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Examples of Nondistinguished Function Spaces $C_p(X)$

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In the pointwise topology, familiar function spaces $C_p(\beta\mathbb{N})$, $C_p(\mathbb{R})$, $C_p\,[0,1]$, and C_p(Cantor set) are not distinguished.

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1. Introduction

A locally convex space E is *distinguished* if its strong dual $(E', \beta(E', E))$ is barrelled. The prequel [1] shows that when the Tychonoff space X is countable, the locally convex space $C_p(X)$ of continuous real-valued functions on X endowed with the pointwise topology is always distinguished. So is $C_p(X)$ for every (uncountable) discrete X, and so is the strong dual of $C_p(X)$ for arbitrary X, as are all higher iterates of strong duals [1, Proof of Theorem 3.9]. Here we note a robust class of $C_p(X)$ spaces that are *not* distinguished.

2. Results

Theorem 2.1. *Let X be a Tychonoff space. If there is a neighborhood base $\mathcal{B}$ of the origin in the strong dual $L_\beta(X)$ of $C_p(X)$ with $|\mathcal{B}| \leq |X|$, then $C_p(X)$ is not distinguished.*

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Proof. We conventionally view X as a Hamel basis for the dual $L(X)$ of $C_p(X)$. Always, $C_p(X)$ is distinguished if and only if $L_\beta(X)$ has its strongest locally convex topology [1, Corollary 3.4]. With $\mathcal{B}$ given as above, for each $V \in \mathcal{B}$ we choose $x_V \in X$ such that $x_V \neq x_U$ for distinct $V, U \in \mathcal{B}$. There exist $\epsilon_V > 0$ such that $\epsilon_V \cdot x_V \in V$. We choose $\epsilon_V > c_V > 0$ and let W be the absolutely convex hull of the Hamel basis

$$\{c_V \cdot x_V : V \in \mathcal{B}\} \cup \{x \in X : x \notin \{x_V : V \in \mathcal{B}\}\}.$$

Since each $\epsilon_V \cdot x_V \in V \setminus W$, no basic neighborhood V is contained in W. Therefore the absolutely convex absorbing set W is not a neighborhood of the origin in $L_\beta(X)$. Hence $L_\beta(X)$ does not have its strongest locally convex topology. □

Corollary 2.2. *The space $C_p(\beta\mathbb{N})$ is not distinguished, where $\beta\mathbb{N}$ is the Stone-Čech compactification of the natural numbers $\mathbb{N}$.*

Proof. Since $\beta\mathbb{N}$ is separable, $|C(\beta\mathbb{N})| = |\mathbb{R}^{\mathbb{N}}| = 2^{\aleph_0} = \mathfrak{c}$. We conclude that $2^{\mathfrak{c}} = |\text{power set of } C_p(\beta\mathbb{N})| \geq |\mathcal{A}|$, where $\mathcal{A}$ is the set of all bounded subsets of $C_p(\beta\mathbb{N})$. Now the set $\{A^\circ : A \in \mathcal{A}\}$ of polars is a neighborhood base $\mathcal{B}$ of the origin in $L_\beta(\beta\mathbb{N})$ with $|\mathcal{B}| \leq |\mathcal{A}| \leq 2^{\mathfrak{c}}$, and it is known that $2^{\mathfrak{c}} = |\beta\mathbb{N}|$. □

Lemma 2.3. *Each separable metric space S contains no more than $\mathfrak{c}$ closed sets.*

Proof. Since the number of subsequences of a fixed dense sequence in S cannot exceed $\mathfrak{c}$, we have $|S| \leq \mathfrak{c}$. Hence the number of sequences in S cannot exceed $\mathfrak{c}^{\aleph_0} = \mathfrak{c}$. Now every subset of S is separable, so every closed set in S is the closure of a sequence in S, and the result follows. □

Michael proved a Tychonoff space X is cosmic, i.e., is a continuous image of a separable metric space, if and only if $C_p(X)$ is cosmic [3, Propositions 10.2, 10.5].

Corollary 2.4. *If X is cosmic with $|X| = \mathfrak{c}$, then $C_p(X)$ is not distinguished.*

Proof. Let $\mathcal{F}$ be the set of closed and bounded sets in $C_p(X)$. The cosmic hypothesis ensures $C_p(X)$ is a continuous image of some separable metric space S (*cf.* [2, Corollary 1.6]). Pre-images of distinct members of $\mathcal{F}$ are distinct closed sets in S, so the Lemma yields $|\mathcal{F}| \leq \mathfrak{c}$. As $\mathfrak{c} = |X|$ and $\mathcal{B} := \{A^\circ : A \in \mathcal{F}\}$ is a base of neighborhoods of the origin in $L_\beta(X)$ with $|\mathcal{B}| \leq |\mathcal{F}|$, the Theorem applies. □

Corollary 2.5. *The familiar spaces $C_p[0,1]$, $C_p(\mathbb{R})$, and $C_p($Cantor set$)$ are not distinguished.*

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