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Zero-Scale Asymptotic Functions and Quasiconvex Optimization

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We introduce the notion of a zero-scale asymptotic function. In contrast to the usual asymptotic function, which is related to the slopes of a function at infinity along a given direction, the new function is related to the jumps of the function along that direction. Applications are given to the unconstrained and the constrained optimization of quasiconvex functions. Also, the problem of quasiconvex maximization is discussed. Further, a class of quasiconvex problems is introduced, that is shown to have zero duality gap. Finally, new results on quasiconvex quadratic programming are obtained.

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1. Introduction

Asymptotic analysis means the study of the behavior of sets or functions at infinity. So, it is suitable when dealing with problems lacking coercivity or involving unbounded sets, and also when standard convexity assumptions are avoided, since properties of local nature are not global.

Usually, the asymptotic behavior of a function is described via its slopes corresponding to a sequence of points approaching infinity, when the directions of the points approach a direction v. To be more precise, the asymptotic function f^∞ of

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a function f is given by the formula (see [1])

$$\begin{aligned} f^\infty(v) &= \inf\left\{\liminf_{k\to+\infty} \frac{f(x_k)}{t_k} : t_k \to +\infty,\ \frac{x_k}{t_k} \to v\right\} \\ &= \inf\left\{\liminf_{k\to+\infty} \frac{f(x_k)-f(x)}{t_k} : t_k \to +\infty,\ \frac{x_k - x}{t_k} \to v\right\} \end{aligned}$$

where x is any point in $\operatorname{dom} f$. If f is convex and lsc, it is enough to consider slopes along a single line with direction v, that is,

$$f^\infty(v) = \lim_{t\to+\infty} \frac{f(x+tv)-f(x)}{t} = \sup_{t>0} \frac{f(x+tv)-f(x)}{t}.$$

For nonconvex functions, the following q-asymptotic function has some important advantages [12] (already introduced in [13]):

$$f_q^\infty(v) = \sup_{x\in\operatorname{dom} f} \sup_{t>0} \frac{f(x+tv)-f(x)}{t}. \tag{1}$$

In a sense, in the above formulas, the asymptotic behavior of f is compared to the asymptotic behavior of t. If f grows at infinity faster that αt for all $\alpha \in \mathbb{R}$, for example if $f(x) = \|x\|^2$, then $f^\infty(v) = +\infty$ for $v \neq 0$. Likewise, if f is nonnegative but grows slower than αt for all $\alpha \in \mathbb{R}$, such as $\sqrt{\|x\|}$, then $f^\infty(v) = 0$. In such cases, the estimates given by f^∞ are rather rough, and some information is lost. It can then be fruitful to compare f not with t, but rather with t^α for some $\alpha \geq 0$, using scaled "slopes" like $(f(x+tv)-f(x))/t^\alpha$. This approach was followed in [17], see also [10] to study nonlinear multivalued complementarity problems.

In this paper, we analyze the asymptotic behavior of functions by taking $\alpha = 0$. That is, instead of the slopes, we consider the jumps of the function along a direction, starting at any point of its domain. This is the reason the name "zero-scale" was chosen. We will see that properties such as boundedness or being nonincreasing can be described via our zero-scale asymptotic function. This will allow us to introduce some special classes of well-behaved functions, and derive results regarding, for instance, the closedness of the image of a closed convex set, that improve similar results that appear in the literature. Since we are interested mainly for nonconvex functions, our starting point will be a formula inspired from f_q^∞ rather than f^∞.

The outline of the paper is as follows. Section 2 introduces the notation and basic definitions. In Section 3 the new notion of zero-scale asymptotic function is introduced, and its main properties are presented. Relationships with the notions of qx-asymptotic function introduced in [19] and f_q^∞ mentioned above, are established in Section 3. Section 4 applies the results in previous sections to discuss the existence of minima for unconstrained and constrained quasiconvex minimization problems, along with a discussion of the problem of maximizing quasiconvex functions. Section 5 introduces a new class of quasiconvex problems having zero duality gap, and Section 6 specializes the situation to the quadratic quasiconvex objective function, and yields new results.

2. Notations and basic definitions

Throughout this paper the underlying space will be finite dimensional, even if some of the results remain valid in any normed vector space. As usual in convex analysis, given a function $f\colon \mathbb{R}^n \to \overline{\mathbb{R}} \doteq \mathbb{R}\cup\{\pm\infty\}$, we use the following notation:

$$\text{dom } f \doteq \{x \in \mathbb{R}^n : f(x) < +\infty\}, \quad \text{epi } f \doteq \{(x,t) \in \mathbb{R}^n \times \mathbb{R} : f(x) \leq t\},$$

$$S_\lambda(f) \doteq \{x \in \mathbb{R}^n : f(x) \leq \lambda\}, \;\; \hat{S}_\lambda(f) \doteq \{x \in \mathbb{R}^n : f(x) < \lambda\}.$$

Given $x, y \in \mathbb{R}^n$, $]x, y[$ stands for the open segment joining x and y, i.e., without the end points x and y. In case $n = 1$, that denotes the open interval.

We say that $f\colon \mathbb{R}^n \to \overline{\mathbb{R}}$ is:

(i) *convex* if epi f is a convex set in $\mathbb{R}^n\times\mathbb{R}$, i.e., for all x, y satisfying $f(x) < +\infty$, $f(y) < +\infty$, one has

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) \;\; \forall\, t \in \;]0,1[.$$

(ii) *quasiconvex* if for every $x, y \in \mathbb{R}^n$, one has $f(z) \leq \max\{f(x), f(y)\}$ for all $z \in \;]x, y[$. In other words, if each sublevel set, $S_\lambda(f)$, is convex for all $\lambda \in \mathbb{R}$, or equivalently, $\hat{S}_\lambda(f)$ is convex for all $\lambda \in \mathbb{R}$.

(iii) *semistrictly quasiconvex* if for every $x, y \in \mathbb{R}^n$ satisfying $f(x) \neq f(y)$, one has $f(z) < \max\{f(x), f(y)\}$ for all $z \in \;]x, y[$.

Under quasiconvexity on f, we deduce that dom f is convex since

$$\text{dom } f = \bigcup_{k\in\mathbb{N}} \Big\{x \in \mathbb{R}^n : \;\, f(x) \leq k\Big\}$$

and $S_{\lambda_1}(f) \subseteq S_{\lambda_2}(f)$ whenever $\lambda_1 \leq \lambda_2$.

We define the *asymptotic cone* of $C \subseteq \mathbb{R}^n$ as the closed cone

$$C^\infty = \Big\{v \in \mathbb{R}^n : \;\exists\; t_k \to +\infty, \exists\; x_k \in C, \;\frac{x_k}{t_k} \to v\Big\},$$

with the convention that $\emptyset^\infty = \emptyset$. When C is closed and convex, one obtains

$$C^\infty = \bigcap_{t>0} t(C - \bar{x}), \tag{2}$$

which is independent of $\bar{x} \in C$. For some of its properties, we refer to [1].

Another notion already mentioned at the introduction is the asymptotic function. It is defined as that function $f^\infty\colon \mathbb{R}^n \to \overline{\mathbb{R}}$ satisfying epi $f^\infty = (\text{epi } f)^\infty$. In case f is convex and lsc, it is known that the following representation holds:

$$f^\infty(v) = \sup_{t>0} \frac{f(\bar{x} + tv) - f(\bar{x})}{t}, \;\; (\bar{x} \in \text{dom } f).$$

Again, for a detailed description of such notions and its uses, we refer to the monograph [1].

3. The zero-scale asymptotic function

3.1. Basic properties

Let $f\colon \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$. We define the *zero-scale asymptotic function*

$$f^{\infty}_{0,q}(v) = \sup_{x \in \operatorname{dom} f} \sup_{t>0} \left(f(x+tv) - f(x)\right). \tag{3}$$

This value, $f^{\infty}_{0,q}(v)$, measures the largest jump of f along the direction v. A simple remark coming from the very definition is:

$$\operatorname{dom} f + t \operatorname{dom} f^{\infty}_{0,q} \subseteq \operatorname{dom} f, \ \forall\, t \geq 0. \tag{4}$$

Thus $\operatorname{dom} f^{\infty}_{0,q} \subseteq (\operatorname{dom} f)^{\infty}$.

The following proposition collects some of the basic properties of the previous definition.

Proposition 3.1. *Let $f\colon \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a proper function. The following assertions hold.*

(a) *$f^{\infty}_{0,q}$ is positively homogeneous of degree zero, i.e., $f^{\infty}_{0,q}(\lambda v) = f^{\infty}_{0,q}(v)$ for all $\lambda > 0$ and all $v \in \mathbb{R}^n$, and so* dom $f^{\infty}_{0,q}$ *is a cone;*

(b) *$f^{\infty}_{0,q}(v) \geq 0$ for all $v \in \mathbb{R}^n$; therefore we may replace $t > 0$ by $t \geq 0$ in* (3)*;*

(c) *$f^{\infty}_{0,q}$ is subadditive, i.e., for every $v_1, v_2 \in \mathbb{R}^n$,*

$$f^{\infty}_{0,q}(v_1 + v_2) \leq f^{\infty}_{0,q}(v_1) + f^{\infty}_{0,q}(v_2).$$

Hence, the sets $\operatorname{dom} f^{\infty}_{0,q}$ *and* $R_q \doteq \left\{v \in \mathbb{R}^n : f^{\infty}_{0,q}(v) = 0\right\}$ *are convex cones;*

(d) *if f lsc then $f^{\infty}_{0,q}$ is lsc too, and*

$$\sup_{t>0} f(x+tv) = \sup_{t\geq 0} f(x+tv), \ \forall\, x \in \operatorname{dom} f, \ \forall\, v \in \mathbb{R}^n.$$

(e) *if f is quasiconvex then $f^{\infty}_{0,q}$ is quasiconvex too.*

(f) *if f is lsc and convex then $f^{\infty}_{0,q}(v) \in \{0, +\infty\}$ for all $v \in \mathbb{R}^n$.*

(g) *if $0 < f^{\infty}_{0,q}(v) < +\infty$ for some $v \in \mathbb{R}^n$, then the restriction of $f^{\infty}_{0,q}$ on the half line $\mathbb{R}_+ v$ is not convex.*

Proof. (a) is straightforward.

(b): We only need to check for $v \in \operatorname{dom} f^{\infty}_{0,q}$. Fix $x_0 \in \operatorname{dom} f$. Given $m \in \mathbb{N}$, define $x_k = x_0 + \frac{k}{m} v$, $k = 1, 2, \ldots, m$. Then for all k, (4) implies that $x_k \in \operatorname{dom} f$ and

$$f(x_{k+1}) - f(x_k) = f(x_k + \frac{1}{m} v) - f(x_k) \leq f^{\infty}_{0,q}(v).$$

Adding these inequalities for all $k = 0, 1, \ldots, m-1$, we obtain $f(x_0 + v) - f(x_0) \leq m f^{\infty}_{0,q}(v)$. This holds for all $m \in \mathbb{N}$, so it implies that $f^{\infty}_{0,q}(v) \geq 0$.

(c): Let $v_1, v_2 \in \mathbb{R}^n$. It is enough to assume that $f_{0,q}^{\infty}(v_1) < +\infty$. Then for each $x \in \operatorname{dom} f$, $t > 0$ one has $x + tv_1 \in \operatorname{dom} f$. We deduce

$$\begin{aligned}
f\left(x + t\left(v_1 + v_2\right)\right) - f\left(x\right) &= f\left(\left(x + tv_1\right) + tv_2\right) - f(x + tv_1) + f(x + tv_1) - f(x) \\
&\leq \sup_{\substack{x' \in \operatorname{dom} f \\ t > 0}} \left(f(x' + tv_2) - f(x')\right) + \sup_{\substack{x' \in \operatorname{dom} f \\ t > 0}} \left(f(x' + tv_1) - f(x')\right) \\
&= f_{0,q}^{\infty}(v_2) + f_{0,q}^{\infty}(v_1).
\end{aligned}$$

Taking the supremum over $x \in \operatorname{dom} f$ and $t > 0$ we obtain subadditivity.

Since $\operatorname{dom} f_{0,q}^{\infty}$ is a cone such that $v_1, v_2 \in \operatorname{dom} f_{0,q}^{\infty}$ implies $v_1 + v_2 \in \operatorname{dom} f_{0,q}^{\infty}$ by subadditivity, we deduce that $\operatorname{dom}\ f_{0,q}^{\infty}$ is convex. The same holds for R_q.

(d): This follows since the supremum of lsc functions is also lsc. The last part is direct.

(e): This is similar to that given in [13, Proposition 3.28(b)], since the supremum of quasiconvex functions is also quasiconvex.

(f): This results from

$$\begin{aligned}
f_{0,q}^{\infty}(v) &= \sup_{t \geq 0} \sup_{x \in \operatorname{dom} f} \left(f(x + tv) - f(x)\right) = \sup_{t \geq 0} f^{\infty}(tv) \\
&= \sup_{t \geq 0}\ t f^{\infty}(v) \in \{0, +\infty\}.
\end{aligned} \tag{5}$$

(g): By using $f_{0,q}^{\infty}(0) = 0$ and $f_{0,q}^{\infty}(tv) = f_{0,q}^{\infty}(v)$ for all $t > 0$, we deduce

$$f_{0,q}^{\infty}(v) = f_{0,q}^{\infty}(tv) > \frac{1}{2} f_{0,q}^{\infty}(0) + \frac{1}{2} f_{0,q}^{\infty}(2tv) = \frac{1}{2} f_{0,q}^{\infty}(v),$$

showing that $f_{0,q}^{\infty}$ is non convex on $\mathbb{R}_+ v$. □

We point out some remarks concerning the previous proposition.

Remark 3.2. (i) From (g) of the preceding proposition $f_{0,q}^{\infty}$ is never convex, unless $f_{0,q}^{\infty}(v) \in \{0, +\infty\}$ for all $v \in \mathbb{R}^n$. Example 3.3 below shows that $f_{0,q}^{\infty}$ is not necessarily quasiconvex.

(ii) Note that a function g which is quasiconvex, homogeneous of degree zero and nonnegative, is subadditive. Indeed,

$$g(v_1 + v_2) = g\left(\frac{v_1 + v_2}{2}\right) \leq \max\left\{g(v_1), g(v_2)\right\} \leq g(v_1) + g(v_2).$$

But we do not need this result for $g = f_{0,q}^{\infty}$ since it is always subadditive, whereas for quasiconvexity of $f_{0,q}^{\infty}$ we need to assume that f is quasiconvex. □

In spite of Proposition 3.1, we show that $f_{0,q}^{\infty}$ is in general non quasiconvex.

Example 3.3. Consider the function $f\colon \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x_1, x_2) = \begin{cases} 0, & \text{if } (x_1, x_2) = (0,0); \\ 2, & \text{if } x_1 > 0, x_2 = 0; \\ 1, & \text{otherwise.} \end{cases}$$

It is easy to check that $f^\infty_{0,q} = f$, so $f^\infty_{0,q}$ is not quasiconvex. As expected, it is subadditive and homogeneous of degree zero. □

The next example shows that, in general, the reverse implication of (f) is not true.

Example 3.4. The quasiconvex and lsc function

$$f(x) = \begin{cases} \dfrac{1}{1+x^2}, & \textit{if } x \geq 0; \\ +\infty, & \textit{if } x < 0, \end{cases}$$

satisfies $f^\infty_{0,q}(v) = 0 \ \forall\, v \geq 0;\ f^\infty_{0,q}(v) = +\infty, \ \ \forall\, v < 0.$ □

The next remark shows the geometrical interpretation of $f^\infty_{0,q}$ at v (see Figure 3.1).

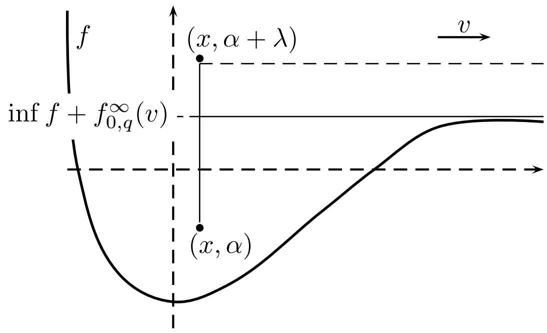

Figure 3.1: If from a point (x, α) of the epigraph we go up by λ, then we can move along v by staying always in the epigraph. The value of $f^\infty_{0,q}(v)$ is the smallest λ for which this can be done for all $(x, \alpha) \in \operatorname{epi} f$.

Remark 3.5. We check the epigraph of $f^\infty_{0,q}$: For $\lambda \in \mathbb{R}$,

$$\begin{aligned}
(v, \lambda) \in \operatorname{epi} f^\infty_{0,q} \iff\ & f^\infty_{0,q}(v) \leq \lambda; \\
\iff\ & \forall\, x \in \operatorname{dom} f,\ \forall\, t \geq 0, f(x+tv) \leq f(x) + \lambda; \\
\iff\ & \forall\, x \in \operatorname{dom} f,\ \forall\, t \geq 0, (x+tv, f(x)+\lambda) \in \operatorname{epi}\ f; \\
\iff\ & \forall\, x \in \operatorname{dom} f,\ \forall\, t \geq 0, (x, f(x)+\lambda) + t(v,0) \in \operatorname{epi}\ f; \\
\iff\ & \forall\, (x, \alpha) \in \operatorname{epi} f,\ \forall\, t \geq 0,\ \ (x+tv, \alpha+\lambda) \in \operatorname{epi}\ f \\
\iff\ & \forall\, (x, \alpha) \in \operatorname{epi} f,\ \forall\, t \geq 0, (x, \alpha+\lambda) + t\,(v,0) \in \operatorname{epi} f.
\end{aligned}$$

As a consequence of the previous remark, we conclude

$$f^\infty_{0,q}(v) = \inf \Big\{ \lambda \geq 0 : \forall\, x \in \operatorname{dom} f,\ \forall\, t \geq 0,\ (x, f(x)+\lambda) + t(v,0) \in \operatorname{epi}\ f \Big\}.$$

We also recall the definition of $f^{qx}\colon \mathbb{R}^n \to \overline{\mathbb{R}}$ for every lsc quasiconvex function f, introduced in [19], which is

$$f^{qx}(v) \doteq \inf\{\lambda \in \mathbb{R} :\ v \in (S_\lambda(f))^\infty\}, \tag{6}$$

where we adopt the convention, as before, that $(\emptyset)^\infty = \emptyset$.

For any $\lambda \in \mathbb{R}$ such that $S_\lambda(f) \neq \emptyset$, we get

$$S_\lambda(f^{qx}) = \bigcap_{\gamma>\lambda}(S_\gamma(f))^\infty = \left(\bigcap_{\gamma>\lambda} S_\gamma(f)\right)^\infty = (S_\lambda(f))^\infty. \tag{7}$$

Thus, from (7), f^{qx} is lsc and quasiconvex as well. In addition, f^{qx} is positively homogeneous of degree zero. The following properties are taken from [19]:

$$f^{qx}(v) = \inf_{x\in\mathbb{R}^n} \sup_{t\geq 0} f(x+tv) \tag{8}$$

and

$$f^{qx}(v) = \inf_{x\in\mathbb{R}^n} \lim_{t\to+\infty} f(x+tv) \tag{9}$$

From (8), one deduces that, in contrast to our definition of $f^\infty_{0,q}$, $f^{qx}(v)$ measures the smallest height of f along the direction v; we also obtain

$$f^{qx}(0) = \inf_{x\in\mathbb{R}^n} f(x) = \inf_{v\in\mathbb{R}^n} f^{qx}(v). \tag{10}$$

On the other hand, assuming that $S_\lambda(f) \neq \emptyset$, from Proposition 4.6 in [13] we have

$$\begin{aligned}
(S_\lambda(f))^\infty &= \bigcap_{x\in\mathbb{R}^n}\Big\{v \in \mathbb{R}^n :\ f(x+tv) \leq \max\{f(x),\lambda\},\ \forall\, t>0\Big\} && (11)\\
&= \Big\{v \in \mathbb{R}^n :\ \sup_{x\in S_\lambda(f)} \sup_{t>0} f(x+tv) \leq \lambda\ \Big\} && (12)\\
&= \Big\{v \in \mathbb{R}^n :\ \sup_{t>0} f(x+tv) \leq \lambda\ \Big\}\ \ (x \in S_\lambda(f)). && (13)
\end{aligned}$$

Take any $v \in (S_\lambda(f))^\infty = S_\lambda(f^{qx})$ with $f^{qx}(v) \in \mathbb{R}$. Since for all $x' \in S_\lambda(f)$

$$f^{qx}(v) = \inf_{x\in\mathbb{R}^n} \sup_{t>0} f(x+tv) \leq \sup_{t>0} f(x'+tv) \leq \sup_{x\in S_\lambda(f)} \sup_{t>0} f(x+tv) \leq \lambda, \tag{14}$$

we get for all x satisfying $f(x) \leq f^{qx}(v)$ and for $\lambda = f^{qx}(v)$

$$f^{qx}(v) = \sup_{t>0} f(x+tv) = \sup_{x\in S_\lambda(f)} \sup_{t>0} f(x+tv). \tag{15}$$

This is part of the next result.

Proposition 3.6. *Let $f\colon \mathbb{R}^n \to \mathbb{R}\cup\{+\infty\}$ be quasiconvex, lsc and proper. Then, for each $x \in \operatorname{dom} f$, $v \in (\operatorname{dom}\ f)^\infty$,*

$$\sup_{t\geq 0}\left(f(x+tv) - f(x)\right) = \max\{0, f^{qx}(v) - f(x)\}. \tag{16}$$

Consequently, if $\sup_{t\geq 0}(f(x_0+tv) - f(x_0)) = +\infty$ for some $x_0 \in \operatorname{dom} f$, then

$$\sup_{t\geq 0}\left(f(x+tv) - f(x)\right) = +\infty,\quad \forall\, x \in \operatorname{dom}\ f.$$

Proof. For each $x \in \operatorname{dom} f$ we consider three possibilities:

(a) If $f(x) \geq f^{qx}(v)$, then $v \in S_{f(x)}(f^{qx}) = \big(S_{f(x)}(f)\big)^{\infty}$. Since $x \in S_{f(x)}(f)$ we deduce that $x + tv \in S_{f(x)}(f)$ for all $t \geq 0$, i.e., $f(x+tv) \leq f(x)$. Therefore, $\sup_{t\geq 0} f(x+tv) \leq f(x)$. On the other hand, obviously $\sup_{t\geq 0} f(x+tv) \geq f(x)$, so $\sup_{t\geq 0} f(x+tv) = f(x)$. Hence

$$f(x) \geq f^{qx}(v) \Rightarrow \sup_{t\geq 0} f(x+tv) = f(x). \tag{17}$$

Thus in this case $\sup_{t\geq 0}(f(x+tv) - f(x)) = 0 = \max\{0, f^{qx}(v) - f(x)\}$.

(b) If $f(x) \leq f^{qx}(v) < +\infty$, then (15) implies

$$\sup_{t\geq 0}(f(x+tv) - f(x)) = f^{qx}(v) - f(x) = \max\{0, f^{qx}(v) - f(x)\}.$$

(c) If $f^{qx}(v) = +\infty$, from (8) we get immediately $\sup\limits_{t\geq 0} f(x+tv) = +\infty$. □

The proposition shows that the "jump" $\sup_{t\geq 0}(f(x+tv) - f(x))$ of f from x in a given direction v, depends only on $f(x)$. In fact this jump is equal to 0 for all x such that $f(x) \geq f^{qx}(v)$; and it is equal to $f^{qx}(v) - f(x)$ for all x such that $f(x) \leq f^{qx}(v)$.

We have the immediate corollary:

Corollary 3.7. *Let $f\colon \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be quasiconvex, lsc and proper. Then, for each $v \in (\operatorname{dom} f)^{\infty}$ and each $x, y \in \operatorname{dom} f$ such that $f(y) \leq f(x)$, one has*

$$\sup_{t\geq 0}(f(y+tv) - f(y)) \geq \sup_{t\geq 0}(f(x+tv) - f(x)). \tag{18}$$

If $f(y) < f(x)$ then the inequality (18) is also strict, unless either $f(y) \geq f^{qx}(v)$, in which case both sides of (18) are equal to 0, or $f^{qx}(v) = +\infty$, in which case both sides are equal to $+\infty$.

It follows that the smaller is $f(x)$, the larger is $\sup_{t\geq 0}(f(x+tv) - f(x))$.

Corollary 3.8. *Let $f\colon \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be lsc, quasiconvex and proper. Then, for every $v \in (\operatorname{dom} f)^{\infty}$ with $0 < f^{\infty}_{0,q}(v) < +\infty$,*

$$\operatorname*{argmin}_{\mathbb{R}^n} f = \operatorname*{argmax}_{\operatorname{dom} f} \left(\sup_{t\geq 0}(f(x+tv) - f(x))\right). \tag{19}$$

Proof. Define a function $g\colon \operatorname{dom} f \to [0, +\infty]$ by $g(x) = \sup_{t\geq 0}(f(x+tv) - f(x))$. If $x \in \operatorname{argmin} f$, then for every $y \in \operatorname{dom} f$ we have $f(x) \leq f(y)$ so by Corollary 3.7, $g(x) \geq g(y)$. Thus, $x \in \operatorname{argmax}_{\operatorname{dom} f} g$.

Conversely, let $x \in \operatorname{argmax}_{\operatorname{dom} f} g$. Assume that $x \notin \operatorname{argmin} f$. Then there exists y such that $f(y) < f(x)$. Given that $g(x) = \sup_{\operatorname{dom} f} g = f^{\infty}_{0,q}(v) \neq \{0, +\infty\}$, Corollary 3.7 implies $g(y) > g(x)$. This contradicts $x \in \operatorname{argmax}_{\operatorname{dom} f} g$. Thus, $x \in \operatorname{argmin} f$. □

Remark 3.9. If f is convex, then according to (5), $f^{\infty}_{0,q}(v) \in \{0, +\infty\}$. Thus (19) does not hold in general for convex functions.

The following proposition connects $f^{\infty}_{0,q}$ with f^{qx}.

Proposition 3.10. *Let $f\colon \mathbb{R}^n \to \mathbb{R}\cup\{+\infty\}$ be lsc, quasiconvex and proper. Then*

$$f^{\infty}_{0,q}(v) = f^{qx}(v) - f^{qx}(0) \tag{20}$$

whenever $f^{qx}(v)$ and $\inf f$ are not both equal to $-\infty$. However, when we have $f^{qx}(v) = f^{qx}(0) = \inf f = -\infty$, then $f^{\infty}_{0,q}(v) = 0$.

Proof. If $f^{qx}(v) = -\infty$, then necessarily $\inf f = -\infty$ because $\inf f = \inf f^{qx}$ (see (10)). For each $x \in \operatorname{dom} f$ the right-hand side of (16) is 0, so (3) gives that $f^{\infty}_{0,q}(v) = 0$. If $f^{qx}(v) > -\infty$, then $\inf f = \inf f^{qx} \leq f^{qx}(v)$. So (16) gives

$$\begin{aligned} f^{\infty}_{0,q}(v) &= \sup_{x\in\operatorname{dom} f}\ \sup_{t\geq 0}\,(f(x+tv) - f(x)) = \sup_{x\in\operatorname{dom} f}\ \max\{0, f^{qx}(v) - f(x)\} \\ &= \max\{0, f^{qx}(v) - \inf f\} = f^{qx}(v) - \inf f = f^{qx}(v) - f^{qx}(0). \end{aligned}$$ □

The proposition has some interesting consequences:

Proposition 3.11. *Let $f\colon \mathbb{R}^n \to \mathbb{R}\cup\{+\infty\}$ be lsc, quasiconvex and proper. Then, f is bounded if and only if $f^{\infty}_{0,q}$ is finite-valued.*

Proof. If f is bounded, then the definition of $f^{\infty}_{0,q}$ implies that $f^{\infty}_{0,q}(v)$ is real for all $v \in \mathbb{R}^n$. Conversely, assume that $f^{\infty}_{0,q}(v) \in \mathbb{R}$ for all $v \in \mathbb{R}^n$. We consider two cases:

(a): If $\inf f > -\infty$, then Proposition 3.10 shows that f^{qx} is real-valued. This implies that f is bounded by Proposition 4.2 in [19].

(b): If $\inf f = -\infty$, then again according to Proposition 3.10, for all $v \in \mathbb{R}^n$, $f^{\infty}_{0,q}(v)$ is finite only if $f^{\infty}_{0,q}(v) = 0$. The definition of $f^{\infty}_{0,q}(v)$ then implies that for all $x \in \operatorname{dom} f$ and all $v \in \mathbb{R}^n$, $t \geq 0$, one has $f(x+tv) \leq f(x)$. Therefore for all $y \in \mathbb{R}^n$,

$$f(y) = f(x + (y-x)) \leq f(x).$$

This means that f is constant, and contradicts $\inf f = -\infty$. Consequently, f is bounded. □

The following proposition is an immediate consequence of Proposition 3.10, together with Proposition 4.3 of [19]. Compare also with Theorem 8.6 of [27].

Proposition 3.12. *Let $f\colon \mathbb{R} \to \mathbb{R}\cup\{+\infty\}$ be quasiconvex, lsc and proper, and $v \in \mathbb{R}^n$, $v \neq 0$. The following conditions are equivalent:*

(a) $f^{\infty}_{0,q}(v) = 0$;

(b) *for all $x \in \operatorname{dom} f$, the function $t \mapsto f(x+tv)$, $t \geq 0$ is nonincreasing.*

3.2. An earlier relative to the zero-scale asymptotic function

A close asymptotic notion was introduced in [13], but it goes back to [9]. The q-asymptotic function $f_q^\infty\colon \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is defined by

$$f_q^\infty(v) = \sup_{\substack{x\in\mathrm{dom}\ f\\ t>0}} \frac{f(x+tv)-f(x)}{t}, \quad \forall\, v \in \mathbb{R}^n. \tag{21}$$

Its use and importance are outlined in the next section. Certainly, as in the case of $f_{0,q}^\infty$, one has

$$\mathrm{dom}\ f + t\ \mathrm{dom}\ f_q^\infty \subseteq \mathrm{dom}\ f,\ \forall\, t \geq 0. \tag{22}$$

Thus $\mathrm{dom}\ f_q^\infty \subseteq (\mathrm{dom}\ f)^\infty$. Observe also that $f_q^\infty = f^\infty$ whenever f is convex and lsc.

Remark 3.13. (i) One can easily check that, whenever a function f is proper

$$f_q^\infty(v) \leq 0 \Longleftrightarrow f_{0,q}^\infty(v) = 0, \tag{23}$$

and, in case f is bounded from below, then

$$f_q^\infty(v) = 0 \Longleftrightarrow f_{0,q}^\infty(v) = 0.$$

(ii) From Proposition 3.8 in [19], f_q^∞ is convex, and it is lsc whenener f is so (see [13, Proposition 3.28(a)]).

4. Quasiconvex optimization

This section concerns the study of quasiconvex optimization problems: first without constraints, and secondly under a convex, closed constraint set. Finally, a formula for the set of maxima of quasiconvex functions on the constrained set, is provided.

4.1. Unconstrained minimization

We now deal with the problem of minimizing a quasiconvex and lsc function $f\colon \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ with $\mathrm{dom}\ f \neq \emptyset$:

$$\min_{x\in\mathbb{R}^n} f(x). \tag{24}$$

The analysis will be based on the notion of zero-scale asymptotic function and on the results of previous sections, so no coercivity assumption will be imposed.

Set
$$R_q \doteq \{v \in \mathbb{R}^n :\ f_{0,q}^\infty(v) = 0\} = \{v \in \mathbb{R}^n :\ f_q^\infty(v) \leq 0\}, \tag{25}$$

where the last equality comes from (23). This set is closed and convex. Cones like R_q have been introduced in [9] for general equilibrium problems. See also [2].

As expected, one must be interested in studying the behaviour of f along directions belonging to R_q. Thus, the condition we impose on f along those directions is expressed in condition (C_q) below:

(C_q) if the sequence $x_k \in \operatorname{dom} f$, $\|x_k\| \to +\infty$ is such that $\frac{x_k}{\|x_k\|} \to v$ with $v \in R_q$ and for all $y \in \operatorname{dom} f$ it exists k_y such that

$$f(y) \geq f(x_k) \quad \text{when} \quad k \geq k_y,$$

then there exist u and $\bar{k}$ such that $\|u\| < \|x_{\bar{k}}\|$ satisfying $f(u) \leq f(x_{\bar{k}})$.

(C_q) is satisfied vacuously if f is coercive in the sense that all of its level sets are bounded (it is enough that $S_\lambda(f)$ is nonempty and bounded for some $\lambda > \inf f$), or, if $R_q = \{0\}$, since there are no sequences satisfying the premises of (C_q).

The next theorem (in view of (25) is [13, Theorem 4.8]) establishes a characterization of the nonemptiness of the solution set to problem (24). We provide here a rather different proof.

Theorem 4.1. *Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be quasiconvex and lsc with* $\operatorname{dom} f \neq \emptyset$. *The following assertions are equivalent:*

(a) $\operatorname{argmin}_{\mathbb{R}^n} f \neq \emptyset$*;*

(b) (C_q) *is satisfied.*

Proof. It remains to check that (b) implies (a).

(b) $\Rightarrow$ (a): for every $k \in \mathbb{N}$, set $K_k = \{x \in \mathbb{R}^n : \|x\| \leq k\}$ and suppose, without loss of generality, that $K_k \cap \operatorname{dom} f \neq \emptyset$ for all $k \in \mathbb{N}$. For every $k \in \mathbb{N}$, consider $x_k \in \operatorname{argmin}\{\|x\| : x \in A_k\}$, where $A_k \doteq \operatorname{argmin}_{K_k} f$. Let us prove that $\{x_k\}$ is bounded. If not, we may assume without loss of generality, that $\|x_k\| \to +\infty$ and $(x_k/\|x_k\|) \to v$. We now show that $v \in R_q$. Indeed, for fixed $x \in \operatorname{dom} f$, we get $f(x_k) \leq f(x)$ for all k sufficiently large. Hence $v \in (S_{f(x)}(f))^\infty = S_{f(x)}(f^{qx})$. Thus $f^{qx}(v) \leq f(x)$ for all $x \in \operatorname{dom} f$, which yields $f^{qx}(v) \leq \inf\ f$, that is, $f^\infty_{0,q}(v) = 0$ by virtue of (10) and Proposition 3.10. By applying assumption (C_q), we reach a contradiction. Hence $\{x_k\}$ is bounded, and so standard arguments show that any cluster point of $\{x_k\}$ is a minimizer for f. $\square$

When R_q is a vector subspace, the following proposition holds: Part (a) with f^∞_q instead of $f^\infty_{0,q}$, may be found in Lemma 5.1 of [12] under the extra assumption of semistrict quasiconvexity on f; whereas (b) provides a sufficient condition for $\operatorname{argmin} f$ to be nonempty and not necessarily bounded.

Proposition 4.2. *Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be proper, lsc and quasiconvex.*

(a) *Assume that $v \in \mathbb{R}^n \backslash \{0\}$. Then,*

$$f^\infty_{0,q}(v) = f^\infty_{0,q}(-v) = 0 \iff f(x + tv) = f(x) \quad \forall\, x \in \operatorname{dom} f,\ \forall\, t \in \mathbb{R}.$$

(b) *Assume that the set $R_q \doteq \{v \in \mathbb{R}^n : f^\infty_{0,q}(v) = 0\}$ satisfies $-R_q \subseteq R_q$ (or equivalently R_q is a vector subspace). Then* $\operatorname{argmin}_{\mathbb{R}^n} f \neq \emptyset$.

Proof. (a): Since $f^\infty_{0,q}(v) = f^\infty_{0,q}(-v) = 0$, from the definition of $f^\infty_{0,q}$ we deduce that $f(x + tv) \leq f(x)$ for all $t \in \mathbb{R}$, so in particular $x + tv \in \operatorname{dom} f$. Also $f(x) = f(x + tv - tv) \leq f(x + tv)$. Thus, f is constant on the line $x + \mathbb{R}v$.

(b): We check that (C_q) is satisfied. Indeed, take any sequence $x_k \in \operatorname{dom} f$ such that $(x_k/\|x_k\|) \to v \in R_q$. By (a), $f(x_k - \|x_k\|v) = f(x_k)$ for all $k \in \mathbb{N}$; and clearly $\|x_k - \|x_k\|v\| < \|x_k\|$ for all k sufficiently large. □

The case $R_q = \{0\}$ yields a more precise result. Some equivalences already appear elsewhere: the equivalence between (a) and (d) is [13, Theorem 4.7]; whereas that between (a) and (c) already appears in [19].

Theorem 4.3. *Let f be proper, quasiconvex and lsc. The following conditions are equivalent:*

(a) $\operatorname{argmin}_{\mathbb{R}^n} f$ *is nonempty and compact.*

(b) $f^{\infty}_{0,q}(v) > 0$ *for all* $v \neq 0$.

(c) $f^{qx}(v) > \inf_{\mathbb{R}^n} f$ *for all* $v \neq 0$.

(d) $f^{\infty}_{q}(v) > 0$ *for all* $v \neq 0$.

Proof. The equivalence between (a) and (d) is Theorem 4.7 in [13], which amounts to writing $R_q = \{0\}$, and therefore the equivalence between (b) and (d) follows in view of (25); whereas the remaining part comes from Proposition 3.10. □

Concerning a sufficient condition implying condition (b) of Proposition 4.2, next remark provides an answer.

Remark 4.4. It is simple to verify that the following class of quasiconvex functions $\mathcal{C}_0$ satisfies the assumption in (b) of Proposition 4.2: A quasiconvex function $f\colon \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is said to be in $\mathcal{C}_0$ if for all $v \in \mathbb{R}^n$, $v \neq 0$, either $f^{\infty}_{0,q}(v) = 0$ or $f^{\infty}_{0,q}(-v) = +\infty$. In case $f^{\infty}_{0,q}(-v) = +\infty$ we also have $f^{\infty}_{0,q}(v) = +\infty$. □

We end this subsection by stating another simple result.

Lemma 4.5. *Let f be a differentiable quasiconvex function, and $u \in \mathbb{R}^n$, $u \neq 0$. Then, $f^{\infty}_{0,q}(u) = 0$ if, and only if*

$$\sup_{x \in \mathbb{R}^n} \sup_{t \geq 0} \nabla f(x + tu)^{\top} u \leq 0.$$

Proof. Simply recall that $f^{\infty}_{0,q}(u) = 0$ if, and only if for all $x \in \operatorname{dom} f$, $0 < t \mapsto f(x + tu)$ is nonincreasing, which means that $f_x(t) \doteq f(x + tu)$ satisfies $0 \geq f'_x(t) = \nabla f(x + tu)^{\top} u$. □

4.2. Constrained minimization

Let us consider the minimization problem

$$\min_{x \in K} f(x), \tag{26}$$

where $K \subseteq \mathbb{R}^n$ is a nonempty closed and convex set, and $f\colon \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is any lsc and quasiconvex proper function, satisfying $\operatorname{dom} f \cap K \neq \emptyset$.

By denoting $f_K \doteq f + \iota_K$, where ι_K stands for the indicator function of the set K, one gets: dom $f_K = K \cap \text{dom } f$, f_K is quasiconvex, and

$$(f_K)^\infty_{0,q}(v) = \begin{cases} \sup\limits_{x \in K\cap \text{dom } f} \sup\limits_{t\geq 0}(f(x+tv) - f(x)), & \text{if } v \in (K \cap \text{dom } f)^\infty, \\ +\infty, & \text{elsewhere.} \end{cases}$$

We simply write $f^\infty_{K,0,q} = (f_K)^\infty_{0,q}$ and $f^\infty_{0,q} = f^\infty_{K,0,q}$ whenever $K = \mathbb{R}^n$. In particular,

$$R_q \doteq \Big\{ v \in \mathbb{R}^n : \ f^\infty_{K,0,q}(v) = 0 \Big\} = \Big\{ v \in K^\infty : \ f^\infty_{K,0,q}(v) = 0 \Big\}. \tag{27}$$

With this in mind, Assumption (C_q) of the previous section becomes:

(C_q) if the sequence $x_k \in K \cap \text{dom } f$, $\|x_k\| \to +\infty$ is such that $(x_k/\|x_k\|) \to v$ with $v \in R_q$ and for all $y \in K \cap \text{dom } f$ there exists k_y such that

$$f(y) \geq f(x_k) \quad \text{when} \quad k \geq k_y,$$

then there exist $u \in K \cap \text{dom } f$ and $\bar{k}$ such that $\|u\| < \|x_{\bar{k}}\|$ satisfying $f(u) \leq f(x_{\bar{k}})$.

Thus, one can write down Theorems 4.1 and 4.3, and Proposition 4.2 for our problem (26).

Looking at R_q in (27), sometimes to compute $f^\infty_{K,0,q}$ is more difficult than $f^\infty_{0,q}$. To overcome this difficulty a class of functions is needed. This class was introduced in [14] to determine closedness of images of closed and convex sets via vector functions having components in that class, under semistrict quasiconvexity; it was later employed in [12] to identify quasiconvex optimization problems having zero duality gap.

Definition 4.6. We say that the function f belongs to $\mathcal{C}$ if for all $x \in \text{dom } f$ and all $v \in (\text{dom } f)^\infty$, $v \neq 0$, one has either

(i) $0 \leq t \mapsto f(x+tv)$ is nonincreasing, or

(ii) $\lim\limits_{t\to+\infty} f(x+tv) = +\infty$.

Note that, under quasiconvexity and lower semicontinuity on f, Propositions 3.6 and 3.12 allow us to write that $f \in \mathcal{C}$ if, and only if for all $v \in (\text{dom } f)^\infty$, $v \neq 0$, either $f^\infty_{0,q}(v) = 0$ or $f^{qx}(v) = +\infty$ if, and only if $f^{qx}(v) = f^{qx}(0)$ or $f^{qx}(v) = +\infty$ (see [19, Proposition 4.4]).

Now, by using $\mathcal{C}$, we can substitute $f^\infty_{K,0,q}$ by $f^\infty_{0,q}$ in R_q.

Theorem 4.7. *Let $f\colon \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be lsc, quasiconvex and proper; K be a nonempty closed convex set in $\mathbb{R}^n$ such that* dom $f \cap K \neq \emptyset$. *Assume in addition that f belongs to $\mathcal{C}$. Set $R \doteq \{v \in K^\infty : \ f^\infty_{0,q}(v) = 0\}$.*

Then $R = R_q$. As a consequence, if $R \subseteq -R$ then $\text{argmin}_K f \neq \emptyset$.

Proof. Evidently $R \subseteq R_q$. Let $v \in K^\infty$ be such that $f^\infty_{K,0,q}(v) = 0$. Then, for each $t \geq 0$ and $x \in K \cap \text{dom}\, f$ one has (this follows from Proposition 3.1(b))

$$\sup_{t\geq 0} (f(x+tv) - f(x)) = 0.$$

It follows from the first part of Proposition 3.6 that $f^{qx}(v) \neq +\infty$. Since $f \in \mathcal{C}$, we get $f^{\infty}_{0,q}(v) = 0$, and so $u \in R$. This proves $R_q = R$. The second part is a consequence of Proposition 4.2(b) and (27). $\square$

In most applications the feasible set K is given by some inequality constraints and an additional geometric constraint set C. The requirement of the convexity and closedness of K is guaranteed by the closedness and convexity of C and the lsc and quasiconvexity of each g_i:

$$K \doteq \{x \in C : \ g_i(x) \leq 0 \ \forall \ i \in J\}, \quad J \doteq \{1, \ldots, m\}.$$

Then (see (7)) $\qquad K^{\infty} = \{v \in C^{\infty} : \ (g_i)^{qx}(v) \leq 0, \ \forall \ i \in J\}.$

With this in mind, Theorem 4.7 becomes

Corollary 4.8. *Let $f, g_i \colon \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be lsc, quasiconvex and proper for $i \in J$; K be as above such that* $\operatorname{dom} f \cap K \neq \emptyset$. *Assume in addition that f belongs to $\mathcal{C}$. If the set*

$$R \doteq \{v \in C^{\infty} : \ f^{\infty}_{0,q}(v) = 0; \ (g_i)^{qx}(v) \leq 0 \ \forall \ i \in J\} \tag{28}$$

is a vector subspace, i.e., $R \subseteq -R$, then $\operatorname{argmin}_K f \neq \emptyset$.

A set related to R is:

$$R_0 \doteq \left\{v \in C^{\infty} : \ f^{\infty}_{0,q}(v) = 0, \ (g_i)^{\infty}_{0,q}(v) = 0, \quad \forall \ i \in J\right\}, \tag{29}$$

A relation between the both sets is given by the following proposition.

Proposition 4.9. *Let $f, g_i \colon \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be lsc, quasiconvex and proper for $i \in J$; K be as above such that* $\operatorname{dom} f \cap K \neq \emptyset$. *Assume in addition that f belongs to $\mathcal{C}$. Then,*

(a) $R_0 \subseteq R$;

(b) *if additionaly each $g_i \in \mathcal{C}$ for $i \in J$, one has $R_0 = R$.*

Proof. (a): Let $v \in R_0$. Pick $x_0 \in K$ (which is nonempty by assumption). By definition of $(g_i)^{\infty}_{0,q}$, one obtains $g_i(x_0 + tv) \leq g_i(x_0)$. Hence, $\sup_{t \geq 0} g_i(x_0 + tv) = g_i(x_0) \leq 0$. By (8), $(g_i)^{qx}(v) \leq 0$ so $v \in R$.

(b): Let $v \in R$, then for all $i \in J$, $(g_i)^{qx}(v) \neq +\infty$, so necessarily $(g_i)^{\infty}_{0,q}(v) = 0$. Hence, $v \in R_0$, so $R \subseteq R_0$. Since $R_0 \subseteq R$ is always true, we infer $R_0 = R$. $\square$

We have to point out that there is not any relationship between R being a subspace and R_0 being a subspace. For example, take $C = \mathbb{R}$, $f = 1$, $m = 1$, and for $\alpha, \beta \geq -1$ set

$$g(x) = \begin{cases} \min\{|x| - 1, \alpha\} & \text{if } x \geq 0; \\ \min\{|x| - 1, \beta\} & \text{if } x < 0. \end{cases}$$

If $\alpha = 1, \beta = 0$, then $R = (-\infty, 0]$ (not a subspace) and $R_0 = \{0\}$ (a subspace).

If $\alpha = 0, \beta = -1$, then $R = \mathbb{R}$ (a subspace) while $R_0 = (-\infty, 0]$ (not a subspace).

4.3. Quasiconvex maximization

This section is motivated by the important class of problems called the fixed charge problem. It arises, roughtly speaking, in a situation which involves the planning of a large number of interdependent activities, where some (or all) of them have set-up time charges: this is a model of minimizing a concave function, see [20, 30]. However, following our setting, we will discuss the problem of maximizing a quasiconvex lower semicontinuous function, by means of similar arguments developed in previous sections. Thus, the results obtained here may be considered as supplementary to those in [13].

Proposition 4.10. *Let K be a nonempty, closed and convex subset of $\mathbb{R}^n$ and $f\colon K \to \mathbb{R}$ be a quasiconvex, lsc function.*

(a) *If $K = \mathbb{R}^n$ and $\operatorname{argmax}_{\mathbb{R}^n} f$ is non-empty, then $\operatorname{argmax}_{\mathbb{R}^n} f$ must contain a halfspace.*

(b) *If f cannot be extended to all $\mathbb{R}^n$ as a quasiconvex finite-valued function, then $\sup_{x\in K} f(x) = +\infty$.*

Proof. (a): Assume $x_0 \in \operatorname{argmax}_{\mathbb{R}^n} f$ and consider the set $A \doteq \{x : f(x) < f(x_0)\}$. So, there is $p \in \mathbb{R}^n$, $p \neq 0$, such that $\langle p, x_0\rangle \leq \langle p, x\rangle$ for all $x \in A$, which implies that $\{x : \langle p, x\rangle < \langle p, x_0\rangle\}$ is contained in $\operatorname{argmax}_{\mathbb{R}^n} f$.

(b): If on the contrary $\sup_K f = \lambda < +\infty$, we consider $\tilde{f}(x) = f(x)$ if $x \in K$, and $\tilde{f}(x) = \lambda$ elsewhere. Then $\tilde{f}$ is quasiconvex and finite-valued defined on $\mathbb{R}^n$, which cannot happen if (b) is assumed. □

In view of the previous proposition, given any function $f\colon K \to \mathbb{R}$, we shall assume $\sup_K f < +\infty$ with $K \neq \mathbb{R}^n$. Thus, the next result arises.

Proposition 4.11. *Let K be a nonempty, closed and convex subset of $\mathbb{R}^n$ and $f\colon K \to \mathbb{R}$ be a quasiconvex, lsc function bounded from above. Let $\hat{f}\colon \mathbb{R}^n \to \mathbb{R}$ be the extension of f defined by*

$$\hat{f}(x) = \begin{cases} f(x), & \text{if } x \in K; \\ \alpha, & \text{if } x \notin K, \end{cases}$$

with $+\infty > \alpha \geq \sup_K f$. For $v \in \mathbb{R}^n$ define $g_v\colon \mathbb{R}^n \to \mathbb{R}_+$ corresponding to $\hat{f}$, i.e.,

$$g_v(x) = \sup_{t\geq 0} \left(\hat{f}(x+tv) - \hat{f}(x)\right).$$

Then $$\operatorname*{argmax}_K f = \bigcap_{v\in\mathbb{R}^n} \operatorname*{argmin}_K g_v = \bigcap_{v\notin K^\infty} \operatorname*{argmin}_K g_v = \operatorname*{argmin}_K g_u \quad \forall\, u \notin K^\infty.$$

Proof. Clearly $\hat{f}$ is also quasiconvex and lsc. If $x \in \operatorname{argmax}_K f$, then for all $y \in K$, $\hat{f}(y) \leq \hat{f}(x)$. By Corollary 3.7 for all $v \in \mathbb{R}^n$, $g_v(x) \leq g_v(y)$. Thus, $x \in \operatorname{argmin}_K g_v$. On the other hand, for each $v \notin K^\infty$ and all $x \in K$, there exists $t > 0$ such that $x + tv \notin K$; hence $g_v(x) = \alpha - f(x)$. It follows that $\operatorname{argmin}_K g_v = \operatorname{argmax}_K f$.

Hence, by the previous remarks, for all $u \notin K^\infty$,

$$\operatorname*{argmax}_K f \subseteq \bigcap_{v\in\mathbb{R}^n} \operatorname*{argmin}_K g_v \subseteq \bigcap_{v\notin K^\infty} \operatorname*{argmin}_K g_v = \operatorname*{argmin}_K g_u = \operatorname*{argmax}_K f. \qquad \square$$

The next example shows that one has to consider also directions $v \notin K^\infty$ to compute $\operatorname{argmax}_K f$.

Example 4.12. Take $K = \mathbb{R}^2_+ \subseteq \mathbb{R}^2$ and $f(x_1, x_2) = 1/(1 + x_1 + x_2)$. Then for every $x \in K$ and $v \in K^\infty = K$, $g_v(x) = 0$. Thus,

$$\{0\} = \operatorname*{argmax}_K f \neq \bigcap_{v\in K^\infty} \operatorname*{argmin}_K g_v = K. \qquad \square$$

5. A class of quasiconvex minimization problems having zero duality gap

With the notation of the last part of Section 4.2, we consider the following problem

$$\mu \doteq \inf_{x\in K} f(x), \tag{30}$$

where $f\colon \mathbb{R}^n \to \mathbb{R}\cup\{+\infty\}$, $g_i\colon \mathbb{R}^n \to \mathbb{R}$ $(i \in J)$, are lsc and quasiconvex functions, $C \subseteq \mathbb{R}^n$ is any nonempty convex and closed set, and K is the feasible set

$$K \doteq \{x \in C : \ g_i(x) \le 0 \ \forall\, i \in J\}.$$

The (Lagrangian) dual problem to (30) is

$$\nu \doteq \sup_{\Lambda\in\mathbb{R}^m_+} \inf_{x\in C} \left\{f(x) + \Lambda^\top G(x)\right\}$$

where $G \doteq (g_1, g_2, \ldots, g_m)$. We say that there is no duality gap, if $\mu = \nu$. It is well-known that if $\mu = -\infty$ then $\nu = -\infty$, and so there is no duality gap. Hence, we consider the case $\mu \in \mathbb{R}$. The *optimal value function* $\psi\colon \mathbb{R}^n \to \overline{\mathbb{R}}$ is defined by

$$\psi(a) = \inf_{x\in K(a)} f(x)$$

where $K(a) \doteq \{x \in C : g_i(x) \le a_i \ \forall\, i \in J\}$, $a = (a_1, \ldots, a_m)$, and we use the convention that the infimum of f over the empty set is $+\infty$.

The closed convex cone R_0 introduced in (29) plays an important role in what follows. Set $F \doteq (f, g_1, \ldots, g_m)$, $J_0 \doteq \{0, 1, \ldots, m\}$.

We start by a lemma which has its own interest.

Lemma 5.1. *Let $f\colon \mathbb{R}^n \to \mathbb{R}\cup\{+\infty\}$, $g_i\colon \mathbb{R}^n \to \mathbb{R}$, $i \in J$, be proper, lsc, quasiconvex functions belonging to $\mathcal{C}$, with $C \subseteq \mathbb{R}^n$ being closed and convex. Assume that $R_0 \subseteq -R_0$ (or, equivalently, R_0 is a linear subspace). If (x_k) is a sequence in C such that the sequences $(f(x_k))$ and $(g_i(x_k))$, $i \in J$, are bounded from above, then there exist $\bar{x} \in C$ and a subsequence (x_{k_l}) such that $f(\bar{x}) \le \liminf_l f(x_{k_l})$ and $g_i(\bar{x}) \le \liminf_l g_i(x_{k_l})$, $i \in J$. In case f and g_i are continuous, then we have equalities.*

Proof. We write, for every k, the decomposition $x_k = x_k^+ + x_k^-$, with $x_k^+ \in R_0$ and $x_k^- \in R_0^\perp$. Since $R_0 \subseteq C^\infty \cap (-C^\infty)$, one gets $x_k^- = x_k - x_k^+ \in C + C^\infty = C$. Set $g_0 = f$. By Proposition 4.2, it follows that

$$g_i(x_k^-) = g_i(x_k - x_k^+) = g_i(x_k), \quad \forall\, i \in J_0. \tag{31}$$

We have two possibilities: either (x_k^-) is unbounded or not. Let us suppose first that $\sup_k \|x_k^-\| = +\infty$. Then, we get, up to a subsequence, that $\|x_k^-\| \to +\infty$ and $(x_k^-/\|x_k^-\|) \to v$ such that $\|v\| = 1$. Then $v \in R_0^\perp \cap C^\infty$, $v \in (\mathrm{dom}\, f)^\infty \cap (\mathrm{dom}\, g_i)^\infty$ $(i \in J)$. In particular, $v \notin R_0$ since otherwise $v \in R_0 \cap R_0^\perp$, which would imply $v = 0$. This means that $(g_j)^\infty_{0,q}(v) > 0$ for some $j \in J_0$; so $(g_j)^{qx}(v) = +\infty$ since $g_j \in \mathcal{C}$. Let M_j be an upper bound of $\{g_j(x_k)\}$. Fix any $x_0 \in \mathrm{dom}\, f$, the previous equality implies that

$$\sup_{t>0} g_j(x_0 + tv) = +\infty. \tag{32}$$

On the other hand, for a fixed $t > 0$ and all k sufficiently large, by quasiconvexity and (31), we obtain

$$\begin{aligned} g_j\Big((1 - \frac{t}{\|x_k^-\|})x_0 + \frac{t}{\|x_k^-\|}x_k^-\Big) &\le \max\{g_j(x_0), g_j(x_k^-)\} \\ &= \max\{g_j(x_0), g_j(x_k)\} \le \max\{g_j(x_0), M_j\}; \end{aligned}$$

which, by lower semicontinuity, yields $g_j(x_0 + tv) \le \max\{g_j(x_0), M_j\}$. Comparing with (32), we reach a contradiction. This shows that (x_k^-) necessarily is bounded.

We may assume, up to a subsequence, that $x_k^- \to \bar{x} \in C$. Then

$$g_i(\bar{x}) \le \liminf_k g_i(x_k^-) = \liminf_k g_i(x_k)$$

for all $i \in J_0$, as desired. The second part of the lemma is now obvious. □

Part (b) of the next theorem improves Corollary 4.1 in [19] in two directions: we delete the assumption on the semistrict quasiconvexity of f and g_i; and the linear structure on R_0 is weaker than the same requirement on each $L_i \doteq \{v \in C^\infty : \ (g_i)^\infty_{0,q}(v) = 0\}$ and on $\{v \in C^\infty : \ f^\infty_{0,q}(v) = 0\}$. Similar remarks apply for the result in [14]. In addition, we must mention that the same result is proved in [22] under a slightly weaker assumption than semitrict quasiconvexity. Finally, in case when every function is convex, the result was proved already in [7].

We recall that $R_0 \doteq \Big\{v \in C^\infty : f^\infty_{0,q}(v) = 0, \ (g_i)^\infty_{0,q}(v) = 0, \ \ \forall\, i \in J\Big\}$.

Theorem 5.2. *Let us consider problem* (30), $\mu \in \mathbb{R}$, $f\colon \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, $g_i\colon \mathbb{R}^n \to \mathbb{R}$, $i \in J$, *be lsc, quasiconvex functions belonging to* $\mathcal{C}$, *with* $C \subseteq \mathbb{R}^n$ *being closed and convex. Assume also that* $R_0 \subseteq -R_0$ (*or, equivalently* R_0 *is a linear subspace*). *The following assertions hold.*

(a) *The function* ψ *is lsc at* 0. *Thus, if additionally* $\overline{F(C) + \mathbb{R}_+ \times \mathbb{R}^m_+}$ *is convex, then there is no duality gap.*

(b) *If, additionally,* f *and each* g_i *is continuous, then* $F(C)$ *is closed.*

Proof. (a): According to Theorem 5.1 in [12], it is enough to show that for each $i_0 \in J$ fixed and $x_k \in C$ satisfying $g_i(x_k) + q_k^i \to 0$, $q_k^i \geq 0$, $i \in J$, $i \neq i_0$; $0 < g_{i_0}(x_k) \to 0$ and $f(x_k) < \mu$, we have $\limsup f(x_k) = \mu$. Note that for $i \neq i_0$, $\limsup g_i(x_k) \leq \limsup (g_i(x_k) + q_k^i) = 0$, so all sequences $\{f(x_k)\}$ and $\{g_i(x_k)\}$, $i \in J$ are bounded from above.

Applying Lemma 5.1 we deduce that there exists $\bar{x} \in C$ such that we get $f(\bar{x}) \leq \liminf_k f(x_k) \leq \limsup_k f(x_k) \leq \mu$ and $g_i(\bar{x}) \leq \liminf_k g_i(x_k) \leq 0$, $i \in J$. Thus, $\bar{x} \in K$, so $\mu = f(\bar{x}) = \lim f(x_k)$, which is the desired conclusion.

(b): Let $x_k \in C$ be such that $F(x_k) \to z = (z_0, z_1, \ldots, z_m)$, i.e., $f(x_k) \to z_0$, $g_j(x_k) \to z_j$, $j = 1, \ldots, m$. Using the second part of Lemma 5.1 we deduce the existence of $\bar{x} \in C$ such that $f(\bar{x}) = \lim f(x_k)$ and $g_i(\bar{x}) = \lim g_i(x_k)$ for all $i \in J$. Thus, $F(\bar{x}) = \lim_k F(x_k) = z$, proving that $F(C)$ is closed. $\square$

Remark 5.3. A rather general assumption to get the convexity of $\overline{F(C) + \mathbb{R}_+^{1+m}}$ is almost-convexity, which is implied by nearly-convexity, of $F(C) + \mathbb{R}_+^{1+m}$. For such notions we refer to [4, 5, 6].

About the convexity of $F(\mathbb{R}^n) + \mathbb{R}_+^2$, that is, when $m = 1$, $C = \mathbb{R}^n$ and f, g_1 are (not necessarily homogeneous) quadratic functions, the result appears in [16, Theorem 4.19] (here, a generalization of the Dines theorem [8] is proved), although such a result was stated in [29, Corollary 10] but its proof is wrong. In case f is a quadratic strictly convex function and each g_i is affine, the convexity of $F(\mathbb{R}^n)$ was obtained in [3] (under additional assumptions). This result was improved in [31, Theorem 9]. When f and each g_i is a homogeneous quadratic function and all the involved matrices are Z-matrices, the convexity of $F(\mathbb{R}^n) + \operatorname{int} \mathbb{R}_+^{1+m}$ was established in [23]. A recent result on the convexity of $F(\mathbb{R}^n) + \mathbb{R}_+^{1+m}$, in which f is any quadratic function, g_1 is any quadratic strictly convex function and the other g_i are affine, was obtained in [24], under further assumptions.

For non quadratic functions regarding the convexity of $F(C) + \mathbb{R}_+^{1+m}$, a result involving $*$-quasiconvexity of F (as introduced in [25]), or equivalently, naturally quasiconvexity of F introduced independently in [28], was established in [26], see also [15].

6. Quasiconvex quadratic programming

We now deal with the case when the objective function is quadratic. Let K be a nonempty closed and convex set in $\mathbb{R}^n$, and consider

$$h(x) = \begin{cases} \frac{1}{2} x^\top A x + a^\top x + \alpha, & \text{if} \quad x \in K, \\ +\infty, & \text{if} \quad x \notin K. \end{cases}, \ \operatorname{dom} h = K.$$

By using the identity

$$\begin{aligned} h(x + tv) - h(x) &= t \nabla h(x)^\top v + \frac{t^2}{2} v^\top A v \\ &= \frac{1}{2} v^\top A v \Big[\Big(t + \frac{\nabla h(x)^\top v}{v^\top A v} \Big)^2 - \Big(\frac{\nabla h(x)^\top v}{v^\top A v} \Big)^2 \Big], \end{aligned} \tag{33}$$

we obtain, without imposing any further assumption on h,

$$h_q^\infty(v) = \begin{cases} \sup\limits_{x\in K}\nabla h(x)^\top v, & \text{if} \quad v \in K^\infty,\ v^\top Av \leq 0; \\ +\infty, & \quad\quad \text{elsewhere} \end{cases}$$

and

$$h_{0,q}^\infty(v) = \begin{cases} 0 & \text{if } -v \in [\nabla h(K)]^*, v^\top Av \leq 0, v \in K^\infty; \\ -\dfrac{1}{2v^\top Av}\left(\sup\limits_{x\in K}\nabla h(x)^\top v\right)^2 & \text{if } -v \notin [\nabla h(K)]^*, v^\top Av < 0, v \in K^\infty; \\ +\infty & \text{elsewhere.} \end{cases}$$

In what follows we collect some important properties about quasiconvex quadratic functions.

Lemma 6.1. *Let $K \subseteq \mathbb{R}^n$ be a closed and convex set; h be a quadratic function as above with* dom $h = K$. *The following assertions hold:*

(a) *h is semistrictly quasiconvex on K if, and only if it is quasiconvex on K;*

(b) *if h is quasiconvex on K, then*

$$v \in K^\infty,\ v^\top Av < 0 \implies -v \in [\nabla h(K)]^*. \tag{34}$$

(c) *if h is quasiconvex but nonconvex on K, then*

$$v \in K^\infty,\ v^\top Av = 0 \implies -v \in [\nabla h(K)]^*. \tag{35}$$

Proof. (a): This follows since if it is constant along any line segment then it is constant along the entire line lying in K.

(b): From (33), the only possibility to get the quasiconvexity of $0 \leq t \mapsto h(x+tv)$ is that $\nabla h(x)^\top v \leq 0$ for all $x \in K$, provided $v^\top Av < 0$, and so the conclusion follows.

(c): We consider first the case int $K \neq \emptyset$. Then, (35) results from (45) in [12] for instance, which is a consequence of the necessary condition for h to be quasiconvex on K, provided int $K \neq \emptyset$ (see [18, Proposition 2.16]).

Assume now that int $K = \emptyset$. Fix $x_0 \in K$ and let $K_1 = K - x_0$, then aff $K = V + x_0$ where $V =$ aff K_1 is a subspace. Set $h_1(x) = h(x + x_0)$, $x \in K_1$. Then h_1 is quasiconvex and nonconvex. Let P be the projection of $\mathbb{R}^n$ onto V and $A_1 = PAP$. We have

$$h_1(x) = \frac{1}{2}(x + x_0)^\top A(x + x_0) + a^\top(x + x_0) + \alpha$$
$$= \frac{1}{2}x^\top Ax + (Ax_0 + a)^\top x + \left(\alpha + \frac{1}{2}x_0^\top Ax_0 + a^\top x_0\right) = \frac{1}{2}x^\top A_1x + a_1^\top x + \alpha_1$$

where $a_1 = P\,(Ax_0 + a)$, $\alpha_1 = \alpha + \frac{1}{2}x_0^\top Ax_0 + a^\top x_0$.

Let $v \in K^\infty$ be such that $v^\top Av = 0$. Then $v \in K_1^\infty = K^\infty \subseteq V$ and $v^\top A_1 v = 0$. Since the interior of K_1 relative to the vector space V is nonempty, by (35) we get $v^\top(A_1x + a_1) \le 0$ for all $x \in K_1$. Using the self-adjointness of P and $Px = x$ we get

$$0 \ge v^\top(PAPx + P(Ax_0 + a)) = v^\top(A(x + x_0) + a) = v^\top \nabla h(x + x_0).$$

That is, $-v \in [\nabla h(K)]^*$. □

Assuming quasiconvexity of h on K the expression for $h^\infty_{0,q}$ reduces by (34) to

$$h^\infty_{0,q}(v) = \begin{cases} 0 & \text{if } -v \in [\nabla h(K)]^*, v^\top Av \le 0,\ v \in K^\infty; \\ +\infty & \text{elsewhere.} \end{cases}$$

Hence,
$$R_q = \{v \in K^\infty :\ v^\top Av \le 0,\ \ -v \in [\nabla h(K)]^*\}. \tag{36}$$

If, additionally K is a cone (implying $K^\infty = K$), then (36) reduces to:

$$R_q = \{v \in K :\quad -Av \in K^*,\ a^\top v \le 0\}. \tag{37}$$

The equivalence between (a) and (c) was already established in [21, Theorem 4.6].

Proposition 6.2. *Let K be closed and convex, and h quasiconvex on K. The following assertions are equivalent:*

(a) $\operatorname{argmin}_K h$ *is nonempty and bounded (so compact);*

(b) *A is copositive on K^∞, and the following implication holds*

$$[v^\top Av = 0,\ v \in K^\infty,\ \ -v \in [\nabla h(K)]^* \Longrightarrow v = 0].$$

(c) $[v^\top Av \le 0,\ v \in K^\infty,\ \ -v \in [\nabla h(K)]^* \Longrightarrow v = 0]$.

If additionally h is not convex on K, they are also equivalent to:

(d) *A is copositive on K^∞, and the following implication holds*

$$[v^\top Av = 0,\ v \in K^\infty \Longrightarrow v = 0];$$

(e) $[v^\top Av \le 0,\ v \in K^\infty \ \Longrightarrow v = 0]$.

Proof. (a) ⇒ (b) This corresponds to (g) implying (h) in [11, Theorem 4.1] (this implication does not require K to be asymptotically linear), which is valid for any (not necessarily quasiconvex) function h.

(b) ⇒ (c): This is straightforward.

(c) ⇒ (a): This is already known: simply write down $R_q = \{0\}$.

(d) and (e): These are consequences of (34) and (35). □

We can go on if K is determined by inequality constraints involving quasiconvex quadratic functions.

The next example shows that the sole assumption $R_q = \{0\}$ does not imply the nonemptiness of $\operatorname{argmin}_K h$.

Example 6.3. We reconsider Example 4.2 from [11] and therefore take the function $h(x_1, x_2) = x_1^2 - x_2^2$ defined in $K = \{(x_1, x_2) \in \mathbb{R}^2 : |x_1 - x_2| \leq 1\}$. Thus we have $K^\infty = \{(t, t) \in \mathbb{R}^2\colon t \in \mathbb{R}\}$, and conditions (b) and (c) are satisfied, but $\operatorname{argmin}_K h = \emptyset$ since $h(x_1, x_1 - 1) = 2x_1 - 1$ for all $x_1 \in \mathbb{R}$. Clearly, h is not quasiconvex on K: indeed,

$$0 = h(0, 0) > \max\{h(0, -1), h(0, 1)\} = -1.$$

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