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Convexity of the Distance Function to Convex Subsets of Riemannian Manifolds

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A characterization of the proximal normal cone is obtained and a separation theorem for convex subsets of Riemannian manifolds is established. Moreover, the convexity of the distance function d_S for a convex subset S in the cases where the boundary of S contains a geodesic segment, the boundary of S is C^2 or the boundary of S is not regular is discussed. Furthermore, a nonsmooth version of positive semi-definiteness of the Hessian of convex functions on Riemannian manifolds is established.

Keywords: Distance function, proximal normal cone, convexity, Riemannian manifold.

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1. Introduction

Convexity is an old and important notion which has a key role in many areas of geometry such as properties of projection map [3, 17], geometrical and topological restriction of Riemannian manifolds [8, 18] and Monge-Ampère equations [4, 10]. Since the development of the theory of nonsmooth analysis, the notion of convexity has been studied widely from different points of view and its various applications have been employed in many branches of mathematics such as weak solutions of partial differential equations, variational analysis and optimization; see [13, 14] and the references therein.

This paper addresses the question of how the curvature affects the convexity of the distance function. Indeed, it is well known that if M is a Hadamard manifold and $S \subseteq M$ is a closed convex subset, then the distance function d_S is convex; see [16]. Hence it is natural to ask in an arbitrary Riemannian manifold M (without any assumptions on the curvature of M) whether the distance function d_S is

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convex. Considering a short segment of a great circle on the sphere as a convex set S, one can see that d_S is not convex on any open neighborhoods of S. Similar examples can be found on every manifold of positive curvature. In this paper, we explore why the convexity of the distance function d_S fails in these examples and by imposing some restrictions on S, the convexity of the distance function d_S is discussed. For this aim, we first study the behavior of closed convex subsets of a Riemannian manifold M and to deal with the boundary of S we use the tools from nonsmooth analysis such as the proximal normal cone. Indeed, we show that the proximal normal cone is a generalization of the normal bundle of Riemannian submanifolds of M for arbitrary subsets of M.

By using this fact, we characterize the elements of the proximal normal cone by projection map and derive the nonsmooth analogous of tubular neighborhood theorem in Riemannian manifolds. We refer the reader to [2], where the concept of the proximal normal cone was considered in Hilbert spaces. Studying this problem leads us to obtain a separation theorem for convex subsets of Riemannian manifolds which help us to study the convexity of the distance function by means of the support principle for convexity. Moreover, the well-known result regarding the positive definiteness of the Hessian of a C^2 convex function on Riemannian manifolds is generalized to continuous convex functions by using the second order superjets.

The paper is organized as follows. Section 2 is devoted to the study of the proximal normal cone and the metric projection in Riemannian manifolds. We present a characterization of the proximal normal cone which provides a separation theorem in the setting of manifolds. In Section 3, by imposing certain conditions on the boundary of locally convex subsets of Riemannian manifolds, we discuss the convexity of the distance function to these subsets.

Let us give a quick review of notations and concepts we need in the sequel. In this paper, we use the standard notations and known results of Riemannian manifolds; see, e.g., [5, 16]. Throughout this paper, $(M, \langle ., . \rangle)$ is a complete finite dimensional Riemannian manifold and $d(x, y)$ is the Riemannian distance on M. If X and Y are smooth vector fields, then $\nabla_X Y$ will stand for the covariant derivative of X with respect to Y. For a smooth vector field V along a smooth curve γ we will often denote $V'(t) = \nabla_{\dot{\gamma}(t)} V(t)$. The interior, closure and boundary of a subset A of M will be denoted by A°, $\overline{A}$ and bdry A, respectively.

A subset S of M is said to be *convex* (or *strongly convex*) if for every $x, y \in S$ there exists a unique minimizing geodesic from x to y lying entirely in S. By the Whitehead theorem there exists a convex neighborhood around each point in M. In such a neighborhood we have *parallel translation* $l_{xy} : T_x M \to T_y M$, that is, a linear isometry which sends each vector v_x to its unique parallel vector w_y. It is easy to see that $l_{xy} \exp_x^{-1} y = -\exp_y^{-1} x$. Besides one can show that the covariant derivative of a vector field $V(t)$ along a short geodesic $\gamma(t)$ can be recovered from parallel translation, by the following formula

$$V'(t_0) = \lim_{t \to t_0} \frac{l_{\gamma(t)\gamma(t_0)} V(t) - V(t_0)}{t - t_0}. \tag{1}$$

The *convexity radius* of M at x is denoted by $r(x)$. Indeed,

$$r(x) = \sup \left\{ r > 0 : \begin{array}{l} \text{any metric ball in } B(x,r) \text{ is convex and any geodesic seg-} \\ \text{ment in } B(x,r) \text{ is a minimal geodesic joing its end points} \end{array} \right\}.$$

A closed subset S is called *locally convex* if for every $x \in S$, there exists an $\varepsilon(x)$ such that $0 < \varepsilon(x) \leq r(x)$ and $S \cap B(x, \varepsilon(x))$ is convex. Note that by the Cartan-Hadamrd theorem, the notions of convexity and local convexity agree in a Hadamard manifold. A real valued function f on an open set $U \subset M$ is called *convex* if $f \circ \alpha$ is convex for every geodesic α in U. One can see that a C^2 function f is convex on U if and only if the Hessian $d^2 f(x) \geq 0$ for every $x \in U$. Recall that the Hessian of a C^2 function f on a Riemannian manifold M is defined by

$$d^2 f(v, w) = \langle \nabla_v \nabla f, w \rangle,$$

where ∇f is the gradient of f and $v, w \in T_x M$.

We also use the notion of Jacobi fields. A vector field along a geodesic satisfying the Jacobi equation is called a *Jacobi field.* If $\alpha\colon [0,a] \times [-\varepsilon, \varepsilon] \to M$ is a variation, the length and energy functionals are defined by

$$L(s) := L(\alpha_s) = \int_0^a \|\dot{\alpha}_s(t)\| dt, \quad \text{and} \quad E(s) := E(\alpha_s) = \int_0^a \|\dot{\alpha}_s(t)\|^2 dt.$$

Notice that if α is a variation through geodesics then

$$L(s)^2 = aE(s). \tag{2}$$

The following lemma which its proof is similar to [5, Lemma 12.3.1] with a slight modification, is substantial in the next section.

Lemma 1.1. *Suppose that M is a complete Riemannian manifold such that all its sectional curvature is bounded above by a constant $\delta > 0$. Suppose that U is a convex subset of M and a, b, and c are three distinct points in U which do not lie on a geodesic segment. Let $d(a,c) < \pi/\sqrt{\delta}$ and $d(b,c) < \pi/\sqrt{\delta}$. These three points determine a unique geodesic triangle $\triangle(a,b,c)$ in U with vertices a, b and c. Let α, β and γ be the angles of vertices a, b and c, respectively and let A, B and C be the lengths of the sides opposite the vertices a, b and c, respectively. Then*

$$\cos(\sqrt{\delta}A)\cos(\sqrt{\delta}B) + \sin(\sqrt{\delta}A)\sin(\sqrt{\delta}B)\cos(\gamma) \geq \cos(\sqrt{\delta}C).$$

2. Proximal normal cone and metric projection

Let us begin this section with the definition of proximal normal cones of subsets of Riemannian manifolds; see [11].

Definition 2.1. Let S be a closed subset of a Riemannian manifold M and $x \in S$. We define the *proximal normal cone* of S at x by

$$N_S^P(x) = \left\{ v \in T_x M : \begin{array}{l} \langle v, \exp_x^{-1} y \rangle \leq \sigma \| \exp_x^{-1} y \|^2 \\ \text{for some } \sigma \geq 0 \text{ and } y \in U \cap S \end{array} \right\},$$

where U is a normal neighborhood of x in M.

In fact $N^P_S(x) = N^P_{\exp_x^{-1}(S\cap U)}(0)$ where $N^P_{\exp_x^{-1}(S\cap U)}(0)$ is defined in [2]. Let us recall here the *metric projection* of a point $x \in M$ to a closed subset $S \subseteq M$ as

$$\mathrm{proj}_S x = \{y \in S : d(y,x) = d_S(x)\},$$

where $d_S(x) = \inf\{d(y,x) : y \in S\}$.

Since in every complete finite dimensional Riemannian manifold the closed balls are compact, we have the following lemma.

Lemma 2.2. *For every $x \in M \setminus S$, $proj_S x \neq \emptyset$ and $\{s \in proj_S x : x \in M \setminus S\}$ is dense in bdry S.*

The following theorem states the relationship between proximal normal cone and metric projection map; see also [15].

Theorem 2.3. *For every $x \in$ bdry S,*

$$N^P_S(x) = \left\{ \dot\gamma(0) \;\middle|\; \begin{array}{l} \gamma\colon [0,\varepsilon] \to M \textit{ s.t. } \gamma(0) = x,\ x \in proj_S\gamma(\varepsilon) \\ \textit{and } \gamma \textit{ is a minimizing geodesic} \end{array} \right\}.$$

Proof. Let $z \in M \setminus S$ and $x \in \mathrm{proj}_S z$. Suppose that $\gamma : [0,\varepsilon] \to M$ is a minimizing geodesic joining x to z. Let $U = B(x,r)$ be a convex neighborhood of x and $0 < t_0 \leq \varepsilon$ be such that $y_0 := \gamma(t_0) \in B(x,r)$. For $y \in U$ define $\varphi(y) = d^2(y,y_0)$. Since γ is a minimizing geodesic, it is obvious that $x \in \mathrm{proj}_S y_0$ and we get $\varphi(x) \leq \varphi(y)$ for all $y \in S \cap U$. By considering Taylor's theorem, for all $y \in U$ we have

$$\varphi(y) = \varphi(x) + \langle d\varphi(x), \exp_x^{-1} y\rangle + o(\|\exp_x^{-1} y\|).$$

Note that $d\varphi(x) = -2\exp_x^{-1} y_0$ (see[16, p.108]). Since φ is smooth on U there exits $M \geq 0$ such that for every $y \in U$ we have

$$o(\|\exp_x^{-1} y\|) \leq M\|\exp_x^{-1} y\|^2,$$

which implies

$$\varphi(y) - \varphi(x) \leq -2\,\langle \exp_x^{-1} y_0, \exp_x^{-1} y\rangle + M\|\exp_x^{-1} y\|^2 \quad \text{for all } y \in S \cap U.$$

This confirms the proximal normal inequality for $\exp_x^{-1} y_0$. Since $N^P_S(x)$ is a cone, we conclude that $\dot\gamma(0) \in N^P_S(x)$.

To prove the converse inclusion, suppose that $v \in N^P_S(x)$. By Definition 2.1 there exists $\sigma \geq 0$ such that for a convex neighborhood $U = B(x,R)$ we have

$$\langle v, \exp_x^{-1} y\rangle \leq \sigma\|\exp_x^{-1} y\|^2 \qquad \text{for all } y \in S \cap U. \tag{3}$$

Take $\gamma(t) := \exp_x tv$ and $y_0 := \exp_x \varepsilon v \in U$. Notice that

$$d_S(y_0) \leq d(y_0,x) = \varepsilon\|v\| := r.$$

Let ε be small enough such that $B(y_0,2r) \subseteq U$. We claim that $x \in \mathrm{proj}_S y_0$. Define the function $\varphi(y) := d^2(y,y_0)$ for every $y \in U$. By using Taylor's theorem, we get

$$\varphi(y) = \varphi(x) + d\varphi(x)(\exp_x^{-1} y) + \frac{1}{2}d^2\varphi(x)(\exp_x^{-1} y)^2 + o(\|\exp_x^{-1} y\|^2).$$

Since $d\varphi(x) = -2\exp_x^{-1} y_0$ we get $d\varphi(x)(\exp_x^{-1} y) = -2\varepsilon\langle v, \exp_x^{-1} y\rangle$.

Put $w := \exp_x^{-1} y$ and $\theta(s) = \exp_x sw$ and consider the variation α through geodesics by $\alpha_s(0) = y_0$ and $\alpha_s(r) = \theta(s)$. Therefore $L(s) = d(\theta(s), y_0)$ and according to (2) we have

$$d^2\varphi(x)(w)^2 = \frac{d^2}{ds^2}\bigg|_{s=0} \varphi(\theta(s)) = \frac{d^2}{ds^2}\bigg|_{s=0} L^2(s) = rE''(0).$$

Now we apply the second variation formula for the variation α and its variation field $V(t) := \frac{\partial\alpha}{\partial s}(0,t)$ and get

$$\frac{1}{2}\,E''(0) = -\int_0^r \langle V, V'' + R(\dot{\alpha_0}, V)V\rangle dt + \langle V(t), V'(t)\rangle\big|_{t=0}^{t=r} + \langle (\frac{\partial\alpha}{\partial s})'(0,t), \dot{\alpha_0}(t)\rangle\big|_{t=0}^{t=r}.$$

Note that V is a Jacobi field, because α is a variation through geodesics and thus

$$V'' + R(\dot{\alpha_0}, V)V = 0.$$

Moreover,
$$(\frac{\partial\alpha}{\partial s})'(0,0) = (\frac{\partial\alpha}{\partial s})'(0,r) = 0.$$

It remains to compute $V'(r)$. Assume that $\sum w_i\partial_i$ is the coordinate representation of w in a normal coordinate chart around y_0. Since V is the unique Jacobi field along α_0 with $V(0) = 0$ and $V(r) = w$, we can write it in this normal coordinate chart by

$$V(t) = \sum \frac{t}{r} w_i\partial_i\big|_{\alpha_0(t)}.$$

Hence $V'(t) = \frac{1}{r}\sum w_i\partial_i\big|_{\alpha_0(t)} + \chi(t)$, where $\chi(t) = \frac{t}{r}\big(\sum w_i\partial_i\big)'$. This leads to

$$d^2\varphi(x)(w)^2 = rE''(0) = r\langle V(r), V'(r)\rangle = \|w\|^2 + r\langle w, \chi(r)\rangle$$

and then

$$\varphi(y) = \varphi(x) - 2\varepsilon\langle v, \exp_x^{-1} y\rangle + \|\exp_x^{-1} y\|^2 + r\langle \exp_x^{-1} y, \chi(r)\rangle + o(\|\exp_x^{-1} y\|^2). \quad (4)$$

Since χ is a smooth vector field along α_0 and $\chi(0) = 0$, by taking r small enough, there exists $-\frac{1}{2} < \delta < \frac{1}{2}$ such that

$$r\langle \exp_x^{-1} y, \chi(r)\rangle = \delta\|\exp_x^{-1} y\|^2. \quad (5)$$

Note that φ is smooth on U and therefore there exists $M > 0$ such that for all $y \in U$

$$|o(\|\exp_x^{-1} y\|^2)| < M\|\exp_x^{-1} y\|^3. \quad (6)$$

By shrinking $U = B(x, R)$ if necessary one can assume that $1 + \delta - MR > 0$ which implies $2\sigma\varepsilon \leq 1 + \delta - MR$ for small enough ε and then

$$2\sigma\varepsilon \leq 1 + \delta - M\|\exp_x^{-1} y\|.$$

It follows from (4)–(6) that

$$\begin{aligned} 2\sigma\varepsilon\|\exp_x^{-1} y\|^2 &\leq \|\exp_x^{-1} y\|^2 + \delta\|\exp_x^{-1} y\|^2 - M\|\exp_x^{-1} y\|^3 \\ &\leq \|\exp_x^{-1} y\|^2 + r\langle \exp_x^{-1} y, \chi(r)\rangle + o(\|\exp_x^{-1} y\|^2) \\ &= \varphi(y) - \varphi(x) + 2\varepsilon\langle v, \exp_x^{-1} y\rangle. \end{aligned}$$

According to (3), one can deduce that for $y \in S \cap U$

$$2\sigma\varepsilon\|\exp_x^{-1} y\|^2 \leq \varphi(y) - \varphi(x) + 2\sigma\varepsilon\|\exp_x^{-1} y\|^2.$$

Thus $d(x, y_0) \leq d(y, y_0)$ for $y \in S \cap U$ and consequently $x \in \mathrm{proj}_{S\cap U} y_0$. Since $d(x, y_0) = r$ and $B(y_0, 2r) \subseteq U$, we conclude that $x \in \mathrm{proj}_S y_0$. □

In the following proposition, we consider the codimension-1 submanifolds of a Riemannian manifold M.

Proposition 2.4. *Let $f : M \to \mathbb{R}$ be a C^1 function and let $S := f^{-1}(c)$ for some $c \in \mathbb{R}$. If $\nabla f(x) \neq 0$ for every $x \in S$, then $N_S^P(x) \subseteq \{\nabla f(x)\}$.*

Proof. Since $N_S^P(x) = N^P_{\exp_x^{-1}(S\cap U)}(0)$ and $d\exp_x(0) = Id$, according to [2, Proposition 1.9] the proof is straightforward. □

Note that Proposition 2.4 does not guarantee that $N_S^P(x) \neq \{0\}$, although we know that $N_S^P(x) = \{\nabla f(x)\}$ if f is a C^2 function. Since every C^2 submanifold is locally a level set of a C^2 function, we have the following corollary.

Corollary 2.5. *Let W be a C^2 codimension-1 submanifold of a Riemannian manifold M and N_xW be the fiber of the normal bundle of W at $x \in W$. Then $N_W^P(x) = N_xW$ for every $x \in W$.*

For more details regarding the proximal normal cone of $C^{1,1}$ subsets of Riemannian manifolds see [12].

We know that each point in the boundary of a convex set $S \subseteq \mathbb{R}^n$ lies in some hyperplane such that S is contained in one of the associated hyperspaces. The following propositions show that we have the same result for convex subsets of Riemannian manifold M in the tangent space T_xM of every $x \in$ bdry S.

Proposition 2.6. *Let S be locally convex and $x \in$ bdry S, then $v \in N_S^P(x)$ if and only if*

$$\langle v, \exp_x^{-1} y\rangle \leq 0 \quad \textit{for all } y \in S \cap B(x, \varepsilon(x)).$$

Proof. The "if" part follows from Definition 2.1. To prove the converse statement, let $v \in N_S^P(x)$. This means that there exists $\sigma \geq 0$ such that for all $y \in S \cap B(x, \varepsilon(x))$ we have

$$\langle v, \exp_x^{-1} y\rangle \leq \sigma\|\exp_x^{-1} y\|^2.$$

The convexity of $S \cap B(x, \varepsilon(x))$ implies $\exp_x t\exp_x^{-1} y \in S \cap B(x, \varepsilon(x))$ for all $y \in S \cap B(x, \varepsilon(x))$ and $t \in [0, 1]$. Thus

$$\langle v, \exp_x^{-1} y\rangle \leq \sigma t\|\exp_x^{-1} y\|^2.$$

Letting $t \to 0$ completes the proof of the proposition. □

Corollary 2.7. *Let S be a closed convex subset of a Riemannian manifold M. Then*

$$v \in N_S^P(x) \text{ if and only if } \langle v, \exp_x^{-1} y\rangle \leq 0, \qquad \text{for all } y \in S.$$

Proposition 2.8. *Let S be a closed locally convex subset of a complete Riemannian manifold M. Then for every $x \in bdry\, S$, $N_S^P(x) \neq \{0\}$.*

Proof. Since $N_S^P(x) = N_{S\cap\bar{U}}^P(x)$ for every closed convex neighborhood $\bar{U}$ of $x \in \operatorname{bdry} S$, then we can assume that S is strongly convex. By Lemma 2.2 and Theorem 2.3 there exists a sequence $\{x_i\} \subseteq \operatorname{bdry} S \cap U$ such that $x_i \to x$ and $N_S^P(x_i) \neq \{0\}$, where U is a convex neighborhood of x. Suppose that $\xi_i \in N_S^P(x_i)$ with $\|\xi_i\| = 1$. Corollary 2.7 yields

$$\langle \xi_i, \exp_{x_i}^{-1} y\rangle \leq 0 \qquad \text{for all } y \in S.$$

Put $\xi_i' := l_{x_ix}\xi_i$ and since $\{\xi_i'\}$ is a bounded sequence in T_xM, we extract a subsequence converging to ξ_0, without relabeling. Thus

$$\langle \xi_i', l_{x_ix} \exp_{x_i}^{-1} y\rangle \leq 0 \qquad \text{for all } y \in S.$$

For a fixed point $y \in S$, the maps $z \mapsto l_{zx}$ and $z \mapsto \exp_z^{-1} y$ are continuous on U, hence

$$\langle \xi_0, \exp_x^{-1} y\rangle \leq 0 \qquad \text{for all } y \in S.$$

Therefore by Corollary 2.7 the proof is complete. □

In [17] it was proved that around any convex subset S of M, there exists a tubular neighborhood W of S which the projection map is single-valued on W. In the following proposition we characterize the elements of W by means of the curvature of M around every boundary point of S.

Proposition 2.9. *Let S be a closed locally convex subset of a Riemannian manifold M and $x \in bdry\, S$. Suppose that all sectional curvature of M on $\overline{B(x, \varepsilon(x))}$ is bounded above by a constant $\delta_x > 0$. Put $t_x := \min\{\frac{\pi}{2\sqrt{\delta_x}}, \frac{\varepsilon(x)}{2}\}$ and*

$$W := \{\exp_x tv \big| x \in bdry\, S, v \in N_S^P(x) \text{ with } \|v\| = 1, t < t_x\}.$$

Then the map $proj_S$ is single-valued map on W. Moreover, for every $x \in bdry\, S$ and $v \in N_S^P(x)$ satisfying $\|v\| = 1$ there exists $t \geq 0$ such that $\exp_x tv \in W^\circ$.

Proof. Let $y = \exp_x tv \in W$ and $x_1, x_2 \in \operatorname{proj}_S y$. It is clear from the definition of W that $x_1, x_2 \in B(x, \varepsilon(x))$. Now consider the triangle $\triangle(x_1, x_2, y)$ in the convex neighborhood $B(x, \varepsilon(x))$ with angel α associated with vertex x_1. Let $A := d(x_1, y) = d(x_2, y)$ and $B := d(x_1, x_2)$. Obviously $A < \frac{\pi}{2\sqrt{\delta}}$ and $B \leq 2A < \frac{\pi}{\sqrt{\delta}}$. According to Lemma 1.1

$$\cos(\sqrt{\delta}A)\cos(\sqrt{\delta}B) + \sin(\sqrt{\delta}A)\sin(\sqrt{\delta}B)\cos\alpha \geq \cos(\sqrt{\delta}A).$$

Using Theorem 2.3 one has $\exp_{x_1}^{-1} y \in N_S^P(x_1)$ and therefore by Proposition 2.6 $\cos\alpha \leq 0$ which implies

$$\cos(\sqrt{\delta}A)\cos(\sqrt{\delta}B) \geq \cos(\sqrt{\delta}A).$$

Hence $\cos(\sqrt{\delta}B) = 1$ and then $B = 0$. Therefore proj_S is single valued on W.

Suppose that for every $x \in S$, $\varepsilon(x) \leq r(x)$ is the largest value with the property that $B(x, \varepsilon(x)) \cap S$ is convex and suppose that $K \subseteq \mathrm{bdry}\, S$ is compact. One can show that for every sequence $\{x_i\} \subset K$ satisfying $x_i \to x_0$, if $d(x_i, x_0) < \varepsilon(x_0)/2$ and $r(x_i) > \varepsilon(x_0)/2$ then $\varepsilon(x_i) \geq \varepsilon(x_0)/2$ which implies $\inf_{x\in K} \varepsilon(x) > 0$. Thus $\inf_{x\in K} t_x > 0$. Note that $\sup_{x\in K} \delta_x < +\infty$. Now let $x \in \mathrm{bdry}\, S$ and $v \in N_S^P(x)$ with $\|v\| = 1$. Put $m := \inf_{x\in \overline{B(x,l)}\cap \mathrm{bdry}\, S} t_x > 0$. Take $t > 0$ and $\delta > 0$ such that $t + \delta < m$ and $2\delta + t < \frac{l}{2}$. Put $y := \exp_x tv$ and suppose that $z \in B(y, \delta)$ and let $x_1 \in \mathrm{proj}_S z$. Note that

$$d(z, x_1) \leq d(z, x) \leq d(z, y) + d(y, x) < \delta + t, \quad \text{and}$$

$$d(x, x_1) \leq d(x, y) + d(y, x_1) \leq 2d(y, x_1) \leq 2(d(y, z) + d(z, x_1)) < 2(2\delta + t).$$

Our choices of δ and t make $x_1 \in B(x, l)$ and therefore $d(z, x_1) < t_{x_1}$. Thus $z \in W$ and $y \in W^\circ$. □

Remark 2.10. By a similar observation as in the proof of Corollary 2.9 one can show that for every $y := \exp_x tv \in W$ we have $\mathrm{proj}_S y = \{x\}$.

Corollary 2.11. *Let S be a closed locally convex subset of a Riemannian manifold M. Then there exists an open set W containing S such that proj_S is single-valued continuous mapping on W.*

Definition 2.12. Let S be a closed subset of a complete Riemannian manifold M. Let $x_* \in \mathrm{bdry}\, S$ and $v \in N_S^P(x_*)$ with $\|v\| = 1$. We say that S is *supported at x_* relative to v* when there exist a smooth codimension-1 submanifold H and $r > 0$ such that $x_* \in H$ and for every $x = \exp_{x_*} tv$ with $x_* = \mathrm{proj}_S x$ and $t < r$ we have

$$\text{(i) } d_H(x) = d_S(x), \quad \text{and} \quad \text{(ii) } d_H(y) < d_S(y)$$

for every $y \in U_x$, where U_x is a small enough neighborhood of x.

Although Propositions 2.6 and 2.8 are not about separation of convex subsets on the manifold, they imply the following theorem.

Theorem 2.13. *Let S be a closed locally convex subset of a complete Riemannian manifold M. Suppose that all sectional curvature of M is bounded above by a constant $\delta > 0$. Then S is supported at every $x_* \in \mathrm{bdry}\, S$ relative to $v \in N_S^P(x_*)$ with $\|v\| = 1$.*

Proof. Let $x_* \in \mathrm{bdry}\, S$ and $v \in N_S^P(x_*)$ with $\|v\| = 1$. By Proposition 2.6 we have

$$\langle v, \exp_{x_*}^{-1} y\rangle \leq 0, \qquad \text{for every } y \in S \cap B(x_*, \varepsilon(x_*)).$$

Put $r = \min\{\frac{\varepsilon(x_*)}{2}, \frac{\pi}{2\sqrt{\delta}}\}$, and let H_0 be the hyperplane in $T_{x_*}M$ with the normal vector v, i.e. $H_0 = \{w \in T_{x_*}M : \langle v, w\rangle = 0\}$. Take

$$H := \exp_{x_*} H_0 \cap \overline{B(x_*, 2r)}. \tag{7}$$

Note that $x_* \in H$, H is a smooth codimension-1 submanifold of M and $v \in N_{x_*}H$. Suppose that W is the tubular neighborhood around S mentioned in Proposition 2.9 and let $x = \exp_{x_*} tv \in B(x_*, r) \cap W$ such that $x_* = \mathrm{proj}_S x$. Take

$$U_x := \{y \in B(x_*, r) \cap W : \langle \exp_{x_*}^{-1} y, v\rangle > 0\},$$

and let $y \in U_x$ and $\gamma_1(t)$ be the unique minimizing geodesic joining y and y_*. If we consider $\tilde{\gamma}_1(t) := \exp_{x_*}^{-1} \gamma_1(t)$, then our choice of U_x grantees that $\tilde{\gamma}_1$ intersects $H_0 \cap B(0_{x_*}, 2r)$. Thus $d_H(y) < d_S(y)$ for every $y \in U_x$.

It remains to show that $d_H(x) = d_S(x)$. To do so, note that by Corollary 2.5, $N_H^P(y) = \{\lambda n(y) : \lambda \in \mathbb{R}\}$ where $n(y)$ is a unite vector in normal vector bundle N_yH. We choose $n(y)$ such that $n(x_*) = v$ and $y \mapsto n(y)$ is a continuous vector field on a neighborhood of H. Suppose that $d_H(x) < d_S(x)$. According to Theorem 2.3 there exists $z \in H$ such that $t' := d_H(x) = d(x, z) < t$ and $x = \exp_z t'n(z)$.

Note that x and z are in a convex neighborhood of x_*. We consider the geodesic triangle $\triangle(x, x_*, z)$ in the convex neighborhood $B(x_*, \varepsilon(x_*))$ with the angels α, β and θ at vertices x_*, z and x, respectively. Since $v \in N_{x_*}H$, $n(z) \in N_zH$ and the unique minimizing geodesic joining x_* and z lies entirely in H, it follows that $\alpha = \beta = \frac{\pi}{2}$. Our choices of W and H imply

$$d(x, x_*) = t < \frac{\pi}{2\sqrt{\delta}}, \qquad d(x, z) = t' < \frac{\pi}{2\sqrt{\delta}}, \qquad t'' := d(x_*, z) < \frac{\pi}{\sqrt{\delta}}.$$

By Lemma 1.1 we have

$$\cos(\sqrt{\delta}l) \leq \cos(\sqrt{\delta}l'')\cos(\sqrt{\delta}l') \quad \text{and} \quad \cos(\sqrt{\delta}l') \leq \cos(\sqrt{\delta}l'')\cos(\sqrt{\delta}l).$$

Since $\sqrt{\delta}l < \frac{\pi}{2}$ and $\sqrt{\delta}l' < \frac{\pi}{2}$, we have $\cos(\sqrt{\delta}l) + \cos(\sqrt{\delta}l') > 0$. Thus $\cos(\sqrt{\delta}l'') \geq 1$ and we conclude that $l'' = 0$ and $z = x_*$ which completes the proof. □

Remark 2.14. The proof of Theorem 2.13 shows how curvature affects the radius of the convex neighborhood $B(x_*, 2r)$ used in (7). In fact the greater curvature implies the less radius. Especially if M is a Hadamard manifold, then $W = M \setminus S$ in Proposition 2.9 and $H = \exp_x H_0$ in Theorem 2.13 for every $x \in \mathrm{bdry}\, S$. This means that the separation theorem does hold on Hadamard manifolds like Euclidean spaces.

3. Convexity of the distance function to a convex subset

In this section we study the convexity of distance function d_S where S is a locally convex subset of M. We show that how curvature of M affect this problem.

Theorem 3.1. *Let M be a Riemannian manifold such that its sectional curvature is positive. Suppose that S is a convex subset which its boundary contains a geodesic segment of M. Then there exists a geodesic $\alpha : (-\varepsilon, \varepsilon) \to M \setminus S$ such that $d_S \circ \alpha(t)$ is a concave function.*

Proof. According to Proposition 2.9 there exists a tubular neighborhood W of S such that the the map $\mathrm{proj}_S : W \to S$ is single valued and continuous. Now suppose that $x \in W \setminus S$ and $\mathrm{proj}_S(x) := x_* = \gamma(0)$ where $\gamma : [-\varepsilon, \varepsilon] \to M$ is a minimal geodesic whose image is contained in S. Put $v := l_{x_* x} v_*$ where $v_* := \dot{\gamma}(0)$. Let $\alpha : (-\varepsilon, \varepsilon) \to M$ be a geodesic in M such that $\alpha(0) = x$ and $\dot{\alpha}(0) = v$. For each $s \in (-\varepsilon, \varepsilon)$, let $\beta_s : [0, 1] \to M$ be the unique minimizing geodesic joining the points $\gamma(s)$ and $\alpha(s)$. By letting x close enough to x_* take $\beta_s(t) = \exp_{\gamma(s)} t \exp^{-1}_{\gamma(s)} \alpha(s)$. Let us define the variation $\beta(s, t) = \beta_s(t)$ and $\beta(t) := \beta_0(t)$. Note that

$$l := \|\dot{\beta}(t)\| = \|\dot{\beta}(0)\| = \| \exp^{-1}_{x_*} x\| = d_S(x).$$

If we denote the length of geodesic $\beta(s)$ by $L(s)$, then by the first variation formula we have

$$\left.\frac{d}{ds}\right|_{s=0} L(s) = \frac{1}{l} \langle V(t), \dot{\beta}(t)\rangle \Big|_{t=0}^{t=1} - \frac{1}{l} \int_0^1 \langle V(t), (\dot{\beta}(t))' \rangle dt, \tag{8}$$

where $V(t) = \frac{\partial \beta}{\partial s}(0, t)$ is the variation field of β_s. Since β is a geodesic, the last term in (8) vanishes. Thus

$$\left.\frac{d}{ds}\right|_{s=0} L(s) = \frac{1}{l} \Big[\langle v, l_{x_* x} \exp^{-1}_{x_*} x\rangle - \langle v_*, \exp^{-1}_{x_*} x\rangle \Big] = 0.$$

This shows that $s \mapsto L(s)$ has an extremum at $s = 0$.

To compute the second derivative of $L(s)$, we use the second variation formula and (2) to get

$$\begin{aligned} \frac{1}{2} \left.\frac{d^2}{ds^2}\right|_{s=0} L^2(s) &= \frac{1}{2} E''(0) \\ &= \int_0^1 (\langle V', V'\rangle - \langle R(V, \dot{\beta})V, \dot{\beta}\rangle) dt + \langle (\frac{\partial \beta_s}{\partial s})'(0, t), \dot{\beta}(t)\rangle \Big|_{t=0}^{t=1}. \end{aligned}$$

Clearly the second term in the above statement vanishes. The first term is called the index form of V and denoted by $I(V, V)$. Since V is a Jacobi field along β, then

$$I(V, V) \leq I(W, W)$$

where W is a vector field along β with $W(0) = V(0)$ and $W(1) = V(1)$ (see [16]). Consider W as a parallel vector field along β with $W(0) = v_*$. Obviously $W(1) = v$ and then

$$I(V, V) \leq I(W, W) = - \int_0^1 \langle R(W, \dot{\beta})W, \dot{\beta}\rangle dt.$$

Since the sectional curvature of M is positive, we conclude that

$$\frac{1}{2}\frac{d^2}{ds^2}\bigg|_{s=0} L^2(s) = I(V,V) < 0,$$

which shows that $L(s)$ has a maximum at $s = 0$. Then

$$L(0) = d_S(x) = d_S(\alpha(0)) > L(s) = d(\gamma(s), \alpha(s)) \geq d_S(\alpha(s)).$$

This means that $d_S(\alpha(s))$ has a strict maximum at $s = 0$ and is concave near $s = 0$. □

Let us recall here the definition of a geodesic point. Let S be a Riemannian submanifold of a Riemannian manifold M. We call $x \in S$ a *geodesic point* when for every $v \in T_xS$ the $M-$geodesic γ_v starting from x with velocity v lies entirely in S; see [5].

Corollary 3.2. *Let M be a Riemannian manifold such that its sectional curvature is positive. Suppose that S is a convex submanifold which its boundary contains a geodesic point. Then there exists a geodesic $\alpha : (-\varepsilon, \varepsilon) \to M \setminus S$ such that $d_S \circ \alpha(t)$ is a concave function.*

The next theorem investigates convexity of the distance function d_S on an open neighborhood of S where S is a locally convex subset with regular boundary.

Theorem 3.3. *Let M be a complete Riemannian manifold and $S \subseteq M$ be a closed locally convex set. Suppose that $W =$ bdry S is a C^2 codimension-1 submanifold and $n(x) \in N_xW$ is a unite vector such that $-n(x) \in N_S^P(x)$ for every $x \in W$. Let h_x be the scalar second fundamental form of W at $x \in W$. If $h_x > 0$ for every $x \in W$, then there exists an open neighborhood U of S such that the distance function d_S is convex on U.*

Before proving Theorem 3.3 we show that the hypothesis of positive definiteness of the scalar second form of the boundary is in fact very natural for the locally convex subset with regular boundary.

Proposition 3.4. *Let S be a locally convex subset of a Riemannian manifold M and $W =$ bdry S be a C^2 codimension-1 submanifold. Suppose that $n(x) \in N_xW$ is the inward pointing normal for every $x \in W$. Then $h_x \geq 0$ for every $x \in W$.*

Proof. By definition we require that

$$\langle \nabla_v n(x), v \rangle \leq 0 \qquad \text{for every } x \in W \text{ and } v \in T_xW.$$

Suppose that $x \in W$ and $v \in T_xW$. Let $\gamma : (-\varepsilon, \varepsilon) \to M \cap B(x, \varepsilon(x))$ be a curve in W such that $\gamma(0) = x$ and $\dot{\gamma}(0) = v$. By Proposition 2.6 we have

$$\langle n(x), \exp_x^{-1}\gamma(t)\rangle \geq 0, \quad \text{and} \quad \langle n(\gamma(t)), \exp_{\gamma(t)}^{-1} x\rangle \geq 0. \tag{9}$$

Put $v_t = l_{\gamma(t)x} n(\gamma(t))$. Thus by (9) we have

$$\langle v_t, \exp_x^{-1}\gamma(t)\rangle \leq 0 \leq \langle n(x), \exp_x^{-1}\gamma(t)\rangle.$$

Hence for all t sufficiently near 0,

$$\langle \frac{v_t - n(x)}{t}, \frac{\exp_x^{-1} \gamma(t)}{t} \rangle \leq 0.$$

Put $\tilde{\gamma}(t) = \exp_x^{-1} \gamma(t)$ which is a curve in T_xM passing through 0_x with initial velocity v. By letting $t \to 0$ we conclude that

$$\lim_{t\to 0} \frac{\exp_x^{-1} \gamma(t)}{t} = \dot{\tilde{\gamma}}(0) = v,$$

and by (1) we get
$$\langle \nabla_v n(x), v \rangle \leq 0.$$
□

Proof of Theorem 3.3. Assume that $\widetilde{W}$ is the tubular neighborhood of W such that the projection map is single valued by Proposition 2.9. Put

$$U := \{y \in \widetilde{W} \mid y = \exp_x tn(x) \text{ for } t > 0 \text{ and } x \in W\},$$

and
$$V := \{y \in \widetilde{W} \mid y = \exp_x tn(x) \text{ for } t < 0 \text{ and } x \in W\}.$$

Let us define $\varphi : \widetilde{W} \to \mathbb{R}$ as

$$\varphi(y) = \begin{cases} -d_W(y), & y \in U \cup W; \\ \ \ d_W(y), & y \in V. \end{cases} \tag{10}$$

In fact for t sufficiently small, by Remark 2.10 we have $\varphi(\exp_x tn(x)) = -t$ which implies $\nabla\varphi(x) = -n(x)$. Now suppose that $v, w \in T_xW$. By the definition of the Hessian we have

$$d^2\varphi(x)(v, w) = \langle \nabla_v \nabla\varphi(x), w \rangle = \langle \nabla_v - n(x), w \rangle = \langle \mathbf{S}_x v, w \rangle = h_x(v, w),$$

where $\mathbf{S}$ is the shape operator of W. By simple calculation one can see that for $v \in T_xM$ and its tangential component $\alpha \in T_xW$, we have

$$d^2\varphi(x)(v)^2 = d^2\varphi(x)(\alpha)^2 = h_x(\alpha, \alpha).$$

Therefore, if the scalar second fundamental form h_x of W is positive definite for every $x \in W$, then φ is a convex function in a neighborhood of W. Since $\varphi(y) = d_S(y)$ for $y \in V$, we conclude that under this assumption d_S is a convex function on a neighborhood of S. □

Example 3.5. Consider $M = \mathbb{S}^2$ the unit sphere in $\mathbb{R}^3$ and let us, as usual, denote $S = \{(x, y, z) : z \geq \frac{1}{2}\} \subset M$. In fact $S = \overline{B(N, \frac{\pi}{3})}$ where N is the north pole. It is obvious that S is convex with C^∞ boundary. By choosing inward normal vector $n(p)$ on bdry S, $h_p > 0$ for every $p \in$ bdry S; since otherwise bdry S is a geodesic segment which is a contradiction. Hence by Theorem 3.3 there exist an open neighborhood U of S such that the distance function d_S is convex on U.

Example 3.6. Suppose that M is the paraboloid of the revolution in $\mathbb{R}^3$. Let $S_1 = \{(x, y, z) \in M : z \leq z_0\}$ where $z_0 \in (0, \infty)$. Consider $f\colon M \to \mathbb{R}$, the function with $f(x, y, z) = x^2 + y^2$. If we consider local parametrization of M by $(s, \theta) \mapsto (s\cos\theta, s\sin\theta, s^2)$, then $f(s, \theta) = s^2$ and

$$d^2 f(s, \theta) = \begin{pmatrix} \frac{2}{1+4s^2} & 0 \\ 0 & \frac{2s^2}{1+4s^2} \end{pmatrix}.$$

This shows that $d^2 f > 0$ and therefore S_1 is a locally convex subset of M with $h_p > 0$ for every $p \in \operatorname{bdry} S_1$ with inward pointing normal direction. Thus by Theorem 3.3, d_{S_1} is a convex function in a neighborhood of S_1.

Here we propose two approaches for considering convexity of the function d_S where the boundary of S is not C^2. The first one is the support principle for convexity which we explain it in the following theorem. Recall that a function f is said to be supported by a function g at a point p if $f(p) = g(p)$ and $g(q) \leq f(q)$ for all q in a neighborhood of p. For more details one can see [7].

Theorem 3.7. *A continuous function f on a Riemannian manifold M is convex if it is supported at each point p by a convex function on some neighborhood of p.*

Remark 3.8. Let S be a closed locally convex subset of a Riemannian manifold M. Then by Theorem 2.13 one can find a neighborhood W of S such that for every $x \in W$ there exist a neighborhood $U_x \subseteq W$ and a smooth codimension-1 submanifold H satisfying $d_H(x) = d_S(x)$ and $d_H(y) \leq d_S(y)$ for every $y \in U_x$, i.e. d_S is supported by d_H at x. If we construct H such that $h_{x_*} > 0$ (by imposing inward pointing normal) or equivalently if $\exp_{x_*}^{-1} H$ is strictly convex at 0_{x_*} in the Euclidean sense (see [9, p. 371]) where $x_* = \operatorname{proj}_S x$, then it follows from Theorem 3.3 and the support principle for convexity that d_S is a convex function on W. This means that in the proof of Theorem 2.13, let $v = (0, ..., 0, -1)$ (by using some rotation in normal coordinate system around x_*), then if we take $a > 0$ such that $H_0 = \{(w_1, ..., w_n) \in T_{x_*}M | w_n = a(\sum_{i=1}^{n-1} w_i^2)\}$, then by Theorem 2.13 d_S is supported by convex function d_H which makes d_S to be a convex function in a neighborhood of S.

The next approach is more general and can be applied to every function on Riemannian manifolds. At first let us define the second superjet of a function f at every point x in the domain of f, denoted by $J^{2,+} f(x)$, as

$$\big\{\big(d\varphi(x), d^2\varphi(x)\big) \mid \varphi \in C^2(U), f - \varphi \text{ attains a local maximum at } x\big\},$$

where U is an open subset of M including the domain of f. Let Ω be an open convex subset of M and $f : \Omega \to \mathbb{R}$ be a convex function. Suppose that φ is a C^2 function such that $f(x_0) = \varphi(x_0)$ and $f - \varphi$ attains its maximum at $x_0 \in \Omega$. Let $\gamma : [0, 1] \to \Omega$ be a geodesic such that for some $t_0 \in (0, 1)$ we have $\gamma(t_0) = x_0$. The convexity of f implies that $f(\gamma(t)) \leq (1 - t) f(\gamma(0)) + t f(\gamma(1)) := g(t)$ for $t \in [0, 1]$. If $f(\gamma(t_0)) < g(t_0)$, then by the continuity of φ we can deduce that $\varphi(\gamma(t)) < (1-t)\varphi(\gamma(0)) + t\varphi(\gamma(1))$ for t sufficiently close to t_0. If $f(\gamma(t_0)) = g(t_0)$, the convex function $f \circ \gamma - g$ admits a local maximum at $t = t_0$ which means that $(f \circ \gamma - g)(t) = (f \circ \gamma - g)(t_0) = 0$ for all $t \in [0, 1]$ and thus the function $\varphi \circ \gamma - g$ attains its minimum at $t = t_0$ which implies that $d^2\varphi(x_0)(\dot{\gamma}(t_0), \dot{\gamma}(t_0)) \geq 0$. These observations show that if f is convex and $(\xi, X) \in J^{2,+}f(x)$, then $X \geq 0$. The next theorem states that the converse statement is true on manifolds with positive sectional curvature.

In the proof of the next theorem we use a special coordinate system called Fermi coordinates. Let us explain the definition of this coordinate system. Suppose that $\gamma : [a, b] \to M$ is a unit speed minimizing geodesic in M and assume that $\{e_i\}_{i=0}^{n-1}$ is an orthonormal basis for $T_{\gamma(a)}M$ with $e_0 := \dot{\gamma}(a)$. Let $e_i(t)$ be the parallel translation of e_i along $\gamma(t)$. Note that $\{e_i(t)\}$ is an orthonormal frame for $T_{\gamma(t)}M$ for every $t \in [a, b]$. Consider the map $\Gamma : [a, b] \times \mathbb{R}^{n-1} \to M$ given by

$$\Gamma(t, x_1, ..., x_{n-1}) = \exp_{\gamma(t)}(\sum x_i e_i(t)).$$

One can see that $d\Gamma(t, 0, ..., 0)$ is invertible for very $t \in [a, b]$ which implies by the inverse function theorem that there exist $\mu > 0$ and $T_\mu \subseteq M$ such that $\Gamma : [a, b] \times B(0, \mu) \to T_\mu$ is a diffeomorphism. Thus (T_μ, ϕ) where $\phi := \Gamma^{-1}$ is a coordinate system around γ which is called Fermi coordinate system; see [1, 6] for more details. Note that by similar observation as in the proof of Theorem 2.13, one can show that $\text{proj}_S \exp_{\gamma(t)}(\sum x_i e_i(t)) = \gamma(t)$ for every t and $\|(x_1, ..., x_{n-1})\| < \mu$ where $\mu > 0$ is small enough and $S := \gamma[a, b]$.

As previously mentioned, the following theorem is a generalization of positive semi-definiteness of the second order differential of a smooth convex function.

Theorem 3.9. *Let M be a Riemannian manifold with positive sectional curvature, Ω be a convex subset of M and let $f : \Omega \to \mathbb{R}$ be a continuous function. Suppose that $X \geq 0$ for every $(\xi, X) \in J^{2,+}f(x)$ and for every $x \in \Omega$. Then f is a convex function.*

Proof. Suppose that there exist a geodesic $\gamma : [0, 1] \to \Omega$ such that $f \circ \gamma$ is not convex. Let $\lambda_0 \in (0, 1)$ be such that

$$f \circ \gamma(\lambda_0) > (1 - \lambda_0)f(x_0) + \lambda_0 f(y_0),$$

where $x_0 := \gamma(0)$ and $y_0 := \gamma(1)$. By shrinking the geodesic segment $\gamma[0, 1]$ if necessary, one can assume that there exist $k, k_0 \in \mathbb{R}$ such that for every $\lambda \in [0, 1]$ we have

$$f \circ \gamma(\lambda_0) > k > k_0 > (1 - \lambda)f(x_0) + \lambda f(y_0).$$

Since f is continuous, we assume that there exists $\mu > 0$ such that for every $y \in B(x_0, \mu)$ and $y \in B(y_0, \mu)$

$$k_0 > f(y). \tag{11}$$

Suppose that $\mu' > \mu$ is small enough such that $(T_{\mu'}, \phi_{\mu'})$ is a Fermi coordinate system around $\gamma(t)$. Define $\varphi : T_{\mu'} \to \mathbb{R}$ by

$$\varphi(y) = -\delta d^2(y, \gamma(\lambda_0)) + C d_S^2(y) + k$$

where δ is such that

$$k_0 = -\delta(\max\{\lambda_0, 1-\lambda_0\}\|\dot\gamma(0)\| + \mu)^2 + k, \tag{12}$$

and C is large enough such that

$$C\mu^2 + k_0 \geq \sup_{\overline{T_{\mu'}}} f, \tag{13}$$

and $S := \gamma[0,1]$. Note that φ is C^2 on $\overline{T_{\mu'}}$ and $f - \varphi$ is a continuous function on $\overline{T_\mu}$. Thus $f - \varphi$ admits its maximum on $\overline{T_\mu}$. In the following we show that this is a local maximum.

If $y = (0, \xi)$ in Fermi coordinates, then by using the fact that $\mathrm{proj}_S y = x_0$ we have

$$\begin{aligned} d^2(y, \gamma(\lambda_0)) &\leq [d(y, \mathrm{proj}_S y) + d(\mathrm{proj}_S y, \gamma(\lambda_0)]^2 \leq [\mu + \|\dot\gamma(0)\|\lambda_0]^2 \\ &\leq (\mu + \max\{\lambda_0, 1-\lambda_0\}\|\dot\gamma(0)\|)^2. \end{aligned}$$

Thus we have from (11) and (12) that

$$\varphi(y) \geq -\delta(\mu + \max\{\lambda_0, 1-\lambda_0\}\|\dot\gamma(0)\|)^2 + k = k_0 > f(y).$$

If $y = (s, \xi)$ with $\|\xi\| = \mu$ and $s \in (0,1)$, we have by (13)

$$\begin{aligned} \varphi(y) &\geq -\delta[\mu + \max\{\lambda_0, 1-\lambda_0\}\|\dot\gamma(0)\|]^2 + C\mu^2 + k = k_0 + C\mu^2 \\ &\geq \sup_{\overline{T_{\mu'}}} f \geq f(y). \end{aligned}$$

If $y = (1, \xi)$, we conclude similarly that $\varphi(y) \geq f(y)$. Thus for every $y \in \mathrm{bdry}\, T_\mu$ we get $\varphi(y) \geq f(y)$. Note that $\varphi(\gamma(\lambda_0)) = k < f(\gamma(\lambda_0))$ which implies that $f - \varphi$ has a maximum in the interior of T_μ. Let y_0 be the point in T_μ such that $f - \varphi$ gets its maximum at this point. By the definition of the second order superjets we have $(d\varphi(y_0), d^2\varphi(y_0)) \in J^{2,+}f(y_0)$. Note that by Theorem 3.1 there exists $X \neq 0$ in the space of symmetric bilinear forms at y_0 such that $d^2 d_S^2(y_0)(X, X) < 0$. Since $-\delta d^2(y, \gamma(\lambda_0))$ is a concave function near $\gamma(\lambda_0)$, we get $d^2\varphi(y_0)(X, X) < 0$ which is a contradiction. □

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