{u},\tilde{v}(t);z(s)) - \varphi_\lambda(t,\tilde{u},\tilde{v}(t);z(t))$. From (2.49) we have

$$
(A_\lambda(t,\tilde{u},\tilde{v}(t);z(t)), z(s) - z(t))_{\tilde{v}(t)} \le \ell_3(s,t). \qquad (2.99)
$$

From (2.50) and (2.51) we have

$$
\begin{aligned}
\ell_3(s,t) &\le \frac{\lambda}{2}\left(\|A_\lambda(t,\tilde{u},\tilde{v}(t);z(s))\|^2_{\tilde{v}(t)} - \|A_\lambda(t,\tilde{u},\tilde{v}(t);z(t))\|^2_{\tilde{v}(t)}\right) \\
&\qquad + (A_\lambda(t,\tilde{u},\tilde{v}(t);z(s)), J_\lambda(t,\tilde{u},\tilde{v}(t);z(s)) - J_\lambda(t,\tilde{u},\tilde{v}(t);z(t)))_{\tilde{v}(t)} \\
&\le (A_\lambda(t,\tilde{u},\tilde{v}(t);z(s)), z(s) - z(t))_{\tilde{v}(t)}. \qquad (2.100)
\end{aligned}
$$

We see from (1) in Lemma 2.7 and (2.99), (2.100) that

$$
\begin{aligned}
|\ell_3(s,t)| &\le c_2 \max\left\{\|A_\lambda(t,\tilde{u},\tilde{v}(t);z(t))\|_{\tilde{v}(t)}, \|A_\lambda(t,\tilde{u},\tilde{v}(t);z(s))\|_{\tilde{v}(t)}\right\} \\
&\qquad \times \left|\int_s^t \|z'(\tau)\|_H d\tau\right| \qquad (2.101) \\
&\le \frac{c_2}{\lambda}\left(2c_2 \max_{0\le t\le T}\|z(t)\|_H + C_4\right)\left|\int_s^t \|z'(\tau)\|_H d\tau\right|. \qquad (2.102)
\end{aligned}
$$

Hence we see from (2.98) and (2.102) that the function $t \mapsto \varphi_\lambda(t, \tilde{u}, \tilde{v}(t); z(t))$ is absolutely continuous from $[0, T]$ into $\mathbb{R}$.

Next, we show (2.91). From (2.99) and (2.100) we have

$$\begin{aligned}
0 &\leq \ell_3(s,t) - (z(s) - z(t), A_\lambda(t, \tilde{u}, \tilde{v}(t); z(t)), z(t)))_{\tilde{v}(t)} \\
&\leq (A_\lambda(t, \tilde{u}, \tilde{v}(t); z(t)), z(t)) - A_\lambda(t, \tilde{u}, \tilde{v}(t); z(t)), z(t)), z(s) - z(t))_{\tilde{v}(t)} \\
&\leq \frac{c_2^2}{\lambda} \|z(s) - z(t)\|_H^2. \qquad (2.103)
\end{aligned}$$

Dividing both sides of (2.103) by $|s - t|$ ($s \neq t$) and taking the limit $s \to t$, we have

$$\lim_{s \to t} \left\{ \ell_3(s,t) - (z(s) - z(t), A_\lambda(t, \tilde{u}, \tilde{v}(t); z(s)) - A_\lambda(t, \tilde{u}, \tilde{v}(t); z(t)))_{\tilde{v}(t)} \right\} = 0. \quad (2.104)$$

Moreover, we see from (2.98)–(2.100) that we have

$$\begin{aligned}
&\left| \varphi_\lambda(s, \tilde{u}, \tilde{v}(s); z(s)) - \varphi_\lambda(t, \tilde{u}, \tilde{v}(t); z(t)) - (z(s) - z(t), A_\lambda(t, \tilde{u}, \tilde{v}(t); z(t)))_{\tilde{v}(t)} \right| \\
&\leq |\ell_2(s,t)| + \ell_3(s,t) - (z(s) - z(t), A_\lambda(t, \tilde{u}, \tilde{v}(t); z(t)))_{\tilde{v}(t)} \\
&\leq \frac{1}{2\lambda} \left(\max\left\{ \sqrt{|\varphi_\lambda(s, \tilde{u}, \tilde{v}(s); z(s))|}, \sqrt{|\varphi_\lambda(t, \tilde{u}, \tilde{v}(t); z(s))|} \right\} + \sqrt{r_4 C_1} + 1 \right)^2 \\
&\qquad\qquad \times \left| \int_s^t \alpha_{r_4}(\tilde{u}; \tau) d\tau \right|^2 \\
&\quad + \max\{ \|A_\lambda(s, \tilde{u}, \tilde{v}(s); z(s))\|_{\tilde{v}(t)}, \|A_\lambda(t, \tilde{u}, \tilde{v}(t); z(s))\|_{\tilde{v}(s)} \} \\
&\quad \times \left(\max\left\{ \sqrt{|\varphi_\lambda(s, \tilde{u}, \tilde{v}(s); z(s))|}, \sqrt{|\varphi_\lambda(t, \tilde{u}, \tilde{v}(t); z(s))|} \right\} + \sqrt{r_4 C_1} + 1 \right) \\
&\qquad\qquad \times \left| \int_s^t |\alpha_{r_4}(\tilde{u}; \tau)| d\tau \right| \\
&\quad + (\max\{ |\varphi_\lambda(s, \tilde{u}, \tilde{v}(s); z(s))|, |\varphi_\lambda(t, \tilde{u}, \tilde{v}(t); z(s))| \} + r_4 C_1 + 1) \\
&\qquad\qquad \times \left| \int_s^t |\beta_{r_4}(\tilde{u}; \tau)| d\tau \right| \\
&\quad + \frac{c_2 \gamma_0 \lambda}{2} \max\left\{ \|A_\lambda(t, \tilde{u}, \tilde{v}(t); z(s))\|_{\tilde{v}(t)}^2, \|A_\lambda(s, \tilde{u}, \tilde{v}(s); z(s))\|_{\tilde{v}(t)}^2 \right\} \\
&\qquad\qquad \times \left| \int_s^t \|v'(\tau)\|_X d\tau \right|. \qquad (2.105)
\end{aligned}$$

We divide both sides of (2.105) by $|t - s|$ ($s \neq t$) and take the limit $s \to t$. Since for each $t \in [0, T]$ the functions $s \mapsto \varphi_\lambda(t, \tilde{u}, \tilde{v}(t); z(s))$ and $s \mapsto \varphi_\lambda(s, \tilde{u}, \tilde{v}(s); z(s))$ are absolutely continuous from $[0, T]$ into $\mathbb{R}$, we see that the inequality (2.91) holds by using Lemmas 2.8 and 2.9. □

3. Auxiliary problem

For each $\tilde{u} \in \mathcal{W}(u_0, v_0)$ we consider the following initial-value problem of an evolution inclusion $\{(3.1),(3.2),(3.3)\}$, which is denoted by $(\mathrm{P})_{\tilde{u},u_0,v_0}$ as an auxiliary problem of $(\mathrm{P})_{u_0,v_0}$:

$$(\mathrm{P})_{\tilde{u},u_0,v_0} \quad \begin{cases} u'(t) + \partial_{\tilde{v}(t)}\varphi(t, \tilde{u}, \tilde{v}(t); u(t)) \ni f(t) \ \text{ in } H(\tilde{v}(t)), \ \text{a.a. } t \in [0, T], & (3.1) \\ \tilde{v}(t) = S(\tilde{u}; t, 0)v_0 \ \text{ in } X, \ \forall\, t \in [0, T], & (3.2) \\ u(0) = u_0 \ \text{ in } H. & (3.3) \end{cases}$$

Since in $(\mathrm{P})_{\tilde{u},u_0,v_0}$ the function $\tilde{u}$ is prescribed, hence, $\tilde{v}$ is also prescribed by (3.2), this auxiliary problem is regarded as one of the typical examples of evolution inclusions with a variational structure for inner products, which was considered in [4]. However, as stated in the Introduction, in [4] Damlamian only considered the case that the family of effective domains $\{D(\varphi(t, \tilde{u}, \tilde{v}(t))\,;\, 0 \le t \le T\}$ is independent of the time variable t, that is, $D(\varphi(t, \tilde{u}, \tilde{v}(t)) = D(\varphi(0, u_0, v_0))$ for all $t \in [0, T]$. On the other hand, in our setting it depends on t satisfying a certain continuity which is precisely given in Proposition 2.5. Hence, we devote this section to show Theorem 3.1 which is clearly stated below.

Theorem 3.1. *For each $(u_0, v_0) \in H \times A$ satisfying $u_0 \in D(\varphi(0, u_0, v_0))$ the problem $(P)_{\tilde{u},u_0,v_0}$ has one and only one solution $u(\tilde{u}, u_0, v_0) \in W^{1,2}(0, T; H)$. Moreover, the following estimates are satisfied:*

(1) *There exists a constant $C_7 > 0$ depending on $\tilde{u}$ and $\|u_0\|_H$, such that*

$$\max_{0\le t\le T} \|u(\tilde{u}, u_0, v_0; t)\|_{\tilde{v}(t)} + \int_0^T |\varphi(t, \tilde{u}, \tilde{v}(t); u(\tilde{u}, u_0, v_0; t))| \le C_7.$$

(2) *There exists a constant $C_8 > 0$ depending on $\tilde{u}$, $\|u_0\|_H$ and $\varphi(0, \tilde{u}, v_0; u_0)$, such that*

$$\|u(\tilde{u}, u_0, v_0)'\|_{L^2(0,T;H)} + \sup_{0\le t\le T} |\varphi(t, \tilde{u}, \tilde{v}(t); (u(\tilde{u}, u_0, v_0))(t))| \le C_8.$$

Hence, we have $u(\tilde{u}, u_0, v_0; t) \in D(\varphi(t, \tilde{u}, \tilde{v}(t)))$ for all $t \in [0, T]$.

(3) *There exist a constant $r_5 > 0$ depending on $\tilde{u}$ and $\|u_0\|_H$, and nonnegative functions $\alpha_{r_5}(\tilde{u}) \in L^2(0, T)$ and $\beta_{r_5}(\tilde{u}) \in L^1(0, T)$ such that the following property $(*)$ is satisfied:*

$(*)$ *for any $\varepsilon \in (0, 1)$ there exists $C_9 > 0$ depending on ε, $\tilde{u}$ and $\|u_0\|_H$, such that the following inequality is satisfied for all $t \in (0, T)$ and a.a. $s \in (0, T)$ with $s \le t$:*

$$\begin{aligned} &\varphi(t, \tilde{u}, \tilde{v}(t); u(\tilde{u}, u_0, v_0; t)) + \int_s^t ((u(\tilde{u}, u_0, v_0))'(\tau), (u(\tilde{u}, u_0, v_0))'(\tau) - f(\tau))_{\tilde{v}(\tau)} d\tau \\ &\le \varphi(s, \tilde{u}, \tilde{v}(s); u(\tilde{u}, u_0, v_0; s)) + \varepsilon \int_s^t \|(u(\tilde{u}, u_0, v_0))'(\tau) - f(\tau)\|^2_{\tilde{v}(\tau)} d\tau \\ &+ C_9 \int_s^t \{\varphi(\tau, \tilde{u}, \tilde{v}(\tau); u(\tilde{u}, u_0, v_0; \tau)) + 1\} \times \left(|\alpha_{r_5}(\tilde{u}, \tau)|^2 + |\beta_{r_5}(\tilde{u}; \tau)| + \|\tilde{v}'(\tau)\|_X\right) d\tau. \end{aligned}$$

Proof of the uniqueness part of Theorem 3.1. For each $i \in \{1,2\}$ and $(u_{0i}, v_0) \in H \times A$ satisfying $u_{0i} \in D(\varphi(0, u_0, v_0))$ we let $u_i := u(\tilde{u}, u_{0i}, v_0)$ a solution to $(\mathrm{P})_{\tilde{u},u_{0i},v_0}$ on $[0,T]$. Taking $u_1 - u_2 \in W^{1,2}(0,T;H)$ as z in Lemma 2.10, we have

$$\begin{aligned}&\frac{d}{dt}\|(u_1-u_2)(t)\|^2_{\tilde{v}(t)} - 2((u_1-u_2)'(t), (u_1-u_2)(t))_{\tilde{v}(t)}\\ &\leq \gamma_0\|v'(t)\|_X\|(u_1-u_2)(t)\|^2_{\tilde{v}(t)}, \quad \text{a.a. } t \in [0,T]. \qquad (3.4)\end{aligned}$$

Since $\partial_{\tilde{v}(t)}\varphi(t,\tilde{u},\tilde{v}(t))$ is maximal monotone on H, from (3.1) we have

$$((u_1-u_2)'(t), (u_1-u_2)(t))_{\tilde{v}(t)} \leq 0, \quad \text{a.a. } t \in [0,T]. \qquad (3.5)$$

We see from (3.4) and (3.5) that

$$\frac{d}{dt}\|(u_1-u_2)(t)\|^2_{\tilde{v}(t)} \leq (\gamma_0\|\tilde{v}'(t)\|_X + 1)\,\|(u_1-u_2)(t)\|^2_{\tilde{v}(t)}, \quad \text{a.a. } t \in [0,T]. \quad (3.6)$$

Applying Gronwall's lemma to (3.6), we have

$$c_1^2\|(u_1-u_2)(t)\|^2_H \leq \|u_{01}-u_{02}\|^2_{v_0}\exp\left(\int_0^T(\gamma_0\|\tilde{v}'(t)\|_X+1)\,dt\right), \quad \forall\, t\in[0,T],$$

which implies the uniqueness of solutions to $(\mathrm{P})_{\tilde{u},u_0,v_0}$ on $[0,T]$ whenever $u_{01} = u_{02}$ in H. □

In order to show the existence of solutions to $(\mathrm{P})_{\tilde{u},u_0,v_0}$ on $[0,T]$, in the rest of this section for each $\lambda \in (0,1]$ we consider an evolution equation (3.7) as an approximation of the evolution inclusion (3.1) and the Cauchy problem $(\mathrm{P})^\lambda_{\tilde{u},u_0,v_0} := \{(3.2),(3.3),(3.7)\}$ as an approximate problem of $(\mathrm{P})_{\tilde{u},u_0,v_0}$:

$$u'(t) + A_\lambda(t,\tilde{u},\tilde{v}(t);u(t)) = f(t) \quad \text{in } H_{\tilde{v}(t)}, \quad \text{a.a. } t \in (0,T). \qquad (3.7)$$

Applying the general results of evolution equations, we see that $(\mathrm{P})^\lambda_{\tilde{u},u_0,v_0}$ has one and only one solution $u_\lambda(\tilde{u},u_0,v_0) \in W^{1,2}(0,T;H)$, whose proof is omitted in this paper. In the following argumentation, we devote ourselves to derive the uniform estimates of $\{u_\lambda(\tilde{u},u_0,v_0)\,;\,0<\lambda\leq 1\}$, which enable us to argue the limit procedure for the approximate solutions $u_\lambda(\tilde{u},u_0,v_0)$ as $\lambda \searrow 0$.

Lemma 3.2. *There exists a constant $C_{10} > 0$ depending on $\tilde{u}$ and $\|u_0\|_H$, such that for all $\lambda \in (0,1]$*

$$\sup_{0\leq t\leq T}\|u_\lambda(\tilde{u},u_0,v_0;t)\|_{\tilde{v}(t)} + \int_0^T|\varphi_\lambda(t,\tilde{u},\tilde{v}(t);u_\lambda(\tilde{u},u_0,v_0;t))|dt \leq C_{10}.$$

Proof. In the rest of this section, for simplicity we put $u_\lambda := u_\lambda(\tilde{u},u_0,v_0)$. We consider the same function $h(\tilde{u})$ that is constructed in Proposition 2.5. Taking $u_\lambda - h(\tilde{u})$ as z in Lemma 2.10, we have for a.a. $t\in[0,T]$ and all $\lambda\in(0,1]$

$$
\begin{aligned}
&\frac{d}{dt}\|u_\lambda(t)-h(\tilde u;t)\|^2_{\tilde v(t)}\\
&\le 2(u'_\lambda(t)-(h(\tilde u))'(t),u_\lambda(t)-h(\tilde u;t))_{\tilde v(t)}+\gamma_0\|\tilde v'(t)\|_X\|u_\lambda(t)-h(\tilde u;t)\|^2_{\tilde v(t)}\\
&\le -2(A_\lambda(t,\tilde u,\tilde v(t);u_\lambda(t)),u_\lambda(t)-h(\tilde u;t))_{\tilde v(t)}\\
&\quad+(\gamma_0\|\tilde v'(t)\|_X+2)\|u_\lambda(t)-h(\tilde u;t)\|^2_{\tilde v(t)},+c_2^2\left(\|f(t)\|^2_H+\|(h(\tilde u))'(t)\|^2_H\right). \qquad (3.8)
\end{aligned}
$$

From (2.48), (2.49), (2.51), Lemma 2.1, the first estimate in (1) of Lemma 2.7, Proposition 2.5 with Remark 2.6, we get for all $t\in[0,T]$ and all $\lambda\in(0,1]$

$$
\begin{aligned}
&-(A_\lambda(t,\tilde u,\tilde v(t);u_\lambda(t)),u_\lambda(t)-h(\tilde u;t))_{\tilde v(t)}\\
&\le \varphi(t,\tilde u,\tilde v(t);h(\tilde u;t))-\varphi(t,\tilde u,\tilde v(t);J_\lambda(t,\tilde u,\tilde v(t);u_\lambda(t))\\
&\le \varphi(t,\tilde u,\tilde v(t);h(\tilde u;t))+C_1\left\{\|u_\lambda(t)\|_{\tilde v(t)}+C_4+1\right\}\\
&\le \frac12\|u_\lambda(t)-h(\tilde u;t)\|^2_{\tilde v(t)}+C_1\{C_1+2C_3+2C_4+1\}+C_3. \qquad (3.9)
\end{aligned}
$$

Substituting (3.9) into (3.8), we get for a.a. $t\in(0,T)$ and all $\lambda\in(0,1]$

$$
\frac{d}{dt}\|u_\lambda(t)-h(\tilde u;t)\|^2_{\tilde v(t)}\le g_1(\tilde u;t)\|u_\lambda(t)-h(\tilde u;t)\|^2_{\tilde v(t)}+g_2(\tilde u;t), \qquad (3.10)
$$

where the nonnegative functions $g_i(\tilde u)\in L^1(0,T)$ $(i=1,2)$ are given by

$$
\begin{aligned}
&g_1(\tilde u;t)=\gamma_0\|\tilde v'(t)\|_X+3,\\
&g_2(\tilde u;t):=c_2^2\left(\|f(t)\|^2_H+\|h'(t)\|^2_H\right)+C_1(C_1+2C_3+2C_4+1)+C_3.
\end{aligned}
$$

Using Condition 2, we have

$$
\begin{aligned}
&\left|\|u_\lambda(s)-h(\tilde u;s)\|^2_{\tilde v(s)}-\|u_\lambda(t_n(\tilde u))-z_n(\tilde u))\|^2_{\tilde v(t_n(\tilde u))}\right|\\
&\le \gamma_0\|u_\lambda(s)-h(\tilde u;s)\|^2_{\tilde v(t_n(\tilde u))}\left|\int_{t_n(\tilde u)}^s\|\tilde v'(\tau)\|_X d\tau\right|\\
&\quad+\left(\|u_\lambda(s)-h(\tilde u;s)\|_{\tilde v(t_n(\tilde u))}+\|u_\lambda(t_n(\tilde u))-z_n(\tilde u)\|_{\tilde v(t_n(\tilde u))}\right)\\
&\quad\times\left(\|u_\lambda(s)-u_\lambda(t_n(\tilde u))\|_{\tilde v(t_n(\tilde u))}+\|h(s)-z_n(\tilde u)\|_{\tilde v(t_n(\tilde u))}\right)\\
&\le c_2^2\gamma_0\left\{\max_{0\le t\le T}\|u_\lambda(t)\|_H+C_3\right\}^2\left|\int_{t_n(\tilde u)}^s\|\tilde v'(\tau)\|_X d\tau\right|\\
&\quad+2c_2^2\left\{\max_{0\le t\le T}\|u_\lambda(t)\|_H+C_3\right\}(\|u_\lambda(s)-u_\lambda(t_n(\tilde u))\|_H+\|h(s)-z_n(\tilde u)\|_H).
\end{aligned}
$$

Taking the limit $s\searrow t_n(\tilde u)$ in the above inequality and using (2.18) in Proposition 2.5, we have

$$
\lim_{s\searrow t_n(\tilde u)}\|u_\lambda(s)-h(\tilde u;s)\|_{\tilde v(s)}=\|u_\lambda(t_n(\tilde u))-z_n(\tilde u)\|_{\tilde v(t_n(\tilde u))}. \qquad (3.11)
$$

Applying Gronwall's lemma to (3.10) on $[s,t] \subset (t_n(\tilde{u}), t_{n+1}(\tilde{u})]$ and taking the limit $s \searrow t_n(\tilde{u})$ in its result, we see from (3.11) that the following uniform estimate holds for all $\lambda \in (0,1]$: for each $n \in \{0,1,2,\dots,n(\tilde{u})-1\}$

$$\begin{aligned}
&\sup_{t_n(\tilde{u}) \le t \le t_{n+1}(\tilde{u})} \|u_\lambda(t) - h(\tilde{u};t)\|^2_{\tilde{v}(t)} \\
&\le \Big(\|u_\lambda(t_n(\tilde{u})) - z_n(\tilde{u})\|^2_{\tilde{v}(t_n(\tilde{u}))} + \int_{t_n(\tilde{u})}^{t_{n+1}(\tilde{u})} g_2(\tilde{u};t)dt \Big) \exp\Big(\int_{t_n(\tilde{u})}^{t_{n+1}(\tilde{u})} g_1(\tilde{u};t)dt \Big) \\
&\le \Big(2\|u_\lambda(t_n(\tilde{u})) - h(\tilde{u};t_n(\tilde{u})\|^2_{\tilde{v}(t_n(\tilde{u}))} + 2\|h(\tilde{u};t_n(\tilde{u})) - z_n(\tilde{u})\|^2_{\tilde{v}(t_n(\tilde{u}))} \\
&\qquad + \int_{t_n(\tilde{u})}^{t_{n+1}(\tilde{u})} g_2(\tilde{u};t)dt \Big) \exp\Big(\int_{t_n(\tilde{u})}^{t_{n+1}(\tilde{u})} g_1(\tilde{u};t)dt \Big).
\end{aligned} \tag{3.12}$$

Inductively, (3.12) leads to the following uniform estimate for $n \in \{1,2,\dots,n(\tilde{u})\}$:

$$\begin{aligned}
&\sup_{0<\lambda\le 1} \left\{ \max_{0 \le t \le t_n(\tilde{u})} \|u_\lambda(t) - h(\tilde{u};t)\|^2_{\tilde{v}(t)} \right\} \\
&\le 2^n \|u_0 - z_0(\tilde{u})\|^2_{v_0} \exp\left(\int_0^{t_n(\tilde{u})} g_1(\tilde{u};t)dt \right) \\
&\quad + \sum_{k=1}^{n-1} 2^{n-k} \|h(\tilde{u};t_k(\tilde{u})) - z_k(\tilde{u})\|_{\tilde{v}(t_k(\tilde{u}))} \exp\left(\int_{t_k(\tilde{u})}^{t_n(\tilde{u})} g_1(\tilde{u};t)dt \right) \\
&\quad + \sum_{k=0}^{n-1} 2^{n-k-1} \int_{t_k(\tilde{u})}^{t_{k+1}(\tilde{u})} g_2(\tilde{u};t)dt \exp\left(\int_{t_k(\tilde{u})}^{t_n(\tilde{u})} g_1(\tilde{u};t)dt \right).
\end{aligned}$$

Lemma 2.3 and Remark 2.6 imply the following uniform estimate:

$$\begin{aligned}
&\sup_{0<\lambda\le 1} \left\{ \max_{0 \le t \le T} \|u_\lambda(t) - h(\tilde{u};t)\|^2_{\tilde{v}(t)} \right\} \\
&\le 2^{n(\tilde{u})} \left\{ \|u_0 - z_0(\tilde{u})\|^2_{v_0} + \sum_{k=0}^{n(\tilde{u})-1} \|h(\tilde{u};t_k(\tilde{u})) - z_k(\tilde{u})\|^2_{\tilde{v}(t_k(\tilde{u}))} + \int_0^T g_2(\tilde{u};t)dt \right\} \\
&\qquad \times \exp\left(\int_0^T g_1(\tilde{u};t)dt \right) \\
&\le 2^{n(\tilde{u})} \left[2c_2^2 \|u_0\|^2_H + 2\{n(\tilde{u})+1\} C_2(\tilde{u})^2 + 2c_2^2 n(\tilde{u}) C_3(\tilde{u})^2 + \int_0^T g_2(\tilde{u};t)dt \right].
\end{aligned} \tag{3.13}$$

Going back to (3.8) and (3.9), we get for a.a. $t \in (0,T)$ and all $\lambda \in (0,1]$

$$\begin{aligned}
&\frac{d}{dt}\|u_\lambda(t) - h(\tilde{u},t)\|^2_{\tilde{v}(t)} + 2\varphi_\lambda(t,\tilde{u},\tilde{v}(t);u_\lambda(t)) \\
&\le 2|\varphi(t,\tilde{u},\tilde{v}(t);u_\lambda(t))| + (\gamma_0\|\tilde{v}'(t)\|_X + 2)\|u_\lambda(t) - h(\tilde{u};t)\|^2_{\tilde{v}(t)} \\
&\qquad + \|f(t)\|^2_{\tilde{v}(t)} + \|(h(\tilde{u}))'(t)\|^2_{\tilde{v}(t)} \\
&\le C_{10,1}\left(\|f(t)\|^2_H + \|(h(\tilde{u}))'(t)\|^2_H + \|\tilde{v}'(t)\|_X + 1 \right),
\end{aligned} \tag{3.14}$$

where $C_{10,1} > 0$ is a suitable constant satisfying

$$C_{10,1} \geq (\gamma_0 + 2) \sup_{0<\lambda\leq 1} \left\{ \max_{0\leq t\leq T} \|u_\lambda(t) - h(\tilde{u};t)\|^2_{\tilde{v}(t)} \right\} + 2C_3 + c_2^2.$$

Actually, such a constant $C_{10,1}$ can be chosen from the uniform estimate (3.13). After integrating both sides of (3.14) on $[s, t_{n+1}(\tilde{u})]$ $(\subset (t_n(\tilde{u}), t_{n+1}(\tilde{u})])$, we take the limit $s \searrow t_n(\tilde{u})$ in its result and use (3.11) to have

$$\sup_{0<\lambda\leq 1} \left\{ \max_{n\in\{0,1,2,\ldots,n(\tilde{u})-1\}} \int_{t_n(\tilde{u})}^{t_{n+1}(\tilde{u})} \varphi_\lambda(t,\tilde{u},\tilde{v}(t);u_\lambda(t))dt \right\} \leq \frac{1}{2}\|u_\lambda(t_n(\tilde{u})) - z_n(\tilde{u})\|^2_{\tilde{v}(t_n(\tilde{u}))}$$
$$+ \frac{C_{10,1}}{2} \int_{t_n(\tilde{u})}^{t_{n+1}(\tilde{u})} \left(\|f(t)\|_H^2 + \|(h(\tilde{u}))'(t)\|_H^2 + \|\tilde{v}'(t)\|_X + 1 \right) dt,$$

hence, for all $\lambda \in (0,1]$

$$\begin{aligned} &\int_0^T \varphi_\lambda(t,\tilde{u},\tilde{v}(t);u_\lambda(t))dt \\ &\leq \sum_{k=0}^{n(\tilde{u})-1} \left(\|u_\lambda(t_n(\tilde{u})) - h(\tilde{u};t_n(\tilde{u}))\|^2_{\tilde{v}(t_n(\tilde{u}))} + \|h(\tilde{u};t_n(\tilde{u})) - z_n(\tilde{u})\|^2_{\tilde{v}(t_n(\tilde{u}))} \right) \\ &\quad + \frac{C_{10,1}}{2} \int_0^T \left(\|f(t)\|_H^2 + \|(h(\tilde{u}))'(t)\|_H^2 + \|\tilde{v}'(t)\|_X + 1 \right) dt. \end{aligned} \tag{3.15}$$

Then, from (2.51), Lemma 2.1 and the first estimate in (1) of Lemma 2.7, we get

$$\begin{aligned} \varphi_\lambda(t,\tilde{u},\tilde{v}(t);u_\lambda(t)) &\geq \varphi(t,\tilde{u},\tilde{v}(t);J_\lambda(t,\tilde{u},\tilde{v}(t);u_\lambda(t))) \\ &\geq -C_1 \left(\|u_\lambda(t)\|_{\tilde{v}(t)} + C_4 + 1 \right), \quad \text{hence} \end{aligned} \tag{3.16}$$

$$|\varphi_\lambda(t,\tilde{u},\tilde{v}(t);u_\lambda(t))| \leq \varphi_\lambda(t,\tilde{u},\tilde{v}(t);u_\lambda(t)) + 2C_1 \left(\|u_\lambda(t)\|_{\tilde{v}(t)} + C_4 + 1 \right). \tag{3.17}$$

Finally, we see from (3.15) and (3.17) that the following uniform estimate holds:

$$\begin{aligned} &\sup_{0<\lambda\leq 1} \int_0^T |\varphi_\lambda(t,\tilde{u},\tilde{v}(t);u_\lambda(t))|dt \\ &\leq n(\tilde{u}) \sup_{0<\lambda\leq 1} \left\{ \max_{0\leq t\leq T} \|u_\lambda(t) - h(\tilde{u};t)\|^2_{\tilde{v}(t)} \right\} \\ &\quad + \frac{C_{10,1}}{2} \int_0^T \left(\|f(t)\|_H^2 + \|(h(\tilde{u}))'(t)\|_H^2 + \|\tilde{v}'(t)\|_X \right) dt \\ &\quad + T \left\{ 2C_1 \sup_{0<\lambda\leq 1} \left(\max_{0\leq t\leq T} \|u_\lambda(t)\|^2_{\tilde{v}(t)} \right) + 2C_1(C_4+1) + \frac{C_{10,1}}{2} \right\} + 2n(\tilde{u}) \left(C_2^2 + C_3^2 \right). \end{aligned}$$

Using the above uniform estimate with (3.13), we see that this lemma holds. □

Lemma 3.3. *There exists a constant $C_{11} > 0$ depending on $\tilde{u}$, $\|u_0\|_H$ and $\varphi(0, u_0, v_0; u_0)$, such that*

$$\sup_{0<\lambda\leq 1} \int_0^T \|(u_\lambda(\tilde{u}, u_0, v_0))'(t)\|_H^2 dt + \sup_{0<\lambda\leq 1} \left\{ \max_{0\leq t\leq T} |\varphi_\lambda(t, \tilde{u}, \tilde{v}(t); u_\lambda(\tilde{u}, u_0, v_0; t))| \right\} \leq C_{11}.$$

Proof. We fix a constant $r_6 > 0$ such that $r_6 > C_4 + c_2 C_{10} + 1$, where $C_4 > 0$ and $C_{10} > 0$ are the same constants that are obtained in the first estimate in (1) of Lemma 2.7 and Lemma 3.2, respectively. Using Condition 1 and applying Lemma 2.10, we see from that there exist nonnegative functions $\alpha_{r_6}(\tilde{u}) \in L^2(0, T)$ and $\beta_{r_6}(\tilde{u}) \in L^1(0, T)$ such that the following inequality holds for a.a. $t \in (0, T)$ and all $\lambda \in (0, 1]$:

$$\begin{aligned}
&\frac{d}{dt}\varphi_\lambda(t, \tilde{u}, \tilde{v}(t); u_\lambda(t)) + \frac{1}{2}\|u'_\lambda(t)\|^2_{\tilde{v}(t)} \\
&\leq \|A_\lambda(t, \tilde{u}, \tilde{v}(t); u_\lambda(t))\|_{\tilde{v}(t)} \left(\sqrt{|\varphi_\lambda(t, \tilde{u}, \tilde{v}(t); u_\lambda(t))|} + \sqrt{r_6 C_1} + 1 \right) \alpha_{r_6}(\tilde{u}; t) \\
&\qquad + \left(|\varphi_\lambda(t, \tilde{u}, \tilde{v}(t); u_\lambda(t))| + r_6 C_1 + 1\right) \beta_{r_6}(\tilde{u}; t) \\
&\qquad + \frac{\gamma_0 \lambda}{2}\|A_\lambda(t, \tilde{u}, \tilde{v}(t); u_\lambda(t))\|^2_{\tilde{v}(t)} \|\tilde{v}'(t)\|_X, \quad \text{hence,}
\end{aligned}$$

$$\frac{1}{2}\|u'_\lambda(t)\|^2_{\tilde{v}(t)} + \frac{d}{dt}\varphi_\lambda(t, \tilde{u}, \tilde{v}(t); u_\lambda(t)) \leq \frac{c_2^2}{2}\|f(t)\|_H^2 + \sum_{k=1}^{3} I_{\lambda,k}(\tilde{u}; t), \tag{3.18}$$

where

$$I_{\lambda,1}(\tilde{u}; t) := \|A_\lambda(t, \tilde{u}, \tilde{v}(t); u_\lambda(t))\|_{\tilde{v}(t)} \left(\sqrt{|\varphi_\lambda(t, \tilde{u}, \tilde{v}(t); u_\lambda(t))|} + \sqrt{r_6 C_1} + 1 \right) \alpha_{r_6}(\tilde{u}; t),$$

$$I_{\lambda,2}(\tilde{u}; t) := \left(|\varphi_\lambda(t, \tilde{u}, \tilde{v}(t); u_\lambda(t))| + r_6 C_1 + 1\right) \beta_{r_6}(\tilde{u}; t),$$

$$I_{\lambda,3}(\tilde{u}; t) := \frac{\gamma_0 \lambda}{2}\|A_\lambda(t, \tilde{u}, \tilde{v}(t); u_\lambda(t))\|^2_{\tilde{v}(t)} \|\tilde{v}'(t)\|_X.$$

We estimate $I_{\lambda,1}(\tilde{u})$ and $I_{\lambda,2}(\tilde{u})$. Using (3.17), (2.54) and Lemma 3.2, we have

$$\begin{aligned}
I_{\lambda,1}(t) &\leq \frac{1}{4}\|u'_\lambda(t)\|^2_{\tilde{v}(t)} + \frac{c_2^2}{4}\|f(t)\|_H^2 \\
&\qquad + 2\left(\sqrt{|\varphi_\lambda(t, \tilde{u}, \tilde{v}(t); u_\lambda(t))|} + \sqrt{r_6 C_1} + 1 \right)^2 |\alpha_{r_6}(\tilde{u}; t)|^2 \\
&\leq \frac{1}{4}\|u'_\lambda(t)|^2_{v(t)} + \frac{c_2^2}{4}\|f(t)\|_H^2 + 4|\alpha_{r_6}(\tilde{u}; t)|^2 \\
&\qquad \times \left\{ \varphi_\lambda(t, \tilde{u}, \tilde{v}(t); u_\lambda(t)) + 2C_1(C_{10} + C_4 + 1) + \left(\sqrt{r_6 C_1} + 1 \right)^2 \right\}, \text{ and}
\end{aligned} \tag{3.19}$$

$$I_{\lambda,2}(\tilde{u}; t) \leq \left\{ \varphi_\lambda(t, \tilde{u}, \tilde{v}(t); u_\lambda(t)) + 2C_1(C_{10} + C_4 + 1) + r_6 C_1 + 1 \right\} \beta_{r_6}(\tilde{u}; t). \tag{3.20}$$

Moreover, in order to estimate $I_{\lambda,3}(\tilde{u})$, we use (2.51) with (2.54) and Lemma 3.2. Then, we have

$$\begin{aligned} I_{\lambda,3}(\tilde{u};t) &= \gamma_0\{\tilde{\varphi}_\lambda(t;u_\lambda(t)) - \tilde{\varphi}(t;\tilde{J}_\lambda(t;u_\lambda(t)))\}\|\tilde{v}'(t)\|_X \\ &\le \gamma_0\left\{\varphi_\lambda(t,\tilde{u},\tilde{v}(t);u_\lambda(t)) + 2C_1(C_{10}+C_4+1)\right\}\|\tilde{v}'(t)\|_X. \end{aligned} \tag{3.21}$$

Substituting (3.19)–(3.21) into (3.18), we get for a.a. $t \in (0,T)$,

$$\begin{aligned} &\frac{d}{dt}\varphi_\lambda(t,\tilde{u},\tilde{v}(t);u_\lambda(t)) + \frac{1}{4}\|u'_\lambda(t)\|^2_{\tilde{v}(t)} \\ &\le (\gamma_0+4)\left(|\alpha_{r_6}(\tilde{u};t)|^2 + \beta_{r_6}(\tilde{u};t) + \|\tilde{v}'(t)\|_X\right)\varphi_\lambda(t,\tilde{u},\tilde{v}(t);u_\lambda(t)) \\ &\quad + C_{11,1}\left(|\alpha_{r_6}(\tilde{u};t)|^2 + \beta_{r_6}(\tilde{u};t) + \|\tilde{v}'(t)\|_X + \|f(t)\|^2_H\right), \end{aligned} \tag{3.22}$$

where $C_{11,1} > 0$ is a positive constant that is, for example, given by

$$C_{11,1} = 2C_1(\gamma_0+4)(C_{10}+C_4+1) + 4\left(\sqrt{r_6 C_1}+1\right)^2 + \frac{c_2^2}{4}.$$

Applying Gronwall's lemma to (3.22), we have

$$\begin{aligned} &\max_{0\le t\le T}\left\{\varphi_\lambda(t,\tilde{u},\tilde{v}(t);u_\lambda(t)) + \frac{1}{4}\int_0^t \|u'_\lambda(s)\|^2_{\tilde{v}(s)}ds\right\} \\ &\le \left(|\varphi(0,\tilde{u},v_0;u_0)| + C_{11,1}\int_0^T \left(|\alpha_{r_6}(\tilde{u};t)|^2 + \beta_{r_6}(\tilde{u};t) + \|\tilde{v}'(t)\|_X + \|f(t)\|^2_H\right)dt\right) \\ &\quad \times \exp\left(2C_1(\gamma_0+4)\int_0^T \left(|\alpha_{r_6}(\tilde{u};t)|^2 + \beta_{r_6}(\tilde{u};t) + \|\tilde{v}'(t)\|_X\right)dt\right) =: C_{11,2}. \end{aligned} \tag{3.23}$$

Using (2.54), (3.12) and Lemma 3.2, we have

$$\sup_{0<\lambda\le 1}\left\{\max_{0\le t\le T}|\varphi_\lambda(t,\tilde{u},\tilde{v}(t);u_\lambda(t))|\right\} \le C_{11,2} + 2C_1(C_{10}+C_4+1), \tag{3.24}$$

hence, from (3.23) and (3.24)

$$\sup_{0<\lambda\le 1}\int_0^T \|u'_\lambda(t)\|^2_{\tilde{v}(t)}dt \le 2C_{11,2} + 2C_1(C_{10}+C_4+1). \tag{3.25}$$

Hence, we see from (3.24) and (3.25) that the uniform estimates of this lemma hold. □

Before showing the existence of solutions to $(\mathrm{P})_{\tilde{u},u_0,v_0}$, we discuss two lemmas below. For this purpose we consider a Hilbert space $L^2(0,T;H)$ whose inner product is usually given by

$$(\eta_1,\eta_2)_{L^2(0,T;H)} := \int_0^T (\eta_1(t),\eta_2(t))_H dt, \quad \forall\,\eta_1,\eta_2 \in L^2(0,T;H).$$

For $\tilde{u} \in \mathcal{W}(u_0, v_0)$ we define an inner product on $L^2(0,T;H)$ associated to $\tilde{u}$ by

$$(\eta_1, \eta_2)_{L^2(0,T;H(\tilde{v}))} := \int_0^T (\eta_1(t), \eta_2(t))_{\tilde{v}(t)} dt, \quad \forall \eta_1, \eta_2 \in L^2(0,T;H).$$

We denote by $L^2(0,T;H(\tilde{v}))$ a Hilbert space $L^2(0,T;H)$ with the inner product $(\cdot,\cdot)_{L^2(0,T;H(\tilde{v}))}$. Then, we easily see from (A1) that $(\cdot,\cdot)_{L^2(0,T;H(\tilde{v}))}$ is equivalent to $(\cdot,\cdot)_{L^2(0,T;H)}$.

Lemma 3.4. *Consider some sequences* $\{\xi_m\}_{m\in\mathbb{N}}$, $\{\eta_m\}_{m\in\mathbb{N}} \subset L^2(0,T;H)$, *and* $\{\tilde{u}_m\}_{m\in\mathbb{N}} \subset \mathcal{W}(u_0, v_0)$, *and elements* ξ, $\eta \in L^2(0,T;H)$, $\tilde{u} \in \mathcal{W}(u_0, v_0)$ *satisfying the following convergences as* $m \to \infty$*:*

$$\begin{cases} \tilde{u}_m \to \tilde{u} & \textit{in} \quad C([0,T];H), \\ \xi_m \to \xi & \textit{in} \quad L^2(0,T;H(\tilde{v})), \\ \eta_m \to \eta & \textit{weakly in} \quad L^2(0,T;H(\tilde{v})). \end{cases} \tag{3.26}$$

Then, we have $$\lim_{m\to\infty} (\xi_m, \eta_m)_{L^2(0,T;H(\tilde{v}_m))} = (\xi, \eta)_{L^2(0,T;H(\tilde{v}))}. \tag{3.27}$$

Proof. We see from (S3) that the following convergence holds as $m \to \infty$:

$$\tilde{v}_m(\cdot) = S(\tilde{u}_m, \cdot, 0)v_0 \longrightarrow \tilde{v}(\cdot) = S(\tilde{u}, \cdot, 0)v_0 \text{ in } C([0,T];X). \tag{3.28}$$

From (3.26) we have $$\lim_{m\to\infty} (\xi_m, \eta_m)_{L^2(0,T;H(\tilde{v}))} = (\xi, \eta)_{L^2(0,T;H(\tilde{v}))}. \tag{3.29}$$

Repeating the argumentation of the proof of Lemma 2.2 in a similar way and using Condition 2, we have

$$\begin{aligned} &\left| \int_0^T ((\xi_m(t), \eta_m(t))_{\tilde{v}_m(t)} - (\xi_m(t), \eta_m(t))_{\tilde{v}(t)}) dt \right| \\ &\le \frac{3\gamma_0}{2} \int_0^T \|\tilde{v}_m(t) - \tilde{v}(t)\|_X \left(\|\xi_m(t)\|^2_{\tilde{v}(t)} + \|\eta_m(t)\|^2_{\tilde{v}(t)} \right) dt \\ &\le \frac{3\gamma_0}{2} \left(\sup_{m\in\mathbb{N}} \int_0^T \|\xi_m(t)\|^2_{\tilde{v}(t)} dt + \sup_{m\in\mathbb{N}} \int_0^T \|\eta_m(t)\|^2_{\tilde{v}(t)} dt \right) \|\tilde{v}_m - \tilde{v}\|_{C([0,T];X)}. \end{aligned}$$

This estimate with (3.26) and (3.28) implies

$$\lim_{m\to\infty} \int_0^T \left\{ (\xi_m(t), \eta_m(t))_{\tilde{v}_m(t)} - (\xi_m(t), \eta_m(t))_{\tilde{v}(t)} \right\} dt = 0. \tag{3.30}$$

Hence, we see from (3.29) and (3.30) that (3.27) holds. □

Next, we define a multi-valued operator $\mathcal{A}(\tilde{u}) : L^2(0,T;H(\tilde{v})) \to 2^{L^2(0,T;H(\tilde{v}))}$ by

$$\eta^* \in \mathcal{A}(\tilde{u}; \eta) \text{ if and only if } \eta^*(t) \in \partial_{\tilde{v}(t)}\varphi(t, \tilde{u}, \tilde{v}(t); \eta(t)), \text{ a.a. } t \in (0,T). \tag{3.31}$$

The operator $\mathcal{A}(\tilde{u})$ is called the *natural extension* of the family $\{\partial_{\tilde{v}(t)}\varphi(t, \tilde{u}, \tilde{v}(t));\ 0 \le t \le T\}$ to $L^2(0,T;H(\tilde{v}))$ and satisfies the properties in Lemma 3.5 below.

Lemma 3.5. *The natural extension $\mathcal{A}(\tilde{u})$ of $\{\partial_{\tilde{v}(t)}\varphi(t,\tilde{u},\tilde{v}(t))\,;\,0 \le t \le T\}$ to $L^2(0,T;H(\tilde{v}))$, which is defined by* (3.31), *is maximal monotone on $L^2(0,T;H(\tilde{v}))$. Hence, $\mathcal{A}(\tilde{u})$ is demi-closed, that is, if any sequence $\{[\eta_m,\eta_m^*]\}_{m\in\mathbb{N}} \subset G(\mathcal{A}(\tilde{u}))$, the graph of $\mathcal{A}(\tilde{u})$, and any element $[\eta,\eta^*] \in L^2(0,T;H(\tilde{v})) \times L^2(0,T;H(\tilde{v}))$ satisfy $\eta_m \longrightarrow \eta$ in $L^2(0,T;H(\tilde{v}))$ and $\eta_m^* \longrightarrow \eta^*$ weakly in $L^2(0,T;H(\tilde{v}))$ as $m \to \infty$, then we have $[\eta,\eta^*] \in G(\mathcal{A}(\tilde{u}))$.*

Proof. Since it is clear that $\mathcal{A}(\tilde{u})$ is monotone on $L^2(0,T;H(\tilde{v}))$, it is enough to show that $\mathcal{A}(\tilde{u})$ is maximal on $L^2(0,T;H(\tilde{v}))$. We fix any $\lambda \in (0,1]$. For any $\eta^* \in L^2(0,T;H(\tilde{v}))$ we define η by $\eta(t) := J_\lambda(t,\tilde{u},\tilde{v}(t);\eta^*(t))$ for all $t \in [0,T]$. Then, we see from the first estimate in (1) of Lemma 2.7 that

$$\|\eta(t)\|_{\tilde{v}(t)} = \|J_\lambda(t,\tilde{u},\tilde{v}(t);\eta^*(t))\|_{\tilde{v}(t)} \le \|\eta^*(t)\|_{\tilde{v}(t)} + C_4,$$

which implies $\eta \in L^2(0,T;H(\tilde{v}))$. Moreover, from (2.46) we have

$$\frac{1}{\lambda}(\eta^*(t) - \eta(t)) \in \partial_{\tilde{v}(t)}\varphi(t,\tilde{u},\tilde{v}(t);\eta(t)), \quad \forall\, t \in [0,T]. \tag{3.32}$$

From (3.31) and (3.32) we have $\eta^* \in (I + \lambda\mathcal{A}(\tilde{u}))\eta$, which implies $L^2(0,T;H(\tilde{v})) = R(I + \lambda\mathcal{A}(\tilde{u}))$. Hence, $\mathcal{A}(\tilde{u})$ is maximal on $L^2(0,T;H(\tilde{v}))$. □

The next lemma is used to show the strong covergence of $\{u_\lambda\,;\,0 < \lambda \le 1\}$ in $L^2(0,T;H(\tilde{v}))$ as $\lambda \searrow 0$, which was given in [3] and sometimes called Crandall-Pazy's convergence criterion.

Lemma 3.6. (cf. [3]) *Let V be a real Hilbert space. We assume that a sequence $\{a_\lambda\,;\,0 < \lambda \le 1\}$ in V satisfies the following inequality:*

$$(a_\lambda - a_\mu, \lambda a_\lambda - \mu a_\mu)_V \le 0, \quad \forall\, \lambda, \mu \in (0,1].$$

Then, a_λ converges in V as $\lambda \searrow 0$ if and only if $\{a_\lambda\,;\,0 < \lambda \le 1\}$ is bounded in V.

Now we are ready to present the full proof of the existence part in Theorem 3.1.

Proof of the existence part and of (1) in Theorem 3.1.
Using Lemma 2.10 as $z = u_\lambda - u_\mu$, we have

$$\begin{aligned}&\frac{d}{dt}\|u_\lambda(t) - u_\mu(t)\|^2_{\tilde{v}(t)} - 2(u_\lambda'(t) - u_\mu'(t), u_\lambda(t) - u_\mu(t))_{\tilde{v}(t)} \\ &\le \gamma_0\|\tilde{v}'(t)\|_X\|u_\lambda(t) - u_\mu(t)\|^2_{\tilde{v}(t)}, \qquad \text{a.a. } t \in (0,T).\end{aligned}$$

We see from (2.50) that the following inequality holds for a.a. $t \in (0,T)$:

$$\begin{aligned}&\frac{d}{dt}\|u_\lambda(t) - u_\mu(t)\|^2_{\tilde{v}(t)} - \gamma_0\|\tilde{v}'(t)\|_X\|u_\lambda(t) - u_\mu(t)\|^2_{\tilde{v}(t)} + 2(A_\lambda(t,\tilde{u},\tilde{v}(t);u_\lambda(t)) \\ &-A_\mu(t,\tilde{u},\tilde{v}(t);u_\mu(t)), \lambda A_\lambda(t,\tilde{u},\tilde{v}(t);u_\lambda(t)) - \mu A_\mu(t,\tilde{u},\tilde{v}(t);u_\mu(t)))_{\tilde{v}(t)} \le 0.\end{aligned}$$

Applying Gronwall's lemma to the above inequality, we get

$$\|u_\lambda(t) - u_\mu(t)\|^2_{\tilde{v}(t)} \exp\left(-\gamma_0 \int_0^t \|\tilde{v}'(s)\|_X ds\right) + 2\int_0^t (z_\lambda(s) - z_\mu(s), \lambda z_\lambda(s) - \mu z_\mu(s))_{\tilde{v}(s)} ds \le 0, \qquad (3.33)$$

for all $t \in [0,T]$ and all $\lambda, \mu \in (0,1]$, where z_λ is defined by

$$z_\lambda(t) := \exp\left(-\frac{\gamma_0}{2}\int_0^t \|\tilde{v}'(s)\|_X ds\right) A_\lambda(t, \tilde{u}, \tilde{v}(t); u_\lambda(t)), \quad \forall\, t \in [0,T].$$

Therefore, we have

$$(z_\lambda - z_\mu, \lambda z_\lambda - \mu z_\mu)_{L^2(0,T;H(\tilde{v}))} \le 0, \quad \forall\, \lambda,\, \mu \in (0,1]. \qquad (3.34)$$

Moreover, Lemma 3.3 implies that $\{z_\lambda\,;\, 0 < \lambda \le 1\}$ is bounded in $L^2(0,T;H(\tilde{v}))$ because of the following estimate: for all $\lambda \in (0,1]$

$$\begin{aligned}\|z_\lambda\|^2_{L^2(0,T;H(\tilde{v}))} &= \int_0^T \exp\left(-\gamma_0\int_0^t \|\tilde{v}'(s)\|_X ds\right) \|A_\lambda(t,\tilde{u},\tilde{v}(t);u_\lambda(t))\|^2_{\tilde{v}(t)} dt \\ &\le 2\int_0^T \|u'_\lambda(t)\|^2_{\tilde{v}(t)} dt + 2\int_0^T \|f(t)\|^2_{\tilde{v}(t)} dt.\end{aligned}$$

Applying Crandall-Pazy's criterion (cf. Lemma 3.6), we see that there exists an element $z(\tilde{u}) \in L^2(0,T;H(\tilde{v}))$ such that

$$z_\lambda \longrightarrow z(\tilde{u}) \quad \text{in} \quad L^2(0,T;H(\tilde{v})) \quad \text{as} \quad \lambda \searrow 0. \qquad (3.35)$$

Going back to (3.33), we get for all $\lambda, \mu \in (0,1]$

$$\begin{aligned}&c_2^2 \max_{0\le t\le T} \|u_\lambda(t) - u_\mu(t)\|^2_H \\ &\le 2\max_{0\le t\le T}\left\{\exp\left(\gamma_0\int_0^t \|\tilde{v}'(s)\|_X ds\right)\int_0^t \left|(z_\lambda(s) - z_\mu(s), \lambda z_\lambda(s) - \mu z_\mu(s))_{\tilde{v}(s)}\right| ds\right\} \\ &\le 4(\lambda+\mu)\exp\left(\gamma_0\int_0^T \|\tilde{v}'(t)\|_X dt\right)\sup_{0<\lambda\le 1}\int_0^T \|z_\lambda(t)\|^2_{\tilde{v}(t)} dt,\end{aligned}$$

which implies that $\{u_\lambda\,;\, 0 < \lambda \le 1\}$ is a Cauchy sequence in $C([0,T];H)$. Hence, we see that there exists an element $u(\tilde{u}) \in C([0,T];H)$ such that

$$u_\lambda \longrightarrow u(\tilde{u}) \quad \text{in} \quad C([0,T];H) \quad \text{as} \quad \lambda \searrow 0. \qquad (3.36)$$

Hence, the following convergence and estimate hold for all $t \in [0,T]$:

$$u_\lambda(t) \to u(\tilde{u};t) \ \text{ in } H(\tilde{v}(t)) \text{ as } \lambda \searrow 0, \qquad (3.37)$$

$$\varphi(t,\tilde{u},\tilde{v}(t);u(\tilde{u};t)) \le \liminf_{\lambda\searrow 0} \varphi(t,\tilde{u},\tilde{v}(t);u_\lambda(t)). \qquad (3.38)$$

On the other hand, defining $\bar{w}(\tilde{u}) \in L^2(0,T;H(\tilde{v}))$ by

$$\bar{w}(\tilde{u},t) := \exp\left(\frac{\gamma_0}{2}\int_0^t \|\tilde{v}'(s)\|_X ds\right) z(\tilde{u};t), \quad \forall\, t \in [0,T],$$

we have

$$\begin{aligned}
&\|A_\lambda(\cdot,\tilde{u},\tilde{v}(\cdot);u_\lambda) - \bar{w}(\tilde{u})\|^2_{L^2(0,T;H(\tilde{v}))} \\
&= \int_0^T \exp\left(\gamma_0 \int_0^t \|\tilde{v}'(s)\|_X ds\right) \|z_\lambda(t) - z(\tilde{u};t)\|^2_{\tilde{v}(t)} dt \\
&\le \exp\left(\frac{\gamma_0}{2}\int_0^T \|\tilde{v}'(t)\|_X dt\right) \|z_\lambda - z(\tilde{u})\|_{L^2(0,T;H(\tilde{v}))},
\end{aligned}$$

which implies that

$$A_\lambda(\cdot,\tilde{u},\tilde{v}(\cdot);u_\lambda) \longrightarrow \bar{w}(\tilde{u}) \quad \text{in} \quad L^2(0,T;H(\tilde{v})) \quad \text{as} \quad \lambda \searrow 0. \tag{3.39}$$

Since from (2.47) we get for all $t \in [0,T]$ and for all $\lambda \in (0,1]$

$$\|J_\lambda(t,\tilde{u},\tilde{v}(t);u_\lambda(t)) - u(\tilde{u};t)\|_{\tilde{v}(t)} \le \lambda\|A_\lambda(t,\tilde{u},\tilde{v}(t);u_\lambda(t))\|_{\tilde{v}(t)} + \|u_\lambda(t) - u(\tilde{u};t)\|_{\tilde{v}(t)},$$

we see from (3.36) and (3.39) that

$$J_\lambda(\cdot,\tilde{u},\tilde{v}(\cdot);u_\lambda) \longrightarrow u(\tilde{u}) \quad \text{in} \quad L^2(0,T;H(\tilde{v})) \quad \text{as} \quad \lambda \searrow 0. \tag{3.40}$$

Using (2.50), (3.39) and (3.40), and applying Lemma 3.5, we obtain immediately $[u(\tilde{u}), \bar{w}(\tilde{u})] \in G(\mathcal{A}(\tilde{u}))$, hence,

$$\bar{w}(\tilde{u};t) \in \partial_{\tilde{v}(t)}\varphi(t,\tilde{u},\tilde{v}(t);u(\tilde{u};t))), \quad \text{a.a. } t \in (0,T). \tag{3.41}$$

From (3.7) and (3.39) we have

$$u'_\lambda \longrightarrow f - \bar{w}(\tilde{u}) \quad \text{in} \quad L^2(0,T;H(\tilde{v})) \quad \text{as} \quad \lambda \searrow 0. \tag{3.42}$$

We see from (3.36) and (3.42) that

$$(u(\tilde{u}))' = f - \bar{w}(\tilde{u}), \tag{3.43}$$

where $(u(\tilde{u}))'$ is the derivative of $u(\tilde{u})$ with respect to the variable t in the distribution sense. So, we have $u(\tilde{u}) \in W^{1,2}(0,T;H)$ and from (3.41) and (3.43)

$$f(t) - (u(\tilde{u}))'(t) \in \partial_{\tilde{v}(t)}\varphi(t,\tilde{u},\tilde{v}(t);u(t)) \quad \text{in} \quad H(\tilde{v}(t)), \quad \text{a.a. } t \in (0,T).$$

Hence, we see that $u(\tilde{u})$ is a solution to $(\mathrm{P})_{\tilde{u},u_0,v_0}$, and from (3.36) and Lemma 3.2 we have

$$\max_{0\le t\le T} \|u(\tilde{u};t)\|_{\tilde{v}(t)} \le C_{10}. \tag{3.44}$$

Applying Fatou's lemma and using (2.51) and (3.40), we have

$$\begin{aligned}
\int_0^T \varphi(t,\tilde{u},\tilde{v}(t);u(\tilde{u};t))dt &\le \liminf_{\lambda\searrow 0} \int_0^T \varphi(t,\tilde{u},\tilde{v}(t);J_\lambda(t,\tilde{u},\tilde{v};u_\lambda(t)))dt \\
&\le \liminf_{\lambda\searrow 0} \int_0^T \varphi_\lambda(t,\tilde{u},\tilde{v}(t);u_\lambda(t))dt.
\end{aligned} \tag{3.45}$$

Using (3.44), (3.45), Lemmas 2.1 and 3.2, we have

$$\int_0^T |\varphi(t, \tilde{u}, \tilde{v}(t); u(\tilde{u}(t)))| dt \le C_{10} + C_1(C_{10} + 1). \tag{3.46}$$

At last, we see from (3.44) and (3.46) that the boundedness (1) of Theorem 3.1 holds. □

Proof of (2) in Theorem 3.1. From (3.41) we have $u(\tilde{u}; t) \in D(\varphi(t, \tilde{u}, \tilde{v}(t)))$ for a.a. $t \in (0, T)$. Hence, we have for a.a. $t \in (0, T)$

$$J_\lambda(t, \tilde{u}, \tilde{v}(t); u(\tilde{u}; t)) \longrightarrow u(\tilde{u}; t) \quad \text{in} \quad H(\tilde{v}(t)) \quad \text{as} \quad \lambda \searrow 0. \tag{3.47}$$

We define a measurable subset $I_1 \subset (0, T)$ by

$$I_1 := \{t \in (0, T)\, ;\, (3.47) \text{ is satisfied at } t\}.$$

Since $J_\lambda(t, \tilde{u}, \tilde{v}(t))$ is a contraction operator on $H(\tilde{v}(t))$, we have

$$\begin{aligned} &\| J_\lambda(t, \tilde{u}, \tilde{v}(t); u_\lambda(t)) - u(\tilde{u}; t)\|_{\tilde{v}(t)} \\ &\qquad \le \|u_\lambda(t) - u(\tilde{u}; t)\|_{\tilde{v}(t)} + \|J_\lambda(t, \tilde{u}, \tilde{v}(t); u(\tilde{u}; t)) - u(\tilde{u}; t)\|_{\tilde{v}(t)}. \end{aligned} \tag{3.48}$$

Hence, we see from (3.37), (3.47) and (3.48) that

$$\forall\, t \in I_1, \quad J_\lambda(t, \tilde{u}, \tilde{v}(t); u_\lambda(t)) \longrightarrow u(\tilde{u}; t) \quad \text{in} \quad H(\tilde{v}(t)) \quad \text{as} \quad \lambda \searrow 0.$$

Repeating the same argumentation of the derivation of (3.45) and using Lemma 3.3, we get

$$\varphi(t, \tilde{u}, \tilde{v}(t); u(\tilde{u}; t)) \le \liminf_{\lambda \searrow 0} \varphi_\lambda(t, \tilde{u}, \tilde{v}(t); u_\lambda(t)) \le C_{11}, \quad \forall\, t \in I_1. \tag{3.49}$$

Now, we assume that there exists a time $t_0 \in [0, T] \setminus I_1$ such that $u(\tilde{u}; t_0) \notin \overline{D(\varphi(t_0, \tilde{u}, \tilde{v}(t_0)))}$. Then, we see that there exists a constant $\delta_0 > 0$ such that

$$\begin{aligned} &\{z \in H\, ;\, \|z - u(\tilde{u}; t_0)\|_H < \delta_0\} \subset H \setminus \overline{D(\varphi(t_0; , \tilde{u}, \tilde{v}(t_0)))}, \ \text{ that is,} \\ &\qquad \|z - u(\tilde{u}; t_0)\|_H \ge \delta_0, \quad \forall\, z \in \overline{D(\varphi(t_0, \tilde{u}, \tilde{v}(t_0)))}. \end{aligned} \tag{3.50}$$

Applying Condition 1 as $r = r_6 \ge C_7$, we see that there exist nonnegative functions $\alpha_{r_6}(\tilde{u}) \in L^2(0, T)$, $\beta_{r_6}(\tilde{u}) \in L^1(0, T)$ and a family $\{z_s(t_0)\, ;\, s \in I_1\}$ such that

$$\{z_s(t_0)\, ;\, s \in I_1\} \subset D(\varphi(t_0, \tilde{u}, \tilde{v}(t_0))), \tag{3.51}$$

$$\|z_s(t_0) - u(\tilde{u}; s)\|_{\tilde{v}(t_0)} \le \left(\sqrt{C_{11}} + 1\right) \int_s^{t_0} \alpha_{r_6}(\tilde{u}; \tau) d\tau, \quad \forall\, s \in I_1. \tag{3.52}$$

Using (3.52) and $u(\tilde{u}) \in C([0, T]; H)$, we can take a constant $\tau_0 \in (0, \min\{T - t_0, t_0\})$ such that

$$\|z_s(t_0) - u(\tilde{u}; s)\|_{\tilde{u}(t_0)} \le \frac{\delta_0 c_1}{4}, \quad \forall\, s \in I_1 \cap [t_0 - \tau_0, t_0], \ \text{ and} \tag{3.53}$$

$$\|u(\tilde{u}; s) - u(\tilde{u}; t_0)\|_H \le \frac{\delta_0}{4}, \quad \forall\, s \in [t_0 - \tau_0, t_0]. \tag{3.54}$$

Hence, from (3.51), (3.53) and (3.54) we have

$$\|z_s(t_0) - u(\tilde{u}; t_0)\|_H \le \frac{\delta_0}{2}, \quad \forall\, s \in I_1 \cap [t_0 - \tau_0, t_0],$$

which is in contradiction to (3.50). So, we see that $u(\tilde{u}; t) \in \overline{D(\varphi(t, \tilde{u}, \tilde{v}(t)))}$ holds for all $t \in [0, T]$. Hence, (3.47) holds for all $t \in [0, T]$. Repeating the same argumentation that was used in the derivation of (3.49), we get

$$\sup_{0 \le t \le T} \varphi(t, \tilde{u}, \tilde{v}(t); u(\tilde{u}; t)) \le C_{11},$$

hence, from Lemma 2.1

$$\sup_{0 \le t \le T} |\varphi(t, \tilde{u}, \tilde{v}(t); u(\tilde{u}; t))| \le C_{11} + C_1(C_7 + 1). \tag{3.55}$$

Moreover, from Lemma 3.3 and (3.42), (3.43) we have

$$\|(u(\tilde{u}))'\|^2_{L^2(0,T;H(\tilde{v}))} \le C_{11}. \tag{3.56}$$

Taking $C_8 > 0$ so that $C_8 \ge 2C_{11} + C_1(C_7 + 1)$, we see from (3.55) and (3.56) that the estimate in (2) of Theorem 3.1 is satisfied. □

Proof of (3) in Theorem 3.1. Following (3.39) we can choose a sequence $\{\lambda_n\}_{n \in \mathbb{N}} \subset (0, 1]$ such that $\lambda_n \searrow 0$ and for a.a. $t \in (0, T)$

$$A_{\lambda_n}(t, \tilde{u}, \tilde{v}(t); u_{\lambda_n}(t)) \longrightarrow \bar{w}(\tilde{u}; t) = f(t) - (u(\tilde{u}))'(t) \quad \text{in} \quad H(\tilde{v}(t)) \tag{3.57}$$

as $n \to \infty$. From (2.50) we have

$$\begin{aligned} &\varphi_{\lambda_n}(t, \tilde{u}, \tilde{v}(t); u_{\lambda_n}(t)) \\ &\le \varphi(t, \tilde{u}, \tilde{v}(t); u(\tilde{u}; t)) + (A_{\lambda_n}(t, \tilde{u}, \tilde{v}(t); u_{\lambda_n}(t)), u_{\lambda_n}(t) - u(\tilde{u}; t))_{\tilde{v}(t)}. \end{aligned} \tag{3.58}$$

Taking $\limsup_{n \to \infty}$ in both sides of (3.58) and using (3.36), (3.57), we have

$$\limsup_{n \to \infty} \varphi_{\lambda_n}(t, \tilde{u}, \tilde{v}(t); u_{\lambda_n}(t)) \le \varphi(t, \tilde{u}, \tilde{v}(t); u(\tilde{u}; t)), \quad \text{a.a. } t \in (0, T). \tag{3.59}$$

Now, we define a subset I_2 of $(0, T)$ by $I_2 := \{t \in (0, T)\,;\, (3.59) \text{ is satisfied at } t.\}$. Then, we see from (3.49) and (3.59) that

$$\lim_{n \to \infty} \varphi_{\lambda_n}(t, \tilde{u}, \tilde{v}(t); u_{\lambda_n}(t)) = \varphi(t, \tilde{u}, \tilde{v}(t); u(\tilde{u}; t)), \quad \forall\, t \in I_2. \tag{3.60}$$

In the following argumentation we fix a constant $r_7 > 0$ so that $r_7 \ge C_4 + C_7 + 1$. Using Condition 1 as $r = r_7$ and applying Lemma 2.11, we see that there exist nonnegative functions $\alpha_{r_7}(\tilde{u}) \in L^2(0, T)$ and $\beta_{r_7}(\tilde{u}) \in L^1(0, T)$ such that the following inequality holds for a.a. $t \in (0, T)$:

$$\begin{aligned} &\frac{d}{dt}\varphi_{\lambda_n}(t, \tilde{u}, \tilde{v}(t); u_{\lambda_n}(t)) - (u'_{\lambda_n}(t), A_{\lambda_n}(t, \tilde{u}, \tilde{v}(t); u_{\lambda_n}(t)))_{\tilde{v}(t)} \\ &\le \varepsilon \|A_{\lambda_n}(t, \tilde{u}, \tilde{v}(t); u_{\lambda_n}(t))\|_{\tilde{v}(t)} \left(\sqrt{|\varphi_{\lambda_n}(t, \tilde{u}, \tilde{v}(t); u_{\lambda_n}(t))|} + \sqrt{r_7 C_1} + 1\right) \alpha_{r_7}(\tilde{u}; t) \\ &\qquad + (|\varphi_{\lambda_n}(t, \tilde{u}, \tilde{v}(t); u_{\lambda_n}(t))| + r_7 C_1 + 1)\, \beta_{r_7}(\tilde{u}; t) \\ &\qquad + \frac{\gamma_0 \lambda}{2} \|A_{\lambda_n}(t, \tilde{u}, \tilde{v}(t); u_{\lambda_n}(t))\|^2_{\tilde{v}(t)} \|\tilde{v}'(t)\|_X. \end{aligned}$$

Using (2.51), (3.16) and (3.17), for each $\varepsilon > 0$ we have for a.a. $t \in (0, T)$

$$
\begin{aligned}
&\frac{d}{dt}\varphi_{\lambda_n}(t, \tilde{u}, \tilde{v}(t); u_{\lambda_n}(t)) - (u'_{\lambda_n}(t), A_{\lambda_n}(t, \tilde{u}, \tilde{v}(t); u_{\lambda_n}(t)))_{\tilde{v}(t)} \\
&\le \varepsilon \|A_{\lambda_n}(t, \tilde{u}, \tilde{v}(t); u_{\lambda_n}(t))\|^2_{\tilde{v}(t)} + \frac{1}{2\varepsilon}\left(|\varphi_{\lambda_n}(t, \tilde{u}, \tilde{v}(t); u_{\lambda_n}(t))| + 2r_7 C_1 + 2\right)|\alpha_{r_6}(\tilde{u}; t)|^2 \\
&\quad + \left(|\varphi_{\lambda_n}(t, \tilde{u}, \tilde{v}(t); u_{\lambda_n}(t))| + r_7 C_1 + 1\right)\beta_{r_7}(\tilde{u}; t) \\
&\quad + \gamma_0 \left\{\varphi_{\lambda_n}(t, \tilde{u}, \tilde{v}(t); u_{\lambda_n}(t)) - \varphi_{\lambda_n}(t, \tilde{u}, \tilde{v}(t); J_{\lambda_n}(t, \tilde{u}, \tilde{v}(t); u_{\lambda_n}(t)))\right\}\|\tilde{v}'(t)\|_X \\
&\le \varepsilon \|A_{\lambda_n}(t, \tilde{u}, \tilde{v}(t); u_{\lambda_n}(t))\|^2_{\tilde{v}(t)} + \left(\frac{1}{2\varepsilon} + \gamma_0 + 1\right)\left(|\alpha_{r_7}(\tilde{u}; t)|^2 + \beta_{r_7}(\tilde{u}; t) + \|\tilde{v}'(t)\|_X\right) \\
&\quad \times \left\{\varphi_{\lambda_n}(t, \tilde{u}, \tilde{v}(t); u_{\lambda_n}(t)) + 2C_1(C_4 + C_8 + 1) + r_7 C_1 + 1\right\}. \qquad (3.61)
\end{aligned}
$$

Choosing $C_9 > 0$ such that

$$
C_9 \ge \left(\frac{1}{2\varepsilon} + \gamma_0 + 1\right)\{2C_1(C_4 + C_8 + 1) + r_7 C_1 + 1\}, \quad \text{and}
$$

integrating both sides of (3.61) on $[s, t] \subset [0, T]$ with $s \le t$ and $s \in I_2$, we get

$$
\begin{aligned}
&\varphi_{\lambda_n}(t, \tilde{u}, \tilde{v}(t); u_{\lambda_n}(t)) - \int_s^t (u'_{\lambda_n}(\tau), A_{\lambda_n}(\tau, \tilde{u}, \tilde{v}(\tau); u_{\lambda_n}(\tau)))_{\tilde{v}(\tau)} d\tau \\
&\le \ \varphi_{\lambda_n}(s, \tilde{u}, \tilde{v}(s); u_{\lambda_n}(s)) + \varepsilon \int_s^t \|A_{\lambda_n}(\tau, \tilde{u}, \tilde{v}(\tau); u_{\lambda_n}(\tau))\|^2_{\tilde{v}(\tau)} d\tau \\
&+ R_3 \int_s^t \{\varphi_{\lambda_n}(\tau, \tilde{u}, \tilde{v}(\tau); u_{\lambda_n}(\tau)) + 1\}\left(|\alpha_{r_7}(\tilde{u}; \tau)|^2 + \beta_{r_7}(\tilde{u}; \tau) + \|\tilde{v}'(\tau)\|_X\right) d\tau. \qquad (3.62)
\end{aligned}
$$

Taking $\liminf_{n\to\infty}$ in both sides of (3.62) and applying Lebesgue's dominated convergence theorem with (3.36), (3.42), (3.49), (3.57), (3.60), we see that (3) of Theorem 3.1 holds. □

At the end of this section, we consider a function $\Phi^T(\tilde{u})$ on $L^2(0, T; H(\tilde{v}))$ defined by

$$
\Phi^T(\tilde{u}; \eta) := \int_0^T \varphi(t, \tilde{u}(t), \tilde{v}(t); \eta(t)) dt, \quad \forall\, \eta \in L^2(0, T; H(\tilde{v})).
$$

Then, we have the following lemmas.

Lemma 3.7. *The function $\Phi^T(\tilde{u})$ is proper, l.s.c. and convex on $L^2(0, T; H(\tilde{v}))$.*

Proof. From Proposition 2.5 we have the function $h(\tilde{u}) \in \mathcal{W}(u_0, v_0)$ such that $\varphi(\cdot, \tilde{u}, \tilde{v}(\cdot); h(\tilde{u})) \in L^\infty(0, T)$, hence, $\Phi^T(\tilde{u})$ is proper on $L^2(0, T; H(\tilde{v}))$. Since $\varphi(t, \tilde{u}, \tilde{v}(t))$ are l.s.c. and convex on $H(\tilde{v}(t))$ for all $t \in [0, T]$, we see that $\Phi^T(\tilde{u})$ is also l.s.c. and convex on $L^2(0, T; H(\tilde{v}))$. Its proof is quite standard, so, we omit it in this paper. □

Lemma 3.8. *The subdifferential $\partial_{\tilde{v}}\Phi^T(\tilde{u})$ of $\Phi^T(\tilde{u})$ on $L^2(0,T;H(\tilde{v}))$, which is defined in the following way: $\eta^* \in \partial_{\tilde{v}}\Phi^T(\tilde{u};\eta)$ if and only if*

$$(\eta^*, \xi-\eta)_{L^2(0,T;H(\tilde{v}))} \leq \Phi^T(\tilde{u};\xi)-\Phi^T(\tilde{u};\eta), \quad \forall\, \xi \in L^2(0,T;H(\tilde{v})), \tag{3.63}$$

coincides with the natural extension $\mathcal{A}(\tilde{u})$ of the family $\{\partial_{\tilde{v}(t)}\varphi(t,\tilde{u},\tilde{v}(t))\,;\, 0 \leq t \leq T\}$, which is defined by (3.31)*.*

Proof. Since we see from Lemmas 3.5 and 3.7 that both $\mathcal{A}(\tilde{u})$ and $\partial_{\tilde{v}}\Phi^T(\tilde{u})$ are maximal monotone on $L^2(0,T;H(\tilde{v}))$, it is enough to show $G(\mathcal{A}(\tilde{u})) \subset G(\partial_{\tilde{v}}\Phi^T(\tilde{u}))$. Let η^* be any element of $\mathcal{A}(\tilde{u};\eta)$. Then, from (3.31) we have

$$\eta^*(t) \in \partial_{\tilde{v}(t)}\varphi(t,\tilde{u},\tilde{v}(t);\eta(t)), \quad \text{a.a. } t \in (0,T).$$

Hence, from (2.45) we get for all $y \in H(\tilde{v}(t))$ and a.a. $t \in (0,T)$

$$(\eta^*(t), y-\eta(t))_{\tilde{v}(t)} \leq \varphi(t,\tilde{u},\tilde{v}(t);y) - \varphi(t,\tilde{u},\tilde{v}(t);\eta(t)). \tag{3.64}$$

For any $\xi \in L^2(0,T;H(\tilde{v}))$ we substitute $y = \xi(t)$ in (3.64) and integrate its result on $(0,T)$. Then, we have (3.63), that is, $\eta^* \in \partial_{\tilde{v}}\Phi^T(\tilde{u};\eta)$. $\square$

As a direct consequence of Lemma 3.8, we derive Theorem 3.9 below, which gives the characterization of the evolution inclusion in $(\mathrm{P})_{\tilde{u},u_0,v_0}$.

Theorem 3.9. *A function $w \in W^{1,2}(0,T;H)$ satisfies the evolution inclusion in $(P)_{\tilde{u},u_0,v_0}$ if and only if w satisfies $f-w' \in \partial_{\tilde{v}}\Phi^T(\tilde{u};w)$, i.e., $f-w' \in D(\Phi^T(\tilde{u}))$ and*

$$(w'-f, w-\xi)_{L^2(0,T;H(\tilde{v}))} \leq \Phi^T(\tilde{u};\xi) - \Phi^T(\tilde{u};w), \quad \forall\, \xi \in L^2(0,T;H(\tilde{v})).$$

4. Existence of local-in-time solutions

We devote this section to show the existence of local-in-time solutions to $(\mathrm{P})_{u_0,v_0}$ by using Schauder's fixed point argumentation. In order to do this, we consider two subsets of $\mathcal{W}(u_0,v_0)$, which are defined by

$$\mathcal{V}(u_0,v_0) := \left\{ \tilde{u} \in \mathcal{W}(u_0,v_0)\,;\, \|\tilde{u}'\|^2_{L^2(0,T;H)} + \sup_{0\leq t\leq T} |\varphi(\tilde{u}(t))| < \infty \right\},$$

and for each $C > 0$

$$\mathcal{V}_C(u_0,v_0) := \left\{ \tilde{u} \in \mathcal{V}(u_0,v_0)\,;\, \|\tilde{u}'\|^2_{L^2(0,T;H)} + \max_{0\leq t\leq T} |\varphi(\tilde{u}(t))| \leq C \right\}.$$

From Theorem 3.1 we have already shown that for each $\tilde{u} \in \overline{\mathcal{V}(u_0,v_0)}$, which denotes the closure of $\mathcal{V}(u_0,v_0)$ in $C([0,T];H)$, the auxiliary Cauchy problem $(\mathrm{P})_{\tilde{u},u_0,v_0}$ has a unique solution $u(\tilde{u},u_0,v_0)$. Hence, we can define a solution operator Λ from $\overline{\mathcal{V}(u_0,v_0)}$ into $C([0,T];H)$ by $\Lambda(\tilde{u}) := u(\tilde{u},u_0,v_0)$ for all $\tilde{u} \in \overline{\mathcal{V}(u_0,v_0)}$, and denote by Λ_C the restriction of Λ on $\overline{\mathcal{V}_C(u_0,v_0)}$. All estimates, which have been obtained in Section 3, depend on the prescribed datum $\tilde{u} \in \mathcal{W}(u_0,v_0)$. So, we cannot use them for the fixed point argumentation to show the existence of local-in-time solutions to $(\mathrm{P})_{u_0,v_0}$. Roughly speaking, first of all we have to derive the uniform estimates of the solutions $\Lambda_C(\tilde{u}) = \Lambda(\tilde{u})$, which are independent of the choice of $\tilde{u} \in \overline{\mathcal{V}_C(u_0, v_0)}$. In order to do this, we assume that Condition 3 below is satisfied, whose original one was proposed in [9].

Condition 3. We give a constant $C_{12} > 0$ by

$$C_{12} := \left(\frac{4}{c_1^2} + 1\right)\left(|\varphi(0, u_0, v_0; u_0)| + c_2 C_1 \|u_0\|_H + C_1\right). \tag{4.1}$$

There exist a constant $C_{13} > C_{12}$ and a family of positive numbers M_r, denoted by $\{M_r\,;\, 0 < r < \infty\}$, such that the following properties are satisfied:

(1) For each $\tilde{u} \in \overline{\mathcal{V}_{C_{13}}(u_0, v_0)}$ the family $\{\varphi(t, \tilde{u}, \tilde{v}(t))\,;\, 0 \le t \le T\}$ satisfies Conditions 1 and 2.

(2) For each $\tilde{u} \in \overline{\mathcal{V}_{C_{13}}(u_0, v_0)}$ we let $\{\alpha_r(\tilde{u})\,;\, 0 < r < \infty\} \subset L^2(0,T)$ and $\{\beta_r(\tilde{u})\,;\, 0 < r < \infty\} \subset L^1(0,T)$ the same families of nonnegative functions that are given in Condition 1. Then, we have

$$\forall\, r \in (0, \infty), \qquad \sup_{\tilde{u} \in \overline{\mathcal{V}_{C_{13}}(u_0,v_0)}} \left\{\|\alpha_r(\tilde{u})\|_{L^2(0,T)} + \|\beta_r(\tilde{u})\|_{L^1(0,T)}\right\} \le M_r.$$

(3) For each $r > 0$ and $\varepsilon > 0$ there exists a constant $\delta_{r,\varepsilon} > 0$ such that

$$\sup_{\tilde{u} \in \overline{\mathcal{V}_{C_{13}}(u_0,v_0)}} \int_0^{\delta_{r,\varepsilon}} \left\{|\alpha_r(\tilde{u}; s)|^2 + |\beta_r(\tilde{u}; s)| + \|\tilde{v}'(s)\|_X\right\} ds < \varepsilon.$$

(4) There exists a constant $C_{14} > 0$ such that

$$\sup_{\tilde{u} \in \overline{\mathcal{V}_{C_{13}}(u_0,v_0)}} \|\tilde{v}\|_{W^{1,1}(0,T;X)} = \sup_{\tilde{u} \in \overline{\mathcal{V}_{C_{13}}(u_0,v_0)}} \|S(\tilde{u}, \cdot, 0)v_0\|_{W^{1,1}(0,T;X)} \le C_{14}. \tag{4.2}$$

The main theorem in this section is Theorem 4.1 below, which guarantees the existence of local-in-time solutions to $(\mathrm{P})_{u_0,v_0}$.

Theorem 4.1. *In addition to all assumptions in Section* 2 *we assume that Condition* 3 *is satisfied. Then, for each* $u_0 \in D(\varphi(0, u_0, v_0))$ *there exists* $T_1 \in (0, T]$ *such that* $(P)_{u_0,v_0}$ *has at least one solution* $u(u_0, v_0)$ *on* $[0, T_1]$. *Moreover, there exists a constant* $C_{15} > 0$ *depending on* $\|u_0\|_H$ *and* $\varphi(0, u_0, v_0; u_0)$, *such that*

$$\|u'\|_{L^2(0,T_1;H(v))} + \sup_{0 \le t \le T_1} |\varphi(t, u, v(t); u(t))| \le C_{15}.$$

In the following argumentation we show Theorem 4.1 by applying Schauder's fixed point theorem. For this purpose we first prove Propositions 4.2 and 4.3 below, which give uniform estimates of solutions $u(\tilde{u}, u_0, v_0)$ to $(\mathrm{P})_{\tilde{u},u_0,v_0}$ for all $\tilde{u} \in \overline{\mathcal{V}_{C_{13}}(u_0, v_0)}$, and enable us to proceed with the fixed point argumentation.

Proposition 4.2. *For each $C > 0$ such that $C \geq |\varphi(u_0)|$ the set $\mathcal{V}_C(u_0, v_0)$ is nonempty, convex and relatively compact in $C([0,T]; H)$ as well as relatively weakly compact in $W^{1,2}(0,T;H)$.*

Proof. Since it is clear that for each $C \geq |\varphi(u_0)|$ the set $\mathcal{V}_C(u_0, v_0)$ is not only nonempty and convex in $C([0,T]; H)$ but also relatively weakly compact $W^{1,2}(0,T;H)$, it is enough to show that it is relatively compact in $C([0,T]; H)$. Using (C1) and applying Ascoli-Arzerà's compactness theorem, we see that $\mathcal{V}_C(u_0, v_0)$ is relatively compact in $C([0,T]; H)$. $\square$

Proposition 4.3. *The operator $\Lambda_{C_{13}} : \overline{\mathcal{V}_{C_{13}}(u_0, v_0)} \longrightarrow C([0,T]; H)$ is continuous.*

In order to show Proposition 4.3, we prove Lemmas 4.4–4.6 below. As a direct consequence of Lemma 4.4, we see that all constants $C_i > 0$ $(i = 2, 3, \dots, 6)$, that are obtained in Section 2, can be chosen so that they are independent of the choice of $\tilde{u} \in \overline{\mathcal{V}_{C_{13}}(u_0, v_0)}$. This fact is essential and important to obtain the uniform estimates of solutions $\Lambda_{C_{13}}(\tilde{u})$ to $(\mathrm{P})_{\tilde{u},u_0,v_0}$ with respect to $\tilde{u} \in \overline{\mathcal{V}_{C_{13}}(u_0, v_0)}$ under Condition 3.

Lemma 4.4. *Let $\left\{ h(\tilde{u}) \in \mathcal{W}(u_0, v_0)\,;\, \tilde{u} \in \overline{\mathcal{V}_{C_{13}}(u_0, v_0)} \right\}$ be a family of functions $h(\tilde{u})$, which are constructed in Proposition 2.5. Then, there exists a constant $C_{16} > 0$ depending on $\|u_0\|_H$ and $\varphi(0, u_0, v_0; u_0)$, such that*

$$n(\tilde{u}) + \sup_{0 \leq t \leq T} \|h(\tilde{u}; t)\|_H + \sup_{0 \leq t \leq T} |\varphi(t, \tilde{u}, \tilde{v}(t); h(\tilde{u}; t))| + \sum_{n=0}^{n(\tilde{u})-1} \|(h(\tilde{u}))'\|^2_{L^2(t_n(\tilde{u}), t_{n+1}(\tilde{u}); H)} \leq C_{16}, \quad \forall\, \tilde{u} \in \overline{\mathcal{V}_{C_{13}}(u_0, v_0)},$$

where $n(\tilde{u})$ is given by (2.29) in the proof of Proposition 2.5.

Proof. Let $r_1 > 0$, $\alpha_{r_1}(\tilde{u}) \in L^2(0,T)$, $\beta_{r_1}(\tilde{u}) \in L^1(0,T)$ and $\{z_t(\tilde{u})\,;\, 0 \leq t \leq T\}$ be the same constant, nonnegative functions and a family in H, respectively, that are obtainded in Lemma 2.3. Using (1) and (2) of Condition 3, we have

$$\begin{aligned} \|z_t(\tilde{u})\|_{\tilde{v}(t)} &\leq \|u_0\|_{v_0} + \left(\sqrt{|\varphi(0, \tilde{u}, v_0; u_0)|} + 1 \right) \int_0^t \alpha_{r_1}(\tilde{u}; s) ds \\ &\leq \|u_0\|_{v_0} + \sqrt{T} \left(\sqrt{|\varphi(0, u_0, v_0; u_0)|} + 1 \right) M_{r_1}, \text{ and} \\ |\varphi(t, \tilde{u}, \tilde{v}(t); z_t(\tilde{u}))| &\leq |\varphi(0, \tilde{u}, v_0; u_0)| + (|\varphi(0, \tilde{u}, v_0; u_0)| + 1) \int_0^t \beta_{r_1}(\tilde{u}; s) ds \\ &\leq (|\varphi(0, u_0, v_0; u_0)| + 1)\,(1 + M_{r_1}). \end{aligned}$$

Hence, we see that there exists a constant $C_{16,1} > 0$ depending on $\|u_0\|_H$ and $\varphi(0, u_0, v_0; u_0)$, such that

$$\sup_{\tilde{u} \in \overline{\mathcal{V}_{C_{13}}(u_0, v_0)}} \left[\sup_{0 \leq t \leq T} \left\{ \|z_t(\tilde{u})\|_{\tilde{v}(t)} + |\varphi(t; \tilde{u}, \tilde{v}(t); z_t(\tilde{u}))| \right\} \right] \leq C_{16,1}.$$

This estimate especially implies that the constant C_2, which is obtained in Lemma 2.3, is independent of the choice of $\tilde{u} \in \overline{\mathcal{V}_{C_{13}}(u_0, v_0)}$. Repeating the argumentation similar to the proof of Proposition 2.5, we see that this lemma holds. $\square$

Lemma 4.5. *Consider sequences $\{\tilde{u}_m\}_{m\in\mathbb{N}} \subset C([0,T];H)$, $\{\xi_m\}_{m\in\mathbb{N}} \subset L^2(0,T;H)$ and elements $\tilde{u} \in C([0,T];H)$, $\xi \in L^2(0,T;H)$ satisfying $\tilde{u}_m \longrightarrow \tilde{u}$ in $C([0,T];H)$ and $\xi_m \longrightarrow \xi$ in $L^2(0,T;H)$ as $m \to \infty$. Then, there exists a subsequence $\{m_k\}_{k\in\mathbb{N}}$ of $\{m\}_{m\in\mathbb{N}}$ such that*

$$\Phi^T(\tilde{v};\xi) \le \liminf_{k\to\infty} \Phi^T(\tilde{v}_{m_k};\xi_{m_k}).$$

Proof. We choose a subsequence $\{m_k\}_{k\in\mathbb{N}}$ of $\{m\}_{m\in\mathbb{N}}$ such that

$$\begin{aligned} &\xi_{m_k}(t) \longrightarrow \xi(t) \quad \text{in} \quad H, \quad \text{a.a. } t \in (0,T) \quad \text{as} \quad k \to \infty, \ \text{ hence,} \\ &\xi_{m_k}(t) \longrightarrow \xi(t) \quad \text{in} \quad H(\tilde{v}(t)), \quad \text{a.a. } t \in (0,T) \quad \text{as} \quad k \to \infty. \end{aligned} \tag{4.3}$$

Since $\varphi(t, \tilde{u}_m, \tilde{v}_m(t)) \longrightarrow \varphi(t, \tilde{u}, \tilde{v}(t))$ on $H(\tilde{v}(t))$ in the sense of Mosco as $m \to \infty$, from (4.3) we have

$$\varphi(t, \tilde{u}, \tilde{v}(t); \xi(t)) \le \liminf_{k\to\infty} \varphi(t, \tilde{u}_{m_k}, \tilde{v}_{m_k}(t); \xi_{m_k}(t)), \quad \text{a.a. } t \in (0,T). \tag{4.4}$$

Integrating both sides of (4.4) on $(0,T)$ and using Fatou's lemma, we see that this lemma holds. $\square$

The next lemma means that any function $\xi \in D(\Phi^T(\tilde{u}))$ can be approximated by a suitable subsequence of $\{\xi_m\}_{m\in\mathbb{N}}$ such that $\xi_m \in D(\Phi^T(\tilde{u}_m))$ for all $m \in \mathbb{N}$ whenever $\tilde{u}_m \longrightarrow \tilde{u}$ in $C([0,T];H)$ as $m \to \infty$. Although the original lemma has already been shown in [10, Proposition 2.7.1], we give its proof in detail here because it is much more complicate than that of [10], and which comes from the variational structure for inner products $(\cdot,\cdot)_{\tilde{v}(t)}$.

Lemma 4.6. *Consider a sequence $\{\tilde{u}_m\}_{m\in\mathbb{N}} \subset \overline{\mathcal{V}_{C_{13}}(u_0, v_0)}$ and an element $\tilde{u} \in \overline{\mathcal{V}_{C_{13}}(u_0, v_0)}$ satisfying $\tilde{u}_m \longrightarrow \tilde{u}$ in $C([0,T];H)$ as $m \to \infty$. Then, for any $\xi \in D(\Phi(\tilde{u}))$ there exists a sequence $\{\xi_k\}_{k\in\mathbb{N}} \subset L^2(0,T;H)$ such that*

$$\xi_k \longrightarrow \xi \quad \text{in} \quad L^2(0,T;H(\tilde{v})) \quad \text{as} \quad k \to \infty, \tag{4.5}$$

$$\lim_{k\to\infty} \Phi^T(\tilde{u}_{m_k};\xi_k) = \Phi^T(\tilde{u};\xi). \tag{4.6}$$

Proof. At first, we consider the case that there exists a constant $\rho > 0$ such that

$$\sup_{0\le t\le T} \|\xi(t)\|_{\tilde{v}(t)} \le \rho, \qquad \sup_{0\le t\le T} |\varphi(t, \tilde{u}, \tilde{v}(t); \xi(t))| \le \rho. \tag{4.7}$$

We fix a number $\varepsilon \in (0,1)$. Applying Lusin's theorem, we see that there exists a closed set $E(\varepsilon) \subset [0,T]$ such that the function $t \mapsto \varphi(t, \tilde{u}, \tilde{v}(t); \xi(t))$ is continuous on $E(\varepsilon)$ and $|[0,T]\setminus E(\varepsilon)| \le \varepsilon$. Then, we can take out a partition $\{E_n(\varepsilon)\,;\, n = 1,2,\dots,\ell(\varepsilon)\}$ of $E(\varepsilon)$ such that

$$E(\varepsilon) = \bigcup_{n=1}^{\ell(\varepsilon)} E_n(\varepsilon), \quad E_i(\varepsilon)\cap E_j(\varepsilon) = \emptyset \quad \text{if } i \neq j,\ \forall\, i,j \in \{1,2,\dots,\ell(\varepsilon)\}, \tag{4.8}$$

$$t_n(\varepsilon) := \inf E_n(\varepsilon) \in E_n(\varepsilon),\ \ \sup E_n(\varepsilon) \le t_{n+1}(\varepsilon),\ \ \sup E_n(\varepsilon) - t_n(\varepsilon) \le \varepsilon, \tag{4.9}$$

$$\sup_{n\in\{1,2,\dots,\ell(\varepsilon)\}} \left\{ \sup_{t\in E_n(\varepsilon)} \|\xi(t) - \xi(t_n(\varepsilon))\|_{\tilde v(t)} \right\} \le \varepsilon, \tag{4.10}$$

$$\sup_{n\in\{1,2,\dots,\ell(\varepsilon)\}} \left\{ \sup_{t\in E_n(\varepsilon)} |\varphi(t,\tilde u,\tilde v(t);\xi(t)) - \varphi(t_n(\varepsilon),\tilde u,\tilde v(t_n(\varepsilon));\xi(t_n(\varepsilon)))| \right\} \le \varepsilon. \tag{4.11}$$

Since $\varphi(t,\tilde u_m,\tilde v_m(t)) \longrightarrow \varphi(t,\tilde u,\tilde v(t))$ on $H(\tilde v(t))$ in the sense of Mosco as $m\to\infty$ for all $t\in[0,T]$, we see that for any $n\in\{1,2,\dots,\ell(\varepsilon)\}$ there exists a sequence $\{z(m,\varepsilon,n)\}_{m\in\mathbb{N}} \subset H$ such that

$$z(m,\varepsilon,n) \longrightarrow \xi(t_n(\varepsilon)) \quad \text{in} \quad H(\tilde v(t_n(\varepsilon))) \quad \text{as} \quad m\to\infty, \tag{4.12}$$

$$\lim_{m\to\infty} \varphi(t_n(\varepsilon),\tilde u_m,\tilde v_m(t_n(\varepsilon)); z(m,\varepsilon,n)) = \varphi(t_n(\varepsilon),\tilde u,\tilde v(t_n(\varepsilon));\xi(t_n(\varepsilon))). \tag{4.13}$$

Hence, we see from (4.7), (4.12) and (4.13) that there exits a number $m(\varepsilon)\in\mathbb{N}$ such that

$$\sup_{m\ge m(\varepsilon)} \left\{ \sup_{n\in\{1,2,\dots,\ell(\varepsilon)\}} \|z(m,\varepsilon,n) - \xi(t_n(\varepsilon))\|_{\tilde v(t_n(\varepsilon))} \right\} \le \varepsilon, \tag{4.14}$$

$$\begin{aligned}\sup_{m\ge m(\varepsilon)} \Big\{ \sup_{n\in\{1,2,\dots,\ell(\varepsilon)\}} &|\varphi(t_n(\varepsilon),\tilde u_m,\tilde v_m(t_n(\varepsilon)); z(m,\varepsilon,n)) \\ &-\varphi(t_n(\varepsilon),\tilde u,\tilde v(t_n(\varepsilon));\xi(t_n(\varepsilon)))| \Big\} \le \varepsilon. \end{aligned} \tag{4.15}$$

In the following argumentation, for each $\varepsilon\in(0,1)$, $n\in\{1,2,...,\ell(\varepsilon)\}$ and $m\in\mathbb{N}$ we consider the Cauchy problem $\{(4.16),(4.17),(4.18)\}$, which is denoted by $(\mathrm{P})_{m,\varepsilon,n}$ here, with a variational structure for inner products:

$$\begin{aligned} u'(t) + \partial_{\bar v_{m,\varepsilon,n}(t)}\bar\varphi(t,(\tilde u_m)_{t_n(\varepsilon)},\bar v_{m,\varepsilon,n}(t);u(t)) \ni 0 \\ \text{in } H(\bar v_{m,\varepsilon,n}(t)),\ \text{a.a. } t\in(0,T), \end{aligned} \tag{4.16}$$

$$\bar v_{m,\varepsilon,n}(t) = S((\tilde u_m)_{t_n(\varepsilon)},t,0)\tilde v_m(t_n(\varepsilon)) \quad \text{in} \quad X, \quad \forall\, t\in[0,T], \tag{4.17}$$

$$u(0) = z(m,\varepsilon,n) \quad \text{in} \quad H(\tilde v_m(t_n(\varepsilon))). \tag{4.18}$$

From (3.2), (4.15), (S5) and (S6) we have for all $t\in[0,T-t_n(\varepsilon)]$

$$z(m,\varepsilon,n) \in D(\bar\varphi(0,(\tilde u_m)_{t_n(\varepsilon)},\bar v_{m,\varepsilon,n}(0))) = D(\varphi(t_n(\varepsilon),\tilde u_m(t_n(\varepsilon)),\tilde v(t_n(\varepsilon)))),$$

$$\bar v_{m,\varepsilon,n}(t) = (S(\tilde u_m,t+t_n(\varepsilon),t_n(\varepsilon))\circ S(\tilde u_m,t_n(\varepsilon),0))v_0 = \tilde v_m(t+t_n(\varepsilon)).$$

Hence, we see that the family $\{\bar\varphi(t,(\tilde u_m)_{t_n(\varepsilon)},\bar v_{m,\varepsilon,n}(t))\,;\,0\le t\le T\}$ satisfies Condition 1. Indeed, for each $r>0$ the following nonnegative functions $\alpha_r(m,\varepsilon,n)\in L^2(0,T)$ and $\beta_r(m,\varepsilon,n)\in L^1(0,T)$ can be chosen as $\alpha_r(\tilde u)$ and $\beta_r(\tilde u)$ in Condition 1:

$$\alpha_r(m,\varepsilon,n;t) := \begin{cases} \alpha_r(\tilde u_m; t+t_n(\varepsilon)) & \text{if} \quad t\in[0,T-t_n(\varepsilon)], \\ 0 & \text{if} \quad t\in(T-t_n(\varepsilon),T], \end{cases}$$

$$\beta_r(m,\varepsilon,n;t) := \begin{cases} \beta_r(\tilde u_m; t+t_n(\varepsilon)) & \text{if} \quad t\in[0,T-t_n(\varepsilon)], \\ 0 & \text{if} \quad t\in(T-t_n(\varepsilon),T]. \end{cases}$$

We see from Condition 3 that the following uniform estimates are satisfied:

$$\sup_{\varepsilon\in(0,1)}\left[\sup_{m\in\mathbb N}\left\{\sup_{n\in\{1,2,\dots,\ell(\varepsilon)\}}\left(\|\alpha_r(m,\varepsilon,n)\|_{L^2(0,T)}+\|\beta_r(m,\varepsilon,n)\|_{L^1(0,T)}\right)\right\}\right]\le M_r.$$

From Theorem 3.1 we see that $(\mathrm P)_{m,\varepsilon,n}$ has one and only one solution $u(m,\varepsilon,n)\in W^{1,2}(0,T;H)$. Further, we use (4.14), (4.15) and the function $\bar h(m,\varepsilon,n)$ defined by

$$\bar h(m,\varepsilon,n)(t) := \begin{cases} h(\tilde u_m; t+t_n(\varepsilon)) & \text{if} \quad t\in[0,T-t_n(\varepsilon)], \\ h(\tilde u_m;T) & \text{if} \quad t\in(T-t_n(\varepsilon),T], \end{cases}$$

where the functions $h(\tilde u_m)$ satisfy the following uniform estimates (cf. Lemma 4.4):

$$\begin{aligned}\sup_{m\in\mathbb N}\Big\{ n(\tilde u_m) &+ \sup_{0\le t\le T}\|h(\tilde u_m;t)\|_{\tilde v(t)} + \sup_{0\le t\le T}|\varphi(t,\tilde u_m,\tilde v_m(t);h(\tilde u_m;t))| \\ &+ \sum_{n=0}^{n(\tilde u_m)-1}\|(h(\tilde u_m))'\|^2_{L^2(t_n(\tilde u_m),t_{n+1}(\tilde u_m);H)}\Big\}\le C_{16,3}\end{aligned}$$

for a suitable constant $C_{16,3}>0$, and repeat the argumention similarly as in the proofs of Lemmas 3.2 and 3.3. Then, we see from (1) and (2) of Theorem 3.1 that there exists a constant $C_{8,1}>0$ such that

$$\begin{aligned}\sup_{\varepsilon\in(0,1)}\Big[\sup_{m\ge m(\varepsilon)}\Big\{\sup_{n\in\{1,2,\dots,\ell(\varepsilon)\}}\Big(&\|(u(m,\varepsilon,n))'\|^2_{L^2(0,T;H)}+\sup_{0\le t\le T}\|u(m,\varepsilon,n;t)\|_H \\ &+\sup_{0\le t\le T}|\bar\varphi(t,(\tilde u_m)_{t_n(\varepsilon)},\bar v_{m,\varepsilon,n}(t);u(m,\varepsilon,n;t))|\Big)\Big\}\Big]\le C_{8,1}. \qquad (4.19)\end{aligned}$$

Using the solutions $u(m,\varepsilon,n)$ to $(\mathrm P)_{m,\varepsilon,n}$, we define the functions $\xi(m,\varepsilon)$ by

$$\xi(m,\varepsilon;t) := \begin{cases} u(m,\varepsilon,n;t-t_n(\varepsilon)) & \text{if} \quad t\in E_n(\varepsilon),\ n\in\{1,2,\dots,\ell(\varepsilon)\}, \\ h(\tilde u_m;t) & \text{if} \quad t\in[0,T]\setminus E(\varepsilon). \end{cases}$$

Then, we see from (4.9), (4.10), (4.14) and (4.19) that

$$\|\xi(t)-\xi(m,\varepsilon;t)\|_{\tilde v(t)} \le \|\xi(t)-\xi(t_n(\varepsilon))\|_{\tilde v(t)} + \frac{c_2}{c_1}\|\xi(t_n(\varepsilon))-z(m,\varepsilon,n)\|_{\tilde v(t_n(\varepsilon))}$$
$$+c_2\int_0^{t-t_n(\varepsilon)}\|(u(m,\varepsilon,n))'(\tau)\|_H d\tau \le \Big(\frac{c_2}{c_1}+1\Big)\varepsilon + c_2\sqrt{\varepsilon C_{8,1}}$$

for all $t\in E_n(\varepsilon)$ and all $m\ge m(\varepsilon)$, hence, from Lemma 4.4

$$\begin{aligned}
&\sup_{m\ge m(\varepsilon)}\|\xi-\xi(m,\varepsilon)\|^2_{L^2(0,T;H(\tilde v))}\\
&\le \sup_{m\ge m(\varepsilon)}\Big\{\int_{[0,T]\setminus E(\varepsilon)}\big(\|\xi(t)\|_{\tilde v(t)}+\|h(\tilde u_m;t)\|_{\tilde v(t)}\big)^2 dt\\
&\qquad\qquad + \sum_{k=1}^{\ell(\varepsilon)}\int_{E_n(\varepsilon)}\|\xi(t)-\xi(m,\varepsilon;t)\|^2_{\tilde v(t)}dt\Big\}\\
&\le \varepsilon(\rho+c_2R_4)^2+\Big\{\Big(\frac{c_2}{c_1}+1\Big)\varepsilon+c_2\sqrt{\varepsilon C_{8,1}}\Big\}^2 T, \qquad \forall\, m\ge m(\varepsilon).
\end{aligned}\tag{4.20}$$

Moreover, applying (3) of Theorem 3.1 as $\varepsilon=1$, we see that there exist constants $r_8>0$ and $C_{9,1}>0$ such that the following estimate holds for all $t\in[t_n(\varepsilon),t_{n+1}(\varepsilon)]$, all $s=t-t_n(\varepsilon)$, all $m\ge m(\varepsilon)$, and all $n\in\{1,2,\dots,\ell(\varepsilon)\}$:

$$\begin{aligned}
&\varphi(t,\tilde u_m,\tilde v_m(t);u(m,\varepsilon,n;t-t_n(\varepsilon)))-\varphi(t_n(\varepsilon),\tilde u_m,\tilde v_m(t_n(\varepsilon));z(m,\varepsilon,n))\\
&=\bar\varphi(s,(\tilde u_m)_{t_n(\varepsilon)},\bar v_{m,\varepsilon,n}(s);u(m,\varepsilon,n;s))-\bar\varphi(0,(\tilde u_m)_{t_n(\varepsilon)},\bar v_{m,\varepsilon,n}(0);u(m,\varepsilon,n;0))\\
&\le C_{9,1}\int_0^s\big\{\bar\varphi(\tau,(\tilde u_m)_{t_n(\varepsilon)},\bar v_{m,\varepsilon,n}(\tau);u(m,\varepsilon,n;\tau))+1\big\}\\
&\qquad\times\big(|\alpha_{r_8}(m,\varepsilon,n;\tau)|^2+|\beta_{r_8}(m,\varepsilon,n;\tau)|+\|(\bar v_{m,\varepsilon,n})'(\tau)\|_X\big)\,d\tau\\
&\le C_{9,1}(C_{8,1}+1)\int_{t_n(\varepsilon)}^{t_{n+1}(\varepsilon)}\big(|\alpha_{r_8}(\tilde u_m;\tau)|^2+|\beta_{r_8}(\tilde u_m;\tau)|+\|\tilde v_m'(\tau)\|_X\big)\,d\tau.
\end{aligned}\tag{4.21}$$

Using (2) of Condition 3 and (4.10), (4.14), (4.21), we have

$$\begin{aligned}
&\int_{E_n(\varepsilon)}\varphi(t,\tilde u_m,\tilde v_m(t);\xi(m,\varepsilon;t))dt-\int_{E_n(\varepsilon)}\varphi(t,\tilde u,\tilde v(t);\xi(t))dt\\
&\le\sum_{n=1}^{\ell(\varepsilon)}\int_{E_n(\varepsilon)}\big\{\varphi(t,\tilde u_m,\tilde v_m(t);u(m,\varepsilon,n;t-t_n(\varepsilon)))\\
&\qquad\qquad-\varphi(t_n(\varepsilon),\tilde u_m,\tilde v_m(t_n(\varepsilon));z(m,\varepsilon,n))\big\}dt\\
&\quad+\sum_{n=1}^{\ell(\varepsilon)}\int_{E_n(\varepsilon)}|\varphi(t_n(\varepsilon),\tilde u_m,\tilde v_m(t_n(\varepsilon));z(m,\varepsilon,n))\\
&\qquad\qquad-\varphi(t_n(\varepsilon),\tilde u_m,\tilde v_m(t_n(\varepsilon));\xi(t_n(\varepsilon)))|dt
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{n=1}^{\ell(\varepsilon)}\int_{E_n(\varepsilon)}|\varphi(t_n(\varepsilon),\tilde{u}_m,\tilde{v}_m(t_n(\varepsilon));\xi(t_n(\varepsilon)))-\varphi(t,\tilde{u},\tilde{v}(t);\xi(t))|dt\\
&\le \varepsilon\left\{2\sum_{n=1}^{\ell(\varepsilon)}|E_n(\varepsilon)|+\sum_{n=1}^{\ell(\varepsilon)}C_{9,1}(C_{8,1}+1)\right.\\
&\qquad\left.\times\int_{t_n(\varepsilon)}^{t_{n+1}(\varepsilon)}\left(|\alpha_{r_8}(\tilde{u}_m;s)|^2+|\beta_{r_8}(\tilde{u}_m;s)|+\|\tilde{v}_m'(s)\|_X\right)ds\right\}\\
&\le \varepsilon\left\{2T+C_{9,1}(C_{8,1}+1)(M_{r_8}+C_{14})\right\},\qquad \forall\, m\ge m(\varepsilon).
\end{aligned}\tag{4.22}
$$

From Lemma 4.4 and (4.19), (4.22), we have

$$
\begin{aligned}
&\Phi^T(\tilde{u}_m;\xi(m,\varepsilon))-\Phi^T(\tilde{u};\xi)\\
&\le \varepsilon\left\{2T+C_{9,1}(C_{8,1}+1)(M_{r_8}+C_{14})\right\}\\
&\qquad+\int_{[0,T]\setminus E(\varepsilon)}\{\varphi(t,\tilde{u}_m,\tilde{v}_m(t);h(\tilde{u}_m;t))-\varphi(t,\tilde{u},\tilde{v}(t);\xi(t))\}dt\\
&\le \varepsilon\left\{2T+C_{9,1}(C_{8,1}+1)(M_{r_8}+C_{14})+C_{13}+\rho\right\},\qquad \forall\, m\ge m(\varepsilon).
\end{aligned}\tag{4.23}
$$

We see from (4.20) and (4.23) that for any $\delta>0$ there exist numbers $\varepsilon(\delta)>0$ and $m(\delta)\in\mathbb{N}$ such that for all $m\ge m(\delta)$

$$
\|\xi(m,\varepsilon(\delta))-\xi\|_{L^2(0,T;H(\tilde{v}))}\le\delta,\quad \Phi^T(\tilde{u}_m;\xi(m,\varepsilon(\delta)))\le\Phi^T(\tilde{u};\xi)+\delta. \tag{4.24}
$$

Hence, from (4.24) we can take out a subsequence $\{\tilde{u}_{m_j}\}_{j\in\mathbb{N}}$ of $\{\tilde{u}_m\}_{m\in\mathbb{N}}$ and construct a sequence $\{\xi_j\}_{j\in\mathbb{N}}\subset L^2(0,T;H(\tilde{v}))$ such that

$$
\xi_j\longrightarrow\xi\quad\text{in}\quad L^2(0,T;H(\tilde{v}))\quad\text{as}\quad j\to\infty, \tag{4.25}
$$

$$
\limsup_{j\to\infty}\Phi^T(\tilde{u}_{m_j};\xi_j)\le\Phi^T(\tilde{u};\xi). \tag{4.26}
$$

Applying Lemma 4.5, we can take out a subsequence of $\{j\}_{j\in\mathbb{N}}$, which is denoted by the same notation $\{j\}_{j\in\mathbb{N}}$ here, such that

$$
\lim_{j\to\infty}\Phi^T(\tilde{u}_{m_j};\xi_j)=\Phi^T(\tilde{u};\xi). \tag{4.27}
$$

From (4.25) and (4.27), this sequence $\{\xi_j\}_{j\in\mathbb{N}}$ is as desired.

Next, we consider the case that (4.7) does not hold. Then, we can take out a sequence $\{F_n\}_{n\in\mathbb{N}}$ of closed subsets F_n of $[0,T]$ and a sequence $\{\rho_n\}_{n\in\mathbb{N}}$ of positive numbers ρ_n such that $\rho_n\longrightarrow\infty$ as $n\to\infty$, and for all $n\in\mathbb{N}$,

$$
|[0,T]\setminus F_n|\le\frac{1}{n},\quad \sup_{t\in F_n}\|\xi(t)\|_{\tilde{v}(t)}\le\rho_n,\quad \sup_{t\in F_n}|\varphi(t,\tilde{u},\tilde{v}(t);\xi(t))|\le\rho_n,
$$

and define a function ξ_n by

$$\xi_n(t) := \begin{cases} \xi(t) & \text{if} \quad t \in F_n, \\ h(\tilde{u};t) & \text{if} \quad t \in [0,T] \setminus F_n, \end{cases}$$

which satisfies
$$\xi_n \longrightarrow \xi \quad \text{in} \quad L^2(0,T;H(\tilde{v})) \quad \text{as} \quad n \to \infty, \tag{4.28}$$

$$\lim_{n\to\infty} \Phi^T(\tilde{u};\xi_n) = \Phi^T(\tilde{u};\xi). \tag{4.29}$$

Moreover, since for each $n \in \mathbb{N}$ the function ξ_n satisfies (4.7) as $\rho = \max\{\rho_n, C_{13}\}$, we see that for each $n \in \mathbb{N}$ there exists a sequence $\{\xi_{n,i}\}$ such that

$$\xi_{n,i} \longrightarrow \xi_n \quad \text{in} \quad L^2(0,T;H(\tilde{v})) \quad \text{as} \quad i \to \infty. \tag{4.30}$$

$$\lim_{i\to\infty} \Phi^T(\tilde{u}_{m_i};\xi_{n,i}) = \Phi^T(\tilde{u};\xi_n). \tag{4.31}$$

From (4.28) and (4.30) we can choose a subsequence $\{\xi_k\}_{k\in\mathbb{N}}$ of $\{\xi_{n,i}\,;\, n,i \in \mathbb{N}\}$ such that (4.5) and (4.6) are satisfied. Hence the proof is complete. $\square$

Proof of Proposition 4.3. Let a sequence $\{\tilde{u}_m\}_{m\in\mathbb{N}} \subset \overline{\mathcal{V}_{C_{13}}(u_0,v_0)}$ and an element $\tilde{u} \in \overline{\mathcal{V}_{C_{13}}(u_0,v_0)}$ satisfy $\tilde{u}_m \longrightarrow \tilde{u}$ in $C([0,T];H)$ as $m \to \infty$. We put $\bar{u}_m := \Lambda_{C_{13}}(\tilde{u}_m)$. We see from (C1), Theorem 3.1 and Lemma 4.4 that there exists a constant $C_{8,2} > 0$ such that

$$\sup_{m\in\mathbb{N}} \left\{ \|\bar{u}_m'\|^2_{L^2(0,T;H)} + \sup_{0\le t\le T} |\varphi(\bar{u}_m(t))| \right\} \le C_{8,2}. \tag{4.32}$$

Since we see from (4.15) and (C1) that $\{\bar{u}_m\}_{m\in\mathbb{N}}$ is not only relatively compact in $C([0,T];H)$ but also relatively weakly compact in $W^{1,2}(0,T;H)$, we can choose a subsequence $\{\bar{u}_{m_k}\}_{k\in\mathbb{N}}$ of $\{\bar{u}_m\}_{m\in\mathbb{N}}$ and an element $\bar{u} \in W^{1,2}(0,T;H)$ such that the following convergence holds as $k \to \infty$:

$$\bar{u}_{m_k} \longrightarrow \bar{u} \quad \text{in } C([0,T];H) \text{ and weakly in } W^{1,2}(0,T;H). \tag{4.33}$$

Applying Theorem 3.9, we have for all $\eta \in L^2(0,T;H(\tilde{v}_{m_k}))$

$$(\bar{u}_{m_k}' - f, \bar{u}_{m_k} - \eta)_{L^2(0,T;H(\tilde{v}_{m_k}))} \le \Phi^T(\tilde{u}_{m_k};\eta) - \Phi^T(\tilde{u}_{m_k};\bar{u}_{m_k}). \tag{4.34}$$

Let ξ be any element in $D(\Phi^T(\tilde{u}))$. Applying Lemma 4.6, we see that there exists a subsequence of $\{m_k\}_{k\in\mathbb{N}}$, which is denoted by the same notation $\{m_k\}_{k\in\mathbb{N}}$, and a sequence $\{\xi_k\}_{k\in\mathbb{N}} \subset L^2(0,T;H(\tilde{v}))$ such that (4.5) and (4.6) are satisfied. We substitute $\eta = \xi_k$ in (4.34) and take $\liminf_{k\to\infty}$ in both sides of its result. Then, we use Lemmas 3.4, 4.4 and 4.6 with (4.5), (4.6), (4.34) to obtain

$$\Phi^T(\tilde{u};\bar{u}) + (\bar{u}' - f, \bar{u} - \xi)_{L^2(0,T;H(\tilde{v}))} \le \Phi^T(\tilde{v};\xi), \quad \forall\, \xi \in L^2(0,T;H(\tilde{v})).$$

Applying Theorem 3.9 again, we see that $\bar{u}$ is a solution to $(\mathrm{P})_{\tilde{u},u_0,v_0}$, that is, $\bar{u} = \Lambda_{C_{13}}(\tilde{u})$. $\square$

At last, we are ready to give the proof of Theorem 4.1.

Proof of Theorem 4.1. For any $\tilde{u} \in \overline{\mathcal{V}_{C_{13}}(u_0,v_0)}$ we put $u(\tilde{u}) = \Lambda_{C_{13}}(\tilde{u})$. We repeat the argumentation similar to that in Section 3, and use Condition

3 and Lemma 4.4. Then, we see from Theorem 3.1 that there exist constants $C_{7,1} > 0$ (depending on C_{13} and $\|u_0\|_H$), $C_{8,3} > 0$ (depending on C_{13}, $\|u_0\|_H$ and $\varphi(0, u_0, v_0; u_0)$), $C_{9,2} > 0$, (depending on C_{13} and $\|u_0\|_H$), and $r_9 > 0$ (depending on $\|u_0\|_H$) such that

$$\sup_{\tilde{u} \in \overline{\mathcal{V}_{C_{13}}(u_0,v_0)}} \left\{ \sup_{0 \le t \le T} \|u(\tilde{u}; t)\|_{\tilde{v}(t)} \right\} \le C_{7,1}, \tag{4.35}$$

$$\sup_{\tilde{u} \in \overline{\mathcal{V}_{C_{13}}(u_0,v_0)}} \left\{ \|(u(\tilde{u}))'\|^2_{L^2(0,T;H)} + \sup_{0 \le t \le T} |\varphi(t, \tilde{u}, \tilde{v}(t); u(\tilde{u}; t))| \right\} \le C_{8,3}, \tag{4.36}$$

$$\begin{aligned} &\varphi(t, \tilde{u}, \tilde{v}(t); u(\tilde{u}; t)) + \int_0^t ((u(\tilde{u}))'(s), (u(\tilde{u}))'(s) - f(s))_{\tilde{v}(s)} ds \\ &\le C_{9,2} \int_0^t \{\varphi(s, \tilde{u}, \tilde{v}(s); u(\tilde{u}; s)) + 1\} \left(|\alpha_{r_9}(\tilde{u}; s)|^2 + |\beta_{r_9}(\tilde{u}; s)| + \|\tilde{v}'(s)\|_X\right) ds \\ &\quad + \varphi(0, u_0, v_0; u_0) + \frac{1}{4} \int_0^t \|u'(\tilde{u}; s) - f(s)\|^2_{\tilde{v}(s)} ds, \ \ \forall \, \tilde{u} \in \overline{\mathcal{V}_{C_{13}}(u_0, v_0)}. \end{aligned} \tag{4.37}$$

Moreover, using (3) of Condition 3, we see that for each $\varepsilon > 0$ there exists a time $t(\varepsilon) \in (0, T]$ such that

$$\begin{aligned} &\frac{c_2^2 C_1^2}{c_1^2} t(\varepsilon) + \frac{c_2^2}{2} \int_0^{t(\varepsilon)} \|f(s)\|^2_H ds \\ &+ C_{9,2}(C_{8,3} + 1) \int_0^{t(\varepsilon)} \left(|\alpha_{r_9}(\tilde{u}; s)|^2 + |\beta_{r_9}(\tilde{u}; s)| + \|\tilde{v}'(s)\|_X\right) ds \le \varepsilon, \end{aligned} \tag{4.38}$$

where $C_1 > 0$ is the same constant that is obtained in Lemma 2.1. From (4.35)–(4.37) we have for all $t \in [0, T]$

$$\begin{aligned} &\varphi(t, \tilde{u}, \tilde{v}(t); u(\tilde{u}; t)) + \frac{1}{2} \int_0^t \|(u(\tilde{u}))'(s)\|^2_{\tilde{v}(s)} ds \le |\varphi(0, u_0, v_0; u_0)| \\ &+ \frac{c_2^2}{2} \int_0^t \|f(s)\|^2_H ds + C_{9,2}(C_{8,3} + 1) \int_0^t \left(|\alpha_{r_9}(\tilde{u}; s)|^2 + |\beta_{r_9}(\tilde{u}; s)| + \|\tilde{v}'(s)\|_X\right) ds. \end{aligned} \tag{4.39}$$

In addtion, we use the following estimate repeatedly: for each $\delta > 0$

$$\begin{aligned} \|u(\tilde{u}; t)\|_{\tilde{v}(t)} &\le \int_0^t \|(u(\tilde{u}))'(s)\|_{\tilde{v}(t)} ds + \|u_0\|_{\tilde{v}(t)} \\ &\le \delta \int_0^t \|(u(\tilde{u}))'(s)\|^2_{\tilde{v}(s)} ds + \frac{c_2^2 t}{4\delta c_1^2} + c_2 \|u_0\|_H, \quad \forall \, t \in [0, T]. \end{aligned} \tag{4.40}$$

Using Lemma 2.1 and (4.40) as $\delta = \frac{1}{4C_1}$, we have for all $t \in [0, T]$

$$-\varphi(t, \tilde{u}, \tilde{v}(t); u(\tilde{u}; t)) \le \frac{1}{4} \int_0^t \|(u(\tilde{u}))'(s)\|^2_{\tilde{v}(s)} ds + \frac{c_2^2 C_1^2 t}{c_1^2} + c_2 C_1 \|u_0\|_H + C_1. \tag{4.41}$$

We substitute (4.41) into (4.39) and use (4.38) to have

$$\frac{1}{4}\int_0^{t(\varepsilon)} \|(u(\tilde{u}))'(s)\|^2_{\tilde{v}(s)} ds \le \varepsilon + |\varphi(0,u_0,v_0;u_0)| + c_2 C_1 \|u_0\|_H + C_1. \tag{4.42}$$

Using (4.38), (4.39) and Lemma 2.1 with (4.40) as $\delta = \frac{1}{2C_1}$ again, we get

$$\begin{aligned}
&|\varphi(u(\tilde{u};t))| \le \varphi(t,\tilde{u},\tilde{v}(t);u(\tilde{u};t)) + C_1\left(\|u_{\tilde{u}}(t)\|_{\tilde{v}(t)} + 1\right)\\
\le&\varphi(t,\tilde{u},\tilde{v}(t);u(\tilde{u};t)) + \frac{1}{2}\int_0^t \|(u(\tilde{u}))'(s)\|^2_{\tilde{v}(s)} ds + \frac{c_2^2 C_1^2}{2c_1^2} t + c_2 C_1 \|u_0\|_H + C_1\\
\le&\varepsilon + |\varphi(0,u_0,v_0;u_0)| + c_2 C_1\|u_0\|_H + C_1, \quad \forall\, t \in [0,t(\varepsilon)]. \qquad (4.43)
\end{aligned}$$

We see from (4.1), (4.42) and (4.43) that

$$\int_0^{t(\varepsilon)} \|u(\tilde{u};t)\|^2_H dt + \sup_{0\le t\le t(\varepsilon)} |\varphi(u(\tilde{u};t))| \le \left(\frac{4}{c_1^2}+1\right)\varepsilon + C_{12}. \tag{4.44}$$

Now, we fix $\varepsilon_1 > 0$ such that $\left(\frac{4}{c_1^2}+1\right)\varepsilon_1 + C_{12} = C_{13}$, and take $T_1 = t(\varepsilon_1) \in (0,T]$. Then, we have

$$\|(u(\tilde{u}))'\|^2_{L^2(0,T_1;H)} + \sup_{0\le t\le T_1} |\varphi(u(\tilde{u};t))| \le C_{13}. \tag{4.45}$$

We define a continuous operator Λ_1 from $C([0,T];H)$ into $\overline{\mathcal{V}_{C_{13}}(u_0,v_0)}$ by

$$(\Lambda_1(z))(t) := \begin{cases} z(t) & \text{if} \quad t \in [0,T_1],\\ z(T_1) & \text{if} \quad t \in (T_1;T]. \end{cases}$$

From Proposition 4.3 we see that the composite operator $\Lambda_1 \circ \Lambda_{C_{13}}$ is continuous from $\overline{\mathcal{V}_{C_{13}}(u_0,v_0)}$ into itself with respect to the topology of $C([0,T];H)$. Applying Schauder's fixed point theorem, the operator $\Lambda_1 \circ \Lambda_{C_{13}}$ has at least one fixed point $u^* \in \overline{\mathcal{V}_{C_{13}}(u_0,v_0)}$, that is, $(\Lambda_1 \circ \Lambda_{C_{13}})(u^*) = u^*$.

We denote by u the restriction of the fixed point u^* of $\Lambda_1 \circ \Lambda_{C_{13}}$ on $[0,T_1]$. Then, we easily see from the definition of Λ_1 that this function u is a solution to $(\mathrm{P})_{u_0,v_0}$ on $[0,T_1]$. □

At the end of this section, we prove the boundedness of the solutions to $(\mathrm{P})_{u_0,v_0}$.

Theorem 4.7. *For each $(u_0,v_0) \in H \times A$ with $u_0 \in D(\varphi(0,u_0,v_0))$ we let u a solution to $(P)_{u_0,v_0}$ on $[0,\tau]$. Then, there exist constants $C_{17} > 0$, $C_{18} > 0$ and $C_{19} > 0$, which are independent of u, such that the following estimates are satisfied for any $h \in W^{1,2}(0,\tau;H)$ with $\varphi(\cdot,u,v;h(\cdot)) \in L^1(0,\tau)$:*

$$\begin{aligned}
&\sup_{0\le t\le\tau} \|u(t)-h(t)\|^2_{v(t)} \le C_{17}\exp\left(C_{18}\int_0^\tau \left(\|v'(t)\|_X + 1\right) dt\right)\Bigg\{\|u_0 - h(0)\|^2_{v_0}\\
&+\int_0^\tau \left(|\varphi(t,u,v(t);h(t))| + \|f(t)\|^2_H + \|h(t)\|^2_H + \|h'(t)\|^2_H + 1\right) dt\Bigg\} \qquad (4.46)
\end{aligned}$$

and
$$\int_0^\tau |\varphi(t,u,v(t);u(t))|dt$$
$$\leq C_{19}\bigg\{\|u_0-h(0)\|_{v_0}^2+\int_0^\tau (\|v'(t)\|_X+1)\,\|u(t)-h(t)\|_{v(t)}^2 dt$$
$$+\int_0^\tau \left(|\varphi(t,u,v(t);h(t))|+\|f(t)\|_H^2+\|h(t)\|_H^2+\|h'(t)\|_H^2+1\right)dt\bigg\}. \tag{4.47}$$

Proof. Using Lemma 2.10 as $z=u-h$ and (1.1),(2.45), we have for a.a. $t\in(0,\tau)$
$$\frac{d}{dt}\|u(t)-h(t)\|_{v(t)}^2+2\varphi(t,u,v(t);u(t)) \tag{4.48}$$
$$\leq \gamma_0\|v'(t)\|_X\|u(t)-h(t)\|_{v(t)}^2+2|\varphi(t,u,v(t);h(t))|$$
$$+2\left(\|f(t)\|_{v(t)}+\|h'(t)\|_{v(t)}\right)\|u(t)-h(t)\|_{v(t)}$$
$$\leq 2|\varphi(t,u,v(t);h(t))|+(\gamma_0\|v'(t)\|_X+2)\,\|u(t)-h(t)\|_{v(t)}^2+\|f(t)\|_{v(t)}^2+\|h'(t)\|_{v(t)}^2.$$
Next, we use Lemma 2.1 to get for all $t\in[0,\tau]$
$$2\varphi(t,u,v(t);u(t))\geq -\|u(t)-h(t)\|_{v(t)}^2-\|h(t)\|_{v(t)}^2-2C_1(C_1+1). \tag{4.49}$$
Substituting (4.49) into (4.48), we obtain for a.a. $t\in[0,\tau]$
$$\frac{d}{dt}\|u(t)-h(t)\|_{v(t)}^2-(\gamma_0\|v'(t)\|_X+3)\,\|u(t)-h(t)\|_{v(t)}^2 \tag{4.50}$$
$$\leq 2|\varphi(t,u(t),v(t);h(t))|+c_2^2\left(\|f(t)\|_H^2+\|h(t)\|_H^2+\|h'(t)\|_H^2\right)+2C_1(C_1+1).$$
Applying Gronwall's lemma to (4.50), we get for all $t\in[0,\tau]$
$$\|u(t)-h(t)\|_{v(t)}^2\leq\Big\{\|u_0-h(0)\|_{v_0}^2+2\int_0 \tau|\varphi(t,u,v(t);h(t))|dt+2C_1(C_1+1)\tau$$
$$+c_2^2\int_0 \tau\left(\|f(t)\|_H^2+\|h(t)\|_H^2+\|h'(t)\|_H^2\right)dt\Big\}\exp\left(\int_0^\tau(\gamma_0\|v'(t)\|_X+3)\,dt\right),$$
which implies (4.46).

Going back to (4.48) and integrating the terms on $[0,\tau]$, we obtain
$$\int_0^\tau \varphi(t,u,v(t);u(t))dt\leq\frac{1}{2}\|u_0-h(0)\|_{v_0}^2$$
$$+\frac{1}{2}\int_0^\tau(\gamma_0\|v'(t)\|_X+2)\,\|u(t)-h(t)\|_{v(t)}^2dt$$
$$+\int_0^\tau|\varphi(t,u,v(t);h(t))|dt+\frac{c_2^2}{2}\int_0^\tau\left(\|f(t)\|_H^2+\|h'(t)\|_H^2\right)dt. \tag{4.51}$$
Using Lemma 2.1 again, we get for all $t\in[0,\tau]$
$$|\varphi(t,u,v(t);u(t))|$$
$$\leq\varphi(t,u,v(t);u(t))+\|u(t)-h(t)\|_{v(t)}^2+\frac{c_2^2}{2}\|h(t)\|_H^2+\frac{3C_1^2}{4}+C_1. \tag{4.52}$$

From (4.51) and (4.52) we have

$$\int_0^\tau |\varphi(t,u,v(t);u(t))|dt \le \frac{1}{2}\|u_0 - h(0)\|^2_{v_0} + \int_0^\tau \left(\frac{\gamma_0}{2}\|v'(t)\|_X + 2\right)\|u(t)-h(t)\|^2_{v(t)}dt + \int_0^\tau |\varphi(t,u,v(t);h(t))|dt + \frac{c_2^2}{2}\int_0^\tau \left(\|f(t)\|_H^2 + \|h(t)\|_H^2 + \|h'(t)\|_H^2\right)dt + \left(\frac{3C_1^2}{4} + C_1\right)\tau,$$

which implies that (4.47) is satisfied. □

5. Existence of global-in-time solutions

We give a subset $\mathcal{U}(u_0,v_0)$ of $\mathcal{W}(u_0,v_0)$ by

$$\mathcal{U}(u_0,v_0) := \left\{\tilde u \in \mathcal{W}(u_0,v_0)\,;\ \sup_{0\le t<T}\|\tilde u(t)\|_H + \int_0^T |\varphi(\tilde u(t))|dt < \infty\right\},$$

and for each $C>0$ we denote by $\mathcal{U}_C(u_0,v_0)$ a subset of $\mathcal{U}(u_0,v_0)$ given by

$$\mathcal{U}_C(u_0,v_0) := \left\{\tilde u \in \mathcal{U}(u_0.v_0)\,;\ \sup_{0\le t<T}\|\tilde u(t)\|_H + \int_0^T |\varphi(\tilde u(t))|dt \le C\right\}.$$

In order to obtain the existence of global-in-time solutions to $(\mathrm{P})_{u_0,v_0}$, we assume that Condition 4 below is satisfied instead of Condition 3:

Condition 4. Let $C_{12}>0$ be the same constant that is given in Condition 3. For each $C>C_{12}$ there exists a family $\{M_{r,C}\,;\,0<r<\infty\}$ of positive numbers $M_{r,C}$ such that the following properties are satisfied:

(1) For $\tilde u \in \mathcal{U}_C(u_0,v_0)$ we take the same families $\{\alpha_r(\tilde u)\,;\,0<r<\infty\}\subset L^2(0,T)$ and $\{\beta_r(\tilde u)\,;\,0<r<\infty\}\subset L^1(0,T)$ as in Condition 1. Then

$$\forall\, r>0, \qquad \sup_{\tilde u\in\mathcal{U}_C(u_0,v_0)}\left\{\|\alpha_r(\tilde u)\|_{L^2(0,T)} + \|\beta_r(\tilde u)\|_{L^1(0,T)}\right\} \le M_{r,C}.$$

(2) For each $r>0$ and $\varepsilon>0$ there exists a constant $\delta_{r,\varepsilon}>0$ such that

$$\sup_{\tilde u\in\mathcal{U}_C(u_0,v_0)}\left\{\sup_{t\in[0,T]}\int_t^{\min\{t+\delta_{r,\varepsilon},T\}}\left(|\alpha_r(\tilde u;s)|^2 + |\beta_r(\tilde u;s)| + \|\tilde v'(s)\|_X\right)ds\right\} \le \varepsilon.$$

(3) There exist a constant $C_{20}>0$ and a family $\{h(\tilde u)\in W^{1,2}(0,T;H);\tilde u\in\mathcal{U}(u_0,v_0)\}$ such that

$$\|(h(\tilde u))'\|^2_{L^2(0,T;H)} + \sup_{0\le t\le T}\|h(\tilde u;t)\|_{\tilde v(t)} + \int_0^T |\varphi(t,\tilde u,\tilde v(t);h(\tilde u;t))|dt \le C_{20}. \tag{5.1}$$

(4) There exists a constant $C_{21}>0$ such that

$$\sup_{\tilde u\in\mathcal{U}_C(u_0,v_0)}\|\tilde v'\|_{L^1(0,T;X)} \le C_{21}. \tag{5.2}$$

Theorem 5.1 below guarantees the existence of global-in-time solutions to $(\mathrm{P})_{u_0,v_0}$.

Theorem 5.1. *Assume that all assumptions except Condition* 3 *in Theorem* 4.1 *and Condition* 4 *are fulfilled. Then, for each* $(u_0, v_0) \in H \times A$ *such that* $u_0 \in D(\varphi(0, u_0, v_0))$ *the Cauchy problem* $(\mathrm{P})_{u_0,v_0}$ *has at least one global-in-time solution* u.

In order to show Theorem 5.1, we define a set $Z(u_0, v_0)$ by

$$Z(u_0, v_0) := \Big\{(u, v, \tau)\,;\, u \text{ is a solution to } (\mathrm{P})_{u_0,v_0} \text{ on } [0, \tau] \text{ and } v(\cdot) = S(u, \cdot, 0)v_0\Big\}$$
$$\subset \bigcup_{0\le\tau\le T} C([0, \tau]; H) \times C([0, \tau]; X) \times \{\tau\},$$

and induce an order relation $\preceq$ on $Z(u_0, v_0)$ by

$$(u_1, v_1, \tau_1) \preceq (u_2, v_2, \tau_2) \quad \text{if and only if} \quad \tau_1 \le \tau_2 \;\text{ and }\; u_1 = u_2 \;\text{ in }\; C([0, \tau_1]; H).$$

From (S5) we have $v_1 = v_2$ in $C([0, \tau_1]; X)$. We see from Theorem 4.1 that $Z(u_0, v_0)$ is non-empty. In order to show Theorem 5.1, first of all we show Proposition 5.2 below.

Proposition 5.2. *The ordered set* $(Z(u_0, v_0), \preceq)$ *is an inductively ordered set. That is, any linearly ordered subset* Y *of* $Z(u_0, v_0)$ *is bounded above.*

Proof. Let Y be any linearly ordered subset of $Z(u_0, v_0)$. We define $\hat\tau := \sup\{\tau \in (0, T]\,;\, (u, v, \tau) \in Y\}$ and a pair $(\hat u, \hat v)$ by

$$(\hat u(t), \hat v(t)) = (u(t), v(t)) \quad \text{if and only if} \quad (u, v, \tau) \in Y \text{ and } t \in [0, \tau]. \tag{5.3}$$

Since Y is a linearly ordered subset of $Z(u_0, v_0)$, we see that the triplet $(\hat u, \hat v, \hat\tau)$ is uniquely determined. Actually, the time $\hat\tau$ is uniquely determined. So, it is enough to show that the pair $(\hat u, \hat v)$ is also uniquely determined. We assume that $\hat u_i$ $(i = 1, 2)$ satisfies (5.3). Then, we have $\hat u_1(t) = \hat u_2(t)$, hence, from (S3) $\hat v_1(t) = S(\hat u_1, t, 0)v_0 = S(\hat u_2, t, 0)v_0 = \hat v_2(t)$ for all $t \in [0, \hat\tau)$.

Next, from the definition of $\hat\tau$ we take out a sequence $\{\tau_n\}_{n\in\mathbb{N}} \subset (0, \hat\tau)$ such that $\tau_n \longrightarrow \hat\tau$ as $n \to \infty$, and consider a sequence $\{(\hat u, \hat v, \tau_n)\}_{n\in\mathbb{N}}$. Using this sequence $\{(\hat u, \hat v, \tau_n)\}_{n\in\mathbb{N}}$, we define a sequence $\{(\bar u_n, \bar v_n)\}_{n\in\mathbb{N}} \subset C([0, T]; H) \times C([0, T]; X)$:

$$(\bar u_n(t), \bar v_n(t)) := \begin{cases} (\hat u(t), \hat v(t)) & \text{if } \; t \in [0, \tau_n], \\ (\hat u(\tau_n), \hat v(\tau_n)) & \text{if } \; t \in (\tau_n, T]. \end{cases}$$

Since $\hat u$ is a solution to $(\mathrm{P})_{u_0,v_0}$ on $[0, \tau_n]$, we have

$$\begin{aligned} &\sup_{0\le t\le T} \|\bar u_n(t)\|_H + \int_0^T |\varphi(t, \bar u_n, \bar v_n(t); \bar u_n(t))|dt \\ \le\; &\sup_{0\le t\le \tau_n} \|\hat u(t)\|_H + \int_0^{\tau_n} |\varphi(t, \hat u, \hat v(t); \hat u(t))|dt + (T - \tau_n)|\varphi(\tau_n, \hat u, \hat v(\tau_n); \hat u(\tau_n))| < \infty, \end{aligned}$$

which implies $\bar{u}_n \in \mathcal{U}(u_0, v_0)$ for all $n \in \mathbb{N}$. Using (3), (4) of Condition 4 and applying Theorem 4.7, we have

$$\begin{aligned}
\sup_{0\le t\le \tau_n} \|\hat{u}(t) - h_n(t)\|^2_{\hat{v}(t)} &\le C_{17} \exp\left(C_{18} \int_0^T (\|\bar{v}_n'(t)\|_X + 1)\, dt \right) \qquad (5.4)\\
&\quad \times \left\{ \|u_0 - h(\bar{u}_n; t)\|^2_{v_0} + \int_0^T |\varphi(t, \bar{u}_n, \bar{v}_n(t); h(\bar{u}_n; t))| \right.\\
&\quad \left. + \int_0^T \left(\|h(\bar{u}_n; t)\|^2_H + \|(h(\bar{u}_n))'(t)\|^2_H + \|f(t)\|^2_H + 1 \right) dt \right\}\\
&\le C_{17} \left\{ 2\|u_0\|^2_{v_0} + (2c_2^2 + T + 1) C_{20} + \|f\|^2_{L^2(0,T;H)} + T \right\} \exp\left(C_{18} C_{21} + C_{18} T \right)
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^{\tau_n} |\varphi(t, \hat{u}, \hat{v}(t); \hat{u}(t))| dt\\
&\le C_{19} \left\{ \|u_0 - h(\bar{u}_n; 0)\|^2_{v_0} + \sup_{0\le t\le \tau_n} \|\hat{u}(t) - h(\bar{u}_n; t)\|^2_{\hat{v}(t)} \int_0^T (\|\bar{v}'(t)\|_X + 1)\, dt \right.\\
&\quad \left. + \int_0^T \left(|\varphi(t, \bar{u}_n, \bar{v}_n(t); h(\bar{u}_n; t)| + \|f(t)\|^2_H + \|h(\bar{u}_n; t)\|^2_H + \|(h(\bar{u}_n))'(t)\|^2_H + 1 \right) dt \right\}\\
&\le C_{19} \left\{ 2\|u_0\|^2_{v_0} + (2c_2^2 + T + 1) C_{20} + \|f\|^2_{L^2(0,T;H)} + T \right\}\\
&\quad + C_{17}(C_{21} + T) \sup_{0\le t\le \tau_n} \|\hat{u}(t) - h(\bar{u}_n; t)\|^2_{\hat{v}(t)}. \qquad (5.5)
\end{aligned}$$

We see from (5.4) and (5.5) that there exists a constant $C_{22} > 0$ such that

$$\sup_{n\in\mathbb{N}} \left\{ \sup_{0\le t\le \tau_n} \|\hat{u}(t)\|_H + \int_0^{\tau_n} |\varphi(t, \hat{u}(t), \hat{v}(t); \hat{u}(t))| dt \right\} \le C_{22}. \qquad (5.6)$$

Taking the limit $n \to \infty$ in (5.6), we have

$$\sup_{0\le t<\hat{\tau}} \|\hat{u}(t)\|_H + \int_0^{\hat{\tau}} |\varphi(t, \hat{u}, \hat{v}(t); \hat{u}(t))| dt \le C_{22},$$

which implies $\hat{u} \in \mathcal{U}(u_0, v_0)$ as $T = \hat{\tau}$. Applying Theorem 3.1, we see that the Cauchy problem $(\mathrm{P})_{\hat{u},u_0,v_0}$ below has one and only one solution $w \in W^{1,2}(0, \hat{\tau}; H)$:

$$(\mathrm{P})_{\hat{u},u_0,v_0} \begin{cases} w'(t) + \partial_{\tilde{v}(t)} \varphi(t, \hat{u}, \tilde{v}(t); w(t)) \ni f(t) \quad \text{in} \quad H(\tilde{v}(t)), \quad \text{a.a. } t \in (0, \hat{\tau}),\\ \tilde{v}(t) = S(\hat{u}, t, 0) v_0 \quad \text{in} \quad X, \quad \forall t \in [0, \hat{\tau}],\\ w(0) = u_0 \quad \text{in} \quad H(v_0). \end{cases}$$

Since $(w(t), \tilde{v}(t)) = (\hat{u}(t), \hat{v}(t))$ for all $t \in [0, \hat{\tau})$, we see that

$$(\hat{u}(\tau_n), \hat{v}(\tau_n)) = (w(\tau_n), \tilde{v}(\tau_n)) \longrightarrow (w(\hat{\tau}), \tilde{v}(\hat{\tau})) \ \text{ in } H \times X \text{ as } n \to \infty.$$

Hence, we see that $(w, \tilde{v}, \hat{\tau})$ is an upper bound of Y. □

In the rest of this section, we give the proof of Theorem 5.1.

Proof of Theorem 5.1. Applying Zorn's lemma, the inductively orderd set $(Z, \preceq)$ has at least one maximal element (u^*, v^*, τ^*). If $\tau^* = T$ is proved, from the definition of $(Z, \preceq)$ it is clear that (u^*, v^*) is a solution to $(\mathrm{P})_{u_0,v_0}$ on $[0, T]$. Hence, it is enough to show $\tau^* = T$. Assume $\tau^* < T$. From Theorem 4.1 we have already shown that there exists a constant $C_{23} > 0$ depending on $\|u_0\|_H$ and $\varphi(0, u_0, v_0; u_0)$, such that

$$\|(u^*)'\|_{L^2(0,\tau^*;H(v^*))} + \sup_{0\le t\le\tau^*} |\varphi(t, u^*, v^*(t); u^*(t))| \le C_{23}. \tag{5.7}$$

Now, we consider the Cauchy problem $\{(5.8), (5.9), (5.10)\}$ below, which is denoted by $(\mathrm{P})^*$ here:

$$w'(t)+\partial_{\tilde v(t)}\varphi^*(t, w(t), \tilde v(t); w(t)) \ni f(t+\tau^*) \text{ in } H(\tilde v(t)), \text{ a.a. } t\in[0, T-\tau^*], \tag{5.8}$$

$$(\mathrm{P})^* \qquad \tilde v(t) = S(w, t, 0)v^*(\tau^*) \quad \text{in} \quad X, \quad \forall\, t \in [0, T-\tau^*], \tag{5.9}$$

$$w(0) = u^*(\tau^*) \quad \text{in} \quad H, \tag{5.10}$$

where $\varphi^*(t, w, \tilde v(t)) = \varphi(t + \tau_*, w, \tilde v(t))$ for all $t \in [0, T-\tau^*]$. From (5.7) we have

$$\begin{aligned} u^*(\tau^*) \in D(\varphi(\tau^*, u^*, v^*(\tau^*))) &= D(\varphi(\tau^*, [u^*(\tau^*)], v^*(\tau^*))) \\ &= D(\varphi^*(0, u^*(\tau^*), v^*(\tau^*))). \end{aligned}$$

Applying Theorem 4.1, we see that there exists a finite time $T_1 \in (0, T-\tau^*]$ such that $(\mathrm{P})^*$ has at least one solution $w \in W^{1,2}(0, T_1; H)$ on $[0, T_1]$.

Then, we define a pair $(u, v) \in W^{1,2}(0, \tau^* + T_1; H) \times W^{1,1}(0, \tau^* + T_1; X)$ by

$$(u(t), v(t)) := \begin{cases} (u^*(t), v^*(t)) & \text{if} \quad t \in [0, \tau^*], \\ (w(t-\tau^*), \tilde v(t-\tau^*)) & \text{if} \quad t \in (\tau^*, \tau^* + T_1]. \end{cases}$$

Then, we easily see that u satisfies (1.1) and (1.3).

Moreover, from (S5) and (S6) we have for all $t \in (\tau^*, \tau^* + T_1]$

$$\begin{aligned} S(u, t, 0)v_0 &= (S(u, t, \tau^*) \circ S(u, \tau^*, 0))v_0 = S(u, t, \tau^*)v^*(\tau^*) \\ &= S(w_{-\tau^*}, t, \tau^*)v^*(\tau^*) = S(w; t-\tau^*, 0)v^*(\tau^*) \\ &= \tilde v(t-\tau^*) = v(t), \end{aligned}$$

which implies that v satisfies (1.2). Hence, u is a solution to $(\mathrm{P})_{u_0,v_0}$ on $[0, \tau^*+T_1]$, which is in contradiction with the definition of (u^*, v^*, τ^*). So, we see that $\tau^* = T$ must hold. □

6. A phase field model with quasi-variational boundary condition

We consider the following phase field model $\{(6.1)$ - $(6.7)\}$ of Fix-Caginalp type, which is denoted by $(\mathrm{PF})_{e_0,w_0}$ here, with a quasi-variationel structure for a boundary condition:

$$e_t - \Delta(e-w) = f \quad \text{a.e. in} \quad \Omega \times (0, T), \tag{6.1}$$

$$w_t - \Delta w + g(w) + \xi + w - e = 0 \quad \text{a.e. in} \quad \Omega \times (0,T), \tag{6.2}$$

$$\xi \in \beta(w) \quad \text{a.e. in} \quad \Omega \times (0,T), \tag{6.3}$$

$$(\mathrm{PF})_{e_0,w_0} \qquad \nabla(e-w)\cdot\nu + n(\bar{w})(e-w) = h \quad \text{a.e. on} \quad \Gamma \times (0,T), \tag{6.4}$$

$$\nabla w \cdot \nu = 0 \quad \text{a.e. on} \quad \Gamma \times (0,T), \tag{6.5}$$

$$e(0) = e_0 \quad \text{a.e. in} \quad \Omega, \tag{6.6}$$

$$w(0) = w_0 \quad \text{a.e. in} \quad \Omega, \tag{6.7}$$

where Ω is a bounded domain in $\mathbb{R}^N$ $(N = 1, 2, 3)$ with a smooth boundary $\Gamma := \partial\Omega$; ν are outer unit normal vectors on Γ, and for each $z \in L^2(\Omega)$ the value $\bar{z} \in \mathbb{R}$ is defined by

$$\bar{z} := \frac{1}{|\Omega|}\int_\Omega z(x)dx.$$

The equations (6.1) and (6.2) are originally proposed and analyzed mathematically in [2, 5] in order to describe solid-liquid phase transition phenomena under the nonconstant relative temperature $\theta := e - w$ in $(\mathrm{PF})_{e_0,w_0}$, where the unknown functions e and w imply the internal energy density and the order parameter describing the phase state of a material, respectively. We entrust the detail explanation of this system from the physical point of view to [2, 5] and omit it in this paper. It is important and essential that the coefficient $n(\bar{w})$ of $e - w$ in (6.4) depends on the order parameter w, which is unknown in $(\mathrm{PF})_{e_0,w_0}$. As you see from the settings below, this property gives $(\mathrm{PF})_{e_0,w_0}$ the quasi-variational structure for inner products.

In order to analyze $(\mathrm{PF})_{e_0,w_0}$, we assume that the following conditions are satisfied:

(P1) β is a maximal monotone graph $\mathbb{R} \times \mathbb{R}$. There exist constants $\sigma_*,\ \sigma^* \in \mathbb{R}$ such that $\sigma_* < \sigma^*$ and $\overline{D(\beta)} = [\sigma_*, \sigma^*]$. Moreover, there exists a nonnegative proper l.s.c. convex function $\hat{\beta}$ on $\mathbb{R}$ such that the subdifferential of $\hat{\beta}$ coincides with β.

(P2) n is a nonconstant C^1−function from $\mathbb{R}$ into itself. There exist constants $n_1 > 0$ and $n_2 > 0$ such that

$$n_1 \le n(r) \le n_2, \quad \forall\, r \in [\sigma_*, \sigma^*].$$

Then, we get for all $r_1, r_2 \in [\sigma_*, \sigma^*]$

$$|n(r_1) - n(r_2)| \le n_3|r_1 - r_2|, \quad \text{with} \quad n_3 := \max_{r\in[\sigma_*,\sigma^*]} |n'(r)|.$$

(P3) g is a continuous function from $\mathbb{R}$ into itself. From (P2) without loss of generality we can assume that g is globally Lipschitz continuous on $\mathbb{R}$ with a Lipschitz constant $L_g > 0$. Moreover, we fix the primitive of g, denoted by $\hat{g}$, such that $\hat{g}(r) \ge 0$ for all $r \in [\sigma_*, \sigma^*]$.

(P4) $f \in L^2(0,T;L^2(\Omega))$ and $h \in L^2(0,T;L^2(\Gamma))$.

(P5) $e_0 \in L^2(\Omega)$.

(P6) $w_0 \in H^1(\Omega)$ with $\hat{\beta}(w_0) \in L^1(\Omega)$, where $H^1(\Omega)$ is a Hilbert space with the usual inner product $(\cdot,\cdot)_{H^1(\Omega)}$ given by

$$(z_1, z_2)_{H^1(\Omega)} = \int_\Omega \nabla z_1(x) \cdot \nabla z_2(x) dx + \int_\Omega z_1(x) z_2(x) dx, \ \forall\, z_1, z_2 \in H^1(\Omega).$$

In order to apply Theorem 5.1 to the Cauchy problem $(\mathrm{PF})_{e_0,w_0}$, we consider V' as a real Hilbert space H, which is the dual space of $V := H^1(\Omega)$ with an inner product below:

$$(z_1, z_2)_V = \int_\Omega \nabla z_1(x) \cdot \nabla z_2(x) dx + \int_\Gamma z_1(x) z_2(x) d\Gamma, \quad \forall\, z_1, z_2 \in V, \tag{6.8}$$

which is equivalent to $(\cdot,\cdot)_{H^1(\Omega)}$, that is, there exist constants $d_1 > 0$ and $d_2 > 0$ such that

$$d_1 \|z\|_{H^1(\Omega)} \le \|z\|_V \le d_2 \|z\|_{H^1(\Omega)}, \quad \forall\, z \in H^1(\Omega). \tag{6.9}$$

We denote by $\langle z^*, z\rangle$ the duality pairing between $z^* \in V'$ and $z \in V$, and by F the duality map from V onto V', respectively. According to [12, Definition 32.19, Example 32.20], we have $F = \partial J$, where J is defined by

$$J(z) := \frac{1}{2}\|z\|_V^2, \quad \forall\, z \in V, \tag{6.10}$$

and $\partial J : V \longrightarrow V'$ is the single-valued subdifferential of J defined by

$$z^* \in \partial J(z) \quad \text{if and only if} \quad \langle z^*, y - z\rangle \le J(y) - J(z), \quad \forall\, y \in V. \tag{6.11}$$

Then, we see that V' is a real Hilbert space with an inner product

$$(z_1^*, z_2^*)_{V'} = \langle z_1^*, F^{-1} z_2^*\rangle, \quad \forall\, z_1^*, z_2^* \in V', \tag{6.12}$$

and the following equalities hold:

$$\langle F z_1, z_2\rangle = (z_1, z_2)_V, \quad \forall\, z_1, z_2 \in V, \tag{6.13}$$

$$\langle z_1, z_2\rangle = (z_1, z_2)_{L^2(\Omega)}, \quad \forall\, z_1 \in L^2(\Omega), \ \forall\, z_2 \in V. \tag{6.14}$$

Moreover, we consider $L^2(\Omega)$ as a Banach space X and define a nonempty subset of $L^2(\Omega)$ by

$$A := \left\{ w \in H^1(\Omega)\, ;\, \hat{\beta}(w) \in L^1(\Omega) \right\}.$$

For each $w \in A$ we define an inner product $(\cdot,\cdot)_{V(w)}$ in V by

$$(z_1, z_2)_{V(w)} = \int_\Omega \nabla z_1(x) \cdot \nabla z_2(x) dx + n(\bar{w}) \int_\Gamma z_1(x) z_2(x) d\Gamma, \ \forall\, z_1, z_2 \in V. \tag{6.15}$$

We also denote by F_w the duality map from V with the inner product given by (6.15) onto V'. Then, we have $F_w = \partial J_w$, where J_w is defined by

$$J_w(z) := \frac{1}{2}\|z\|_{V(w)}^2, \quad \forall\, z \in V, \tag{6.16}$$

and $\partial J_w : V \longrightarrow V'$ is the single-valued subdifferential of J_w defined by

$$z^* \in \partial J_w(z) \quad \text{if and only if} \quad \langle z^*, y - z\rangle \le J_w(y) - J_w(z), \quad \forall\, y \in V. \tag{6.17}$$

Then, for each $w \in A$ we see that V' is also a real Hilbert space with an inner product

$$(z_1^*, z_2^*)_w := \langle z_1^*, F_w^{-1} z_2^*\rangle, \quad \forall\, w \in A,\ \forall\, z_1^*, z_2^* \in V', \tag{6.18}$$

which is denoted by $(V(w))'$ throughout this section, and have

$$\langle F_w z_1, z_2\rangle = (z_1, z_2)_{V(w)}, \quad \forall\, w \in A,\ \forall\, z_1, z_2 \in V. \tag{6.19}$$

For $(\mathrm{PF})_{e_0,w_0}$ we have Theorem 6.1 below.

Theorem 6.1. *Assume that all conditions* (P1)–(P6) *are satisfied. Then, $(PF)_{e_0,w_0}$ has at least one global-in-time solution (e, w) such that the following properties are satisfied:*

(1) $e \in W^{1,2}(0, T; V') \cap L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; V)$;

(2) $w \in W^{1,2}(0, T; L^2(\Omega)) \cap L^\infty(0, T; H^1(\Omega)) \cap L^2(0, T; H^2(\Omega))$;

(3) (6.1) *is satisfied in the following quasi-variational sense for a.a. $t \in (0, T)$:*

$$\begin{aligned}\langle e'(t), z\rangle &+ \int_\Omega \nabla(e - w)(x, t)\cdot\nabla z(x)dx + n(\bar{w})\int_\Gamma (e - w)(x, t)z(x)d\Gamma \\ &= \langle f^*(t), z\rangle := \int_\Omega f(x, t)z(x)dx + \int_\Gamma h(x, t)z(x)d\Gamma, \quad \forall\, z \in V,\end{aligned} \tag{6.20}$$

that is, for a.a. $t \in (0, T)$

$$e'(t) + F_{w(t)}(e - w)(t) = f^*(t) \quad \text{in} \quad (V(w(t)))'; \tag{6.21}$$

(4) *There exists a function $\xi \in L^2(0, T; L^2(\Omega))$ and* (6.2) *is satisfied in the following sense for a.a. $t \in (0, T)$:*

$$w'(t) - \Delta_N w(t) + g(w(t)) + \xi(t) + w(t) = e(t) \ \text{in}\ L^2(\Omega); \tag{6.22}$$

(5) $e(0) = e_0$ *in* V';

(6) $w(0) = w_0$ *in* $L^2(\Omega)$.

We build on the general theory of duality maps of [12] and start the argumentation beginning with Lemma 6.2 below.

Lemma 6.2. (A1) *is satisfied, that is, there exist constants $c_1 > 0$ and $c_2 > 0$ such that*

$$c_1\|z^*\|_{V'} \le \|z^*\|_w \le c_2\|z^*\|_{V'}, \quad \forall\, w \in A,\ \forall\, z^* \in V'.$$

Proof. In the following argumentation, we let v, z and z^* any elements in A, V and V', respectively. From (P2) we have

$$\min\{1, n_1\}\|z\|_V^2 \le \|z\|_{V(w)}^2 \le \max\{1, n_2\}\|z\|_V^2,$$

so, from (6.19)

$$\min\{1, n_1\}\|z\|_V^2 \le \langle F_w z, z\rangle \le \max\{1, n_2\}\|z\|_V^2, \tag{6.23}$$

which implies

$$\min\{1, n_1\}\|z\|_V \le \|F_w z\|_{V'} \le \max\{1, n_2\}\|z\|_V. \tag{6.24}$$

Substituting $z = F_w^{-1} z^*$ into (6.23) and (6.24), and using (6.18), we get

$$\sqrt{\min\{1, n_1\}}\, \|F_w^{-1} z^*\|_V^2 \le \|z^*\|_w^2 \le \sqrt{\max\{1, n_2\}}\, \|F_w^{-1} z^*\|_V^2, \text{ and} \tag{6.25}$$

$$\min\{1, n_1\} \|F_w^{-1} z^*\|_V \le \|z^*\|_{V'} \le \max\{1, n_2\} \|F_w^{-1} z^*\|_V. \tag{6.26}$$

Taking constants $c_1 > 0$ and $c_2 > 0$, for example, as

$$c_1 := \frac{\sqrt{\min\{1, n_1\}}}{\max\{1, n_2\}} \quad \text{and} \quad c_2 := \frac{\sqrt{\max\{1, n_2\}}}{\min\{1, n_1\}},$$

we see from (6.25) and (6.26) that this lemma holds. □

Lemma 6.3. (A2) *is satisfied, that is, there exists a constant* $\gamma_0 > 0$ *such that*

$$\left| \|z^*\|_{w_1}^2 - \|z^*\|_{w_2}^2 \right| \le \gamma_0 \|w_1 - w_2\|_{L^2(\Omega)} \|z^*\|_{w_2}^2, \quad \forall\, w_1, w_2 \in A, \ \forall\, z^* \in V'.$$

Hence, Condition 2 *is also satisfied.*

Proof. Using (P2) and (6.15), (6.19), we get for all $w_1, w_2 \in A$ and all $z_1, z_2 \in V$

$$\begin{aligned} \langle (F_{w_1} - F_{w_2}) z_1, z_2 \rangle &\le |n(\overline{w_1}) - n(\overline{w_2})| \int_\Gamma |z_1(x)||z_2(x)| d\Gamma \\ &\le \frac{n_3}{\sqrt{|\Omega|}} \|w_1 - w_2\|_{L^2(\Omega)} \|z_1\|_V \|z_2\|_V, \end{aligned}$$

which implies for all $w_1, w_2 \in A$

$$\|F_{w_1} - F_{w_2}\|_{\mathcal{L}(V,V')} \le \frac{n_3}{\sqrt{|\Omega|}} \|w_1 - w_2\|_{L^2(\Omega)}. \tag{6.27}$$

Using (6.18), (6.24), (6.27) and Lemma 6.2, we get for all $w_1, w_2 \in A$ and $z^* \in V'$

$$\begin{aligned} \left| \|z^*\|_{w_1}^2 - \|z^*\|_{w_2}^2 \right| &\le \left\| F_{w_1}^{-1} (F_{w_2} - F_{w_1}) F_{w_2}^{-1} z^* \right\|_V \|z^*\|_{V'} \\ &\le \left\| F_{w_1}^{-1} \right\|_{\mathcal{L}(V',V)} \|F_{w_2} - F_{w_1}\|_{\mathcal{L}(V,V')} \left\| F_{w_2}^{-1} \right\|_{\mathcal{L}(V',V)} \|z^*\|_{V'}^2 \\ &\le \frac{n_3 (\max\{1, n_2\})^2}{\sqrt{|\Omega|} (\min\{1, n_1\})^3} \, \|w_1 - w_2\|_{L^2(\Omega)} \|z^*\|_{w_2}^2, \end{aligned}$$

which completes the proof of this lemma. □

For each $\tilde{e} \in C([0,T]; V') \cap L^2(0,T; L^2(\Omega))$, $s \in [0,T]$ and $\tilde{w}_s \in A$ we consider the Cauchy problem $(\mathrm{P})_{\tilde{e},\tilde{w}_s} := \{(6.2), (6.3), (6.5), (6.6)\}$ in which e, 0 and w_0 are replaced by $\tilde{e}$, s and $\tilde{w}_s$, respectively. Applying the general theory established in [10], we have already obtained the following lemma.

Lemma 6.4. *For each* $\tilde{e} \in C([0,T]; V') \cap L^2(0,T; L^2(\Omega))$, $s \in [0,T]$ *and* $\tilde{w}_s \in A$ *the Cauchy problem* $(P)_{\tilde{e},\tilde{w}_s}$ *has a unique solution*

$$\tilde{w} := \tilde{w}(\tilde{e}, \tilde{w}_s) \in W^{1,2}(s,T; L^2(\Omega)) \cap L^\infty(s,T; H^1(\Omega)) \cap L^2(s,T; H^2(\Omega))$$

such that (4) *and* (6) *of Theorem* 6.1 *are satisfied in which* e, w, 0 *and* w_0 *are replaced by* $\tilde{e}$, $\tilde{w}$, s *and* $\tilde{w}_s$, *respectively.*

Moreover, there exists a constant $C_{24} > 0$ *depending on* $\|\tilde{e}\|_{L^2(0,T;L^2(\Omega))}$, $\|\tilde{w}_s\|_{H^1(\Omega)}$ *and* $\|\hat{\beta}(\tilde{w}_s)\|_{L^1(\Omega)}$, *such that*

$$\|(\tilde{w}(\tilde{e},\tilde{w}_s))'\|_{L^2(s,T;L^2(\Omega))} + \sup_{s\le t\le T}\|\tilde{w}(\tilde{e},\tilde{w}_s;t)\|_{H^1(\Omega)} + \sup_{s\le t\le T}\|\hat{\beta}(\tilde{w}(\tilde{e},\tilde{w}_s;t)\|_{L^1(\Omega)} \le C_{24}.$$

Due to Lemma 6.4, for each $\tilde{e} \in C([0,T];V') \cap L^2(0,T;L^2(\Omega))$ we can define an evolution system $\{S(\tilde{e},t,s)\,;\, 0 \le s \le t \le T\}$ on A by

$$\forall\, s,t \in [0,T] \text{ with } s \le t, \quad S(\tilde{e},t,s)\tilde{w}_s := \tilde{w}(\tilde{e},\tilde{w}_s;t), \quad \forall\, \tilde{w}_s \in A,$$

which satisfies the conditions (S1), (S4) and (S5).

Lemma 6.5. (S2) *and* (S3) *are satisfied.*

Proof. For each $\tilde{e}_i \in W^{1,2}(0,T;V') \cap L^2(0,T;L^2(\Omega))$ $(i = 1,2)$, $s \in [0,T]$ and $w_{i,s} \in A$ $(i = 1,2)$ we put $\tilde{w}_i(t) := S(\tilde{e}_i,t,s)w_{i,s}$ for all $t \in [s,T]$ throughout this proof. Then, we have

$$\begin{aligned}(\tilde{w}_1 - \tilde{w}_2)'(t) - \Delta_N(\tilde{w}_1 - \tilde{w}_2)(t) + g(\tilde{w}_1(t)) - g(\tilde{w}_2(t)) + \tilde{\xi}_1(t) - \tilde{\xi}_2(t)&\\ +\tilde{w}_1(t) - \tilde{w}_2(t) = \tilde{e}_1(t) - \tilde{e}_2(t) \quad \text{in} \quad L^2(\Omega), \text{ a.a. } t \in (s,T);& \qquad (6.28)\end{aligned}$$

$$\tilde{\xi}_i(t) \in \beta(\tilde{w}_i(t) \quad \text{in} \quad L^2(\Omega), \quad \text{a.a. } t \in (s,T),\ i = 1,2; \qquad (6.29)$$

$$\tilde{w}_i(s) = w_{i,s} \quad \text{in} \quad L^2(\Omega), \quad i = 1,2,. \qquad (6.30)$$

We take the inner product in $L^2(\Omega)$ in both sides of (6.28), and use (P3), (6.14) and the monotonicity of β in $L^2(\Omega)$ with (6.29). Then, we get for a.a. $t \in (s,T)$

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|(\tilde{w}_1 - \tilde{w}_2)(t)\|^2_{L^2(\Omega)} + \|(\tilde{w}_1 - \tilde{w}_2)(t)\|^2_{H^1(\Omega)}\\ &\le \langle(\tilde{e}_1 - \tilde{e}_2)(t), (\tilde{w}_1 - \tilde{w}_2)(t)\rangle + L_g\|(\tilde{w}_1 - \tilde{w}_2)(t)\|^2_{L^2(\Omega)},\end{aligned}$$

hence from (6.9)

$$\begin{aligned}&\frac{d}{dt}\|(\tilde{w}_1 - \tilde{w}_2)(t)\|^2_{L^2(\Omega)} + \|(\tilde{w}_1 - \tilde{w}_2)(t)\|^2_{H^1(\Omega)}\\ &\le 2L_g\|(\tilde{w}_1 - \tilde{w}_2)(t)\|^2_{L^2(\Omega)} + d_2^2\|(\tilde{e}_1 - \tilde{e}_2)(t)\|^2_{V'}. \qquad (6.31)\end{aligned}$$

Applying Gronwall's lemma to (6.31), we get for all $t \in [s,T]$

$$\begin{aligned}&\|(\tilde{w}_1 - \tilde{w}_2)(t)\|^2_{L^2(\Omega)} + \int_s^t \|(\tilde{w}_1 - \tilde{w}_2)(\tau)\|^2_{H^1(\Omega)}d\tau\\ &\le e^{2L_g(t-s)}\left(\|w_{1s} - w_{2s}\|^2_{L^2(\Omega)} + \frac{{d_2}^2}{2L_g}\max_{0\le t\le T}\|(\tilde{e}_1 - \tilde{e}_2)(t)\|^2_{V'}\right), \text{ hence,}\end{aligned}$$

$$\max_{s \le t \le T} \|(\tilde{w}_1 - \tilde{w}_2)(t)\|^2_{L^2(\Omega)} + \|\tilde{w}_1 - \tilde{w}_2\|^2_{L^2(s,T;H^1(\Omega)}$$

$$\le 2e^{2L_g T} \left(\|w_{1s} - w_{2s}\|^2_{L^2(\Omega)} + \frac{{d_2}^2}{2L_g} \max_{0 \le t \le T} \|(\tilde{e}_1 - \tilde{e}_2)(t)\|^2_{V'} \right). \tag{6.32}$$

The estimation (6.32) directly implies that (S2) and (S3) are satisfied. □

Next, for each $(e_0, w_0) \in L^2(\Omega) \times A$, $t \in [0,T]$ and $\tilde{e} \in C([0,T]; V') \cap L^2(0,T; L^2(\Omega))$ such that $\tilde{e}(0) = e_0$ we define a time-dependent proper l.s.c. convex function $\varphi(\tilde{e}, \tilde{w}(t))$ on V' by

$$\varphi(\tilde{e}, \tilde{w}(t); z^*) := \begin{cases} \frac{1}{2} \int_\Omega |z^*(x) - \tilde{w}(x,t)|^2 dx, & \text{if} \quad z^* \in L^2(\Omega), \\ +\infty, & \text{if} \quad z^* \in V' \setminus L^2(\Omega), \end{cases} \tag{6.33}$$

where $\tilde{w}(t) = S(\tilde{e}, t, 0)w_0$ for all $t \in [0,T]$.

Lemma 6.6. (C1) *and* (C2) *are satisfied.*

Proof. We see from Lemma 6.5 that (C2) is satisfied. So, in this proof it is enough to show that (C1) is satisfied. In order to do this, for each $C > 0$ we define a proper l.s.c. convex function φ_C on V' by

$$\varphi_C(z) := \begin{cases} \frac{1}{4} \int_\Omega |z^*(x)|^2 dx - C, & \text{if} \quad z^* \in L^2(\Omega), \\ +\infty, & \text{if} \quad z^* \in V' \setminus L^2(\Omega). \end{cases}$$

From Lemma 6.4 we have $\sigma_* \le \tilde{w}(t) \le \sigma^*$ a.a. in Ω for all $t \in [0,T]$ and

$$D(\varphi_C) = D(\varphi(\tilde{e}, \tilde{w}(t))) = L^2(\Omega), \quad \forall\, t \in [0,T].$$

Hence, we get for all $z^* \in L^2(\Omega)$

$$\varphi(\tilde{e}, \tilde{w}(t); z^*) \ge \frac{1}{4} \int_\Omega |z^*(x)|^2 dx - \frac{1}{2} |\Omega| \max \left\{ |\sigma_*|^2, |\sigma^*|^2 \right\}. \tag{6.34}$$

Taking $C_{25} > 0$ such that $C_{25} \ge \frac{1}{2}|\Omega| \{|\sigma_*|^2, |\sigma^*|^2\}$, we see that (C1) is satisfied for $\varphi := \varphi_{C_{25}}$. □

Lemma 6.7. (C3) *is satisfied.*

Proof. Assume that the sequences $\{\tilde{e}_m\}_{m\in\mathbb{N}} \subset C([0,T]; V') \cap L^2(0,T; L^2(\Omega))$, $\{w_{0m}\}_{m\in\mathbb{N}} \subset A$ and the elements $\tilde{e} \in C([0,T]; V') \cap L^2(0,T; L^2(\Omega))$, $w_0 \in A$, satisfy

$$\tilde{e}_m \longrightarrow \tilde{e} \quad \text{in} \quad C([0,T]; V'), \quad w_{0m} \longrightarrow w_0 \quad \text{in} \quad L^2(\Omega) \quad \text{as} \quad m \to \infty.$$

We see from (6.32) in the proof of Lemma 6.5 that

$$\tilde{w}_m \longrightarrow \tilde{w} \quad \text{in} \quad C([0,T]; L^2(\Omega)) \cap L^2(0,T; H^1(\Omega)) \quad \text{as} \quad m \to \infty, \tag{6.35}$$

where $\tilde{w}_m(t) = S(\tilde{e}_m, t, 0)w_{0m}$ and $\tilde{w}(t) = S(\tilde{e}, t, 0)w_0$ for all $t \in [0, T]$. From (6.35) we have

$$\forall\, z^* \in L^2(\Omega),\ \forall\, t \in [0, T], \quad \lim_{m\to\infty} \varphi(\tilde{e}_m, \tilde{w}_m(t); z^*) = \varphi(\tilde{e}, \tilde{w}(t); z^*),$$

which implies that (1) of (C3) is satisfied.

Next, we consider any subsequences $\{\tilde{e}_{m_k}\}_{k\in\mathbb{N}}$ and $\{w_{0m_k}\}_{k\in\mathbb{N}}$ of $\{\tilde{e}\}_{m\in\mathbb{N}}$ and $\{w_{0m}\}_{m\in\mathbb{N}}$, respectively, and a sequence $\{z_k^*\}_{k\in\mathbb{N}} \subset V'$ and an element $z^* \in V'$ such that

$$z_k^* \longrightarrow z^* \quad \text{weakly in} \quad (V(\tilde{w}(t)))' \quad \text{as} \quad k \to \infty. \tag{6.36}$$

It is enough to show the case that $d := \liminf_{k\to\infty} \varphi(\tilde{e}_{m_k}, \tilde{w}_{m_k}(t); z_k^*) < +\infty$. First of all, from (6.34) we have

$$\begin{aligned}
&|\varphi(\tilde{e}_{m_k}, \tilde{w}_{m_k}(t); z_k^*) - \varphi(\tilde{e}, \tilde{w}(t); z_k^*)| \\
&= \left| \frac{1}{2}\int_\Omega |z_k^*(x)||(\tilde{w}_{m_k} - \tilde{w})(x,t)|dx + \frac{1}{2}\int_\Omega \left(|\tilde{w}_{m_k}(x,t)|^2 - |\tilde{w}(x,t)|^2\right) dx \right| \\
&\le \|(\tilde{w}_{m_k} - \tilde{w})(t)\|_{L^2(\Omega)} \left(\|z_k^*\|_{L^2(\Omega)} + |\Omega|^{\frac{1}{2}} \max\{|\sigma_*|, |\sigma^*|\} \right) \\
&= \|(\tilde{w}_{m_k} - \tilde{w})(t)\|_{L^2(\Omega)} \left(\varphi(\tilde{e}, \tilde{w}(t); z_k^*) + 2\sqrt{|\Omega|} \max\{|\sigma_*|, |\sigma^*|\} + 1 \right).
\end{aligned} \tag{6.37}$$

From (6.35) and (6.37) we have

$$\lim_{k\to\infty} \{\varphi(\tilde{e}_{m_k}, \tilde{w}_{m_k}(t); z_k^*) - \varphi(\tilde{e}, \tilde{w}(t); z_k^*)\} = 0. \tag{6.38}$$

Moreover, since $\varphi(\tilde{e}, \tilde{w}(t))$ is weakly sequentially l.s.c. on $(V(\tilde{w}(t)))'$, from (6.36) we have

$$\varphi(\tilde{e}, \tilde{w}(t); z^*) \le \liminf_{k\to\infty} \varphi(\tilde{e}, \tilde{w}(t); z_k^*). \tag{6.39}$$

From (6.38) and (6.39) we have

$$\begin{aligned}
\liminf_{k\to\infty} \varphi(\tilde{e}_{m_k}, \tilde{w}_{m_k}(t); z_k^*) &= \lim_{k\to\infty} \{\varphi(\tilde{e}_{m_k}, \tilde{w}_{m_k}(t); z_k^*) - \varphi(\tilde{e}, \tilde{w}(t); z_k^*)\} \\
&\quad + \liminf_{k\to\infty} \varphi(\tilde{e}, \tilde{w}(t); z_k^*) \ge \varphi(\tilde{e}, \tilde{w}(t); z^*).
\end{aligned} \tag{6.40}$$

Hence, we see from (6.39) and (6.40) that (2) of (C3) is satisfied. □

Lemma 6.8. *Condition* 1 *is satisfied.*

Proof. For $\tilde{e} \in C([0, T]; V') \cap L^2(0, T; L^2(\Omega))$ we have $D(\varphi(\tilde{e}, \tilde{w}(t))) = L^2(\Omega)$ for all $t \in [0, T]$. Now, we define functions $\alpha(\tilde{e})$ and $\beta(\tilde{e})$, which are independent of $r > 0$, by

$$\alpha(\tilde{e}; t) = 0, \qquad \beta(\tilde{e}; t) = \|\tilde{w}'(t)\|_{L^2(\Omega)}, \qquad \forall\, t \in [0, T].$$

Then, we see from (6.34) that there exists a constant $C_{26} > 0$ such that

$$\begin{aligned}
&|\varphi(\tilde{e}, \tilde{w}(t); z^*) - \varphi(\tilde{e}, \tilde{w}(s); z^*)| \\
\leq\ & \int_\Omega (|z^*(x)| + |\tilde{w}(x,t)| + |\tilde{w}(x,s)|)|\tilde{w}(x,t) - \tilde{w}(x,s)|dx \\
\leq\ & \left(\int_\Omega (|z^*(x)| + 2\max\{|\sigma_*|, |\sigma^*|\})^2 dx \right)^{\frac{1}{2}} \|\tilde{w}(t) - \tilde{w}(s)\|_{L^2(\Omega)} \\
\leq\ & C_{26} \{\varphi(\tilde{e}, \tilde{w}(t); z^*) + 1\} \left| \int_s^t \|\tilde{w}'(\tau)\|_{L^2(\Omega)} d\tau \right|.
\end{aligned}$$

Hence, we see that Condition 1 is satisfied. □

Lemma 6.9. *Condition* 4 *is satisfied.*

Proof. We consider subsets $\mathcal{W}(e_0, w_0)$ and $\mathcal{U}_C(e_0, w_0)$ of $C([0,T]; V')$ given by

$$\mathcal{W}(e_0, w_0) := \{\tilde{e} \in C([0,T]; V'); \tilde{e}(0) = e_0 \text{ and } \tilde{w}(t) = S(\tilde{e}, t, 0)w_0,\ \forall\, t \in [0,T]\},$$

$$\mathcal{U}_C(e_0, w_0) := \left\{ \tilde{e} \in \mathcal{W}(e_0, w_0)\,;\ \sup_{0 \leq t \leq T} \|\tilde{e}(t)\|_{V'} + \int_0^T |\varphi(\tilde{e}(t))| dt \leq C \right\}.$$

We take the inner products in $L^2(\Omega)$ between $\tilde{w}(t)$ and both sides of (6.22), in which e, w and ξ are replaced by $\tilde{e}$, $\tilde{w}$ and $\tilde{\xi}$, respectively. Then, we get for a.a. $t \in (0,T)$

$$\begin{aligned}
\frac{1}{2}\|\tilde{w}'(t)\|^2_{L^2(\Omega)} + \frac{d}{dt}\left\{ \frac{1}{2}\|\tilde{w}(t)\|^2_{H^1(\Omega)} + \int_\Omega \hat{\beta}(\tilde{w}(x,t))dx + \int_\Omega \hat{G}(\tilde{w}(x,t))dx \right\} \\
\leq 2|\varphi(\tilde{e}(t))| + \frac{1}{2}C_{25}, \quad \text{hence}
\end{aligned}$$

$$\begin{aligned}
\|\tilde{w}'\|^2_{L^2(0,T;L^2(\Omega))} &\leq 4C + C_{25}T + \|w_0\|^2_{H^1(\Omega)} \\
&\quad + 2\int_\Omega \hat{\beta}(w_0(x))dx + 2\int_\Omega \hat{G}(w_0(x))dx =: \tilde{M}_C, \qquad (6.41)
\end{aligned}$$

which implies (1) and (4) of Condition 4. Next, from (6.41) we get

$$\sup_{\tilde{e} \in \mathcal{U}_C(e_0,w_0)} \left\{ \sup_{t \in [0,T]} \int_t^{\min\{t+\delta, T\}} \|\tilde{w}'(s)\|_{L^2(\Omega)} ds \right\} \leq \sqrt{\delta}\|\tilde{w}'\|_{L^2(0,T;L^2(\Omega))} \leq \sqrt{\delta \tilde{M}_C},$$

which implies that (3) of Condition 4 is satisfied.

Finally, because of $D(\varphi(\tilde{e}, \tilde{w}(t))) = L^2(\Omega)$ for all $t \in [0,T]$, we easily see that (3) of Condition 4 is satisfied. Actually, we can choose the function $h(\tilde{e}; t) = 0$ for all $t \in [0,T]$. □

In order to rewrite $(\mathrm{PF})_{e_0,w_0}$ into one of the examples of $(\mathrm{P})_{u_0,v_0}$, we show Proposition 6.10 below which definitely gives the expression of the subdifferential of $\varphi(\tilde{e}, \tilde{w}(t))$ in $(V(\tilde{w}(t)))'$.

Proposition 6.10. *The following properties are equivalent:*

(1) $\xi^* \in \partial_{\tilde{w}(t)}\varphi(\tilde{e}, \tilde{w}(t); z^*)$.

(2) $z^* \in V$ *and* $\xi^* = F_{\tilde{w}(t)}(z^* - \tilde{w}(t))$.

Proof. First of all, we show that (1) implies that (2) is satisfied. Since $F_{\tilde{w}(t)}$ is one-to-one from V onto V', we see that there exists a unique element $\bar{z} \in V$ such that $\xi^* = F_{\tilde{w}(t)}\bar{z}$. Using the definition of $\partial_{\tilde{w}(t)}\varphi(\tilde{e}, \tilde{w}(t))$, we have $z^* \in L^2(\Omega)$ and

$$\begin{aligned}(y^* - z^*, \bar{z})_{L^2(\Omega)} &= \langle y^* - z^*, \bar{z}\rangle = (F_{\tilde{w}(t)}\bar{z}, y^* - z^*)_{\tilde{w}(t)} \\ &= (\xi^*, y^* - z^*)_{\tilde{w}(t)} \le \varphi(\tilde{e}, \tilde{w}(t); y^*) - \varphi(\tilde{e}, \tilde{w}(t); z^*), \quad \forall\, y^* \in L^2(\Omega).\end{aligned}$$

Substituting $y^*+z^* \in L^2(\Omega)$ as y^* in the above inequality, we get for all $y^* \in L^2(\Omega)$

$$\int_\Omega \Big[-\frac{1}{2}\{y^*(x) + z^*(x) - \bar{z}(x) + \tilde{w}(x,t)\}^2 + \frac{\{z^*(x) - \bar{z}(x) + \tilde{w}(x,t)\}^2}{4}\Big] dx \le 0. \tag{6.42}$$

From (6.42) we have $\bar{z} = z^* - \tilde{w}(t)$ a.e. in Ω, so, $\bar{z} = z^* - \tilde{w}(t)$ in V. Because of $\bar{z} \in V$ as well as $\tilde{w}(t) \in H^1(\Omega)$, we have $z^* \in V$.

Conversely, we show that (2) implies that (1) is satisfied. From $z^* \in V$ we have

$$\begin{aligned}(\xi^*, y^* - z^*)_{\tilde{w}(t)} &= (F_{\tilde{w}(t)}(z^* - \tilde{w}(t)), y^* - z^*)_{\tilde{w}(t)} = \langle y^* - z^*, z^* - \tilde{w}(t)\rangle \\ &= (y^* - z^*, z^* - \tilde{w}(t))_{L^2(\Omega)} \\ &= (y^* - \tilde{w}(t), z^* - \tilde{w}(t))_{L^2(\Omega)} - \|z^* - \tilde{w}(t)\|^2_{L^2(\Omega)} \\ &\le \varphi(\tilde{e}, \tilde{w}(t); y^*) - \varphi(\tilde{e}, \tilde{w}(t); z^*), \quad \forall\, y^* \in L^2(\Omega),\end{aligned}$$

which implies that (1) is satisfied. □

By Proposition 6.10 we can rewrite $(\mathrm{PF})_{e_0,w_0}$ as the following Cauchy problem of an evolution inclusion with a quasi-variational structure for inner products of V':

$$\begin{cases} e'(t) + \partial_{w(t)}\varphi(e, w(t); e(t)) \ni f^*(t) \quad \text{in } (V(w(t)))', \text{ a.a. } t \in (0,T), \\ w(t) = S(e,t,0)w_0 \quad \text{in } L^2(\Omega), \ \forall\, t \in [0,T], \\ e(0) = e_0 \quad \text{in } (V(w_0))'. \end{cases}$$

Theorem 6.1 is a direct consequence, which is obtained by applying Theorem 5.1, except for the regularities $e \in L^2(0,T;V)$ and $w \in L^2(0,T;H^2(\Omega))$. But these regularities are obtained from (6.20) and (6.22).

References

[1] H. Brézis: *Opérateurs Maximaux Monotones et Semi-Groupes de Contractions dans les Espaces de Hilbert*, North-Holland, Amsterdam (1973).

[2] G. Caginalp: *An analysis of a phase field model of a free boundary*, Arch. Rat. Mech. Anal. 92 (1986) 205–245.

[3] M. G. Crandall, A. Pazy: *Semigroups of nonlinear contractions and dissipative sets*, J. Funct. Analysis 3 (1969) 376–418.

[4] A. Damlamian: *Non linear evolution equations with variable norms*, Thesis, Harvard University, Cambridge (1974).

[5] G. J. Fix: *Phase field methods for free boundary problems*, in: *Free Boundary Problems*, Pitman Research Notes in Mathematics 79 (1983) 580–589.

[6] R. Kano: *The existence of solutions for tumor invasion models with time and space dependent diffusion*, Discrete Contin. Dyn. Syst. Ser. S 7 (2014) 63–74.

[7] R. Kano, A. Ito: *The existence of time global solutions for tumor invasion models with constraints*, in: *Dynamical Systems, Differential Equations and Applications*, 8th AIMS Conference, Discrete Contin. Dyn. Syst., Suppl. Vol. II (2011) 774–783.

[8] R. Kano, A. Ito: *Tumor invasion model with a random motility of tumor cells depending upon extracellular matrix*, Adv. Math. Sci. Appl. 23 (2013) 397–411.

[9] R. Kano, N. Kenmochi, Y. Murase: *Nonlinear evolution equations generated by subdifferentials with nonlocal constraints*, in: *Nonlocal and Abstract Parabolic Equations and their Applications*, Banach Center Publication 86, P. B. Mucha, M. Niezgódka, P. Rybka (eds.), Polish Academy of Sciences, Institute of Mathematics, Warsaw (2009) 175–194.

[10] N. Kenmochi: *Solvability of nonlinear evolution equations with time-dependent constraints and applications*, Bull. Fac. Education, Chiba Univ. 30 (1981) 1–87.

[11] U. Mosco: *Convergence of convex sets and solutions of variational inequalities*, Advances Math. 3 (1969) 510–585.

[12] E. Zeidler: *Nonlinear Functional Analysis and its Applications. II/B: Nonlinear Monotone Operators*, Springer, New York (1990).