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Convex Bodies Associated to Tensor Norms

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We determine when a convex body in $\mathbb{R}^d$ is the closed unit ball of a reasonable crossnorm on $\mathbb{R}^{d_1} \otimes \cdots \otimes \mathbb{R}^{d_l}$, $d = d_1 \cdots d_l$. We call these convex bodies "tensorial bodies". We prove that, among them, the only ellipsoids are the closed unit balls of Hilbert tensor products of Euclidean spaces. It is also proved that linear isomorphisms on $\mathbb{R}^{d_1} \otimes \cdots \otimes \mathbb{R}^{d_l}$ preserving decomposable vectors map tensorial bodies into tensorial bodies. This leads us to define a Banach-Mazur type distance between them, and to prove that there exists a Banach-Mazur type compactum of tensorial bodies.

Keywords: Convex body, tensor norm, Minkowski space, Banach-Mazur distance, tensor product of convex sets, linear mappings on tensor spaces.

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1. Introduction

Tensor products of finite dimensional spaces play a fundamental role in a wide range of problems in applications. They arise, among others, in quantum computing [31], in theoretical computer science [10], and in the use of tensor decompositions to extract and explain properties from data arrays (see [17] and the references therein). This fact has motivated the current research into their geometric, topologic and algebraic properties, as can be seen in [6, 12, 14, 18, 28].

On the other hand, there is a well developed theory of norms defined on tensor products of Banach spaces. This theory was established by A. Grothendieck [13]. It has had a great impact in the Geometry of Banach spaces, as can be traced in [7, 8, 9, 22, 24, 27]. Indeed, its impact extends even beyond Mathematical Analysis. By way of example, we refer to the survey [16] where applications of Grothendieck's theorem (usually called Grothendieck's inequality) to the design of polynomial time algorithms for computing approximate solutions of NP problems are detailed. In the other direction, we refer to [5] where results from theoretical computer science are used to prove that for some indices p_1, p_2, p_3, the space $\ell_{p_1} \hat{\otimes}_\pi \ell_{p_2} \hat{\otimes}_\pi \ell_{p_3}$ fails to have non trivial cotype. The interested reader can consult

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[1, 23, 26] for further information about tensor products of Banach spaces and their applications.

In the case of finite dimensions, the Minkowski functional enables the use of convex geometry to study finite dimensional Banach spaces (also known as Minkowski spaces) and vice versa. With it, a bijection between norms and 0-symmetric convex bodies in $\mathbb{R}^d$ is established. This result was originally due to H. Minkowski [21], and nowadays is a standard result (see [25, Remark 1.7.7] for a modern statement). Thus, in the context of tensors of finite dimensional spaces, a natural question to ask is if it is possible to determine the convex bodies that are the unit balls of tensor normed spaces, as well as 0-symmetric convex bodies are the unit balls of normed spaces. The main result of this paper, Theorem 3.2, provides an affirmative answer to this question.

This work, as well as [3], lies between the theory of tensor norms and convex geometry. In [3], G. Aubrun and S. Szarek establish connections between tensor norms on finite dimensions and convex geometry to estimate the volume of the set of separable mixed quantum states.

We now briefly expose our results. Bringing together the theory of tensor norms and convex geometry, we immediately obtain that the the convex bodies $Q \subset \mathbb{R}^d$ that are the unit ball of a reasonable crossnorm defined on $\mathbb{R}^d = \mathbb{R}^{d_1} \otimes \cdots \otimes \mathbb{R}^{d_l}$, $d = d_1 \cdots d_l$, are those $Q \subset \mathbb{R}^{d_1} \otimes \cdots \otimes \mathbb{R}^{d_l}$ for which there exist norms $\|\cdot\|_i$ on $\mathbb{R}^{d_i}$ such that

$$B_{\otimes_\pi\left(\mathbb{R}^{d_i},\|\cdot\|_i\right)} \subseteq Q \subseteq B_{\otimes_\epsilon\left(\mathbb{R}^{d_i},\|\cdot\|_i\right)}, \tag{1}$$

where B denotes the closed unit ball of the projective and the injective tensor norms. In Proposition 3.1, we prove that (1) is equivalent to say that

$$Q_1 \otimes_\pi \cdots \otimes_\pi Q_l \subseteq Q \subseteq Q_1 \otimes_\epsilon \cdots \otimes_\epsilon Q_l, \tag{2}$$

where for each i, $Q_i \subset \mathbb{R}^{d_i}$ is the closed unit ball of $\left(\mathbb{R}^{d_i}, \|\cdot\|_i\right)$, and $\otimes_\pi, \otimes_\epsilon$ are the projective and the injective tensor products of 0-symmetric convex bodies, defined by G. Aubrun and S. Szarek in [3, 4].

Our main result (Theorem 3.2) lies much deeper than Proposition 3.1. There, we establish the conditions on $Q \subset \mathbb{R}^d = \mathbb{R}^{d_1} \otimes \cdots \otimes \mathbb{R}^{d_l}$ that guarantee the existence of the convex sets $Q_i \subset \mathbb{R}^{d_i}$ in (2) and give an explicit description of them.

Proposition 3.1 and Theorem 3.2 allow us to introduce "tensorial bodies": a 0-symmetric convex body $Q \subset \mathbb{R}^{d_1} \otimes \cdots \otimes \mathbb{R}^{d_l}$ is a *tensorial body in* $\mathbb{R}^{\mathbf{d_1}} \otimes \cdots \otimes \mathbb{R}^{\mathbf{d_l}}$ if there exist 0-symmetric convex bodies $Q_i \subset \mathbb{R}^{d_i}$, $i = 1, ..., l$ such that (2) holds (see Definition 3.3).

Corollary 3.4 shows that the reasonable crossnorms on $\mathbb{R}^{d_1} \otimes \cdots \otimes \mathbb{R}^{d_l}$ are the image of the tensorial bodies in $\mathbb{R}^{d_1} \otimes \cdots \otimes \mathbb{R}^{d_l}$, under the bijection given by the Minkowski functional. With it, we can go further with the study of this class of convex sets. We prove that the polar set of a tensorial body is a tensorial body and prove the stability of tensorial bodies by multiplying for positive scalars (Proposition 3.5). We also show that the convex bodies Q_i in (2) are essentially unique (see Proposition 3.6).

In Theorem 3.12 we prove that the subgroup of linear isomorphisms on $\mathbb{R}^{d_1} \otimes \cdots \otimes \mathbb{R}^{d_l}$ preserving decomposable vectors also preserve tensorial bodies. We denote this group by $GL_{\otimes}\left(\otimes_{i=1}^{l}\mathbb{R}^{d_i}\right)$. By means of $GL_{\otimes}\left(\otimes_{i=1}^{l}\mathbb{R}^{d_i}\right)$, we start a geometric study of the set of tensorial bodies, defining the following distance:

$$\delta_{\otimes}^{BM}(P,Q) := \inf\left\{\lambda \geq 1 : Q \subseteq TP \subseteq \lambda Q \text{ for } T \in GL_{\otimes}\left(\otimes_{i=1}^{l}\mathbb{R}^{d_i}\right)\right\},$$

where $P, Q \subset \mathbb{R}^{d_1} \otimes \cdots \otimes \mathbb{R}^{d_l}$ are tensorial bodies. We call $\delta_{\otimes}^{BM}$ the tensorial Banach-Mazur distance. We use it to show that there is a Banach-Mazur type compactum of tensorial bodies in $\mathbb{R}^{d_1} \otimes \cdots \otimes \mathbb{R}^{d_l}$ (Theorem 3.13).

Finally, we apply the ideas developed through the paper to prove that the only ellipsoids that are also tensorial bodies in $\mathbb{R}^{d_1} \otimes \cdots \otimes \mathbb{R}^{d_l}$ are the unit balls of the Hilbert tensor product of Euclidean spaces (see Theorem 4.2 and Corollary 4.3).

The paper is organized as follows: in Subection 1.1, we introduce the notation and basic results that we will use throughout the paper. In Section 2, we recall the main properties of the projective and the injective tensor products of 0-symmetric convex bodies. In Section 3, we prove Theorem 3.2 and establish the fundamental properties of tensorial bodies. There, we exhibit examples of tensorial bodies and show that not every 0-symmetric convex body is of this type. In Subsection 3.2, we establish the relation between $GL_{\otimes}\left(\otimes_{i=1}^{l}\mathbb{R}^{d_i}\right)$ and the set of tensorial bodies, and settle the fundamental properties of $\delta_{\otimes}^{BM}$. We finish this section by giving upper bounds for $\delta_{\otimes}^{BM}$ (Corollary 3.15). In Section 4, we characterize the ellipsoids in the class of tensorial bodies (Theorem 4.2 and Corollary 4.3).

We like to point out that Theorems 3.2 and 3.12 remain true in $\mathbb{C}^d = \otimes_{i=1}^{l}\mathbb{C}^{d_i}$, $d = d_1 ... d_l$, when circled convex bodies (i.e. a convex body $Q \subset \mathbb{C}^d$ s.t. $e^{i\theta}Q = Q$) are considered. As a consequence, it is possible to provide the corresponding notion of "tensorial body in $\otimes_{i=1}^{l}\mathbb{C}^{d_i}$" as well as the definition of the tensorial Banach-Mazur distance. Here, for the sake of transparency we will concentrate in the case of 0-symmetric convex bodies in real spaces.

1.1. Preliminaries

Throughout this paper, X, Y or X_i will denote Banach spaces. The *closed unit ball* of X will be denoted by B_X and its *dual space* by X^*. We write $\mathcal{L}(X,Y)$ to denote the space of bounded linear operators from X to Y.

Let V_i, $i = 1, \ldots, l$ be vector spaces over $\mathbb{R}$. By $\otimes_{i=1}^{l} V_i$ we denote its tensor product, and by $\otimes$ we denote the canonical multilinear map:

$$\otimes : V_1 \times \cdots \times V_l \longrightarrow \otimes_{i=1}^{l} V_i, \quad \left(x^1, \ldots, x^l\right) \to x^1 \otimes \cdots \otimes x^l.$$

In the case of Banach spaces, a norm $\alpha(\cdot)$ on the tensor product $\otimes_{i=1}^{l} X_i$ is called a *reasonable crossnorm* if

(1) $\alpha\left(x^1 \otimes \cdots \otimes x^l\right) \leq \|x^1\| \cdots \left\|x^l\right\|$ for every $x^i \in X_i$ with $i = 1, ..., l$.
(2) If $x_i^* \in X_i^*$ for $i = 1, ..., l$ then $x_1^* \otimes \cdots \otimes x_l^* \in \left(\otimes_{i=1}^{l} X_i, \alpha\right)^*$ and $\|x_1^* \otimes \cdots \otimes x_l^*\| \leq \|x_1^*\| \cdots \|x_l^*\|$.

If $\alpha(\cdot)$ is a reasonable crossnorm on $\otimes_{i=1}^{l} X_i$, $\otimes_{\alpha,i=1}^{l} X_i$ will denote the *normed space* $\left(\otimes_{i=1}^{l} X_i, \alpha\right)$, and $X_1 \hat{\otimes}_\alpha \cdots \hat{\otimes}_\alpha X_l$ its *completion*. For each $u \in \otimes_{i=1}^{l} X_i$, the *projective norm* π and the *injective norm* ϵ are defined by:

$$\pi(u) := \inf \left\{ \sum_{i=1}^{n} \left\|x_i^1\right\| \cdots \left\|x_i^l\right\| : u = \sum_{i=1}^{n} x_i^1 \otimes \cdots \otimes x_i^l \right\}, \quad \text{and}$$

$$\epsilon(u) := \sup \left\{ \left|x_1^* \otimes \cdots \otimes x_l^*(u)\right| : x_i^* \in B_{X_i^*}, \text{ for } i = 1, \ldots, l \right\}.$$

Both the projective and the injective norm are reasonable crossnorms on $\otimes_{i=1}^{l} X_i$. Indeed, these norms provide the next fundamental characterization of reasonable crossnorms: A norm $\alpha(\cdot)$ on $\otimes_{i=1}^{l} X_i$ is a *reasonable crossnorm* if and only if

$$\epsilon(u) \leq \alpha(u) \leq \pi(u) \text{ for every } u \in \otimes_{i=1}^{l} X_i. \tag{3}$$

The proof of this equivalence in the case of two normed spaces can be consulted in [24, Proposition 6.3]. For a deeper discussion about tensor norms we refer to [7].

1.1.1. Convex bodies in Euclidean spaces

Let $\mathbb{E}$ be a real Euclidean space with scalar product $\langle \cdot, \cdot \rangle_{\mathbb{E}}$ and Euclidean ball $B_{\mathbb{E}}$. A subset $P \subset \mathbb{E}$ is called a *convex body* if P is a compact convex set with nonempty interior. Every convex body $P \subset \mathbb{E}$ for which $P = -P$ is called a 0-symmetric (or centrally symmetric) convex body. The set of 0-symmetric convex bodies in $\mathbb{E}$ is denoted by $\mathcal{B}(\mathbb{E})$ (resp. $\mathcal{B}(d)$ if $\mathbb{E} = \mathbb{R}^d$).

If C is a nonempty subset of $\mathbb{E}$, then its *polar set* is defined by

$$C^\circ := \left\{ y \in \mathbb{E} : \sup_{x \in C} \langle x, y \rangle_{\mathbb{E}} \leq 1 \right\}.$$

The *Minkowski functional* (or gauge function) of $P \in \mathcal{B}(\mathbb{E})$ is defined as

$$g_P(x) := \inf \left\{ \lambda > 0 : \frac{1}{\lambda} x \in P \right\} \text{ for } x \in \mathbb{E}.$$

A fundamental result concerning 0-symmetric convex bodies is the bijection between norms defined on $\mathbb{E}$ and 0-symmetric convex bodies in $\mathbb{E}$. This result, originally due to H. Minkowksi [21], will be used throughout the paper without making an explicit reference. We will use it in the following form:

Theorem 1.1. *Let $\mathbb{E}$ be a Euclidean space. If $A \in \mathcal{B}(\mathbb{E})$, then*

$$\|x\|_A := g_A(x) \text{ for } x \in \mathbb{E}$$

defines a norm $\|\cdot\|_A$ on $\mathbb{E}$ for which A is the closed unit ball. Furthermore, for every $x \in \mathbb{E}$ we have

$$\|x\|_{A^\circ} = \left\| \langle \cdot, x \rangle : (\mathbb{E}, \|\cdot\|_A) \to \mathbb{R} \right\|.$$

This statement as well as the theory of convex bodies and convex geometry that will be used in this paper, can be found in [25].

2. Projective and injective tensor products of 0-symmetric convex bodies

To introduce the projective and the injective tensor products of 0-symmetric convex bodies, it is convenient to recall two well known facts about tensor products of Banach spaces. The first one is that

$$B_{X_1\hat{\otimes}_\pi\cdots\hat{\otimes}_\pi X_l} = \overline{\mathrm{conv}}\left\{x^1 \otimes \cdots \otimes x^l : x^1 \in B_{X_1}, \ldots, x^l \in B_{X_l}\right\}. \tag{4}$$

The second one is the duality between the injective and projective tensor product of Banach spaces given by the canonical isometry:

$$X_1 \otimes_\epsilon \cdots \otimes_\epsilon X_l \hookrightarrow \left(X_1^* \otimes_\pi \cdots \otimes_\pi X_l^*\right)^*, \tag{5}$$

which on finite dimensions is an isometric isomorphism (see [7, pp. 27, 46], respectively).

Let $Q_i \subset \mathbb{R}^{d_i}$, $i = 1, \ldots, l$ be 0-symmetric convex bodies with associated Minkowski functionals g_{Q_i}, $i = 1, \ldots, l$. By (4), $\overline{\mathrm{conv}}\left\{x^1 \otimes \cdots \otimes x^l : x^i \in Q_i\right\}$ is the closed unit ball of the projective norm on $\otimes_{i=1}^l \left(\mathbb{R}^{d_i}, g_{Q_i}\right)$. This fact provides a natural way to define the projective tensor product of 0-symmetric convex bodies: the *projective tensor product of* $Q_1, \ldots, Q_l$ is the 0-symmetric convex body in $\otimes_{i=1}^l \mathbb{R}^{d_i}$ defined by:

$$Q_1 \otimes_\pi \cdots \otimes_\pi Q_l := \mathrm{conv}\left\{x^1 \otimes \cdots \otimes x^l \in \otimes_{i=1}^l \mathbb{R}^{d_i} : x^i \in Q_i, i = 1, \ldots, l\right\}.$$

This definition was introduced by G. Aubrun and S. Szarek in [3]. There, the projective tensor product of more general classes of convex sets is considered.

Since $\mathrm{conv}\left\{x^1 \otimes \cdots \otimes x^l \in \otimes_{i=1}^l \mathbb{R}^{d_i} : x^i \in Q_i\right\}$ is compact (see Proposition 2.4), it coincides with its closure. Then,

$$Q_1 \otimes_\pi \cdots \otimes_\pi Q_l = B_{\otimes_{\pi,i=1}^l \left(\mathbb{R}^{d_i}, g_{Q_i}(\cdot)\right)}. \tag{6}$$

The duality between the injective and the projective tensor norms given in (5) gives rise to a notion of injective tensor product of 0-symmetric convex bodies. To be precise, we first fix the scalar products that will be used through the paper.

Given $d \in \mathbb{N}$, we will denote by $\langle \cdot, \cdot \rangle$ the standard scalar product on $\mathbb{R}^d$, and by $\|\cdot\|_2$, B_2^d its associated norm and Euclidean ball respectively.

The scalar product on $\otimes_{i=1}^l \mathbb{R}^{d_i}$ will be the one associated to the Hilbert tensor product $\otimes_{H,i=1}^l \mathbb{R}^{d_i}$, that is, $\langle \cdot, \cdot \rangle_H$ will be the bilinear form determined by the relation

$$\left\langle x^1 \otimes \cdots \otimes x^l, y^1 \otimes \cdots \otimes y^l \right\rangle_H := \Pi_{i=1}^l \left\langle x^i, y^i \right\rangle$$

(see [15, Section 2.5]). The closed unit ball of $\otimes_{H,i=1}^l \mathbb{R}^{d_i}$ will be denoted by $B_2^{d_1,\ldots,d_l}$, and its norm by $\|\cdot\|_H$. In this way, given a 0-symmetric convex body $Q \subset \otimes_{i=1}^l \mathbb{R}^{d_i}$, its polar set acquires the form $Q^\circ = \left\{z \in \otimes_{i=1}^l \mathbb{R}^{d_i} : \sup_{u \in Q} \left|\langle u, z \rangle_H\right| \leq 1\right\}$.

Now, if $Q_i \subset \mathbb{R}^{d_i}$, $i = 1, \ldots, l$ are 0-symmetric convex bodies, the *injective tensor product of* $Q_1, \ldots, Q_l$ is the 0-symmetric convex body in $\otimes_{i=1}^l \mathbb{R}^{d_i}$ defined as follows:

$$Q_1 \otimes_\epsilon \cdots \otimes_\epsilon Q_l := (Q_1^\circ \otimes_\pi \cdots \otimes_\pi Q_l^\circ)^\circ.$$

This definition appeared for the first time in the remarkable monograph [4, Subsection 4.1.4] published in 2017. Later we will use this identity written in the following equivalent ways:

Proposition 2.1. *Let* $Q_i \subset \mathbb{R}^{d_i}$, $i = 1, \ldots, l$ *be* 0*-symmetric convex bodies. Then,*

(1) $(Q_1 \otimes_\epsilon \cdots \otimes_\epsilon Q_l)^\circ = Q_1^\circ \otimes_\pi \cdots \otimes_\pi Q_l^\circ$.

(2) $(Q_1 \otimes_\pi \cdots \otimes_\pi Q_l)^\circ = Q_1^\circ \otimes_\epsilon \cdots \otimes_\epsilon Q_l^\circ$.

Due to the duality between the projective and the injective tensor norms (5), along with (6), we have that

$$Q_1 \otimes_\epsilon \cdots \otimes_\epsilon Q_l = B_{\otimes_{\epsilon,i=1}^l \left(\mathbb{R}^{d_i}, g_{Q_i}(\cdot)\right)}. \tag{7}$$

2.1. The unit balls of ℓ_1^d and ℓ_∞^d.

Proposition 2.2 below, together with (6) and (7) show that the convex bodies B_1^d, B_∞^d, $d = d_1 \cdots d_l$, are the closed unit balls of $\otimes_{\pi,i=1}^l \ell_1^{d_i}$ and $\otimes_{\epsilon,i=1}^l \ell_\infty^{d_i}$, respectively. In effect, let B_p^d be the closed unit ball of ℓ_p^d, $d \in \mathbb{N}$ and $1 \le p \le \infty$. For each $i = 1, ..., l$, let $\left\{e_{j_i}^{d_i}\right\}_{j_i=1,\ldots,d_i}$ be the standard basis of $\mathbb{R}^{d_i}$. Then, the set of vectors $\left\{e_{j_1}^{d_1} \otimes \cdots \otimes e_{j_l}^{d_l}\right\}$ is an orthonormal basis in $\otimes_{H,i=1}^l \mathbb{R}^{d_i}$, and it can be identified with the standard basis of $\mathbb{R}^d$, $d = d_1 \cdots d_l$. Consequently, for each $1 \le p \le \infty$, the sets

$$B_p^{d_1,\ldots,d_l} := \left\{ z \in \otimes_{i=1}^l \mathbb{R}^{d_i} : \sum_{j_1,\ldots,j_l} \left| \left\langle z, e_{j_1}^{d_1} \otimes \cdots \otimes e_{j_l}^{d_l} \right\rangle_H \right|^p \le 1 \right\} \text{ for } p \neq \infty$$

$$\text{and} \qquad B_\infty^{d_1,\ldots,d_l} := \left\{ z \in \otimes_{i=1}^l \mathbb{R}^{d_i} : \max_{j_1,\ldots,j_l} \left| \left\langle z, e_{j_1}^{d_1} \otimes \cdots \otimes e_{j_l}^{d_l} \right\rangle_H \right| \le 1 \right\}$$

are naturally identified with the closed unit balls of ℓ_p^d. Thus, $B_p^d = B_p^{d_1,\ldots,d_l}$ for $1 \le p \le \infty$.

Proposition 2.2. *Let* $d \in \mathbb{N}$. *For every factorization of* d *in natural numbers* $d = d_1 \cdots d_l$, *we have*

$$B_1^d = B_1^{d_1} \otimes_\pi \cdots \otimes_\pi B_1^{d_l} \qquad \textit{and} \qquad B_\infty^d = B_\infty^{d_1} \otimes_\epsilon \cdots \otimes_\epsilon B_\infty^{d_l}.$$

The previous proposition is a well known result, see for instance [24, Excercise 2.6] or [4, pp. 83].

In Subection 3.1, we will treat the case $1 < p < \infty$. We will see that B_p^d is the closed unit ball associated to a reasonable crossnorm on $\otimes_{i=1}^l \ell_p^{d_i}$. In this case it is not the projective nor the injective tensor norm on $\otimes_{i=1}^l \ell_p^{d_i}$.

We finish this section stating without proof two results that will be used throughout the paper. Proposition 2.3 is a well known result (for a proof see [6, Proposition 4.2]). Proposition 2.4 is a direct consequence of the continuity of the canonical multilinear map $\otimes : \mathbb{R}^{d_1} \times \cdots \times \mathbb{R}^{d_l} \to \otimes_{H,i=1}^{l} \mathbb{R}^{d_i}$.

Proposition 2.3. *The set of decomposable vectors*

$$\left\{x^1 \otimes \cdots \otimes x^l \in \otimes_{i=1}^{l} \mathbb{R}^{d_i} : x^i \in \mathbb{R}^{d_i}\right\}$$

is a closed subset of $\otimes_{H,i=1}^{l} \mathbb{R}^{d_i}$.

Proposition 2.4. *If* $A_i \subseteq \mathbb{R}^{d_i}$, $i = 1, ..., l$ *are compact sets then*

$$\otimes (A_1, \ldots, A_l) := \left\{x^1 \otimes \cdots \otimes x^l \in \otimes_{i=1}^{l} \mathbb{R}^{d_i} : x^i \in A_i\right\}$$

is a compact subset of $\otimes_{H,i=1}^{l} \mathbb{R}^{d_i}$.

3. Tensorial bodies

In this section we characterize the convex bodies in $\otimes_{i=1}^{l} \mathbb{R}^{d_i}$ that are the closed unit balls of reasonable crossnorms. They will be called tensorial bodies (Definition 3.3). A main tool to study them is the group of linear isomorphisms that preserve decomposable vectors. With it, we will introduce a Banach-Mazur type distance between tensorial bodies, and prove that there is a Banach-Mazur type compactum associated to them (see Subsection 3.2).

Recall that we have already fixed the scalar product $\langle \cdot, \cdot \rangle_H$ on $\otimes_{i=1}^{l} \mathbb{R}^{d_i}$ and that g_Q denotes the Minkowski functional of a 0-symmetric convex body Q. Whit them, we have:

Proposition 3.1. *Let* $Q \subset \otimes_{i=1}^{l} \mathbb{R}^{d_i}$ *and* $Q_i \subset \mathbb{R}^{d_i}$, $i = 1, \ldots, l$ *be 0-symmetric convex bodies. Then,* $g_Q(\cdot)$ *is a reasonable crossnorm on* $\otimes_{i=1}^{l} \left(\mathbb{R}^{d_i}, g_{Q_i}(\cdot)\right)$ *if and only if*

$$Q_1 \otimes_\pi \cdots \otimes_\pi Q_l \subseteq Q \subseteq Q_1 \otimes_\epsilon \cdots \otimes_\epsilon Q_l. \tag{8}$$

In this case, for every decomposable vector $x^1 \otimes \cdots \otimes x^l \in \otimes_{i=1}^{l} \mathbb{R}^{d_i}$ *we have:*

$$g_Q\left(x^1 \otimes \cdots \otimes x^l\right) = g_{Q_1}\left(x^1\right) \cdots g_{Q_l}\left(x^l\right), \quad \textit{and}$$
$$g_{Q^\circ}\left(x^1 \otimes \cdots \otimes x^l\right) = g_{Q_1^\circ}\left(x^1\right) \cdots g_{Q_l^\circ}\left(x^l\right).$$

Proof. Let $Q_i \subset \mathbb{R}^{d_i}$, $i = 1, \ldots, l$, be 0-symmetric convex bodies. Then, (6) and (7) tell us that $Q_1 \otimes_\pi \cdots \otimes_\pi Q_l$ is the closed unit ball of $\otimes_{\pi,i=1}^{l} \left(\mathbb{R}^{d_i}, g_{Q_i}\right)$, and $Q_1 \otimes_\epsilon \cdots \otimes_\epsilon Q_l$ is the closed unit ball of $\otimes_{\epsilon,i=1}^{l} \left(\mathbb{R}^{d_i}, g_{Q_i}\right)$. Now, the proof of the first part follows from the characterization of a reasonable crossnorm (3). The second part follows using the two properties that define being a reasonable crossnorm. □

This proposition can be understood as the definition of a reasonable crossnorm written in terms of convex bodies. It determines when a 0-symmetric convex body in $\otimes_{i=1}^{l} \mathbb{R}^{d_i}$ is the unit ball of a reasonable crossnorm when the norms g_{Q_i} on each

$\mathbb{R}^{d_i}$ are given. Our next result goes further: it determines when a 0-symmetric convex body in $\otimes_{i=1}^{l}\mathbb{R}^{d_i}$ is the unit ball of a reasonable crossnorm, with respect to some norms (not determined a priori) on the spaces $\mathbb{R}^{d_i}$.

Given a 0-symmetric convex body $Q \subset \otimes_{i=1}^{l}\mathbb{R}^{d_i}$ and a non-zero decomposable vector $\mathbf{a} = a^1 \otimes \cdots \otimes a^l \in \otimes_{i=1}^{l}\mathbb{R}^{d_i}$, let us consider the 0-symmetric convex bodies in $\mathbb{R}^{d_i}$, $i = 1, \ldots, l$, defined as:

$$Q_i^{a^1,\ldots,a^l} := \left\{x^i \in \mathbb{R}^{d_i} : a^1 \otimes \cdots \otimes a^{i-1} \otimes x^i \otimes a^{i+1} \otimes \cdots \otimes a^l \in Q\right\}. \tag{9}$$

Theorem 3.2. *Let $Q \subset \otimes_{i=1}^{l}\mathbb{R}^{d_i}$ be a 0-symmetric convex body. Then, there exist norms $\|\cdot\|_i$ on $\mathbb{R}^{d_i}$, $i = 1, ..., l$, such that Q is the closed unit ball of a reasonable crossnorm on $\otimes_{i=1}^{l}\left(\mathbb{R}^{d_i}, \|\cdot\|_i\right)$ if and only if for an arbitrary decomposable vector $a^1 \otimes \cdots \otimes a^l \in \partial Q$ it holds:*

$$Q_1^{a^1,\ldots,a^l} \otimes_\pi \cdots \otimes_\pi Q_l^{a^1,\ldots,a^l} \subseteq Q \subseteq Q_1^{a^1,\ldots,a^l} \otimes_\epsilon \cdots \otimes_\epsilon Q_l^{a^1,\ldots,a^l}. \tag{10}$$

In such a situation, $Q_i^{a^1,\ldots,a^l} = \|a^i\|_i B_{\left(\mathbb{R}^{d_i},\|\cdot\|_i\right)}$.

Proof. Suppose that Q is the unit ball of a reasonable crossnorm $\alpha(\cdot)$ on the product $\otimes_{i=1}^{l}\left(\mathbb{R}^{d_i}, \|\cdot\|_i\right)$. Clearly $g_Q(\cdot) = \alpha(\cdot)$ and for each $x^1 \otimes \cdots \otimes x^l \in \otimes_{i=1}^{l}\mathbb{R}^{d_i}$ we have:

$$\begin{aligned} &g_Q\left(x^1 \otimes \cdots \otimes x^l\right) = \left\|x^1\right\|_1 \cdots \left\|x^l\right\|_l \quad \text{and} \\ &\left\|\left\langle \cdot, x^1 \otimes \cdots \otimes x^l\right\rangle_H\right\| = \left\|\left\langle \cdot, x^1\right\rangle\right\| \cdots \left\|\left\langle \cdot, x^l\right\rangle\right\|, \end{aligned} \tag{11}$$

where $\langle \cdot, x^i\rangle$ is a linear funtional on $\left(\mathbb{R}^{d_i}, \|\cdot\|_i\right)$, $i = 1, \ldots, l$. Now, if we fix an arbitrary $a^1 \otimes \cdots \otimes a^l \in \partial Q$, then $g_Q\left(a^1 \otimes \cdots \otimes a^l\right) = \|a^1\|_1 \cdots \left\|a^l\right\|_l = 1$, and

$$\begin{aligned} &g_Q\left(a^1 \otimes \cdots a^{i-1} \otimes x^i \otimes a^{i+1} \otimes \cdots \otimes a^l\right) \\ &= \left\|a^1\right\|_1 \cdots \left\|a^{i-1}\right\|_{i-1} \left\|x^i\right\|_i \left\|a^{i+1}\right\|_{i+1} \cdots \left\|a^l\right\|_l = \frac{1}{\left\|a^i\right\|_i} \left\|x^i\right\|_i. \end{aligned}$$

From the definition of $Q_i^{a^1,\ldots,a^l}$, we obtain $g_{Q_i^{a^1,\ldots,a^l}}(x^i) = \frac{1}{\|a^i\|_i}\|x^i\|_i$ for $i = 1, \ldots, l$ and $Q_i^{a^1,\ldots,a^l} = \|a^i\|_i B_{\left(\mathbb{R}^{d_i},\|\cdot\|_i\right)}$, equivalent to $g_{\left(Q_i^{a^1,\ldots,a^l}\right)^\circ}(x^i) = \|a^i\|_i \|\langle\cdot, x^i\rangle\|$, for $i = 1, \ldots, l$. Hence, from (11) and the previous equalities we have:

$$\begin{aligned} g_Q\left(x^1 \otimes \cdots \otimes x^l\right) &= g_{Q_1^{a^1,\ldots,a^l}}\left(x^1\right) \cdots g_{Q_l^{a^1,\ldots,a^l}}\left(x^l\right) \quad \text{and} \\ g_{Q^\circ}\left(x^1 \otimes \cdots \otimes x^l\right) &= \left\|\left\langle \cdot, x^1 \otimes \cdots \otimes x^l\right\rangle_H\right\| = \left\|\left\langle \cdot, x^1\right\rangle\right\| \cdots \left\|\left\langle \cdot, x^l\right\rangle\right\| \\ &= g_{\left(Q_1^{a^1,\ldots,a^l}\right)^\circ}\left(x^1\right) \cdots g_{\left(Q_l^{a^1,\ldots,a^l}\right)^\circ}\left(x^l\right). \end{aligned}$$

Therefore, by Proposition 3.1, (10) holds.

To prove the converse, suppose that Q satisfies (10) for $a^1 \otimes \cdots \otimes a^l \in \partial Q$, then from Proposition 3.1, we conclude that g_Q is a reasonable crossnorm on $\otimes_{i=1}^{l}\left(\mathbb{R}^{d_i}, g_{Q_i^{a^1,\ldots,a^l}}\right)$. This completes the proof. $\square$

Now, we introduce the formal notion of a tensorial body:

Definition 3.3. A 0-symmetric convex body $Q \subset \otimes_{i=1}^{l}\mathbb{R}^{d_i}$ is called a *tensorial body in* $\otimes_{\mathbf{i=1}}^{\mathbf{l}}\mathbb{R}^{\mathbf{d_i}}$ if there exist 0-symmetric convex bodies $Q_i \subset \mathbb{R}^{d_i}$, $i = 1, ..., l$ such that

$$Q_1 \otimes_\pi \cdots \otimes_\pi Q_l \subseteq Q \subseteq Q_1 \otimes_\epsilon \cdots \otimes_\epsilon Q_l.$$

If Q satisfies the inclusions in Definition 3.3, we will say that Q is a *tensorial body with respect to* $Q_1, \ldots, Q_l$. By $\mathcal{B}_\otimes\left(\otimes_{i=1}^{l}\mathbb{R}^{d_i}\right)$ we denote the set of tensorial bodies in $\otimes_{i=1}^{l}\mathbb{R}^{d_i}$. The set of tensorial bodies w.r.t. $Q_1, ..., Q_l$ is denoted by $\mathcal{B}_{Q_1,...,Q_l}\left(\otimes_{i=1}^{l}\mathbb{R}^{d_i}\right)$.

In the next corollary, we summarize the relation between tensorial bodies in $\otimes_{i=1}^{l}\mathbb{R}^{d_i}$ and reasonable crossnorms. We omit its proof, since it follows directly from Proposition 3.1 and Theorem 3.2.

Corollary 3.4. *Let $Q \subset \otimes_{i=1}^{l}\mathbb{R}^{d_i}$ be a 0-symmetric convex body. The following are equivalent:*

(1) *Q is a tensorial body in $\otimes_{i=1}^{l}\mathbb{R}^{d_i}$.*

(2) *Q satisfies* (10) *for any $a^1 \otimes \cdots \otimes a^l \in \partial Q$.*

(3) *There exist norms $\|\cdot\|_i$ on $\mathbb{R}^{d_i}$, $i = 1, ..., l$, such that g_Q is a reasonable crossnorm on $\otimes_{i=1}^{l}\left(\mathbb{R}^{d_i}, \|\cdot\|_i\right)$.*

In this case, $g_{Q_i^{a^1,\ldots,a^l}}(\cdot) = \frac{1}{\|a^i\|_i}\|\cdot\|_i$ for $i = 1, \ldots, l$.

Proposition 3.5. *Let $Q \subset \otimes_{i=1}^{l}\mathbb{R}^{d_i}$ be a tensorial body. Then Q° and λQ, $\lambda > 0$, are tensorial bodies. Indeed, if $Q \in \mathcal{B}_{Q_1,...,Q_l}\left(\otimes_{i=1}^{l}\mathbb{R}^{d_i}\right)$ for some 0-symmetric convex bodies $Q_i \subset \mathbb{R}^{d_i}$, $i = 1, ..., l$, then*

(1) $Q^\circ \in \mathcal{B}_{Q_1^\circ,...,Q_l^\circ}\left(\otimes_{i=1}^{l}\mathbb{R}^{d_i}\right)$.

(2) $\lambda Q \in \mathcal{B}_{Q_1,...,(\lambda Q_k),...,Q_l}\left(\otimes_{i=1}^{l}\mathbb{R}^{d_i}\right)$.

Proof. (1). If $Q \subset \otimes_{i=1}^{l}\mathbb{R}^{d_i}$ is a tensorial body with respect to $Q_1, \ldots, Q_l$ then

$$Q_1 \otimes_\pi \cdots \otimes_\pi Q_l \subseteq Q \subseteq Q_1 \otimes_\epsilon \cdots \otimes_\epsilon Q_l.$$

Thus, $(Q_1 \otimes_\epsilon \cdots \otimes_\epsilon Q_l)^\circ \subseteq Q^\circ \subseteq (Q_1 \otimes_\pi \cdots \otimes_\pi Q_l)^\circ$. By Proposition 2.1, this implies that $Q_1^\circ \otimes_\pi \cdots \otimes_\pi Q_l^\circ \subseteq Q^\circ \subseteq Q_1^\circ \otimes_\epsilon \cdots \otimes_\epsilon Q_l^\circ$ which is equivalent to $Q^\circ \in \mathcal{B}_{Q_1^\circ,...,Q_l^\circ}\left(\otimes_{i=1}^{l}\mathbb{R}^{d_i}\right)$.

(2) We assume, w.l.o.g., that $k = 1$. To prove this part, it is enough to observe that, by definition, for each real number $\lambda > 0$, we have $\lambda(Q_1 \otimes_\pi \cdots \otimes_\pi Q_l) = (\lambda Q_1) \otimes_\pi \cdots \otimes_\pi Q_l$ and

$$\begin{aligned}\lambda Q_1 \otimes_\epsilon \cdots \otimes_\epsilon Q_l &= \lambda\left(Q_1^\circ \otimes_\pi \cdots \otimes_\pi Q_l^\circ\right)^\circ = \left(\lambda^{-1}(Q_1^\circ \otimes_\pi \cdots \otimes_\pi Q_l^\circ)\right)^\circ \\ &= \left((\lambda Q_1)^\circ \otimes_\pi \cdots \otimes_\pi Q_l^\circ\right)^\circ = (\lambda Q_1) \otimes_\epsilon \cdots \otimes_\epsilon Q_l.\end{aligned}$$

Therefore $\lambda Q \in \mathcal{B}_{(\lambda Q_1),...,Q_l}\left(\otimes_{i=1}^{l}\mathbb{R}^{d_i}\right)$, if $Q \in \mathcal{B}_{Q_1,...,Q_l}\left(\otimes_{i=1}^{l}\mathbb{R}^{d_i}\right)$. □

A tensorial body in $\otimes_{i=1}^{l}\mathbb{R}^{d_i}$ is a tensorial body with respect to an essentially unique l-tuple of convex bodies. More precisely:

Proposition 3.6. *Let $P_i, Q_i \subset \mathbb{R}^{d_i}$, $i = 1\ldots, l$ be 0-symmetric convex bodies. If*

$$Q \in \mathcal{B}_{P_1,\ldots,P_l}\left(\otimes_{i=1}^{l}\mathbb{R}^{d_i}\right) \cap \mathcal{B}_{Q_1,\ldots,Q_l}\left(\otimes_{i=1}^{l}\mathbb{R}^{d_i}\right),$$

then there exist real numbers $\lambda_i > 0$, $i = 1,\ldots,l$ such that $\lambda_1\cdots\lambda_l = 1$ and $P_i = \lambda_i Q_i$ for $i = 1,...,l$.

Proof. Let g_Q, g_{Q_i} and g_{P_i} be the Minkowski functionals associated to Q, Q_i and P_i respectively. If Q is a tensorial body with respect to P_i, $i = 1,\ldots,l$, and with respect to Q_i, $i = 1,\ldots,l$, then Proposition 3.1 implies that:

$$g_{P_1}\left(x^1\right)\cdots g_{P_l}\left(x^l\right) = g_Q\left(x^1\otimes\cdots\otimes x^l\right) = g_{Q_1}\left(x^1\right)\cdots g_{Q_l}\left(x^l\right).$$

Therefore, if we fix $a^1\otimes\cdots\otimes a^l \in \partial Q$, then

$$g_{Q_1}\left(a^1\right)\cdots g_{Q_l}\left(a^l\right) = g_{P_1}\left(a^1\right)\cdots g_{P_l}\left(a^l\right) = 1.$$

Analogously, for $a^1\otimes\cdots\otimes a^{i-1}\otimes x^i\otimes a^{i+1}\otimes\cdots\otimes a^l$, $i = 1,...,l$, we have:

$$\begin{aligned}&g_{Q_1}\left(a^1\right)\cdots g_{Q_{i-1}}\left(a^{i-1}\right)g_{Q_i}\left(x^i\right)g_{Q_{i+1}}\left(a^{i+1}\right)\cdots g_{Q_l}\left(a^l\right)\\ &= g_{P_1}\left(a^1\right)\cdots g_{P_{i-1}}\left(a^{i-1}\right)g_{P_i}\left(x^i\right)g_{P_{i+1}}\left(a^{i+1}\right)\cdots g_{P_l}\left(a^l\right).\end{aligned}$$

Now, if we multiply both sides of the above equation by $g_{Q_i}\left(a^i\right)g_{P_i}\left(a^i\right)$, we obtain $g_{P_i}\left(a^i\right)g_{Q_i}\left(x^i\right) = g_{Q_i}\left(a^i\right)g_{P_i}\left(x^i\right)$, equivalent to $g_{P_i}\left(x^i\right) = \frac{g_{P_i}\left(a^i\right)}{g_{Q_i}\left(a^i\right)}g_{Q_i}\left(x^i\right)$. Thus, if $\lambda_i := \frac{g_{Q_i}\left(a^i\right)}{g_{P_i}\left(a^i\right)}$, $i = 1,...,l$ then we have proved that $\lambda_1\cdots\lambda_l = 1$ and $P_i = \lambda_i Q_i$, as required. □

In order to simplify the arguments, we will choose the convex bodies defined (9) in a specific way: for every 0-symmetric convex body $Q \subset \otimes_{i=1}^{l}\mathbb{R}^{d_i}$, Q^i will denote the convex bodies generated by $e_1^{d_1}\otimes\cdots\otimes(\lambda e_1^{d_l})$, $\lambda = 1/(g_Q(e_1^{d_1}\otimes\cdots\otimes e_1^{d_l}))$. That is, $Q^i := Q_i^{e_1^{d_1},\ldots,\lambda e_l^{d_l}}$ for $i = 1,\ldots,l$.

Proposition 3.7. *If Q_n, $n \in \mathbb{N}$, are tensorial bodies in $\otimes_{i=1}^{l}\mathbb{R}^{d_i}$ such that g_{Q_n} converges uniformly on compact sets to g_Q, for some 0-symmetric convex body Q, then Q is a tensorial body in $\otimes_{i=1}^{l}\mathbb{R}^{d_i}$. In this case, $g_{Q_n^i}$ and $g_{(Q_n^i)^\circ}$ converge uniformly on compact sets to g_{Q^i} and $g_{(Q^i)^\circ}$, respectively.*

Proof. Since Q_n, $n \in \mathbb{N}$, are tensorial bodies, then (2) of Corollary 3.4 implies that $Q_n^1\otimes_\pi\cdots\otimes_\pi Q_n^l \subseteq Q_n \subseteq Q_n^1\otimes_\epsilon\cdots\otimes_\epsilon Q_n^l$, for $n \in \mathbb{N}$.

Suppose that we already proved the uniform convergence (on compact sets) of $g_{Q_n^i}$ to g_{Q^i}. From this, it follows that $g_{(Q_n^i)^\circ}$ converges uniformly on compact sets to $g_{(Q^i)^\circ}$. Thus, we get:

$$g_Q\left(x^1\otimes\cdots\otimes x^l\right)=\lim_{n\to\infty}g_{Q_n}\left(x^1\otimes\cdots\otimes x^l\right)=\lim_{n\to\infty}g_{Q_n^1}\left(x^1\right)\cdots g_{Q_n^l}\left(x^l\right)$$
$$=g_{Q^1}\left(x^1\right)\cdots g_{Q^l}\left(x^l\right).$$

Similarly, since the uniform convergence of g_{Q_n} to g_Q implies the convergence $g_{Q_n^\circ}$ to g_{Q°, we have $g_{Q^\circ}\left(x^1\otimes\cdots\otimes x^l\right)=g_{(Q^1)^\circ}\left(x^1\right)\cdots g_{\left(Q^l\right)^\circ}\left(x^l\right)$. Therefore, from Proposition 3.1, Q is a tensorial body w.r.t. Q^i, $i=1,\dots,l$.

Now, we turn to prove that for each $i=1,\dots,l$, $g_{Q_n^i}$ converges pointwise to g_{Q^i}. Then, by [25, Theorem 1.8.12], we know that this implies the uniform convergence. From the convergence of g_{Q_n} to g_Q, and the definition of Q^l, Q_n^l, it follows directly that $g_{Q_n^l}$ converges pointwise to g_{Q^l}. For the case $i=1,\dots,l-1$, it is enough to observe that

$$g_{Q_n^i}\left(x^i\right)=\frac{1}{g_{Q_n}\left(e_1^{d_1}\otimes\cdots\otimes e_1^{d_l}\right)}g_{Q_n}\left(e_1^{d_1}\otimes\cdots\otimes e_1^{d_{i-1}}\otimes x^i\otimes e_1^{d_{i+1}}\otimes\cdots\otimes e_1^{d_l}\right),$$

for $x^i\in\mathbb{R}^{d_i}$. Thus, from the definiton of Q^i and the convergence of g_{Q_n} to g_Q, we know that $g_{Q_n^i}$ converges pointwise g_{Q^i}. □

3.1. Examples of tensorial bodies

3.1.1. The trivial case

Every 0-symmetric convex body $Q\subset\mathbb{R}\otimes\mathbb{R}^d$ is a tensorial body in $\mathbb{R}\otimes\mathbb{R}^d$:

Proposition 3.8. *Let $Q\subset\mathbb{R}\otimes\mathbb{R}^d$ be a 0-symmetric convex body. Then we have $Q=[-1,1]\otimes_\pi\tilde{Q}$ where $[-1,1]=\{\lambda\in\mathbb{R}:-1\le\lambda\le1\}$ and $\tilde{Q}:=\{x\in\mathbb{R}^d: 1\otimes x\in Q\}$.*

Proof. Let $u\in\mathbb{R}\otimes\mathbb{R}^d$, then $u=\sum_{i=1}^N\lambda_i\otimes\ x_i=1\otimes\left(\sum_{i=1}^N\lambda_ix_i\right)$. Thus, $u\in Q$ if and only if $u=1\otimes\tilde{u}$ with $\tilde{u}=\sum_{i=1}^N\lambda_ix_i\in\tilde{Q}$. Therefore, from the definition of $[-1,1]\otimes_\pi\tilde{Q}$ we obtain the desired result. □

The proposition above and (6) show that every 0-symmetric convex body Q in $\mathbb{R}\otimes\mathbb{R}^d$ is the closed unit ball associated to the projective tensor norm on $\mathbb{R}\otimes\left(\mathbb{R}^d,g_{\tilde{Q}}\right)$. It is also worth to notice that on $\mathbb{R}\otimes\mathbb{R}^d$, the projective and the injective tensor product of 0-symmetric convex bodies are equal.

3.1.2. The closed unit balls of ℓ_p^d.

Proposition 3.9. *Let $d\in\mathbb{N}$. For every factorization of d in natural numbers $d=d_1\cdots d_l$ and for every $1\le p\le\infty$, $B_p^d=B_p^{d_1,\dots,d_l}$ is a tensorial body in $\otimes_{i=1}^l\mathbb{R}^{d_i}$. It holds that*

$$B_p^{d_1}\otimes_\pi\cdots\otimes_\pi B_p^{d_l}\subseteq B_p^d=B_p^{d_1,\dots,d_l}\subseteq B_p^{d_1}\otimes_\epsilon\cdots\otimes_\epsilon B_p^{d_l},\ 1\le p\le\infty.$$

In the cases where $1<p<\infty$, $d_i\ge2$, $i=1,\dots,l$, $B_p^{d_1,\dots,d_l}$ is not the projective nor the injective tensor product of $B_p^{d_i}$, $i=1,\dots,l$.

Proof. We will use the notation fixed in Example 2.1. The cases $p = 1, \infty$ were already proved in Proposition 2.2. We will give the proof for $1 < p < \infty$.

Let $x^i \in \mathbb{R}^{d_i}$, $i = 1, \ldots, l$ then:

$$g_{B_p^{d_1,\ldots,d_l}}\left(x^1 \otimes \cdots \otimes x^l\right) = \left(\sum\nolimits_{j_1,\ldots,j_l} \left|\left\langle x^1 \otimes \cdots \otimes x^l, e_{j_1}^{d_1} \otimes \cdots \otimes e_{j_l}^{d_l}\right\rangle_H\right|^p\right)^{\frac{1}{p}}$$
$$= \left\|x^1\right\|_p \cdots \left\|x^l\right\|_p .$$

Thus, from the last equality for p^* and the relation $\left(B_p^{d_1,\ldots,d_l}\right)^\circ = B_{p^*}^{d_1,\ldots,d_l}$ we have that $g_{\left(B_p^{d_1,\ldots,d_l}\right)^\circ}\left(x^1 \otimes \cdots \otimes x^l\right) = \|x^1\|_{p^*} \cdots \left\|x^l\right\|_{p^*}$. Therefore, by Proposition 3.1, $B_p^{d_1,\ldots,d_l}$ is a tensorial body w.r.t. $B_p^{d_i}$, $i = 1 \ldots, l$.

To prove the other statement, first observe that if $d_i = 1$, $i = 1, \ldots, l-1$ then $B_p^{1,\ldots,d_l} = B_p^1 \otimes_\epsilon \cdots \otimes_\epsilon B_p^{d_l} = B_p^1 \otimes_\epsilon \cdots \otimes_\epsilon B_p^{d_l}$. To avoid this case, we assume that each $d_i \geq 2$.

Let $E \subset \otimes_{\pi,i=1}^l \ell_p^{d_i}$ be the vector space generated by $\left\{e_1^{d_1} \otimes \cdots \otimes e_1^{d_l}, e_2^{d_1} \otimes \cdots \otimes e_2^{d_l}\right\}$. Then, from [2, Theorem 1.3,], it follows that E is isometrically isomorphic to ℓ_r^2 for $\frac{1}{r} = min\{1, \frac{l}{p}\}$. Hence, for each $u = a_1 e_1^{d_1} \otimes \cdots \otimes e_1^{d_l} + a_2 e_2^{d_1} \otimes \cdots \otimes e_2^{d_l} \in E$ we have $\pi(u) = (|a_1|^r + |a_2|^r)^{\frac{1}{r}}$. Therefore, $B_p^{d_1,\ldots,d_l} \neq B_p^{d_1} \otimes_\pi \cdots \otimes_\pi B_p^{d_l}$.

Now, suppose that $B_p^{d_1,\ldots,d_l} = B_p^{d_1} \otimes_\epsilon \cdots \otimes_\epsilon B_p^{d_l}$ for some $1 < p < \infty$. Then, from Proposition 2.1, $B_{p^*}^{d_1,\ldots,d_l} = B_{p^*}^{d_1} \otimes_\pi \cdots \otimes_\pi B_{p^*}^{d_l}$. Since the latter equality is not possible, we must have $B_p^{d_1,\ldots,d_l} \neq B_p^{d_1} \otimes_\epsilon \cdots \otimes_\epsilon B_p^{d_l}$ for all $1 < p < \infty$. □

Proposition 3.9 together with Corollary 3.4 imply that B_p^d, $1 \leq p \leq \infty$, is the closed unit ball associated to a reasonable crossnorm on $\otimes_{i=1}^l \ell_p^{d_i}$ for any factorization $d = d_1 \cdots d_l$.

3.1.3. A convex body in $\mathbb{R}^{mn} = \mathbb{R}^m \otimes \mathbb{R}^n$ which is not a tensorial body in $\mathbb{R}^m \otimes \mathbb{R}^n$

Let $m, n \geq 2$ be natural numbers. Let

$$\mathcal{E} = \left\{z = \sum_{i,j=1}^{m,n} z_{ij} e_i^m \otimes e_j^n : \frac{|z_{11}|^2}{3} + \frac{|z_{mn}|^2}{2} + \sum_{(i,j)\neq(1,1),(m,n)} |z_{ij}|^2 \leq 1\right\}.$$

Then $\mathcal{E} \in \mathcal{B}(\mathbb{R}^m \otimes \mathbb{R}^n) \setminus \mathcal{B}_\otimes(\mathbb{R}^m \otimes \mathbb{R}^n)$. To verify this, consider the convex bodies generated by $e_1^m \otimes \sqrt{3} e_1^n$ and $e_m^m \otimes \sqrt{2} e_n^n$, according to the relation (9):

$$\mathcal{E}_1^{e_1^m, \sqrt{3}e_1^n} = \left\{x \in \mathbb{R}^m : x \otimes \sqrt{3} e_1^n \in \mathcal{E}\right\}, \quad \mathcal{E}_1^{e_m^m, \sqrt{2}e_n^n} = \left\{x \in \mathbb{R}^m : x \otimes \sqrt{2} e_n^n \in \mathcal{E}\right\}.$$

We proceed by contradiction. Suppose that $\mathcal{E}$ is a tensorial body in $\mathbb{R}^m \otimes \mathbb{R}^n$, then Corollary 3.4 and Proposition 3.6 imply that there exists $\lambda_1 > 0$ such that $\mathcal{E}_1^{e_1^m, \sqrt{3}e_1^n} = \lambda_1 \mathcal{E}_1^{e_m^m, \sqrt{2}e_n^n}$.

However, for every $x = (x_1, \ldots, x_m) \in \mathbb{R}^m$ we have:

$$g_{\mathcal{E}_1^{e_1^m, \sqrt{3}e_1^n}}(x) = g_{\mathcal{E}}\left(x \otimes \sqrt{3}e_1^n\right) = \sqrt{|x_1|^2 + 3\sum_{i=2}^m |x_i|^2},$$

$$g_{\mathcal{E}_1^{e_m^m, \sqrt{2}e_n^n}}(x) = g_{\mathcal{E}}\left(x \otimes \sqrt{2}e_n^n\right) = \sqrt{|x_m|^2 + 2\sum_{i=1}^{m-1} |x_i|^2}.$$

Thus $\mathcal{E}_1^{e_1^m, \sqrt{3}e_1^n} \neq \lambda \mathcal{E}_1^{e_m^m, \sqrt{2}e_n^n}$ for all $\lambda > 0$. This is a contradiction, hence Q is not a tensorial body in $\mathbb{R}^m \otimes \mathbb{R}^n$.

Analogous examples $\mathcal{E}$ so that $\mathcal{E} \in \mathcal{B}(\otimes_{i=1}^l \mathbb{R}^{d_i}) \setminus \mathcal{B}_\otimes(\otimes_{i=1}^l \mathbb{R}^{d_i})$, can be constructed in $\otimes_{i=1}^l \mathbb{R}^{d_i}$ when $l \geq 2$.

Remark 3.10. The previous example, in contrast with Example 3.1.1 (the trivial case), shows that if $d = mn$ is not a trivial factorization of d (i.e. if $m \neq 1$ or $n \neq 1$) then $\mathcal{B}_\otimes(\mathbb{R}^m \otimes \mathbb{R}^n) \subsetneq \mathcal{B}(\mathbb{R}^m \otimes \mathbb{R}^n)$. As a consequence being a tensorial body depends on the tensor decomposition defined on $\mathbb{R}^d$.

3.2. Linear isomorphisms preserving tensorial bodies

A linear map $T\colon \otimes_{i=1}^l \mathbb{R}^{d_i} \to \otimes_{i=1}^l \mathbb{R}^{d_i}$ *preserves decomposable vectors* if the vector $T\left(x^1 \otimes \cdots \otimes x^l\right)$ is decomposable for every $x^i \in \mathbb{R}^{d_i}$, $i = 1, \ldots, l$. By $GL_\otimes\left(\otimes_{i=1}^l \mathbb{R}^{d_i}\right)$ we denote the set of linear isomorphisms $T : \otimes_{i=1}^l \mathbb{R}^{d_i} \to \otimes_{i=1}^l \mathbb{R}^{d_i}$ preserving decomposable vectors. To shorten notation we usually write $GL_\otimes$.

Linear mappings preserving decomposable vectors have been deeply studied. For an account on this topic as well as for the fundamentals about it, we refer the reader to [19, 20, 29, 30]. In [19, Corollary 2.14], it is proved that if $d_i \geq 2$, $i = 1, \ldots, l$, then for each element $T \in GL_\otimes\left(\otimes_{i=1}^l \mathbb{R}^{d_i}\right)$ there exists a permutation σ on $\{1, ..., l\}$ and linear isomorphisms $T_i : \mathbb{R}^{d_{\sigma(i)}} \to \mathbb{R}^{d_i}$, $i = 1, ..., l$ such that for every $x^1 \otimes \cdots \otimes x^l \in \otimes_{i=1}^l \mathbb{R}^{d_i}$,

$$T\left(x^1 \otimes \cdots \otimes x^l\right) = T_1\left(x^{\sigma(1)}\right) \otimes \cdots \otimes T_l\left(x^{\sigma(l)}\right). \tag{12}$$

Using this characterization and the fact that the set of decomposable vectors is closed in $\otimes_{H,i=1}^l \mathbb{R}^{d_i}$ (see Proposition 2.3), we can easily obtain the next result.

Proposition 3.11. *Let $d_i \geq 2$, $i = 1, \ldots, l$ be natural numbers.*
Then $GL_\otimes\left(\otimes_{i=1}^l \mathbb{R}^{d_i}\right)$ is a closed subgroup of $GL\left(\otimes_{H,i=1}^l \mathbb{R}^{d_i}\right)$.

Theorem 3.12. *($GL_\otimes$ preserves tensorial bodies). Assume $d_i \geq 2$ for $i = 1, ..., l$. If $T \in GL_\otimes\left(\otimes_{i=1}^l \mathbb{R}^{d_i}\right)$ and $Q \in \mathcal{B}_\otimes\left(\otimes_{i=1}^l \mathbb{R}^{d_i}\right)$, then $TQ \in \mathcal{B}_\otimes\left(\otimes_{i=1}^l \mathbb{R}^{d_i}\right)$.*

Proof. Suppose that Q is a tensorial body in $\otimes_{i=1}^l \mathbb{R}^{d_i}$, then (8) holds for suitable 0-symmetric convex bodies $Q_i \subset \mathbb{R}^{d_i}$, $i = 1, \ldots, l$.

On the other hand, let T be an element in $GL_\otimes\left(\otimes_{i=1}^l \mathbb{R}^{d_i}\right)$ and let T_i, $i = 1, ..., l$, be as in (12). Then, by the definition of $\otimes_\pi$, we have:

$$T\left(Q_1 \otimes_\pi \cdots \otimes_\pi Q_l\right) = T_1 Q_{\sigma(1)} \otimes_\pi \cdots \otimes_\pi T_l Q_{\sigma(l)}. \tag{13}$$

Similarly, $$T\left(Q_1 \otimes_\epsilon \cdots \otimes_\epsilon Q_l\right) = \left((T^t)^{-1}\left(Q_1^\circ \otimes_\pi \cdots \otimes_\pi Q_l^\circ\right)\right)^\circ$$
$$= \left((T_1^t)^{-1}Q_{\sigma(1)}^\circ \otimes_\pi \cdots \otimes_\pi (T_l^t)^{-1}Q_{\sigma(l)}^\circ\right)^\circ = T_1 Q_{\sigma(1)} \otimes_\epsilon \cdots \otimes_\epsilon T_l Q_{\sigma(l)}.$$

Therefore, $T_1 Q_{\sigma(1)} \otimes_\pi \cdots \otimes_\pi T_l Q_{\sigma(l)} \subseteq TQ \subseteq T_1 Q_{\sigma(1)} \otimes_\epsilon \cdots \otimes_\epsilon T_l Q_{\sigma(l)}$. This proves that TQ is a tensorial body in $\otimes_{i=1}^l \mathbb{R}^{d_i}$. □

A Banach-Mazur type distance

From now on, we will assume that each space $\mathbb{R}^{d_i}$, $i = 1, \ldots, l$ has dimension $d_i \geq 2$. Using Theorem 3.12 we are able to define a distance $\delta_\otimes^{BM}$ between tensorial bodies in $\otimes_{i=1}^l \mathbb{R}^{d_i}$, which is the analogue, for tensorial bodies, of the Banach-Mazur distance.

Recall that the *Banach-Mazur distance* between isomorphic Banach spaces X and Y is defined as:

$$\delta^{BM}(X,Y) := \inf\left\{\|T\|\left\|T^{-1}\right\| : T \in \mathcal{L}(X,Y) \text{ and } T^{-1} \in \mathcal{L}(Y,X)\right\}.$$

Between 0-symmetric convex bodies in a Euclidean space $\mathbb{E}$, it is defined as:

$$\delta^{BM}(P,Q) := \inf\left\{\lambda \geq 1 : \begin{array}{l} T\colon \mathbb{E} \to \mathbb{E} \text{ is a bijective linear} \\ \text{map and } Q \subseteq TP \subseteq \lambda Q \end{array}\right\}. \tag{14}$$

A complete exposition of the Banach-Mazur distance and its properties can be found in [27].

Let P, Q be tensorial bodies in $\otimes_{i=1}^l \mathbb{R}^{d_i}$. We define the *tensorial Banach-Mazur distance* $\delta_\otimes^{BM}(P,Q)$ as follows:

$$\delta_\otimes^{BM}(P,Q) := \inf\left\{\lambda \geq 1 : Q \subseteq TP \subseteq \lambda Q, \text{ for } T \in GL_\otimes\left(\otimes_{i=1}^l \mathbb{R}^{d_i}\right)\right\}. \tag{15}$$

It is well defined, since for every $P, Q \in \mathcal{B}_\otimes\left(\otimes_{i=1}^l \mathbb{R}^{d_i}\right)$ there exist real numbers $r_1, r_2 > 0$ such that $Q \subseteq r_1 P \subseteq r_2 Q$. It holds that

$$\delta^{BM}(P,Q) \leq \delta_\otimes^{BM}(P,Q) \text{ for } P, Q \in \mathcal{B}_\otimes\left(\otimes_{i=1}^l \mathbb{R}^{d_i}\right). \tag{16}$$

Using Proposition 3.11 and Theorem 3.12, it can be directly proved that for each pair $P, Q \subset \otimes_{i=1}^l \mathbb{R}^{d_i}$ of tensorial bodies, the infimum in (15) attains its value at some $\lambda > 0$ and some $T \in GL_\otimes\left(\otimes_{i=1}^l \mathbb{R}^{d_i}\right)$. Indeed, it is possible to define the following equivalence relation:

For every $P, Q \in \mathcal{B}_\otimes\left(\otimes_{i=1}^l \mathbb{R}^{d_i}\right)$, $P \sim Q$ *if and only if* $\delta_\otimes^{BM}(P,Q) = 1$.

We denote $\mathcal{BM}_\otimes\left(\otimes_{i=1}^l \mathbb{R}^{d_i}\right)$ the set of equivalence classes of tensorial bodies determined by this relation. Elementary arguments show that $\log \delta_\otimes^{BM}$ is a metric on this set. Moreover, this metric gives rise to a Banach-Mazur type compactum of tensorial bodies:

Theorem 3.13. $\left(\mathcal{BM}_{\otimes}\left(\otimes_{i=1}^{l}\mathbb{R}^{d_i}\right), \log\delta_{\otimes}^{BM}\right)$ *is a compact metric space.*

Proof. The proof is essentially a standard argument of compactness. Given a tensorial body $P \subset \otimes_{i=1}^{l}\mathbb{R}^{d_i}$, $[P]$ denotes its associate equivalence class.

Let $\{[P_n]\}$ be a sequence in $\mathcal{BM}_{\otimes}\left(\otimes_{i=1}^{l}\mathbb{R}^{d_i}\right)$ and let P_n^i be the convex sets introduced in Proposition 3.7. By Corollary 3.4, $P_n^1\otimes_\pi\cdots\otimes_\pi P_n^l \subseteq P_n \subseteq P_n^1\otimes_\epsilon\cdots\otimes_\epsilon P_n^l$. Thus, from [11, Proposition 2.4], it follows for every $n\in\mathbb{N}$ that

$$P_n^1\otimes_\pi\cdots\otimes_\pi P_n^l \subseteq P_n \subseteq \frac{d}{d_l}P_n^1\otimes_\pi\cdots\otimes_\pi P_n^l. \tag{17}$$

On the other hand, from a general well known fact, for every P_n^i, $i=1,...,l$ there exists a linear isomorphism $T_{i,n}:\mathbb{R}^{d_i}\to\mathbb{R}^{d_i}$, such that $B_1^{d_i}\subseteq T_{i,n}P_n^i\subseteq d_iB_1^{d_i}$. Hence, applying $T_{1,n}\otimes\cdots\otimes T_{l,n}$ to (17) together with (13), we have

$$B_1^{d_1}\otimes_\pi\cdots\otimes_\pi B_1^{d_l}\subseteq\left(T_{1,n}\otimes\cdots\otimes T_{l,n}\right)P_n\subseteq\frac{d^2}{d_l}B_1^{d_1}\otimes_\pi\cdots\otimes_\pi B_1^{d_l}.$$

Now, for each $n\in\mathbb{N}$ denote by Q_n the tensorial body $\left(T_{1,n}\otimes\cdots\otimes T_{l,n}\right)P_n$. By the Arzela-Ascoli theorem, there is a subsequence $\left\{g_{Q_{n_k}}\right\}$ converging uniformly (on compact sets of $\otimes_{i=1}^{l}\mathbb{R}^{d_i}$) to g_Q for some 0-symmetric convex body Q. Hence, by Proposition 3.7, Q is a tensorial body in $\otimes_{i=1}^{l}\mathbb{R}^{d_i}$.

It remains to show that $[P_{n_k}]$ converges to $[Q]$. To prove this, notice that the uniform convergence of $g_{Q_{n_k}}$ to g_Q implies that the indentity map

$$I_k:\left(\otimes_{i=1}^{l}\mathbb{R}^{d_i}, g_{Q_{n_k}}\right)\to\left(\otimes_{i=1}^{l}\mathbb{R}^{d_i}, g_Q\right)$$

is such that $\lim_{k\to\infty}\|I_k\|\left\|I_k^{-1}\right\|=1$. Thus, $\lim_{k\to\infty}\delta_{\otimes}^{BM}\left(Q_{n_k},Q\right)=1$. Since

$$\delta_{\otimes}^{BM}\left(P_{n_k},Q\right)=\delta_{\otimes}^{BM}\left(Q_{n_k},Q\right),$$

we conclude that $[P_{n_k}]$ converges to $[Q]$ as required. □

We finish this section by giving some upper bounds for the tensorial Banach-Mazur distance $\delta_{\otimes}^{BM}$.

Proposition 3.14. *Let $P_i,Q_i\subset\mathbb{R}^{d_i}$, $i=1,\dots,l$ be 0-symmetric convex bodies. Then,*

(1) $\delta^{BM}\left(P_1\otimes_\pi\cdots\otimes_\pi P_l, Q_1\otimes_\pi\cdots\otimes_\pi Q_l\right)\le\delta^{BM}\left(P_1,Q_1\right)\cdots\delta^{BM}\left(P_l,Q_l\right).$

(2) $\delta^{BM}\left(P_1\otimes_\epsilon\cdots\otimes_\epsilon P_l, Q_1\otimes_\epsilon\cdots\otimes_\epsilon Q_l\right)\le\delta^{BM}\left(P_1,Q_1\right)\cdots\delta^{BM}\left(P_l,Q_l\right).$

Proof. We give the proof only for the projective tensor product of 0-symmetric convex bodies. The proof for $\otimes_\epsilon$ is analogous. First, we will show that for each $i\in\{1,...,l\}$ the following inequality holds:

$$\delta^{BM}(P_1\otimes_\pi...\otimes_\pi P_i\otimes_\pi...\otimes_\pi P_l, P_1\otimes_\pi...\otimes_\pi Q_i\otimes_\pi...\otimes_\pi P_l)\le\delta^{BM}(P_i,Q_i). \tag{18}$$

Let $\lambda \geq \delta^{BM}(P_i, Q_i)$. Then, $Q_i \subseteq T_i(P_i) \subseteq \lambda Q_i$ for some linear isomorphism $T_i : \mathbb{R}^{d_i} \to \mathbb{R}^{d_i}$. By (13) we have

$$\begin{aligned} &P_1 \otimes_\pi \cdots \otimes_\pi T_i P_i \otimes_\pi \cdots \otimes_\pi P_l \\ &= I_{\mathbb{R}^{d_1}} \otimes \cdots \otimes T_i \otimes \cdots \otimes I_{\mathbb{R}^{d_l}} \left(P_1 \otimes_\pi \cdots \otimes_\pi P_i \otimes_\pi \cdots \otimes_\pi P_l\right). \end{aligned}$$

Therefore, if $S = I_{\mathbb{R}^{d_1}} \otimes \cdots \otimes T_i \otimes \cdots \otimes I_{\mathbb{R}^{d_l}}$ then

$$\begin{aligned} &P_1 \otimes_\pi \cdots \otimes_\pi Q_i \otimes_\pi \cdots \otimes_\pi P_l \subseteq S\left(P_1 \otimes_\pi \cdots \otimes_\pi P_i \otimes_\pi \cdots \otimes_\pi P_l\right) \\ &\subseteq P_1 \otimes_\pi \cdots \otimes_\pi \lambda Q_i \otimes_\pi \cdots \otimes_\pi P_l = \lambda\left(P_1 \otimes_\pi \cdots \otimes_\pi Q_i \otimes_\pi \cdots \otimes_\pi P_l\right), \end{aligned}$$

which implies (18). To prove (1), observe that from the multiplicative triangle inequality of δ^{BM} and (18) we have:

$$\begin{aligned} &\delta^{BM}\left(P_1 \otimes_\pi \cdots \otimes_\pi P_l, Q_1 \otimes_\pi \cdots \otimes_\pi Q_l\right) \\ &\leq \prod_{i=1}^{l} \delta^{BM}\left(Q_1 \otimes_\pi \cdots \otimes_\pi Q_{i-1} \otimes_\pi P_i \otimes_\pi P_{i+1} \otimes_\pi \cdots \otimes_\pi P_l,\right. \\ &\qquad \left. Q_1 \otimes_\pi \cdots \otimes_\pi Q_{i-1} \otimes_\pi Q_i \otimes_\pi P_{i+1} \otimes_\pi \cdots \otimes_\pi P_l\right) \\ &\leq \delta^{BM}(P_1, Q_1) \cdots \delta^{BM}(P_l, Q_l). \end{aligned}$$

□

Using the previous proposition and [11, Proposition 2.4], we obtain the following upper bound for the tensorial Banach-Mazur distance.

Corollary 3.15. *For every pair of tensorial bodies $P, Q \subset \otimes_{i=1}^{l} \mathbb{R}^{d_i}$ we have:*

(1) $\delta_{\otimes}^{BM}(P, Q) \leq (d_1 \cdots d_{l-1})^2 \left(\prod_{i=1}^{l} \delta^{BM}(P^i, Q^i)\right)$.

(2) $\delta_{\otimes}^{BM}(P, Q) \leq (d_1 \cdots d_{l-1})^2 (d_1 \cdots d_l)$.

4. Tensorial ellipsoids

In this section we give a complete description of the ellipsoids in $\otimes_{i=1}^{l} \mathbb{R}^{d_i}$ which are also tensorial bodies (Corollary 4.3). To this end, we first introduce some definitions.

Recall that an ellipsoid $\mathcal{E} \subset V$ in a vector space of dimension d is defined as the image of the Euclidean ball B_2^d by a linear isomorphism $T\colon \mathbb{R}^d \to V$. In the case of ellipsoids in $\otimes_{i=1}^{l} \mathbb{R}^{d_i}$, alternatively, we will say that $\mathcal{E} \subset \otimes_{i=1}^{l} \mathbb{R}^{d_i}$ is an *ellipsoid* if $\mathcal{E} = T(B_2^{d_1,\ldots,d_l})$ for some linear isomorphism $T\colon \otimes_{i=1}^{l} \mathbb{R}^{d_i} \to \otimes_{i=1}^{l} \mathbb{R}^{d_i}$, providing we have identified $B_2^d = B_2^{d_1,\ldots,d_l}$ (see Subsection 2.1).

Definition 4.1. An ellipsoid $\mathcal{E} \subset \otimes_{i=1}^{l} \mathbb{R}^{d_i}$ is called a *tensorial ellipsoid in* $\otimes_{\mathbf{i=1}}^{\mathbf{l}} \mathbb{R}^{\mathbf{d_i}}$ if $\mathcal{E}$ is also a tensorial body in $\otimes_{i=1}^{l} \mathbb{R}^{d_i}$. The set of tensorial ellipsoids in $\otimes_{i=1}^{l} \mathbb{R}^{d_i}$ will be denoted by $\mathscr{E}_{\otimes}\left(\otimes_{i=1}^{l} \mathbb{R}^{d_i}\right)$.

If $\mathcal{E}_i = T_i\left(B_2^{d_i}\right)$, $i = 1, \ldots, l$ are ellipsoids in $\mathbb{R}^{d_i}$, $i = 1, ..., l$ respectively, then the *Hilbertian tensor product of* $\mathcal{E}_1, \ldots, \mathcal{E}_l$, introduced in [3], is defined as

$$\mathcal{E}_1 \otimes_2 \cdots \otimes_2 \mathcal{E}_l := T_1 \otimes \cdots \otimes T_l \left(B_2^{d_1,\ldots,d_l}\right).$$

It can be directly proved that $\mathcal{E}_1 \otimes_2 \cdots \otimes_2 \mathcal{E}_l$ is the closed unit ball of the Hilbert tensor product $\otimes_{H,i=1}^{l} \left(\mathbb{R}^{d_i}, g_{\mathcal{E}_i}\right)$. Thus, Hilbertian tensor products of ellipsoids are the first examples of tensorial ellipsoids. In particular for the Euclidean ball we have:

$$B_2^{d_1,\ldots,d_l} = B_2^{d_1} \otimes_2 \cdots \otimes_2 B_2^{d_l} \in \mathscr{E}_{\otimes} \left(\otimes_{i=1}^{l} \mathbb{R}^{d_i}\right).$$

Actually, in Theorem 4.2 we prove that $B_2^{d_1,\ldots,d_l}$ is the only ellipsoid between $B_2^{d_1} \otimes_\pi \cdots \otimes_\pi B_2^{d_l}$ and $B_2^{d_1} \otimes_\epsilon \cdots \otimes_\epsilon B_2^{d_l}$. From this, we obtain that the only tensorial ellipsoids are the Hilbertian tensor product of ellipsoids (Corollary 4.3).

Theorem 4.2. *Let $\mathcal{E} \subset \otimes_{i=1}^{l} \mathbb{R}^{d_i}$ be an ellipsoid such that*

$$B_2^{d_1} \otimes_\pi \cdots \otimes_\pi B_2^{d_l} \subset \mathcal{E} \subset B_2^{d_1} \otimes_\epsilon \cdots \otimes_\epsilon B_2^{d_l}, \tag{19}$$

then $\mathcal{E} = B_2^{d_1,\ldots,d_l}$.

We will give the proof of the theorem at the end of the section. Before, we will prove Corollary 4.3 and several related results.

Corollary 4.3. *If $\mathcal{E}$ is a tensorial ellipsoid in $\otimes_{i=1}^{l} \mathbb{R}^{d_i}$, then there exist linear isomorphisms $T_i : \mathbb{R}^{d_i} \to \mathbb{R}^{d_i}$ for $i = 1, \ldots, l$ such that*

$$\mathcal{E} = T_1 \otimes \cdots \otimes T_l \left(B_2^{d_1,\ldots,d_l}\right) = T_1 \left(B_2^{d_1}\right) \otimes_2 \cdots \otimes_2 T_l \left(B_2^{d_l}\right).$$

Proof. Assume that $\mathcal{E}$ belongs to $\mathscr{E}_{\otimes} \left(\otimes_{i=1}^{l} \mathbb{R}^{d_i}\right)$. Then there exist 0-symmetric convex bodies $A_i \subset \mathbb{R}^{d_i}$, $i = 1, ..., l$ such that

$$A_1 \otimes_\pi \cdots \otimes_\pi A_l \subset \mathcal{E} \subset A_1 \otimes_\epsilon \cdots \otimes_\epsilon A_l.$$

Since $\mathcal{E}$ is an ellipsoid we must have that all A_i, $i = 1, \ldots, l$ are ellipsoids. Thus, there exist linear isomorphisms $T_i : \mathbb{R}^{d_i} \to \mathbb{R}^{d_i}$, $i = 1, \ldots, l$ with $A_i = T_i \left(B_2^{d_i}\right)$. From this and Theorem 3.12, we obtain:

$$T_1 \left(B_2^{d_1}\right) \otimes_\pi \cdots \otimes_\pi T_l \left(B_2^{d_l}\right) \subset \mathcal{E} \subset T_1 \left(B_2^{d_1}\right) \otimes_\epsilon \cdots \otimes_\epsilon T_l \left(B_2^{d_l}\right),$$

$$B_2^{d_1} \otimes_\pi \cdots \otimes_\pi B_2^{d_l} \subset \left(T_1^{-1} \otimes \cdots \otimes T_l^{-1}\right) \mathcal{E} \subset B_2^{d_1} \otimes_\epsilon \cdots \otimes_\epsilon B_2^{d_l}.$$

Therefore, Theorem 4.2 implies that $\left(T_1^{-1} \otimes \cdots \otimes T_l^{-1}\right) \mathcal{E} = B_2^{d_1,\ldots,d_l}$. Thus we have $\mathcal{E} = T_1 \otimes \cdots \otimes T_l \left(B_2^{d_1,\ldots,d_l}\right)$ or equivalently $\mathcal{E} = T_1 \left(B_2^{d_1}\right) \otimes_2 \cdots \otimes_2 T_l \left(B_2^{d_l}\right)$. □

Every ellipsoid $\mathcal{E} = T(B_2^{d_1,\ldots,d_l}) \subset \otimes_{i=1}^{l} \mathbb{R}^{d_i}$ is the closed unit ball associated to the scalar product $\langle \cdot, \cdot \rangle_{\mathcal{E}} := \langle T^{-1}(\cdot), T^{-1}(\cdot) \rangle_H$. In view of this, the following proposition describes the relation between $\langle \cdot, \cdot \rangle_{\mathcal{E}}$ and $\langle \cdot, \cdot \rangle_H$ on decomposable vectors, when $\mathcal{E}$ is a tensorial ellipsoid in $\mathbb{R}^m \otimes \mathbb{R}^n$.

Proposition 4.4. *Let m, n be natural numbers. If $\mathcal{E} = T\left(B_2^{m,n}\right) \subset \mathbb{R}^m \otimes \mathbb{R}^n$ is an ellipsoid, then*

$$B_2^m \otimes_\pi B_2^n \subset \mathcal{E} \subset B_2^m \otimes_\epsilon B_2^n \tag{20}$$

if and only if for $L = T^{-1}, T^t$ the following relations hold :

$$\langle x, z\rangle\langle y, w\rangle = \left[\langle L\left(x\otimes y\right), L\left(z\otimes w\right)\rangle_H + \langle L\left(x\otimes w\right), L\left(z\otimes y\right)\rangle_H\right]/2 \qquad (21)$$

for each $x, z \in \mathbb{R}^m$ and $y, w \in \mathbb{R}^n$.

Proof. Recall, if $\mathcal{E} = T\left(B_2^{m,n}\right) \subset \mathbb{R}^m \otimes \mathbb{R}^n$ is an ellipsoid then $\mathcal{E}^\circ = \left(T^t\right)^{-1}\left(B_2^{m,n}\right)$. Assume that (21) holds for T^{-1} and T^t. Then, if we make $x = y$ and $z = w$ in (21), we have $g_{\mathcal{E}}\left(x\otimes y\right) = \|x\|_2\|y\|_2$ and $g_{\mathcal{E}^\circ}\left(x\otimes y\right) = \|x\|_2\|y\|_2$. Thus, from Proposition 3.1, we get that (20) holds.

Assume that (20) holds. Let $x, z \in \mathbb{R}^m$ and $y, w \in \mathbb{R}^n$. From Proposition 3.1, we know that $g_{\mathcal{E}}\left(x\otimes y\right) = \|x\|_2\|y\|_2$. Thus, $\|T^{-1}\left(x\otimes y\right)\|_H = \|x\|_2\|y\|_2$.

Now, the polarization formula applied to $\langle T^{-1}\left(x\otimes y\right), T^{-1}\left(x\otimes w\right)\rangle_H$ and the latter equality imply:

$$\left\langle T^{-1}\left(x\otimes y\right), T^{-1}\left(x\otimes w\right)\right\rangle_H = \|x\|_2^2\langle y, w\rangle. \qquad (22)$$

From the polarization formula and (22), we have

$$\langle x, z\rangle\langle y, w\rangle = \left[(\|x+z\|_2^2 - \|x-z\|_2^2)/4\right]\langle y, w\rangle.$$

Thus, using (22) in the last equality, we get that (21) holds for $L = T^{-1}$. To finish the proof, observe that $\mathcal{E}^\circ$ also satisfies (20), see Proposition 3.5. Hence, (21) holds for T^t. □

Lemma 4.5. *Let $\mathcal{E}$ be a tensorial ellipsoid in $\otimes_{i=1}^l \mathbb{R}^{d_i}$. For every $z^l \in \partial B_2^{d_l}$, let*

$$i_{z^l} : \mathbb{R}^{d_1}\otimes\cdots\otimes\mathbb{R}^{d_{l-1}} \to \mathbb{R}^{d_1}\otimes\cdots\otimes\mathbb{R}^{d_{l-1}}\otimes\mathbb{R}^{d_l},$$
$$x^1\otimes\cdots\otimes x^{l-1} \to x^1\otimes\cdots\otimes x^{l-1}\otimes z^l,$$

and $\mathcal{E}_{z^l} := i_{z^l}^{-1}\left(\mathcal{E}\right)$. Then, if

$$B_2^{d_1}\otimes_\pi\cdots\otimes_\pi B_2^{d_l} \subset \mathcal{E} \subset B_2^{d_1}\otimes_\epsilon\cdots\otimes_\epsilon B_2^{d_l}$$

one has $$B_2^{d_1}\otimes_\pi\cdots\otimes_\pi B_2^{d_{l-1}} \subset \mathcal{E}_{z^l} \subset B_2^{d_1}\otimes_\epsilon\cdots\otimes_\epsilon B_2^{d_{l-1}}.$$

Proof. Let $\langle\cdot,\cdot\rangle_{\mathcal{E}}$ be the scalar product associated to $\mathcal{E}$. From the definition of $\mathcal{E}_{z^l}$, we know that it is an ellipsoid. By $\langle\cdot,\cdot\rangle_{z^l}$, $g_{\mathcal{E}_{z^l}}(\cdot)$ we denote the scalar product and the Minkowski functional determined by $\mathcal{E}_{z^l}$. Thus, for every $u \in \mathbb{R}^{d_1}\otimes\cdots\otimes\mathbb{R}^{d_{l-1}}$, we have $g_{\mathcal{E}_{z^l}}(u) = g_{\mathcal{E}}\left(i_{z^l}(u)\right)$. Since $\mathcal{E}$ is a tensorial ellipsoid, from Proposition 3.1, $g_{\mathcal{E}_{z^l}}\left(x^1\otimes\cdots\otimes x^{l-1}\right) = \|x^1\|_2\cdots\left\|x^{l-1}\right\|_2$. We also have:

$$\begin{aligned} g_{\left(\mathcal{E}_{z^l}\right)^\circ}\left(x^1\otimes\cdots\otimes x^{l-1}\right) &= \sup_{g_{\mathcal{E}_{z^l}}(a)\le 1}\left|\left\langle a, x^1\otimes\cdots\otimes x^{l-1}\right\rangle_H\right| \\ &= \sup_{g_{\mathcal{E}}\left(i_{z^l}(a)\right)\le 1}\left|\left\langle i_{z^l}(a), x^1\otimes\cdots\otimes x^{l-1}\otimes z^l\right\rangle_H\right| \\ &\le g_{\mathcal{E}^\circ}\left(x^1\otimes\cdots\otimes x^{l-1}\otimes z^l\right) = \left\|x^1\right\|_2\cdots\left\|x^{l-1}\right\|_2. \end{aligned}$$

Therefore, from Proposition 3.1, we know that $\mathcal{E}_{z^l}$ is a tensorial body w. r. t. $B_2^{d_i}$, $i = 1, \ldots, l-1$. $\square$

Proof of Theorem 4.2. The proof will be divided into two parts. First, we will prove the theorem for tensorial ellipsoids in $\mathbb{R}^m \otimes \mathbb{R}^n$. Then, for the general case we will use induction on l, the number of factors on the tensor product $\otimes_{i=1}^l \mathbb{R}^{d_i}$.

Step 1. Take an ellipsoid $\mathcal{E} \subset \mathbb{R}^m \otimes \mathbb{R}^n$ such that $B_2^m \otimes_\pi B_2^n \subset \mathcal{E} \subset B_2^m \otimes_\epsilon B_2^n$. If $\mathcal{E} = T\left(B_2^{m,n}\right)$ for some linear isomorphism on $\mathbb{R}^m \otimes \mathbb{R}^n$, then from Proposition 4.4, (21) holds for T^{-1}, T^t. Thus, for $x, z \in \mathbb{R}^m$, $y, w \in \mathbb{R}^n$ and $S = TT^t$ we have:

$$\langle x, z\rangle \langle y, w\rangle = \left[\left\langle S^{-1}(x\otimes y), z\otimes w\right\rangle_H + \left\langle S^{-1}(x\otimes w), z\otimes y\right\rangle_H\right]/2,$$
$$\langle x, z\rangle \langle y, w\rangle = \left[\left\langle S(x\otimes y), z\otimes w\right\rangle_H + \left\langle S(x\otimes w), z\otimes y\right\rangle_H\right]/2.$$

On the other hand, for the canonical basis $\{e_\sigma\}_{\sigma=1,\ldots,d} \subseteq \mathbb{R}^d$, $\{e_i\}_{i=1,\ldots,m} \subseteq \mathbb{R}^m$ and $\{e_j\}_{j=1,\ldots,n} \subseteq \mathbb{R}^n$ let $\Phi_{(m,n)} : \mathbb{R}^m \otimes \mathbb{R}^n \to \mathbb{R}^d$ be the bijective map such that $\Phi_{(m,n)}(e_i \otimes e_j) = e_{(i-1)n+j}$ for $1 \le i \le m$ and $1 \le j \le n$. Clearly, $\Phi_{(m,n)}$ preserves $\langle\cdot,\cdot\rangle_H$ and the standard scalar product $\langle\cdot,\cdot\rangle$ on $\mathbb{R}^d$. Hence if $\tilde{S} := \Phi_{(m,n)} S \Phi_{(m,n)}^{-1}$ we have:

$$\left\langle e_{(i-1)n+j}, e_{(k-1)n+l}\right\rangle = \left[\left\langle \tilde{S}^{-1} e_{(i-1)n+j}, e_{(k-1)n+l}\right\rangle + \left\langle \tilde{S}^{-1} e_{(i-1)n+l}, e_{(k-1)n+j}\right\rangle\right]/2,$$
$$\left\langle e_{(i-1)n+j}, e_{(k-1)n+l}\right\rangle = \left[\left\langle \tilde{S} e_{(i-1)n+j}, e_{(k-1)n+l}\right\rangle + \left\langle \tilde{S} e_{(i-1)n+l}, e_{(k-1)n+j}\right\rangle\right]/2,$$

for $1 \le i, k \le m$ and $1 \le j, l \le n$. Therefore, for $W = \tilde{S}, \tilde{S}^{-1}$:

$$\left\langle W e_{(i-1)n+j}, e_{(k-1)n+l}\right\rangle = \begin{cases} 1 & \text{if } k = i, l = j \\ 0 & \text{if } k = i, l \neq j \\ 0 & \text{if } k \neq i, l = j \\ -\left\langle W e_{(i-1)n+l}, e_{(k-1)n+j}\right\rangle & \text{if } k \neq i, l \neq j \end{cases} \tag{23}$$

Hence, the positive definite matrices associated to $\tilde{S}, \tilde{S}^{-1}$ can be written using the matrices

$$A_{ki} := \left(\left\langle \tilde{S} e_{(i-1)n+j}, e_{(k-1)n+l}\right\rangle\right)_{l,j} \quad \text{and} \quad B_{ki} := \left(\left\langle \tilde{S}^{-1} e_{(i-1)n+j}, e_{(k-1)n+l}\right\rangle\right)_{l,j}.$$

That is, $\tilde{S} = (A_{ki})_{k,i}$ and $\tilde{S}^{-1} = (B_{ki})_{k,i}$ for $1 \le k, i \le m$. Clearly, $A_{ki}, B_{ki} \in M_{n,n}(\mathbb{R})$ for all $1 \le k, i \le m$. Moreover, from (23), it follows that A_{ki} and B_{ki}, $k \neq i$, are antysimmetric matrices and $A_{kk} = B_{kk} = I_n$, $k = i$ (I_n is the identity matrix). From this and the symmetry of $\tilde{S}, \tilde{S}^{-1}$, we know that $A_{ik} = -A_{ki}$. Thus, $\tilde{S}, \tilde{S}^{-1}$ satisfy (24) (see Lemma 4.6 at the end of this section) and $\tilde{S} = I_d$. The latter implies that the linear isomorphism T is such that $TT^t = I_d$, so it is an orthogonal map on $\otimes_{H,i=1}^l \mathbb{R}^{d_i}$ and $\mathcal{E} = B_2^{m,n}$. This finishes the first part of the proof.

Step 2. As we mentioned at the beginning of the proof, this case will be proved by induction on the number l of factors on the tensor product. To simplify the notation, in this part of the proof we use the symbol $\|\cdot\|_Q$ to denote the Minkowski functional associated to a 0-symmetric convex body Q.

The case $l = 2$ was already proved. Now we assume that the result holds for $l-1$. This means that for every tensorial ellipsoid $\mathcal{E} \subset \otimes_{i=1}^{l-1}\mathbb{R}^{d_i}$ such that

$$B_2^{d_1} \otimes_\pi \cdots \otimes_\pi B_2^{d_{l-1}} \subset \mathcal{E} \subset B_2^{d_1} \otimes_\epsilon \cdots \otimes_\epsilon B_2^{d_{l-1}},$$

we have $\mathcal{E} = B_2^{d_1,\ldots,d_{l-1}}$.

Let $\mathcal{E}$ be a tensorial ellipsoid in $\otimes_{i=1}^{l}\mathbb{R}^{d_i}$ satisfying (19), and let $\|\cdot\|_{\mathcal{E}}$ be its Minkowski functional. By Lemma 4.5, for every $z^l \in \partial B_2^{d_l}$ we have $B_2^{d_1} \otimes_\pi \cdots \otimes_\pi B_2^{d_{l-1}} \subset \mathcal{E}_{z^l} \subset B_2^{d_1} \otimes_\epsilon \cdots \otimes_\epsilon B_2^{d_{l-1}}$. Applying the induction hypothesis we obtain $\mathcal{E}_{z^l} = B_2^{d_1,\ldots,d_{l-1}}$. Therefore, for every $\sum_{i=1}^N x_i^1 \otimes \cdots \otimes x_i^{l-1} \in \otimes_{i=1}^{l-1}\mathbb{R}^{d_i}$ we have

$$\begin{aligned} &\left\|\sum\nolimits_{i=1}^{N} x_i^1 \otimes \cdots \otimes x_i^{l-1} \otimes z^l\right\|_{\mathcal{E}} \\ &= \left\|\sum\nolimits_{i=1}^{N} x_i^1 \otimes \cdots \otimes x_i^{l-1}\right\|_{\mathcal{E}_{z^l}} = \left\|\sum\nolimits_{i=1}^{N} x_i^1 \otimes \cdots \otimes x_i^{l-1}\right\|_{H}. \end{aligned}$$

Since, by Proposition 3.5, $\mathcal{E}^\circ$ also satisfies (19) , we also have

$$\left\|\sum\nolimits_{i=1}^{N} x_i^1 \otimes \cdots \otimes x_i^{l-1} \otimes z^l\right\|_{\mathcal{E}^\circ} = \left\|\sum\nolimits_{i=1}^{N} x_i^1 \otimes \cdots \otimes x_i^{l-1}\right\|_{H}.$$

Now, consider the canonical isomorphism

$$\begin{aligned} \psi : \left(\mathbb{R}^{d_1} \otimes \cdots \otimes \mathbb{R}^{d_{l-1}}\right) \otimes \mathbb{R}^{d_l} &\to \mathbb{R}^{d_1} \otimes \cdots \otimes \mathbb{R}^{d_{l-1}} \otimes \mathbb{R}^{d_l} \\ \left(x^1 \otimes \cdots \otimes x^{l-1}\right) \otimes x^l &\to x^1 \otimes \cdots \otimes x^{l-1} \otimes x^l, \end{aligned}$$

and denote by $\tilde{\mathcal{E}}$ the ellipsoid in $\left(\mathbb{R}^{d_1} \otimes \cdots \otimes \mathbb{R}^{d_{l-1}}\right) \otimes \mathbb{R}^{d_l}$ determined by this isomorphism and $\mathcal{E}$.

Then, for each non-zero $x^l \in \mathbb{R}^{d_l}$ and $u = \sum_{i=1}^N x_i^1 \otimes \cdots \otimes x_i^{l-1} \in \mathbb{R}^{d_1} \otimes \cdots \otimes \mathbb{R}^{d_{l-1}}$ we have

$$\begin{aligned} \left\|u \otimes x^l\right\|_{\tilde{\mathcal{E}}} &= \left\|\sum\nolimits_{i=1}^{N} x_i^1 \otimes \cdots \otimes x_i^{l-1} \otimes x^l\right\|_{\mathcal{E}} \\ &= \left\|\sum\nolimits_{i=1}^{N} x_i^1 \otimes \cdots \otimes x_i^{l-1}\right\|_{H} \left\|x^l\right\|_2 = \|u\|_H \left\|x^l\right\|_2, \quad \text{and} \\ \left\|u \otimes x^l\right\|_{\tilde{\mathcal{E}}^\circ} &= \sup_{a \in \tilde{\mathcal{E}}} \left|\left\langle a, u \otimes x^l\right\rangle_H\right| \\ &= \sup_{\|\psi(a)\|_{\mathcal{E}} \le 1} \left|\left\langle \psi(a), \sum\nolimits_{i=1}^{N} x_i^1 \otimes \cdots \otimes x_i^{l-1} \otimes x^l\right\rangle_H\right| \\ &\le \left\|\sum\nolimits_{i=1}^{N} x_i^1 \otimes \cdots \otimes x_i^{l-1} \otimes x^l\right\|_{\mathcal{E}^\circ} \\ &= \left\|\sum\nolimits_{i=1}^{N} x_i^1 \otimes \cdots \otimes x_i^{l-1}\right\|_{H} \left\|x^l\right\|_2 = \|u\|_H \left\|x^l\right\|_2. \end{aligned}$$

Thus, by Proposition 3.1, $\tilde{\mathcal{E}}$ is a tensorial ellipsoid in $\left(\mathbb{R}^{d_1} \otimes \cdots \otimes \mathbb{R}^{d_{l-1}}\right) \otimes \mathbb{R}^{d_l}$, and $B_2^{d_1,\ldots,d_{l-1}} \otimes_\pi B_2^{d_l} \subset \tilde{\mathcal{E}} \subset B_2^{d_1,\ldots,d_{l-1}} \otimes_\epsilon B_2^{d_l}$.

Now, let $d = d_1 \cdots d_{l-1}$. Then, by the identification $B_2^{d_1,\ldots,d_{l-1}} = B_2^d$ (see Subsection 2.1) and the case of two factors proved in Step 1., we know that $\tilde{\mathcal{E}} = B_2^{d,d_l}$. Finally, since $\mathcal{E} = \psi\left(\tilde{\mathcal{E}}\right)$ and ψ is an orthogonal map, we have that $\mathcal{E} = B_2^{d_1,\ldots,d_l}$ which is the desired result. $\square$

The next lemma is used in the proof of Theorem 4.2. For a given $n \in \mathbb{N}$, $I_n \in M_{n\times n}(\mathbb{R})$ will denote the identity matrix of dimension n.

Lemma 4.6. *Let $m, n \in \mathbb{N}$ and $d = mn$. If $S \in M_{d\times d}(\mathbb{R})$ is a positive definite matrix and there exist antisymmetric matrices $A_{ki}, B_{ki} \in M_{n,n}(\mathbb{R})$, $k = 1, \ldots, m-1$, $i = k+1, \ldots, m$ such that*

$$S = \begin{bmatrix} I_n & A_{12} & \ldots & A_{1,m} \\ -A_{12} & I_n & \ldots & A_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ -A_{1,m} & -A_{2,m} & \ldots & I_n \end{bmatrix}, \; S^{-1} = \begin{bmatrix} I_n & B_{12} & \ldots & B_{1,m} \\ -B_{12} & I_n & \ldots & B_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ -B_{1,m} & -B_{2,m} & \ldots & I_n \end{bmatrix}. \tag{24}$$

Then $S = I_d$.

Proof. Let $n \geq 1$ be fixed. We will prove the result by induction on m.

Step 1. If $m = 1$, then $d = n$. By definition of S, $S = I_d$ and the result is proved. Assume now that $m = 2$. In this case,

$$S = \begin{bmatrix} I_n & A_{12} \\ -A_{12} & I_n \end{bmatrix} \text{ and } S^{-1} = \begin{bmatrix} I_n & B_{12} \\ -B_{12} & I_n. \end{bmatrix}$$

Then
$$SS^{-1} = \begin{bmatrix} I_n - A_{12}B_{12} & A_{12} + B_{12} \\ -A_{12} - B_{12} & I_n - A_{12}B_{12} \end{bmatrix} = \begin{bmatrix} I_n & 0 \\ 0 & I_n \end{bmatrix}.$$

Thus, $B_{12} = -A_{12}$ and $A_{12}^2 = 0$. Since A_{12} is antisymmetric, the latter equality implies that $A_{12}^t A_{12} = 0$ so $A_{12} = 0$ which completes the proof.

Step 2. Assume the result is valid for $m-1$. By E, F, G, H we denote the following matrices:

$$E := \begin{bmatrix} I_n & A_{12} & \ldots & A_{1,m-1} \\ -A_{12} & I_n & \cdots & A_{2,m-1} \\ \vdots & \vdots & \ddots & \vdots \\ -A_{1,m-1} & -A_{2,m-1} & \cdots & I_n \end{bmatrix}_{n(m-1),n(m-1)}, \quad F := \begin{bmatrix} A_{1,m} \\ A_{2,m} \\ \vdots \\ A_{m-1,m} \end{bmatrix}_{n(m-1),n},$$

$$G := \begin{bmatrix} I_n & B_{12} & \ldots & B_{1,m-1} \\ -B_{12} & I_n & \cdots & B_{2,m-1} \\ \vdots & \vdots & \ddots & \vdots \\ -B_{1,m-1} & -B_{2,m-1} & \cdots & I_n \end{bmatrix}_{n(m-1),n(m-1)}, \quad H := \begin{bmatrix} B_{1,m} \\ B_{2,m} \\ \vdots \\ B_{m-1,m} \end{bmatrix}_{n(m-1),n},$$

Clearly, since S is a positive definite matrix then S^{-1}, E, G are positive definite matrices. Also,

$$S = \begin{bmatrix} E & F \\ F^t & I_n \end{bmatrix}, \quad S^{-1} = \begin{bmatrix} G & H \\ H^t & I_n \end{bmatrix}$$

and $$SS^{-1} = \begin{bmatrix} EG + FH^t & EH + FI_n \\ F^tG + I_nH^t & F^tH + I_n \end{bmatrix} = \begin{bmatrix} I_{n(m-1)} & 0_{n(m-1),n} \\ 0_{n,n(m-1)} & I_n \end{bmatrix}.$$

Therefore, $F^tH + I_n = I_n$ and $F^tH = 0_{n,n}$. Since we also have $F^tG + I_nH^t = 0_{n,n(m-1)}$ then $H^t = -F^tG$. This yields

$$H = -GF. \tag{25}$$

From the previous equations we get $0 = F^tH = -F^tGF$ and

$$F^tGF = 0_{n,n}. \tag{26}$$

Now, if we write F_i, $i = 1, 2, \ldots, n$ for the columns of F, then from (26) we have

$$F_i^tGF_i = 0. \tag{27}$$

Since G is a positive definite matrix, from (27) we know that each $F_i = 0$, $i = 1, 2, \ldots, n$ and $F = 0$. This and (25) imply $H = 0$.

Finally, we are in position to apply our inductive hypothesis to E and $E^{-1} = G$. Then, $E = I_{n(m-1)}$ which implies $S = I_d$. □

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