Journal of Convex Analysis
Volume 26 (2019), No. 4, 1337–1346

Metrizable Bounded Sets in $C(X)$ Spaces and Distinguished $C_p(X)$ Spaces*

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Received: September 10, 2018
Revised manuscript received: April 11, 2019
Accepted: April 12, 2019

Quite recently W. Ruess [17] has shown that a wide class of locally convex spaces for which all bounded sets are metrizable enjoy Rosenthal's ℓ_1-dichotomy. Being motivated by this fact we show that for a Tychonoff space X the bounded sets of $C_p(X)$ are metrizable (respectively, the bounded sets of $C_k(X)$ are weakly metrizable) if and only if X is countable. If X is a P-space we show that every bounded set in $C_p(X)$ is metrizable if and only if X is countable and discrete. The second part of the paper deals with distinguished $C_p(X)$ spaces. Among other things we show that $C_p(X)$ is distinguished if and only if the strong topology of the dual coincides with its strongest locally convex topology, and that $C_p(X)$ is always distinguished whenever X is countable.

Keywords: Countable tightness, Fréchet-Urysohn space, strong dual, strongest locally convex topology, distinguished space.

2010 Mathematics Subject Classification: 54C35, 54E15, 46A03.

1. Preliminaries

In what follows all topological spaces are supposed to be Hausdorff. Unless otherwise stated X will stand for a completely regular topological space. We represent by $C_p(X)$ the linear space $C(X)$ of real-valued continuous functions defined on X equipped with the *pointwise* topology τ_p, whose topological dual will be denoted by $L(X)$, or by $L_p(X)$ when equipped with the weak* topology. We shall denote by $C_k(X)$ the space $C(X)$ endowed with the compact-open topology. We shall refer to the set $\delta(X) = \{\delta_x : x \in X\}$ in $L(X)$ as the *canonical bijective copy* of X or as the *homeomorphic copy* of X in $L_p(X)$. The notation $C^b(X)$ stands for the Banach space of all real-valued continuous and bounded functions defined on X, equipped with the supremum-norm. Let us recall that $C_p(X)$ is locally

*Both authors were supported by grant PGC2018-094431-B-100 of the Ministry of Science, Innovation and Universities of Spain. The second author was also supported by GAČR Project 16-34860L and RVO: 67985840.

ISSN 0944-6532 / $ 2.50 © Heldermann Verlag

complete if and only if X is a P-space (see [10, Theorem 1.1]), and that $C_k(X)$ is complete (hence locally complete, a duality invariant property) if X is a $k_{\mathbb{R}}$-space. A locally convex space E is called *distinguished* if its strong dual, that is, its topological dual E' equipped with the strong $\beta(E', E)$ topology, is barrelled (see [15, 23.7.(1)]). Let us mention that in this paper the term *bounded set* is used in the context of topological vector spaces. So, we say that a subset in a topological vector space is bounded if it is absorbed by every neighborhood of zero (see [3, Definition 1.4.5]). The term *bounded* as used by C_p-theorists is rendered here as *functionally bounded.* Other notions on locally convex space theory not defined in this paper can be found in [3, 15].

From now onwards if $\mathcal{N}$ is a uniform structure for a set X we denote by $\tau_{\mathcal{N}}$ the uniform topology defined by $\mathcal{N}$. If $(X, \mathcal{N})$ is a uniform space we shall denote by $C(X, \tau_{\mathcal{N}})$ the ring of real-valued $\tau_{\mathcal{N}}$-continuous functions defined on X. A uniform space $(X, \mathcal{N})$ is called *trans-separable* (Drewnowski, [6]) if for each vicinity N of $\mathcal{N}$ there is a countable subset Z of X with $\bigcup_{z\in Z} U_N(z) = X$, where $U_N(x) = \{y \in X : (x, y) \in N\}$. A locally convex space E is called trans-separable if the uniform space $(E, \mathcal{M})$, where $\mathcal{M}$ is the uniformity generated by its locally convex structure, is trans-separable.

Recall that a topological space X is either *countably tight* or *Fréchet-Urysohn* if $x \in \overline{A} \subseteq X$ implies the existence of a countable set $B \subseteq A$ or a sequence $\{x_n\}_{n=1}^{\infty}$ in A such that $x \in \overline{B}$ or $x = \lim_{n\to\infty} x_n$, respectively. A topological space X is said to have the *Discrete Countable Chain Condition* (DCCC) if every discrete family of open sets is countable. We quote the two following results for later use.

Theorem 1.1. (Cascales-Orihuela, [4, Theorem 4]) *A topological space X has the DCCC if and only if every pointwise bounded equicontinuous subset of $C(X)$ is pointwise metrizable.*

Theorem 1.2. (Ruess, [17, Proposition 3.3]) *Every locally complete locally convex space E whose bounded sets are metrizable has the Rosenthal property.*

Rosenthal's property means that each bounded sequence has a subsequence that either is a Cauchy sequence in the weak topology or is equivalent (in the original topology) to the unit vector basis of ℓ_1. For spaces $C_k(X)$ we have the following Rosenthal-type result.

Proposition 1.3. (Gabriyelyan et al., [13, Lemma 6.3]) *For a Tychonoff space X the space $C_k(X)$ does not contain a copy of ℓ_1 if and only if $C_k(X)$ is Asplund* (*in the sense that each separable Banach subspace has a separable dual*) *if and only if every compact subset of X is scattered.*

On the other hand, since $C^b(X)$, regarded as a linear subspace of $C(X)$, is dense in $C_p(X)$, the topological dual of $C^b(X)$ equipped with the relative topology of $C_p(X)$ coincides with $L(X)$, particularly $L(X) \subseteq C^b(X)^*$, where the latter stands for the dual of the Banach space $C^b(X)$. We shall use the following folklore result (see [5]), which is consequence of the fact that if $x \neq y$ in X then $\|\delta_x - \delta_y\| = 2$ for the norm of $C^b(X)^*$.

Fact 1.4 *The canonical bijective copy $\delta(X)$ of X in $L(X)$ is a discrete space under the norm topology of the Banach space $C^b(X)^*$, which is coarser than the relative strong topology $\beta(L(X), C(X))$ on $\delta(X)$.*

The papers [2, 7, 18] characterize those Fréchet (i. e., metrizable and complete locally convex) spaces E for which every bounded set of the strong dual space $(E', \beta(E', E))$ or of the Mackey* dual $(E', \mu(E', E))$ is metrizable. Both dual spaces enjoy Rosenthal's property via Theorem 1.2. Motivated by these results, we pursue in this paper the metrizability of bounded sets in $C_p(X)$ and $C_k(X)$, as well as the weak metrizability of bounded sets in $C_k(X)$. Although the C_p-case looks easy (see Corollary 2.2 below) and might be known to specialists, we show also that every bounded set in $C_k(X)$ is metrizable in the weak topology of $C_k(X)$ if and only if X is countable. This essentially supplements the well known fact that in a normed space E bounded sets are weakly metrizable if and only if the dual E^* is norm-separable. The second part of the paper deals with distinguished $C_p(X)$ spaces. Among other things we show that *(i)* $C_p(X)$ is distinguished if and only if $\beta(L(X), C(X))$ coincides with the strongest locally convex topology of $L(X)$, and *(ii)* $C_p(X)$ is always distinguished whenever X is countable.

2. Metrizability of bounded sets in $C(X)$ spaces

If X is a web-compact space (see [14, Chapter 4]) a compact set P in $C_p(X)$ is metrizable if and only if P is contained in a separable subset of $C_p(X)$ (see [14, Corollary 4.5]). In particular, if X is analytic then it is submetrizable (Talagrand) and $d(X) \leq \aleph_0$, therefore $C_p(X)$ is separable, so that every compact set P in $C_p(X)$ is metrizable. Of course, if X is analytic $C_p(X)$ is even submetrizable. If X is analytic and $C_p(X)$ is a μ-space, where this latter happens for instance if X has countable $\mathbb{R}$-tightness (see [1, Theorem 3.4.12]), then the closure of every functionally bounded set in $C_p(X)$ is compact, so that in this case every functionally bounded set is metrizable. Moreover, it is known that for metrizable X every compact set in $C_p(X)$ is metrizable if and only if X is second countable (see for instance [19, S.215]). However, as we show next, the requirement that for arbitrary X all bounded sets of $C_p(X)$ are metrizable is strongly demanding, much more than X to have the DCCC, as initially follows from Theorem 1.1. For the next theorem note that if X, Y and Z are completely regular spaces and there is a homeomorphism φ from Y onto Z then the map $f \mapsto \varphi \circ f$ is a homeomorphism from $C_p(X, Y)$ onto $C_p(X, Z)$. For the notation see [1, Chapter 0, Section 3]

Proposition 2.1. *Let X be a completely regular space and P a hereditary topological property for $C_p(X)$. In order that every bounded set in $C_p(X)$ has property P it is necessary and sufficient that $C_p(X)$ has property P.*

Proof. Since the open interval $(-1, 1)$ is homeomorphic to the real line $\mathbb{R}$, the subspace $C(X, (-1, 1))$ of $C_p(X)$ equipped with the relative pointwise topology, i. e., the space $C_p(X, (-1, 1))$, is homeomorphic to $C_p(X) = C_p(X, \mathbb{R})$. Now $C(X, (-1, 1))$ is a pointwise bounded set in $C(X)$, and if $C_p(X, (-1, 1))$ has a topological property P then $C_p(X)$ also has property P. Conversely, if $C_p(X)$ has a hereditary property P, every subspace enjoys the same property. In particular each bounded set in $C_p(X)$ has property P. $\square$

Corollary 2.2. *Every bounded set in $C_p(X)$ is metrizable, countably tight or has the Fréchet-Urysohn property if and only if $C_p(X)$ is respectively metrizable, countably tight or has the Fréchet-Urysohn property. In particular, every bounded set in $C_p(X)$ is metrizable if and only if X is countable.*

Proof. The first statement is consequence of the previous proposition. The second follows from the fact that $C_p(X)$ is metrizable if and only if X is countable. $\square$

If X is a compact set, all bounded sets in the Banach space $C(X)$ equipped with the supremum norm are weakly metrizable if and only if X is scattered and countable. For the 'if' direction note that if X is scattered the weak topology of $C(X)$ coincides on the closed unit ball Q of $C(X)$ with the pointwise topology (see [14, Page 273]), so if X is countable then Q is weakly metrizable. For the 'only if' direction observe that X must be countable and scattered because the dual $C(X)^*$ is norm separable. Further, for each quasibarrelled space $C_k(X)$ every bounded set in the strong dual of $C_k(X)$ is metrizable, as follows from [16, Proposition 8.3.35] bearing in mind that the space $C_k(X)$ is always quasinormable. This provides an additional motivation to the following theorem.

Theorem 2.3. *Bounded sets of $C_k(X)$ are weakly metrizable if and only if X is countable.*

Proof. Let us denote by F the topological dual of $C_k(X)$ and let us assume that the bounded sets of $C_k(X)$ are metrizable under the weak topology of $C_k(X)$. According to [8, Corollary 5], when equipped with the strong topology $\beta(F, C(X))$, the space F becomes a trans-separable locally convex space. In other words, if $\mathcal{P}$ denotes the family of gauges of the polars in F of the bounded sets of $C_k(X)$, then F is trans-separable when equipped with the topology defined by the family $\mathcal{P}$. On the other hand, the linear subspace $C^b(X)$ is dense in $C_k(X)$, so that, if $\widehat{\tau}_k$ stands for the relative compact-open topology induced on $C^b(X)$ by $C_k(X)$, the dual of $\left(C^b(X), \widehat{\tau}_k\right)$ also coincides with F. If Q is the closed unit ball of the Banach space $C^b(X)$, provided with the supremum-norm topology, then Q is a bounded subset of $C_k(X)$. So, if Q^0 stands for the polar of Q in F and $q \in \mathcal{P}$ denotes the gauge of Q^0, by the very definition of trans-separable locally convex space, the pair (F, q) is a separable metric space. In particular, the dual $L(X)$ of $C_p(X)$, regarded as a linear subspace of F with the relative topology $\widehat{\mathcal{T}_q}$ induced by the norm q is separable. Since X is a discrete subspace of the separable metric space $(L(X), \widehat{\mathcal{T}_q})$ by virtue of Fact 1.4, necessarily the space X must be countable.

For the converse, assume that X is countable. Thanks to Corollary 2.2, it suffices to show that on each bounded set in $C_k(X)$ the weak topology of $C_k(X)$ coincides with the pointwise topology of $C(X)$. So, let B be a bounded set in $C_k(X)$. Then B is pointwise bounded, hence pointwise metrizable by the previous corollary. Now choose a pointwise compact subset K of B. We claim that K is weakly compact.

Since $C_p(X)$ is metrizable the space $C_k(X)$ is weakly angelic, so we need only to check that K is sequentially compact in the weak topology of $C_k(X)$. Pick a sequence $\{f_n\}_{n=1}^{\infty}$ in K and a select a subsequence $\{g_n\}_{n=1}^{\infty}$ of $\{f_n\}_{n=1}^{\infty}$ that converges pointwise to some $g \in K$. According to the version of the Lebesgue dominated convergence theorem for (discrete) measures of compact support, the sequence $\{g_n\}_{n=1}^{\infty}$ converges to g in the weak topology of $C_k(X)$, which shows that K is weakly compact, as claimed.

Hence the pointwise topology and the weak topology of $C_k(X)$ have the same compact sets inside of B. But since B is pointwise metrizable, it is a k–space for the pointwise topology, so that the weak topology and the pointwise topology coincide on B. Therefore B must be weakly metrizable. □

If additionally X is compact, the previous theorem implies the well known fact that the dual of the Banach space $C(X)$ is separable if and only if X is scattered and countable.

Example 2.4. If E is a locally convex space whose strong dual $(E', \beta(E', E))$ is separable, the bidual E'' is weak* submetrizable and, consequently, every bounded set in E is weakly metrizable (and separable). The converse statement is not true in general. For example, if X is a completely regular space with $|X| > 2^{\aleph_0}$ the dual $L(X)$ of $C_p(X)$ equipped with the strongest locally convex topology $\beta\left(L(X), \mathbb{R}^X\right)$ enjoys the property that every strongly bounded set is finite-dimensional. However the dual $\mathbb{R}^X$ of the space $\left(L(X), \beta\left(L(X), \mathbb{R}^X\right)\right)$ is not separable due to the fact that the strong topology $\beta\left(\mathbb{R}^X, L(X)\right)$ coincides with the product topology. On the other hand, observe that if E is an infinite-dimensional Banach space with separable dual every bounded set in E is weakly metrizable, but E (weak) is not. □

The approach of Proposition 2.1 holds for $C_k(X)$ since $C(X, (-1, 1))$ is also bounded in $C_k(X)$. In particular, if every bounded set in $C_k(X)$ is metrizable, then $C_k(X)$ must be metrizable as well. The following more constructive proof easily verifies the last statement and also Aren's classic theorem that $C_k(X)$ is metrizable if and only if X is hemicompact [20, Theorem 13.2.4].

Theorem 2.5. *Let X be completely regular. The bounded sets of $C_k(X)$ are metrizable if and only if X is hemicompact.*

Proof. We first assume all bounded sets in $C_k(X)$ are metrizable and show X must be hemicompact. If Y is a subset of X let Y^0 denote the set

$$\left\{f \in C(X) : \sup_{y \in Y} |f(y)| \leq 1\right\}.$$

In the bounded subset X^0 of $C_k(X)$ a base of neighborhoods of the origin consists of all sets $\epsilon K^0 \cap X^0$ with $\epsilon > 0$ and K compact. Metrizability ensures a countable base $\{\epsilon_n K_n^0 \cap X^0 : n \in \mathbb{N}\}$ with $\epsilon_n > 0$ and K_n compact. Thus for a given compact K in X there is some $p \in \mathbb{N}$ such that

$$\epsilon_p K_p^0 \cap X^0 \subseteq 2^{-1} K^0 \cap X^0.$$

We claim $K \subseteq K_p$. If not, there is $x \in K \setminus K_p$. So, we may choose $h \in X^0$ such that $h(x) = 1$ and $h(K_p) = \{0\}$. This contradicts the previous relation, since $h \in \epsilon_p K_p^0 \cap X^0$ but $h \notin 2^{-1}K^0$. Therefore the sequence $\{K_n : n \in \mathbb{N}\}$ swallows all compact sets K of X, i. e., X is hemicompact.

Conversely, if a sequence of compact sets $\{K_n : n \in \mathbb{N}\}$ swallows all compact sets in X, then $\{m^{-1}K_n^0 : m, n \in \mathbb{N}\}$ is a countable base of neighborhoods of the origin in the locally convex space $C_k(X)$, which implies $C_k(X)$ is metrizable. In particular, all its bounded subsets are metrizable. □

The following example collects together some (possibly well known) facts concerning the spaces $C_k(\mathbb{Q})$ and $C_k(\mathbb{R})$ as an application of the previous results.

Example 2.6. If $\mathbb{Q}$ is the set of rational numbers equipped with the relative topology of $\mathbb{R}$, by Theorem 2.3 the bounded sets of $C_k(\mathbb{Q})$ are weakly metrizable, whereas by Theorem 2.5 there are bounded sets that are not metrizable under the compact-open topology. Despite that, $C_k(\mathbb{Q})$ has countable tightness, as follows for instance from [11, Proposition 6]. Theorem 2.5 also implies that the compact-open topology and the weak topology of $C_k(\mathbb{Q})$ do not have the same compact sets, otherwise an argument similar to that of the final part of the proof of Theorem 2.3 would prove that all bounded sets in $C_k(\mathbb{Q})$ are metrizable, contradicting Theorem 2.5. In particular, $C_k(\mathbb{Q})$ does not have the Schur property. The reason for this fact is that $\mathbb{Q}$ has an infinite compact set, as shown in [12, Proposition 2.14]. On the other hand, the fact that $\mathbb{R}$ is uncountable means the bounded sets of $C_k(\mathbb{R})$ are not all weakly metrizable. However $C_k(\mathbb{R})$ is metrizable (indeed a Fréchet space) since $\mathbb{R}$ is an hemicompact $k_{\mathbb{R}}$-space. □

3. Distinguished $C_p(X)$ spaces

Let us recall that a locally convex space F is called *distinguished* if its strong dual $(F', \beta(F', F))$ is barrelled. In what follows we shall denote by E the *bidual* $C_p(X)''$ of $C_p(X)$, i. e., the dual of $L(X)$ when equipped with the strong topology $\beta(L(X), C(X))$. By identifying X with the Hamel basis $\delta(X)$ in $L(X)$, one may regard E as a linear subspace of $\mathbb{R}^X$, the algebraic dual of $L(X)$. Thus $C(X) \subseteq E \subseteq \mathbb{R}^X$. Clearly, the product topology of $\mathbb{R}^X$ induces the weak* topology $\sigma(E, L(X))$ on E. Moreover, $\sigma(E, L(X))$ coincides with the strong topology $\beta(E, L(X))$ since, as shown in [9, page 392], $L(X)$ is feral.

Theorem 3.1. (Ferrando, Kąkol and Saxon, [9]) *If X is completely regular, then each $\beta(L(X), C(X))$-bounded set in $L(X)$ is finite-dimensional. Equivalently, on the bidual E of $C_p(X)$ these three topologies coincide: the weak* topology $\sigma(E, L(X))$, the strong topology $\beta(E, L(X))$, and the relative product topology induced by $\mathbb{R}^X$.*

Corollary 3.2. *The strong dual $(L(X), \beta(L(X), C(X)))$ of $C_p(X)$ is barrelled if and only if it is quasibarrelled.*

Proof. This is because every $\beta(L(X), C(X))$-barrel in $L(X)$ is bornivorous. □

If $\mathsf{Bound}\left(C_p\left(X\right)\right)$ stands for the family of all bounded sets of $C_p\left(X\right)$ we have

$$E=\bigcup\left\{\overline{P}^{\mathbb{R}^X}:P\in\mathsf{Bound}\left(C_p\left(X\right)\right)\right\}=\bigcup\left\{Q^{00}:Q\in\mathsf{Bound}\left(C_p\left(X\right)\right)\right\}.$$

Let us denote by E_0 the linear subspace of $\mathbb{R}^X$ consisting of those functions with finite support (i. e., with finitely many non null coordinates). If Q denotes the closed unit ball of the Banach space $C^b\left(X\right)$, note that $Q^{00}=\delta\left(X\right)^0$, where both the second polar of the left side and the polar of $\delta\left(X\right)$ in the right side are in $\mathbb{R}^X$, the algebraic dual of $L\left(X\right)$. Hence $\overline{Q}^{\mathbb{R}^X}\subseteq E$, which means that $\ell_\infty\left(X\right)\subseteq E$, where $\ell_\infty\left(X\right)$ represents the linear space of all uniformly bounded real-valued functions defined on X. In particular $E_0\subseteq E$.

Theorem 3.3. *Let X be a completely regular space. Then:*

(1) *If X is countable, then $C_p\left(X\right)$ is distinguished.*

(2) *If $C_p\left(X\right)$ is distinguished, the bidual of $C_p\left(X\right)$ coincides with $\mathbb{R}^X$.*

(3) *If X is countable, the bidual of $C_p\left(X\right)$ coincides with $\mathbb{R}^X$.*

Proof. For the first statement, assume that X is countable. Then $C_p\left(X\right)$ is metrizable and consequently $L_\beta\left(X\right)$ is a (DF)-space. In particular $L_\beta\left(X\right)$ is $\aleph_0$-quasibarrelled. A countable Hamel basis $\delta\left(X\right)$ also ensures that $L_\beta\left(X\right)$ is separable. Therefore $L_\beta\left(X\right)$ is quasibarrelled [16, 8.2.20]. According to Corollary 3.2, $L_\beta\left(X\right)$ is barrelled.

For the second statement, let us assume that $L_\beta\left(X\right)$ is barrelled. Then its dual E is a quasicomplete [15, 23.1 (3)] subspace of $\mathbb{R}^X$ containing E_0, and the quasi-completion of E_0 (equipped with the relative product topology) coincides with $\mathbb{R}^X$. Hence $E=\mathbb{R}^X$.

The third statement is a straightforward consequence of the previous two. □

Corollary 3.4. *The space $C_p\left(X\right)$ is distinguished if and only if $\beta\left(L\left(X\right),C\left(X\right)\right)$ coincides with the strongest locally convex topology on $L\left(X\right)$.*

Proof. The 'if' part is obvious, since any linear space equipped with the strongest locally convex topology is barrelled. The 'only if' part is a consequence of the second affirmation of the previous result. For if $L_\beta\left(X\right)$ is barrelled, it has its Mackey topology $\mu\left(L\left(X\right),E\right)=\mu\left(L\left(X\right),\mathbb{R}^X\right)$, the strongest locally convex topology compatible with the pairing between $L\left(X\right)$ and its algebraic dual $\mathbb{R}^X$. □

Corollary 3.5. *The strong dual of $C_p\left(X\right)$ is separable if and only if X is countable.*

Proof. If the strong dual $L_\beta\left(X\right)$ of $C_p\left(X\right)$ is separable, every bounded set in $C_p\left(X\right)$ is metrizable. So we can apply Corollary 2.2 to get that X is countable. Conversely, every countable-dimensional locally convex space is separable. □

Corollary 3.6. *Let X be a P-space. The bounded sets of $C_p\left(X\right)$ are metrizable if and only if X is countable and discrete.*

Proof. If the bounded sets are metrizable, $C_p(X)$ is metrizable and reflexive, so that the previous theorem implies that its strong bidual coincides with $\mathbb{R}^X$. Alternatively, by Corollary 2.2 the bounded sets of $C_p(X)$ are metrizable if and only if $C_p(X)$ is metrizable, and the latter holds if and only if X is countable. Finally, note that every countable P-space is discrete. □

Example 3.7. By the previous theorem, the space $C_p(\mathbb{Q})$ is distinguished. □

The converse of the first statement of the previous theorem does not hold, since if X is uncountable and discrete then $C_p(X) = \mathbb{R}^X$ is distinguished. There are also distinguished $C_p(X)$ spaces with X a P-space which is uncountable and non discrete, as the following example shows.

Example 3.8. If ω_1 is the first uncountable ordinal and L_{ω_1} the one-point Lindelöfication of the discrete space $[0, \omega_1)$ (see [1, Example 4.2.15]), then $C_p(L_{\omega_1})$ is distinguished.

Proof. We put $X = L_{\omega_1}$. As a set, $X = [0, \omega_1]$. A function $g \in \mathbb{R}^X$ belongs to $C(X)$ if and only if it is constant on a tail, i. e., outside some countable subset of $[0, \omega_1)$. Let A be an arbitrary $\sigma(E, L(X))$-bounded set in the bidual E of $C_p(X)$. To show that A is equicontinuous on $L_\beta(X)$, define a set B consisting of all $g \in \mathbb{R}^X$ for which there exist an $f \in A$ and a finite set $S \subseteq [0, \omega_1)$ such that (i) $g(x) = f(x)$ if $x \in S$, and (ii) $g(x) = f(\omega_1)$ if $x \in X \setminus S$. From (ii) it is clear that $B \subseteq C(X)$. Furthermore, B is bounded in $C_p(X)$ since

$$\sup_{g \in B} |g(x)| \leq \max\left\{\sup_{f \in A} |f(x)|, \sup_{f \in A} |f(\omega_1)|\right\}$$

for each $x \in X$. Therefore, the polar B^0 is a neighborhood U of the origin in the strong dual $L_\beta(X)$. Let us regard $\mathbb{R}^X$ as the algebraic dual of $L(X)$, with $C(X) \subseteq E \subseteq \mathbb{R}^X$, and take the bipolar B^{00} in $\mathbb{R}^X$. Now B^{00}, being uniformly bounded on U, is equicontinuous on $L_\beta(X)$ and contains the closure $\overline{B}$ of B in $\mathbb{R}^X$. The definition of B ensures $A \subseteq \overline{B}$. Hence A is equicontinuous on $L_\beta(X)$. □

Theorem 3.9. *The strong bidual E of $C_p(X)$ is a barrelled and distinguished subspace of $\mathbb{R}^X$, and the strong bidual of E coincides with $\mathbb{R}^X$.*

Proof. The strong bidual E has dual $E' = L(X)$ since the topology of E is induced by $\mathbb{R}^X$, in which E is dense. Theorem 3.1 further asserts that $\sigma(E, L(X)) = \beta(E, L(X))$. Hence $\sigma(E, E') = \beta(E, E')$; equivalently, E is barrelled [15, 21.2 (2)]. Next we claim that if E is any subspace of $\mathbb{R}^X$ containing the space E_0 of functions with finite support, then the strong dual $(E', \beta(E', E))$ is barrelled, since it coincides with $L(X)$ equipped with its strongest locally convex topology τ_0. From this it follows that the strong bidual of E is $\mathbb{R}^X$. Then, all the higher strong dual iterations are distinguished, as they alternate between the barrelled spaces $(L(X), \tau_0)$ and $\mathbb{R}^X$. To establish the claim, let U be an absorbing absolutely convex set in $L(X)$. For each $x \in X$ there exists $\epsilon_x > 0$ such that $\epsilon_x x \in U$. Let V be the absolutely convex hull of the set $\{\epsilon_x x : x \in X\}$ and note that $V \subseteq U$ and V is absorbing. Therefore V^0 is pointwise bounded in E and V^{00} is a $\beta(L(X), E)$-neighborhood of the origin in $L(X) = E'$. Moreover, V is

$\sigma(L(X), E)$-closed since $E_0 \subseteq E$. Therefore $V = V^{00}$ by the bipolar theorem, so the superset U is also a $\beta(L(X), E)$-neighborhood of the origin. □

Proposition 3.10. *If X is completely regular, the strong dual of $C_p(X)$ is always complete. Equivalently, each linear functional on $C(X)$ whose restrictions to the bounded sets of $C_p(X)$ are relatively continuous is continuous on the whole of $C_p(X)$.*

Proof. Consider the restriction map $T : C_p(\upsilon X) \to C_p(X)$ given by $Tf = f|_X$. This is a linear continuous map from $C_p(\upsilon X)$ onto $C_p(X)$, so that the adjoint mapping $T^* : L_p(X) \to L_p(\upsilon X)$ is an isomorphism into. If Q is a bounded set in $C_p(X)$, then $T^{-1}(Q) = Q^\upsilon = \{f^\upsilon : f \in Q\}$ is a bounded set of $C_p(\upsilon X)$. Otherwise if there is $y \in \upsilon X$ with $\sup\{|f(y)| : f \in Q\} = +\infty$, we may choose a sequence $\{f_n\}_{n=1}^\infty$ in Q such that $\sup_{n\in\mathbb{N}} |f_n(y)| = +\infty$. But according to [14, Lemma 9.1] there is $x_y \in X$ such that $f_n(y) = f_n(x_y)$ for all $n \in \mathbb{N}$. Hence,

$$\sup\{|f_n(y)| : n \in \mathbb{N}\} = \sup\{|f_n(x_y)| : n \in \mathbb{N}\} < +\infty$$

a contradiction. The fact that $C_p(\upsilon X)$ and $C_p(X)$ have the same bounded sets or, more exactly, that $T^{-1}(Q) \in \mathsf{Bound}(C_p(\upsilon X))$ whenever $Q \in \mathsf{Bound}(C_p(X))$, ensures that $T^*(Q^0) = (Q^\upsilon)^0 \cap T^*(L(X))$ for every bounded set Q in $C_p(X)$, so that T^* is an isomorphism from the strong dual $L_\beta(X)$ of $C_p(X)$ into the strong dual $L_\beta(\upsilon X)$ of $C_p(\upsilon X)$. This means that, considering $L(X)$ as a linear subspace of $L(\upsilon X)$, the relative topology on $L(X)$ induced by the strong topology $\tau_\upsilon = \beta(L(\upsilon X), C(\upsilon X))$ of $L(\upsilon X)$ coincides with the strong topology $\beta(L(X), C(X))$ of $L(X)$.

We claim that $L_\beta(X)$ is (isomorphic to) a closed subspace of $L_\beta(\upsilon X)$. First note that if $e_y \in \mathbb{R}^{\upsilon X}$ with $y \in \upsilon X$ is defined by $e_y(x) = 0$ if $x \neq y$ and $e_y(y) = 1$, then $e_y = \chi_{\{y\}}$ so that e_y belongs to the bidual of $C_p(\upsilon X)$. If we assume by contradiction that there exists $u \in L(\upsilon X) \setminus L(X)$ which is the limit of a net $\{u_d : d \in D\}$ in $L(X)$ under the strong topology τ_υ, there is $y \in \upsilon X \setminus X$ such that $\langle e_y, u\rangle \neq 0$ but $\langle e_y, u_d\rangle \to 0$ at the same time. This contradiction establishes the claim.

Since υX is realcompact, the space $C_p(\upsilon X)$ is bornological by the Buchwalter-Schmets theorem, which implies that the strong dual $L_\beta(\upsilon X)$ is complete (see [15, 28.5.(1)]). So, the fact that $L_\beta(X)$ is (isomorphic to) a closed subspace of $L_\beta(\upsilon X)$, guarantees that $L_\beta(X)$ is complete as well. □

Corollary 3.11. *If X is completely regular then $C_p(X)$ is distinguished if and only if its strong dual is bornological.*

Proof. If $C_p(X)$ is distinguished then $\beta(L(X), C(X))$ coincides with the strongest locally convex topology on $L(X)$ by Corollary 3.4. Hence $(L(X), \beta(L(X), C(X)))$, as the locally convex hull of its finite-dimensional linear subspaces, is bornological. For the converse, since each complete bornological space is barrelled (see [3, Theor. 3.6.19]), Proposition 3.10 guarantees that $(L(X), \beta(L(X), C(X)))$ is barrelled. Or, one could use Corollary 3.2 and the fact that every bornological space is quasibarrelled. □

Acknowledgement. We thank Tomasz Kania for useful comments and suggestions, and Professor Witold Marciszewski for providing us a shorter and effective approach to Corollary 2.2. We are also indebted to the referee, whose excellent work and valuable suggestions have shortened proofs and improved the paper considerably.

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