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Local Properties of the Surface Measure of Convex Bodies

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It is well known that any measure in S^2 satisfying certain simple conditions is the surface measure of a bounded convex body in $\mathbb{R}^3$. It is also known that a local perturbation of the surface measure may lead to a nonlocal perturbation of the corresponding convex body. We prove that, under mild conditions on the convex body, there are families of perturbations of its surface measure forming line segments, with the original measure at the midpoint, corresponding to *local* perturbations of the body. Moreover, there is, in a sense, a huge amount of such families. We apply this result to Newton's problem of minimal resistance for convex bodies.

Keywords: Convex sets, Blaschke addition, Newton's problems of minimal resistance.

2010 Mathematics Subject Classification: 52A15, 52A40, 49Q10, 49Q20.

1. Motivation

The motivation for this study came from the famous Newton problem of minimal resistance. In modern terms and in a generalized form the problem can be formulated as follows.

Let $C_1 \subset C_2 \subset \mathbb{R}^3$ be convex bodies. Denote by $\mathfrak{C}(C_1, C_2)$ the set of convex bodies C satisfying $C_1 \subset C \subset C_2$. Let $f\colon S^2 \to \mathbb{R}$ be a continuous function. The *generalized Newton problem* is as follows:

$$\inf_{C \in \mathfrak{C}(C_1,C_2)} F(C), \qquad \text{where} \quad F(C) = \int_{\partial C} f(n_\xi)\, d\mathcal{H}^2(\xi). \tag{1}$$

Here and in what follows, $\mathcal{H}^2$ designates the 2-dimensional Hausdorff measure in $\mathbb{R}^3$, and n_ξ is the outer normal to C at a regular point $\xi \in \partial C$.

In this form the problem first appears in the paper by Buttazzo and Guasoni [5]. In fact, they consider an even more general case when f depends not only on n_ξ, but also on ξ, and is continuous in both variables, and investigate some

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other classes of convex bodies, along with $\mathfrak{C}(C_1, C_2)$. They prove that the problem always has a solution.

Of special interest is the particular case when C_2 is a right circular cylinder and C_1 is its rear base. In an appropriate orthogonal coordinate system with the coordinates x, y, z, C_1 and C_2 take the form: $C_1 = \Omega \times \{0\}$ and $C_2 = \Omega \times [0, M]$, with $\Omega = \{(x, y) : x^2 + y^2 \le 1\}$ and $M > 0$. Additionally, the function f is taken to be $f(n) = \langle n, e_3\rangle_+^3$, where $e_3 = (0, 0, 1)$ and b_+ is the positive part of $b \in \mathbb{R}$. Here and in what follows, $\langle \cdot\, , \cdot\rangle$ means the scalar product.

In this particular case, a body $C \in \mathfrak{C}(C_1, C_2)$ is the subgraph of a concave function $u_C \colon \Omega \to \mathbb{R}$ satisfying $0 \le u_C(x, y) \le M$, namely,

$$C = \{(x, y, z) : (x, y) \in \Omega,\ 0 \le z \le u_C(x, y)\}.$$

Further, ∂C is the union of the disc $\Omega \times \{0\}$ and the graph of u_C; for $\xi = (x, y, 0) \in \Omega \times \{0\}$ one has $n_\xi = (0, 0, -1)$ and $f(n_\xi) = 0$, and for $\xi = (x, y, u_C(x, y))$ one has

$$n_\xi = \frac{1}{\sqrt{1 + |\nabla u_C(x, y)|^2}} \Big(-\frac{\partial u_C}{\partial x}(x, y),\ -\frac{\partial u_C}{\partial y}(x, y),\ 1\Big),$$

$$f(n_\xi) = \frac{1}{\big(1 + |\nabla u_C(x, y)|^2\big)^{3/2}}, \quad \text{and} \quad d\mathcal{H}^2(\xi) = \sqrt{1 + |\nabla u_C(x, y)|^2}\ dx\, dy.$$

In view of this, Problem (1) can be written as follows: minimize the integral

$$\iint_\Omega \frac{1}{1 + |\nabla u(x, y)|^2}\, dx\, dy \tag{2}$$

in the class of concave functions $u \colon \Omega \to \mathbb{R}$ satisfying $0 \le u(x, y) \le M$. In what follows, Problem (2) will be called the *particular Newton problem*.

The particular problem was first stated by Buttazzo and Kawohl in [6] and has been intensively studied since then; see, e.g., [3, 4, 8, 9, 10, 11, 12]. However, it has not been completely solved yet. It is proved that the solution u exists [4] and has the following properties: first, if ∇u exists at a certain point $(x, y) \in \Omega$ then either $|\nabla u(x, y)| = 0$ or $|\nabla u(x, y)| \ge 1$ [4]; second, if u is in the class C^2 in a neighborhood of (x, y) and $0 < u(x, y) < M$ then the matrix of the second derivative $D^2 u(x, y)$ has a zero eigenvalue [3]. Properties of the solution were numerically examined in [10, 15]. Note that problem (2) in the subclass of concave radially symmetric functions was first considered by I. Newton in his *Principia* (1687).

Different functions $f \colon S^2 \to \mathbb{R}$ correspond to different physical models describing interaction of convex bodies with a sparse flow of point particles. It is always assumed in these models that the flow is extremely rarefied, so that mutual interaction of the flow particles can be neglected.

Consider the case when a homogeneous flow of particles with the same velocity is incident on a convex body and the particles are reflected elastically after hitting the body's surface. Choose a coordinate system so as the z-axis is parallel and counter-directional to the flow velocity. Then the force of resistance of the body to the flow equals $2\rho \mathrm{v}^2\, F(C)$, where $F(C)$ is defined by formula (1), ρ is the density, v is the scalar velocity of the flow, and $f(n) = \langle n,\, e_3\rangle_+^3$. If $C \in \mathfrak{C}(C_1, C_2)$, where C_2 is a right circular cylinder and C_1 is its rear base, $F(C)$ can be expressed in terms of the function $u = u_C$ by formula (2).

If there is a thermal motion of particles in the flow, and consequently, the velocities of incident particles do not coincide, one should take a different function f in (1). The precise form of this function depends on the temperature, the mean velocity, and the composition of the flow. The corresponding class of models and related minimization problems for rotationally symmetric bodies was considered in [14] (see also chapter 3 in the book [13]).

It may also happen that the law of reflection is not elastic. Assume, for example, that the velocity of the reflected particle has the same modulus as the velocity of incidence, is tangent to the body's surface at the point of reflection, and the normal vector at this point is a positive linear combination of the velocity of incidence and the velocity of reflection. If, additionally, the incident particles of the flow have the same velocity, we have $f(n) = (1 - \sqrt{1 - n_3^2})(n_3)_+$, and the integrand in (2) should be substituted with $1 - |\nabla u(x,y)|/\sqrt{1 + |\nabla u(x,y)|^2}$.

2. The surface measure of convex bodies

Let $C \subset \mathbb{R}^3$ be a convex body , that is, a convex compact set with nonempty interior. By $|A|$ or $\mathcal{H}^2(A)$ we denote the 2-dimensional Hausdorff measure of a Borel set $A \subset \partial C$.

Define the map $\mathcal{N}_C \colon \partial' C \to S^2$ by $\mathcal{N}_C(\xi) = n_\xi$, where $\partial' C$ denotes the (full-measure) subset of regular points of ∂C. The pushforward measure $\nu_C := \mathcal{N}_C \# \mathcal{H}^2$ defined in S^2 is called the *surface measure of* C. Thus, for a Borel set $A \subset S^2$ we have $\nu_C(A) = |\mathcal{N}_C^{-1}(A)|$.

The surface measure $\nu = \nu_C$ of a convex body C satisfies the following conditions.

(A1) $\int_{S^2} n\, d\nu(n) = \vec{0}$.

(A2) The linear span of $\operatorname{spt} \nu$ is $\mathbb{R}^3$.

By Alexandrov's theorem [1], conversely, if a measure ν defined in S^2 satisfies (A1) and (A2) then there exists a unique (up to a translation) convex body C with the surface measure equal to ν, that is, $\nu_C = \nu$.

The notion of surface measure generalizes to all dimensions, and Alexandrov's theorem is also valid in any dimension.

The notion of surface measure allows one to define the sum of two convex bodies A and B (Blaschke sum), which is, up to a translation, the convex body $A \# B$ such that $\nu_{A\#B} = \nu_A + \nu_B$.

In terms of surface measure, problem (1) can be rewritten as a minimization problem for a linear functional on a space of measures,

$$\inf_{\nu \in \Upsilon(C_1, C_2)} \mathcal{F}(\nu), \qquad \text{where} \quad \mathcal{F}(\nu) = \int_{S^2} f(n)\, d\nu(n), \tag{3}$$

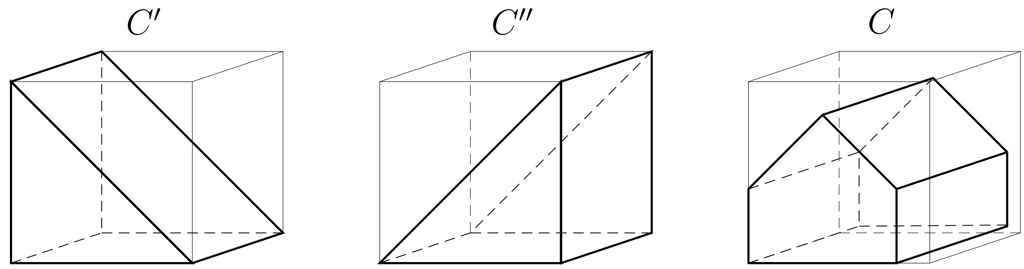
where $\Upsilon(C_1, C_2)$ designates the set of surface measures ν_C with $C_1 \subset C \subset C_2$; that is, $\Upsilon(C_1, C_2) := \{\nu_C : C_1 \subset C \subset C_2\}$. It follows from [5] that this problem always has a solution.

The fact that Newton's problem can be formulated as a linear problem in terms of surface measure was first noticed by Carlier and Lachand-Robert in [7]. They used the linear representation to study the generalized Newton problem for any dimension $d \geq 2$ (i) in the class of convex bodies with fixed $(d-1)$-dimensional volume of the boundary and (ii) in the more restricted class of convex bodies C contained in the lower half-space $\{x = (x_1, \dots, x_d) \in \mathbb{R}^d : x_d \leq 0\}$ with fixed $(d-1)$-dimensional volume of the boundary ∂C, with fixed $(d-1)$-dimensional volume of $\partial C \cap \{x_n = 0\}$ and such that the orthogonal projection of C on the hyperplane $\{x_n = 0\}$ is contained in $\partial C \cap \{x_n = 0\}$.

These classes admit an easy and natural representation in terms of the surface measure. Indeed, the class of surface measures corresponding to (i) contains all measures ν satisfying conditions (A1) $\int_{S^{d-1}} n\, d\nu(n) = \vec{0}$ and (A2) the linear span of sptν is $\mathbb{R}^d$, and having fixed full measure $\nu(S^{d-1})$, while the more restricted class of surface measures corresponding to (ii) contains the measures satisfying conditions (A1) and (A2), having fixed full measure $\nu(S^{d-1})$, and satisfying the additional condition that $\nu\rfloor_{S^{d-1}_+}$ is concentrated at $(0, \dots, 0, 1)$.

On the contrary, the main difficulty of Problem (3) resides in the extreme complexity of the set of measures $\Upsilon(C_1, C_2)$. Nevertheless, we believe that studying this set could trigger further progress in Newton's problem and additionally, extend our knowledge about the surface measure and Blaschke addition.

Generally, the set $\Upsilon(C_1, C_2)$ is not convex. We consider two examples.



(a) (b) (c)

Figure 2.1: The convex bodies from Example 2.1. Here C_2 is the cube, and C_1 is its lower face.

Example 2.1. This example is probably the simplest one. Let C_2 be a unit cube and C_1 be one of its faces, $C_2 = [0,\, 1]^3$ and $C_1 = [0,\, 1] \times [0,\, 1] \times \{0\}$. Take the convex bodies $C' = \{(x, y, z) \in C_2 : x + z \leq 1\}$ and $C'' = \{(x, y, z) \in C_2 : x \geq z\}$; see Fig. 2.1(a,b). Obviously, both C' and C'' contain C_1 and are contained in C_2, and therefore, both $\nu_{C'}$ and $\nu_{C''}$ are elements of $\Upsilon(C_1, C_2)$. We have

$$\nu_{C'} = \frac{1}{2}\, \delta_{e_2} + \frac{1}{2}\, \delta_{-e_2} + \delta_{-e_3} + \delta_{-e_1} + \sqrt{2}\, \delta_{v_R},$$

$$\nu_{C''} = \frac{1}{2}\,\delta_{e_2} + \frac{1}{2}\,\delta_{-e_2} + \delta_{-e_3} + \delta_{e_1} + \sqrt{2}\,\delta_{v_L},$$

where $e_1 = (1,0,0)$, $e_2 = (0,1,0)$, $e_3 = (0,0,1)$, $v_R = (e_1+e_3)/\sqrt{2}$, and $v_L = (-e_1+e_3)/\sqrt{2}$. Therefore,

$$\frac{1}{2}\,(\nu_{C'} + \nu_{C''}) = \frac{1}{2}\,(\delta_{e_2} + \delta_{-e_2}) + \delta_{-e_3} + \frac{1}{2}\,(\delta_{e_1} + \delta_{-e_1}) + \frac{1}{\sqrt{2}}\,(\delta_{v_R} + \delta_{v_L}).$$

One easily sees that $(\nu_{C'} + \nu_{C''})/2 = \nu_C$, where

$$C = \{(x,y,z) : 0 \le x \le a,\ 0 \le y \le b,\ 0 \le z \le \min\{h+x,\ h+a-x\}\}$$

is a "house" (see Fig. 2.1 (c)) with the parameters a, b, h (the lengths and the height of the "walls") satisfying the relations $ab = 1$, $bh = 1/2$, $ah + a^2/4 = 1/2$. Thus, $a = \sqrt{2/3}$, $b = \sqrt{3/2}$, $h = 1/\sqrt{6}$. The width of C in the x-direction is $a = \sqrt{2/3} < 1$, and therefore, a translation of C cannot contain C_1. The width of C in the y-direction is $b = \sqrt{3/2} > 1$, and therefore, a translation of C cannot be contained in C_2. It follows that $(\nu_{C'} + \nu_{C''})/2$ is not an element of $\Upsilon(C_1, C_2)$. Hence $\Upsilon(C_1, C_2)$ is not convex. □

Example 2.2. This example is related to the particular Newton problem, where $C_1 = \Omega \times \{0\}$ and $C_2 = \Omega \times [0,\ M]$. Take $C' = \{(x,y,z) \in C_2 : Mx + 2z \le M\}$ and $C'' = \{(x,y,z) \in C_2 : -Mx + 2z \le M\}$; see Fig. 2.2(a,b). Both C' and C'' contain C_1 and are contained in C_2, and therefore, both $\nu_{C'}$ and $\nu_{C''}$ are elements of $\Upsilon(C_1, C_2)$.

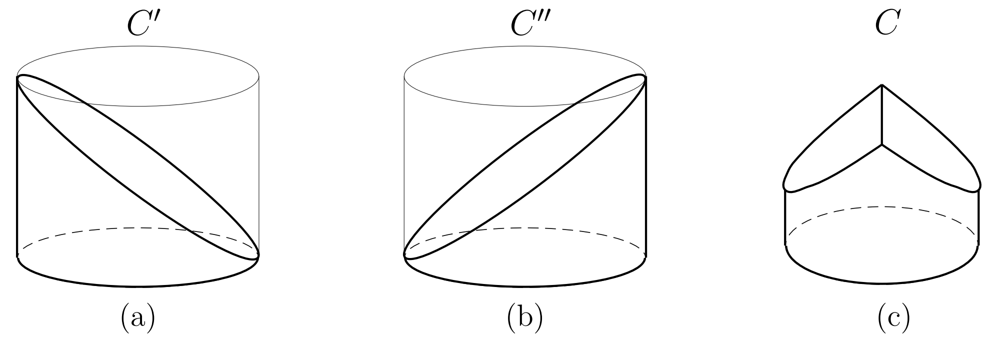


Figure 2.2: The convex bodies from Example 2.2. Here C_2 is the right circular cylinder and C_1 is its lower base.

The surface of C' is composed of 3 parts. The lower part of $\partial C'$ is a flat unit disk with the outer normal $-e_3$. The upper part of $\partial C'$ is the part of the plane $Mx + 2z = M$ bounded by an ellipse, with the area $\pi\sqrt{M^2/4+1}$ and the outer normal $v_M = \frac{1}{\sqrt{M^2+4}}(M,0,2)$. The lateral part of $\partial C'$ is contained in the cylindrical surface $x^2 + y^2 = 1$. Correspondingly, the surface measure of C' is

$$\nu_{C'} = \pi\delta_{-e_3} + \pi\sqrt{M^2/4+1}\,\delta_{v_M} + \nu_1,$$

where ν_1 is supported on the horizontal circle $x^2 + y^2 = 1$, $z = 0$.

Note that C'' is symmetric to the body $C_2 \backslash C' = \{(x,y,z) \in C_2 : Mx+2z > M\}$ with respect to the horizontal plane $z = M/2$; correspondingly, the surface measure $\nu_{C''}$ is symmetric to $\nu_{C_2 \backslash C'}$ with respect to the plane $z = 0$. We have

$$\nu_{C_2 \backslash C'} = \pi \delta_{e_3} + \pi \sqrt{M^2/4+1}\, \delta_{-v_M} + \nu_2,$$

where ν_2 is also supported on the circle $x^2 + y^2 = 1$, $z = 0$. Since e_3 is symmetric to $-e_3$ and $-v_M$ is symmetric to v_{-M} with respect to the plane $z = 0$, we have

$$\nu_{C''} = \pi \delta_{-e_3} + \pi \sqrt{M^2/4+1}\, \delta_{v_{-M}} + \nu_2.$$

Notice that the union of the lateral surfaces of C' and $C_2 \setminus C'$ is the cylindrical surface $x^2 + y^2 = 1$, $0 \le z \le M$, hence the sum of the measures $\nu_1 + \nu_2$ has the linear density M on the horizontal circle $x^2 + y^2 = 1$, $z = 0$.

We have

$$\frac{1}{2}(\nu_{C'} + \nu_{C''}) = \pi \delta_{-e_3} + \frac{\pi}{2} \sqrt{M^2/4+1}\, (\delta_{v_{-M}} + \delta_{v_M}) + \frac{1}{2}(\nu_1 + \nu_2).$$

Consider a convex body C such that $\nu_C = (\nu_{C'} + \nu_{C''})/2$; see Fig. 2.2 (c). We are going to show that the double inclusion

$$C_1 \subset C \subset C_2 \tag{4}$$

is impossible, and therefore, $\Upsilon(C_1, C_2)$ is not convex.

Indeed, the lower part of the surface ∂C is a flat horizontal convex body D with the area π, the upper part of ∂C is the union of two flat convex bodies with the outer normals v_M and v_{-M}, and the lateral part of ∂C is a part of a vertical cylindrical surface; see Fig. 2.2 (c). If D lies below the horizontal plane $z = 0$ then C is not contained in C_2, and if it lies above this plane then C does not contain C_1. In both cases the relation (4) is violated. It remains to consider the case when D is contained in the plane $z = 0$.

If (4) is satisfied then we have $C_1 \subset C \cap \{z = 0\} = D$. Since the area of D coincides with the area of C_1, we conclude that $D = C_1$; that is, the lower part of ∂C is the circle $x^2 + y^2 \le 1$, and consequently, the lateral surface of C is contained in the cylindrical surface $x^2 + y^2 = 1$. The measure $(\nu_1 + \nu_2)/2$ corresponding to the lateral surface is supported on the horizontal circle $x^2 + y^2 = 1$, $z = 0$ and has the linear density $M/2$. It follows that the lateral part of ∂C is a part of a cylindrical surface bounded above by the circumference $x^2 + y^2 = 1$, $z = M/2$, and hence, the upper part of ∂C cannot be the union of two flat inclined surfaces.

We conjecture that $\Upsilon(C_1, C_2)$ is not convex *for all* $C_1 \subsetneqq C_2$. □

Here are some intriguing questions concerning $\Upsilon(C_1, C_2)$, with reasonable sets C_1 and C_2 (for example, a cylinder and one of its bases, or two concentric balls).

(Q1) Characterize the convex hull, $\mathrm{Conv}\Upsilon(C_1, C_2)$, of the set $\Upsilon(C_1, C_2)$ and the class of convex bodies with the surface measure contained in $\mathrm{Conv}\Upsilon(C_1, C_2)$.

(Q2) Characterize the set $\Upsilon(C_1, C_2) \cap \partial\,(\mathrm{Conv}\Upsilon(C_1, C_2))$.[1] Each measure contained in it minimizes a linear functional on $\Upsilon(C_1, C_2)$.

(Q3) Characterize the extreme points of $\mathrm{Conv}\Upsilon(C_1, C_2)$.

It would be interesting to develop a computer algorithm of solving Problem (3). In this connection we mention the papers [2, 16], where numerical algorithms for representation of Blaschke sum of convex polyhedra are proposed.

3. Main results

Here we state the main results of the paper concerning local properties of surface measures, and deduce from them some properties of a solution to a generalized Newton problem.

Let C be a convex body in $\mathbb{R}^3$. Take a point O outside C and consider the closed cone K with the vertex at O circumscribed about C; in other words, K is the union of all rays intersecting C with the origin at O. Clearly, K is convex. Denote by $\partial_- C$ the part of the boundary of C contained in the interior of the convex hull of $C \cup \{O\}$,

$$\partial_- C := \partial C \cap \mathrm{int}\,(\mathrm{Conv}(C \cup \{O\})),$$

and let $\partial_+ C$ be the complementary part of the boundary,

$$\partial_+ C := \partial C \setminus \partial_- C = \partial C \cap \partial(\mathrm{Conv}(C \cup \{O\})).$$

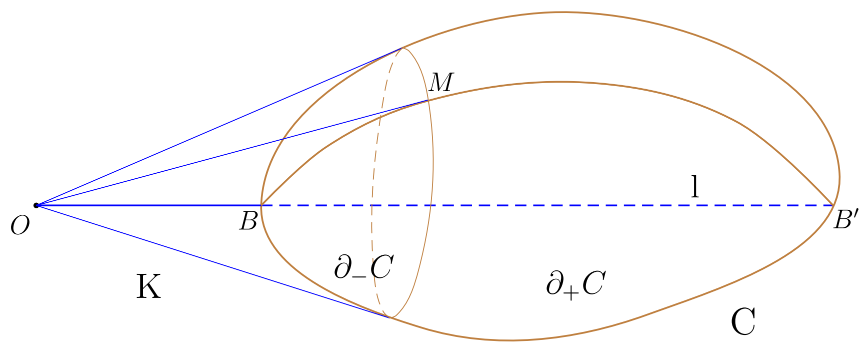


Figure 3.1: The convex body C and the circumscribed cone K.

Draw a line l through O intersecting the interior of C. Let B and B' be the points of intersection of l with ∂C, where the semiopen segment $[O,\, B)$ lies outside C, and the closed segment $[B,\, B']$ is contained in C; see Figure 3.1. The intersection of a half-plane bounded by l with ∂C is a planar convex curve with the endpoints B and B' (the curve BMB' in Figure 3.1), and the intersection of this half-plane with ∂K is the ray of support to this curve with the origin at O (the ray OM in Figure 3.1).

[1]Here $\partial\,(\mathrm{Conv}\Upsilon(C_1, C_2))$ means the *algebraic* boundary, that is, the set of measures ν such that for any finite signed measure η on S^2, there exists $\varepsilon > 0$ such that the line segment of measures $(\nu,\, \nu{+}\varepsilon\eta)$ is contained in $\partial\,(\mathrm{Conv}\Upsilon(C_1, C_2))$, while $(\nu,\, \nu{-}\varepsilon\eta)$ is disjoint with $\partial\,(\mathrm{Conv}\Upsilon(C_1, C_2))$.

Theorem 3.1. *Assume that*

(i) *in each half-plane, the corresponding ray of support is tangent to the curve*[2]*;*

(ii) *there exists $k > 0$ such that in any half-plane, the part of the curve between B and the point of tangency is of class C^1 with the curvature $\le k$.*

Then there exists a family of convex bodies $C(s)$, $s \in [-1, 1]$ such that

(a) $C(0) = C$ *and* $C(1) \ne C$*;*

(b) $\partial_+ C \subset \partial C(s)$*;*

(c) *the corresponding family of surface measures $\nu_{C(s)}$ is a line segment with the midpoint at ν_C; that is, for all $s \in [-1, 1]$*

$$\nu_{C(s)} = \nu_C + s\,(\nu_{C(1)} - \nu_C).$$

In particular, $\nu_C = \frac{1}{2}\big(\nu_{C(-1)} + \nu_{C(1)}\big)$, and the signed measure $\nu_{C(1)} - \nu_C$ is a director vector of the segment.

(d) *Moreover, the aforementioned family of bodies is not unique; there exists a set of families of bodies $C^{(\theta)}(s)$ depending on the parameter $\theta \in \Theta$, with each family satisfying (a), (b), (c), such that the union of 1-dimensional subspaces spanned by the corresponding director vectors, $\cup_{\theta\in\Theta}\{\lambda(\nu_{C^{(\theta)}(1)} - \nu_C) : \lambda \in \mathbb{R}\}$, is an infinite-dimensional subspace of the space of signed measures.*

Remark 3.2. By saying that the curvature of a C^1 curve is $\le k$ we mean that, given a natural parametrization $\gamma(\varsigma)$ of the curve, the angle between any two unit vectors $\gamma'(\varsigma_1)$ and $\gamma'(\varsigma_2)$ does not exceed $k|\varsigma_2 - \varsigma_1|$. $\square$

Remark 3.3. The 2D analogue of Theorem 3.1 is trivial. Namely, take a convex body $C \subset \mathbb{R}^2$, take an arc $\Omega_- \subset S^1$ and the complementary arc $\Omega_+ = S^1 \setminus \Omega_-$, and assume that the set $\operatorname{spt} \nu_C \cap \Omega_-$ contains at least 3 points. This means that the set of normal vectors to C at points of the set $\partial_- C := \mathcal{N}_C^{-1}(\Omega_-) \subset \partial C$ contains at least 3 vectors. Denote also $\partial_+ C = \mathcal{N}_C^{-1}(\Omega_+)$. Then $[\alpha]$ there exists a 1-parameter family of convex bodies $C(s)$ for which claims (a), (b), and (c) of Theorem 3.1 hold true. Moreover, $[\beta]$ if $\operatorname{spt} \nu_C \cap \Omega_-$ contains infinitely many points then there exists a huge set of such families $C^{(\theta)}(s)$ for which claim (d) hold true.

Let us explain the idea of the proof of the part $[\alpha]$ of the statement (the the part $[\beta]$ will not be proved). Suppose that $\operatorname{spt} \nu_C \cap \Omega_-$ contains at least 3 points. Take 3 mutually disjoint set D_1, D_2, $D_3 \subset \Omega_-$ such that the vectors $\int_{D_i} n\, d\nu_C(n)$, $i = 1, 2, 3$ are not collinear. Then there is a nontrivial relation

$$a_1 \int_{D_1} n\, d\nu_C(n) + a_2 \int_{D_2} n\, d\nu_C(n) + a_3 \int_{D_3} n\, d\nu_C(n) = 0, \quad (a_1, a_2 a_3) \ne (0,0,0).$$

Take $\varepsilon > 0$ and consider the line segment of measures

$$\nu(s) = \nu_C + \varepsilon s\big(a_1 \nu_C\rfloor_{D_1} + a_2 \nu_C\rfloor_{D_2} + a_3 \nu_C\rfloor_{D_3}\big), \quad s \in [-1, 1].$$

[2]Note that the intersection of the curve and the ray of support is either a singleton, or a line segment. In the latter case, all points of the segment are points of tangency.

It is not difficult to check that for ε sufficiently small, each measure $\nu(s)$ satisfies conditions (a) and (b) in the beginning of Section 2, and therefore, is the surface measure of a convex body $C(s)$. The measures $\nu(s)\rfloor_{\Omega_+}$ and $\nu_C\rfloor_{\Omega_+}$ coincide; this implies that the curve $\partial_+ C(s) := \mathcal{N}_{C(s)}^{-1}(\Omega_+) \subset \partial C(s)$ coincides, up to a translation, with $\partial_+ C = \mathcal{N}_C^{-1}(\Omega_+)$.

An example of a family of planar convex bodies $C(s)$ corresponding to a line segment of measures on S^1 is shown in Fig. 3.2. The convex body C is bounded by the closed curve ABCD, where AB and CD are line segments intersecting at a point O. The body $C(s)$ is bounded by the closed line $\mathrm{AB}_t\mathrm{C}_t\mathrm{D}$, where the curve $\mathrm{B}_t\mathrm{C}_t$ is obtained from the curve BC by the dilation with the center at O and with the ratio $1 + \varepsilon s$, with $\varepsilon > 0$ sufficiently small. □

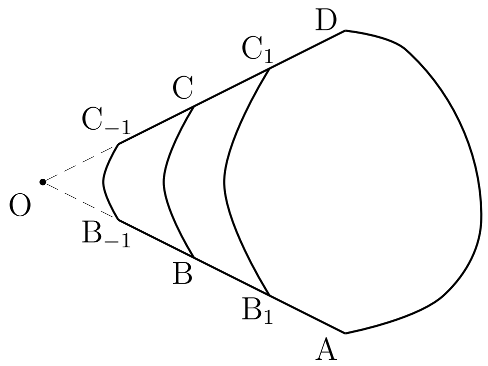


Figure 3.2: A family of planar convex bodies $C(s)$. The bodies $C(1) = \mathrm{AB}_1\mathrm{C}_1\mathrm{D}$, $C(-1) = \mathrm{AB}_{-1}\mathrm{C}_{-1}\mathrm{D}$, and $C = C(0) = \mathrm{ABCD}$ are shown.

The proof of Theorem 3.1 is given in Section 4.

Note that a local perturbation of the surface ∂C of a convex body C leads to a local perturbation of its surface measure ν_C. The converse, however, is not true: it is well known that a local perturbation of the surface measure may lead to a nonlocal perturbation of the body surface. In other words, if we have a domain $\Omega \subset S^2$ and two surface measures ν_{C_1} and ν_{C_2} such that $\nu_{C_1}\rfloor_\Omega = \nu_{C_2}\rfloor_\Omega$, it does not follow that $\mathcal{N}_{C_1}^{-1}(\Omega)$ and $\mathcal{N}_{C_2}^{-1}(\Omega)$ coincide up to a translation.

Indeed, let $e_1 = (1,0,0)$, $e_2 = (0,1,0)$, $e_3 = (0,0,1)$, take $\alpha \neq 0$ and denote $v^+ = (e_3 + \alpha e_1)/\sqrt{1+\alpha^2}$, $v^- = (e_3 - \alpha e_1)/\sqrt{1+\alpha^2}$. One obviously has $e_3 = \sqrt{1+\alpha^2}\,(v^+ + v^-)/2$. The measures

$$\nu_{C_1} = \delta_{e_1} + \delta_{-e_1} + \delta_{e_2} + \delta_{-e_2} + \delta_{e_3} + \delta_{-e_3}$$

and
$$\nu_{C_2} = \delta_{e_1} + \delta_{-e_1} + \delta_{e_2} + \delta_{-e_2} + \frac{\sqrt{1+\alpha^2}}{2}\,(\delta_{v^+} + \delta_{v^-}) + \delta_{-e_3}$$

are surface measures of a cube and a body called "house"; see Fig. 3.3. Let $\Omega \subset S^2$ be a domain that contains e_1, $-e_1$, e_2, $-e_2$, and $-e_3$, and does not contain e_3, v^+, and v^-. One has $\nu_{C_1}\rfloor_\Omega = \nu_{C_2}\rfloor_\Omega$; however, the sets $\mathcal{N}_{C_1}^{-1}(\Omega)$ and $\mathcal{N}_{C_2}^{-1}(\Omega)$ are not translations of each other. Indeed, $\mathcal{N}_{C_1}^{-1}(\Omega)$ is the union of the lateral and the lower faces of the cube, whereas $\mathcal{N}_{C_2}^{-1}(\Omega)$ is the union of the "walls" and the "floor" of the "house".

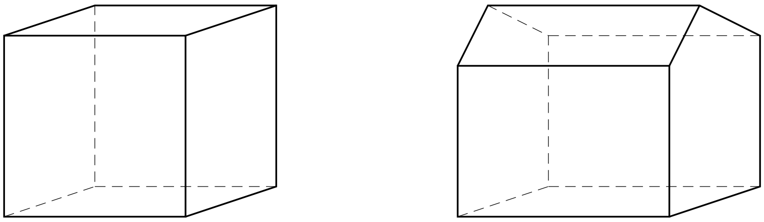


Figure 3.3: A cube and a ”house”.

It is important therefore to find out conditions which would guarantee that a local perturbation of the measure implies a local perturbation of the body surface.

Theorem 3.1 is a step in this direction; it states that, under mild conditions, there is a family of perturbations of the measure forming a line segment, with the original measure at the midpoint, leading to a local perturbation of the body surface. Moreover, there is, in a sense, a huge amount of such perturbations.

Theorem 3.1 can also be interpreted as follows. Consider all possible convex bodies obtained from C by perturbations of its boundary in the subset $\partial_- C$, and consider the set of their surface measures. Then ν_C is not an extreme point of this set.

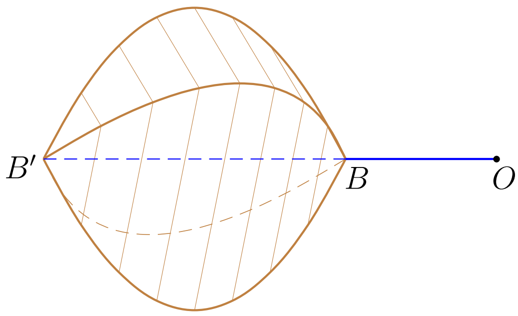


Figure 3.4: A convex body satisfying the hypothesis of Theorem 3.1.

Corollary 3.4. *Assume that an open subset* $\mathcal{O} \subset \partial C$ *of the body's boundary is of class* C^2 *and its gaussian curvature is positive. Take a point* $O \notin C$ *so as the closure of the set* $\partial_- C = \partial C \cap \operatorname{int}(\operatorname{Conv}(C \cup \{O\}))$ *is contained in* $\mathcal{O}$*, and therefore,* $\partial_+ C = \partial C \setminus \partial_- C$ *contains* $\partial C \setminus \mathcal{O}$*. Then the statement of Theorem* 3.1 *holds true.*

Proof. Draw a line l through O intersecting the interior of C, denote by B and B' the points of intersection of l with ∂C, as explained in the beginning of this section, and consider all half-planes bounded by l. Note that all rays of support to C from O intersect ∂C inside $\mathcal{O}$, and therefore, are tangent to C. Thus, condition (i) of the theorem is satisfied.

Consider the concave curves resulting from the intersection of the half-planes with ∂C. The parts of these curves between B and the point of tangency of the corresponding ray of support lie in $\overline{\partial_- C}$, and therefore are of class C^2. Their curvature is a continuous function defined on the compact set with the coordinates (*the distance of the point from* l*, the angle of inclination of the corresponding half-*

plane), and therefore does not exceed a positive constant k. Thus, condition (ii) of Theorem 3.1 is also valid, and therefore, the statement of the theorem holds true. $\square$

Remark 3.5. The hypothesis of Theorem 3.1 is strictly weaker than the hypothesis of Corollary 3.4. Indeed, let C be the convex hull of two planar domains, $C = \mathrm{Conv}(\mathrm{D}_1 \cup \mathrm{D}_2)$, where $D_1 = \{(x, y, z) : |x| \le 1,\ |z| \le 1 - x^2,\ y = 0\}$ and $D_2 = \{(x, y, z) : |x| \le 1,\ |y| \le 1 - x^2,\ z = 0\}$; see Fig. 3.4. Then the hypothesis of Theorem 3.1 holds true for $O = (c, 0, 0)$ with $c > 1$, $B = (1, 0, 0)$, $B' = (-1, 0, 0)$, and $k = 2$; however, the gaussian curvature is zero at any regular point of ∂C. $\square$

From Theorem 3.1 one derives an important statement concerning linear Problem (3), and therefore also the corresponding generalized Newton problem (1).

Theorem 3.6. *Let $C_1 \subset C \subset C_2$, and let the point $O \notin C$ be such that the set $\partial_- C = \partial C \cap \mathrm{int}\,(\mathrm{Conv}(C \cup \{O\}))$ does not intersect ∂C_1 and ∂C_2, that is, $\partial_- C \cap (\partial C_1 \cup \partial C_2) = \emptyset$. Draw a line l through O and the interior of C, and assume that the hypothesis of Theorem 3.1 is satisfied. Then either ν_C is not a solution to Problem (3), or ν_C is included in a huge set of solutions: there exists an infinite-dimensional linear space $\mathcal{M}$ of signed measures such that for any $\nu \in \mathcal{M}$ and some $\varepsilon(\nu) > 0$, all measures of the kind $\nu_C + s\nu$, $s \in [-\varepsilon(\nu), \varepsilon(\nu)]$, are solutions.*

Proof. Indeed, making if necessary the point O sufficiently close to C on the line l, one can assure that, on one hand, the set $\mathrm{Conv}\,(C \cup \{O\})$ is contained in C_2 and, on the other hand, $\mathrm{Conv}\,(\partial_+ C)$ contains C_1. All convex sets, corresponding to the elements of the segment of measures indicated in the statement of Theorem 3.1, contain $\partial_+ C$, and therefore, also contain C_1. Further, all such convex sets lie in the intersection of all half-spaces bounded by planes of support through points of $\partial_+ C$, and this intersection of half-spaces coincides with $\mathrm{Conv}\,(C \cup \{O\})$. This proves that these convex sets are contained in C_2.

If ν_C is a solution to Problem (3) then for any $\theta \in \Theta$, all measures $\nu_{C^{(\theta)}(s)} = \nu_C + s\,(\nu_{C^{(\theta)}(1)} - \nu_C)$, $s \in [-1, 1]$ forming the corresponding linear segment are also solutions. Take $\mathcal{M} = \cup_{\theta \in \Theta} \{\lambda(\nu_{C^{(\theta)}(1)} - \nu_C) : \lambda \in \mathbb{R}\}$; by statement (d) of Theorem 3.1, $\mathcal{M}$ is an infinite-dimensional space. Further, for any $\nu \in \mathcal{M}$ there exist $\lambda \in \mathbb{R}$ and $\theta \in \Theta$ such that $\nu = \lambda(\nu_{C^{(\theta)}(1)} - \nu_C)$, and taking $\varepsilon(\nu) = 1/|\lambda|$, one obtains the statement of Theorem 3.6. $\square$

Going back to the particular Newton problem (2), we first note that the set $\partial C \setminus (\partial C_1 \cup \partial C_2)$ coincides with the part of the graph of u with $(x, y) \in \mathrm{int}(\Omega)$ and $0 < u(x, y) < M$, where $\mathrm{int}(\Omega)$ means the interior of Ω. Recall that to each u one naturally assigns the surface measure of the corresponding convex body $\{(x, y, z) : (x, y) \in \Omega,\ 0 \le z \le u(x, y)\}$.

Take a point $(x_0, y_0) \in \mathrm{int}(\Omega)$ and a value $z_0 > u(x_0, y_0)$. For any φ consider the ray of support $z = z_0 + \kappa_\varphi t,\ t \ge 0,\ x = x_0 + t\cos\varphi,\ y = y_0 + t\sin\varphi,\ t \ge 0$ in the half-plane $x = x_0 + t\cos\varphi,\ y = y_0 + t\sin\varphi,\ t \ge 0$ to the graph of the induced function of one variable $z = u(x_0 + t\cos\varphi,\ y_0 + t\sin\varphi)$, $t \ge 0$, and fix the smallest value $t = t_\varphi > 0$ corresponding to the intersection of the ray of support with the graph.

Applying Theorem 3.6 to Problem (2), we come to the following statement.

Corollary 3.7. *Let* $(x_0, y_0) \in \mathrm{int}(\Omega)$ *and* $0 < u(x_0, y_0) < z_0 < M$*. Suppose that (i) for all* φ*, the corresponding ray of support touches the graph of the corresponding induced function, and (ii) there exists* $k > 0$ *such that* $\frac{d^2}{dt^2} u(x_0 + t\cos\varphi,\, y_0 + t\sin\varphi) \le k$ *for all* φ *and* $0 \le t \le t_\varphi$*. Then either* u *is not a solution to* (2)*, or* u *is included in a huge set of solutions: let* ν *be the measure corresponding to* u*; there exists an infinite-dimensional linear space* $\mathcal{M}$ *of signed measures in* S^2 *such that for any* $\nu' \in \mathcal{M}$ *and some* $\varepsilon(\nu') > 0$*, all measures of the kind* $\nu + s\nu'$*,* $s \in [-\varepsilon(\nu'), \varepsilon(\nu')]$*, correspond to solutions of* (2)*.*

Now using Corollaries 3.4 and 3.7, we arrive at

Corollary 3.8. *Assume that in an open subset of* Ω *the function* u *is of class* C^2 *and the matrix of second derivatives* D^2u *is non-degenerate (and therefore positive definite). Then the statement of Corollary* 3.7 *holds true.*

Remark 3.9. The two Corollaries 3.7 and 3.8 remain true if we replace the integrand $1/(1+|\nabla u(x,y)|^2)$ in (3.7) by $f(\nabla u(x,y))$, where $f\colon \mathbb{R}^2 \to \mathbb{R}$ is an arbitrary bounded continuous function. □

Compare Corollaries 3.7 and 3.8 with the result obtained in [3] (Remark 3.4): if in an open subset of Ω the function u is of class C^2 and the matrix of second derivatives D^2u is non-degenerate, then u is not a solution to the particular Newton problem. Apart from an obvious similarity of these statements, there is some difference. Namely, according to Remark 3.9, our statements remain valid in a more general case when the integrand in (2) is substituted with $f(\nabla u(x,y))$. On the other hand, we do not claim that u is not a solution, but instead, if u is a solution then the set of solutions is extremely large and u is not its extreme point.

4. Proof of the main theorem

We will prove a slightly stronger version of Theorem 3.1, with condition (ii) replaced by a weaker condition (ii)′; see below. Condition (i) of the theorem remains unchanged.

The idea of the proof is as follows. We take a 1-parameter family of cones circumscribed about the body C. The vertices of these cones form a line segment outside C. Then we translate each of these cones along the segment, with the translation continuously depending on the parameter. Under certain conditions, the family of translated cones is tangent to another convex body. Knowing the magnitude of translation and the surface measure of the original body C, one is able to determine the surface measure of the new body. Using the obtained description of the correspondence of measures, one constructs a family of translations of the family of cones that defines a family of convex bodies with their surface measures forming a line segment, with ν_C at the midpoint. This construction depends on a functional parameter.

We introduce some additional notation. Determine an orthogonal coordinate system with the coordinates x, y, z so as the point O is at the origin and the positive z-semiaxis coincides with the ray OB. Consider a C^1 function $\alpha(t)$, $t \in [0, 1]$ such that $\alpha(0) = 0$, $\alpha'(t) > 0$ for all $0 \le t \le 1$, and $(0, 0, \alpha(1))$ coincides with the point B. Denote $B_t = (0, 0, \alpha(t))$; thus, $B_0 = O$ and $B_1 = B$.

Let K_t be the closed cone with the vertex at B_t circumscribed about C. Clearly, all cones K_t are convex, and $K_0 = K$. If B is a regular point of ∂C, then K_1 is a half-space bounded by the tangent plane to C through B.

Using the circumscribed cones, the body C can be represented as

$$C = \big(\cap_{0 \le t \le 1} K_t \big) \cap \mathrm{Conv}\big(C \cup \{O\}\big).$$

For any φ designate by Π_φ the half-plane $x = r\cos\varphi$, $y = r\sin\varphi$, $r \ge 0$; it is bounded by the straight line OB and has the angle of inclination φ with respect to the half-plane of reference $y = 0$, $x \ge 0$; see Fig. 4.1.

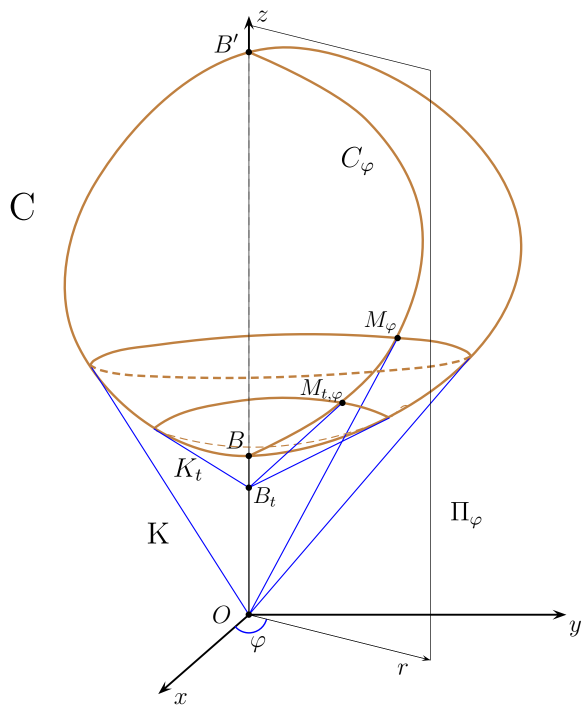


Figure 4.1: The convex body C and some additional notation.

The intersection of C with Π_φ is the 2-dimensional convex body C_φ bounded by a segment of the positive z-semiaxis and by the concave curve $\partial C \cap \Pi_\varphi$. Draw the ray of support from O to this curve and denote by M_φ the closest to O point of intersection of the ray with the curve. Denote by $\partial_- C_\varphi$ the part of the curve between B and M_φ (including B and excluding M_φ). $\partial_- C_\varphi$ is the intersection of Π_φ with $\partial_- C$, and it is the graph of a convex function $r \mapsto z(r, \varphi)$. It can be expressed as

$$\partial_- C_\varphi = \partial C_\varphi \cap \mathrm{int}\big(\mathrm{Conv}(C_\varphi \cup \{O\})\big) = \partial_- C \cap \Pi_\varphi.$$

Denote by $\partial_+ C_\varphi$ the resting part of the curve between M_φ and B'. It can be expressed as

$$\partial_+ C_\varphi = \partial C_\varphi \cap \partial\big(\mathrm{Conv}(C_\varphi \cup \{O\})\big) \setminus [B,\, B') = \partial_+ C \cap \Pi_\varphi.$$

Draw the ray of support in Π_φ from a point B_t to the curve $\partial C \cap \Pi_\varphi$. The intersection of the ray with the curve is a line segment (which may degenerate to a point). Denote by $M_{t,\varphi}$ the endpoint of this segment closest to B_t. In particular, we have $M_{0,\varphi} = M_\varphi$ and $M_{1,\varphi} = B$.

Recall that condition (i) of Theorem 3.1 states that for any φ the ray of support from O to the curve in the half-plane Π_φ touches this curve.

We also consider the following

Condition (ii)′. *For any $0 < \tau < 1$ there exists $k = k(\tau)$ such that for any φ, the part of the curve $\partial_- C_\varphi$ between $M_{\tau,\varphi}$ and M_φ is of class C^1 and has curvature $\leq k$.*

We will prove the following theorem, which is closely related to Theorem 3.1.

Theorem 4.1. *Let conditions* (i) *and* (ii)′ *be satisfied; then statements* (a)–(d) *of Theorem* 3.1 *are true.*

Let us describe the scheme of the proof in some detail. We introduce some auxiliary functions. Namely, for any pair (t, φ) we consider the ray $\partial K_t \cap \Pi_\varphi$ and denote by $\sigma(t, \varphi)$ its slope, by $r(t, \varphi)$ the Hausdorff distance between the z-axis and the intersection of this ray with C (which can be a point or a line segment), and by $H_C(t, \varphi)$ the outward unit normal vector to K_t at a point of this ray (which exists for almost all (t, φ)).

The unit sphere is decomposed into the disjoint union $S^2 = \Omega_+ \cup \Omega_-$, where each normal vector to C at $\partial_+ C$ ($\partial_- C$) lies in Ω_+ (Ω_-). In Lemma 4.7 we derive formula (16) that represents the measure $\nu_C(D)$ of a Borel set $D \subset \Omega_-$ in terms of the functions H_C, σ, and r; it turns out that this value depends linearly on r^2.

Further, we introduce a positive C^1 function $\eta(t)$, $t \in [0,\, 1]$ (see page 1393) and translate each cone K_t in the vertical direction, so as the z-coordinate of the vertex of the resulting cone $\tilde{K}_t$ is $\tilde{\alpha}(t) = \int_0^t \eta(\xi)\, d\alpha(\xi)$. The family of shifted cones $\tilde{K}_t$ defines a new convex body $\tilde{C}$; namely, $\tilde{C}$ is the intersection of $\mathrm{Conv}(C \cup \{O\})$ with $\cap_t \tilde{K}_t$.

If $\eta - 1$ and η' are sufficiently small, one can guarantee that all cones $\tilde{K}_t$ are circumscribed about $\tilde{C}$, and consequently, the auxiliary functions $\tilde{H}_C$ and $\tilde{\sigma}$ for $\tilde{C}$ coincide, respectively, with H_C and σ. We prove that the function $\tilde{r}(t, \varphi) = \eta(t) r(t, \varphi)$ plays the role of $r(t, \varphi)$ for the body $\tilde{C}$. Further, it is proved that formula (16), where r is replaced with $\tilde{r}$, remains valid for $\tilde{C}$.

Finally, we consider the 1-parameter family of convex bodies $\tilde{C} = C(s)$ corresponding to a family of functions η of the form $\eta(t) = \eta(t, s) = \sqrt{1 + s\theta(t)}$ depending on the parameter $s \in [-1,\, 1]$, where the functions θ and θ' are sufficiently small. The restrictions of the corresponding measures $\nu_{C(s)}$ on Ω_+ coincide, and for a fixed Borel set $D \subset \Omega_-$ the value $\nu_{C(s)}(D)$ depends linearly on the function $\tilde{r}^2(t,\varphi) = (1 + s\theta(t))r^2(t,\varphi)$. Having the functional parameters θ, r, σ, H_C fixed, we conclude that the value $\nu_{C(s)}(D)$ is linear in s. It follows that the measures $\nu_{C(s)}$ form a linear segment with the midpoint at ν_C. Varying the function θ, one obtains a huge variety of such segments.

Proof of Theorem 4.1. According to condition (ii)$'$, the function $r \mapsto z(r,\varphi)$ is of class C^1.

Introduce some additional functions associated with the curve $\partial_- C_\varphi$. At each point $(r, z(r,\varphi))$ draw the tangent line to $\partial_- C_\varphi$ in the plane containing the half-plane Π_φ. This line intersects the z-axis at a point B_t; thus, the monotone non-increasing function $r \mapsto t(r,\varphi)$ is defined.

This function can also be found from the implicit equation

$$\alpha(t) = z(r,\varphi) - r\,\frac{\partial z}{\partial r}(r,\varphi).$$

Since the right hand side of this equation is continuous in r and the function α is continuous and strictly increasing, we conclude that the function $r \mapsto t(r,\varphi)$ is continuous.

It is also convenient to use the generalized inverse to the function $t(\cdot,\varphi)$. Namely, for $0 \le t \le 1$ designate by $r(t,\varphi)$ the smallest value of r such that $t(r,\varphi) = t$. The function $t \mapsto r(t,\varphi)$ defined this way is strictly decreasing and right semicontinuous. Additionally, we have $r(1,\varphi) = 0$ and $M_{t,\varphi} = (r(t,\varphi),\ z(r(t,\varphi),\varphi))$.

Further, we denote by $\sigma = \sigma(t,\varphi)$ the slope of the corresponding tangent line. More precisely, $\sigma(t,\varphi)$ is the tangent of the angle of inclination of the ray of support from B_t to the curve $\partial_- C_\varphi$ in the half-plane Π_φ. Obviously, one has $\sigma(t,\varphi) = \frac{\partial z}{\partial r}(r(t,\varphi),\varphi)$. For all φ, the function of one variable $t \mapsto \sigma(t,\varphi)$ is monotone non-increasing.

Lemma 4.2. (a) *The function $\sigma(t,\varphi)$ is continuous.*

(b) *The partial derivative $\frac{\partial \sigma}{\partial \varphi}(t,\varphi)$ exists almost everywhere.*

Proof. For each t the cone K_t is the graph of a convex homogeneous function, which in polar coordinates has the form $(r,\varphi) \mapsto \alpha(t) + r\sigma(t,\varphi)$. This function is continuous and almost everywhere differentiable; it follows that $\sigma(t,\varphi)$ is continuous in φ, and $\frac{\partial \sigma}{\partial \varphi}(t,\varphi)$ exists for almost all values (t,φ). Thus, (b) is proved.

Draw the tangent lines in Π_φ to the curve $\partial_- C_\varphi$ at its endpoints B and M_φ, and fix the corresponding angles of inclination. When the point of tangency passes through all points of the curve from B to M_φ, the angle of inclination of the corresponding tangent line takes all intermediate values. Correspondingly, the monotone function $t \mapsto \sigma(t,\varphi)$ takes all values intermediate between $\sigma(0,\varphi)$ and $\sigma(1,\varphi)$. It follows that the function $\sigma(t,\varphi)$ is continuous in t.

Now having that σ is continuous in each of the variables φ and t and monotone decreasing in t, it is not difficult to conclude that σ is continuous. Thus, (a) is also proved. $\square$

Lemma 4.3. *For any $0 < \tau < 1$ there exists $\gamma = \gamma(\tau) > 0$ such that for all φ and $0 \le t_1 < t_2 \le \tau$,*

$$r(t_2, \varphi) - r(t_1, \varphi) \le -\gamma\,(t_2 - t_1). \tag{5}$$

Proof. The continuous function $\sigma(t, \varphi)$ is defined on the compact set $[0,\, 1] \times [0,\, 2\pi]$, and therefore, $|\sigma(t, \varphi)| \le c$ for a certain constant c. It follows that for any φ the cosine of the angle of inclination, with respect to the r-axis, of the curve $\partial_- C_\varphi$ in Π_φ is greater than or equal to $1/\sqrt{1 + c^2}$.

Take $0 \le t_1 < t_2 \le \tau$ and introduce the shorthand notation $r_1 := r(t_1, \varphi)$, $r_2 := r(t_2, \varphi)$, $\psi_1 := \arctan(\sigma(t_1, \varphi))$, $\psi_2 := \arctan(\sigma(t_2, \varphi))$, $z(r) := z(r, \varphi)$. (In Fig. 4.2, $\psi_1 > 0$ and $\psi_2 < 0$.) We have $t_1 < t_2$, therefore $r_1 > r_2$. The length of the part of curve between the points $M_{t_1,\varphi} = (r_1, z(r_1))$ and $M_{t_2,\varphi} = (r_2, z(r_2))$ does not exceed $\sqrt{1 + c^2}(r_1 - r_2)$, hence the angle between the tangent lines at these points does not exceed $k\sqrt{1 + c^2}(r_1 - r_2)$, that is,

$$0 \le \psi_1 - \psi_2 \le k\sqrt{1 + c^2}(r_1 - r_2).$$

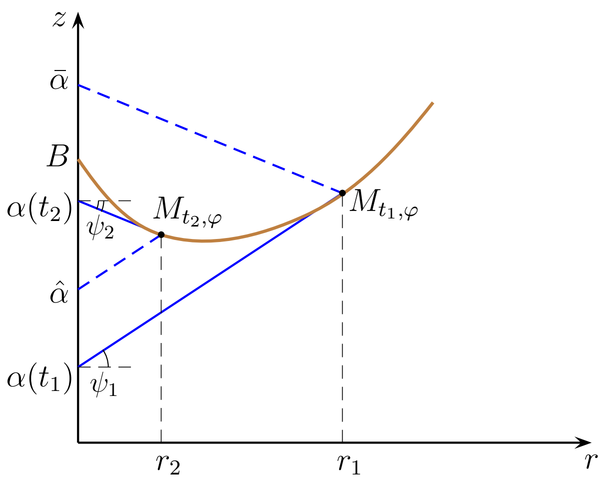


Figure 4.2: Auxiliary construction to Lemmas 4.3 and 4.13.

Draw the line through the point $(r_1, z(r_1))$ parallel to the tangent line at $(r_2, z(r_2))$, and let $\bar{\alpha}$ be the z-coordinate of intersection of this line with the z-axis; see Fig. 4.2. Since the curve is convex, we have $\bar{\alpha} \ge \alpha(t_2)$. Thus, we obtain

$$\alpha(t_2) - \alpha(t_1) \le \bar{\alpha} - \alpha(t_1) = r_1\left(\sigma(t_1, \varphi) - \sigma(t_2, \varphi)\right) = r_1(\tan\psi_1 - \tan\psi_2).$$

Further, taking into account that the points $M_{t_1,\varphi} = (r_1, z(r_1))$ and $B = (0, \alpha(1))$ are contained in C, we have

$$r_1 \le \operatorname{dist}\big((r_1, z(r_1)),\, (0, \alpha(1))\big) \le \operatorname{diam}(C),$$

where $\operatorname{diam}(C)$ denotes the diameter of C, and we get for some intermediate value $\psi_0 \in (\psi_2,\, \psi_1)$,

$$\tan\psi_1 - \tan\psi_2 = \tan'\psi_0\,(\psi_1 - \psi_2) = (1 + \tan^2\psi_0)\,(\psi_1 - \psi_2) \le (1 + c^2)\,(\psi_1 - \psi_2).$$

Finally, we have

$$t_2 - t_1 \le \frac{\alpha(t_2) - \alpha(t_1)}{\min_{t\in[0,\tau]} \alpha'(t)} \le \frac{r_1(\tan\psi_1 - \tan\psi_2)}{\min_{t\in[0,\tau]} \alpha'(t)}$$

$$\le \frac{\mathrm{diam}(C)\,(1+c^2)\,(\psi_1-\psi_2)}{\min_{t\in[0,\tau]} \alpha'(t)} \le \frac{\mathrm{diam}(C)\,(1+c^2)\,k\sqrt{1+c^2}}{\min_{t\in[0,\tau]} \alpha'(t)}\,\big(r_1 - r_2\big).$$

It follows that the statement of the lemma holds true for

$$\gamma = \frac{\min_{t\in[0,\tau]} \alpha'(t)}{\mathrm{diam}(C)\,k\,(1+c^2)^{3/2}}. \qquad \square$$

Using Lemma 4.3, one easily obtains the following statement.

Corollary 4.4. *Consider the map $g\colon (t,\varphi) \mapsto (r(t,\varphi),\varphi)$ from $[0,\,1]\times[0,\,2\pi)$ to $\mathbb{R}^2$. If a planar set A has Lebesgue measure 0, then $g^{-1}(A)$ also has Lebesgue measure* 0.

Proof. Let $0 < \tau < 1$, and take a Borel set $A \subset g([0,\,\tau]\times[0,\,2\pi))$. Using that $|r(t_2,\varphi) - r(t_1,\varphi)| \ge \gamma(\tau)|t_2 - t_1|$ for all φ and $t_1,\,t_2 \in [0,\,\tau]$, one concludes that $|g^{-1}(A)| \le \frac{1}{\gamma(\tau)}\,|A|$. In particular, if $|A| = 0$ then $|g^{-1}(A)| = 0$.

Let now $A \subset \mathbb{R}^2$ have measure 0. Then for all $0 < \tau < 1$ the set $g^{-1}\big(A \cap g([0,\,\tau]\times[0,\,2\pi))\big)$ also has measure 0. Taking $0 < \tau_k < 1,\ k \in \mathbb{N}$ and $\tau_k \to 1$ as $k \to \infty$, one has

$$g^{-1}(A) = \cup_{k\in\mathbb{N}} g^{-1}\big(A \cap g([0,\,\tau_k]\times[0,\,2\pi))\big) \cup \big(\{1\}\times[0,\,2\pi)\big);$$

thus, $g^{-1}(A)$ is the union of countably many sets of Lebesgue measure 0, and therefore, $|g^{-1}(A)| = 0$. $\square$

Denote by $K_t(\varphi)$ the 2-dimensional cone in Π_φ bounded by the z-axis and by the ray of support through B_t; see Fig. 4.3. Clearly, $K_t(\varphi) = K_t \cap \Pi_\varphi$, and $K_t(\varphi)$ is circumscribed about C_φ. In the (r,z)-coordinate system one has

$$K_t(\varphi) = \{(r,z) : r \ge 0,\ z \ge \sigma(t,\varphi)r + \alpha(t)\}.$$

The body C_φ can be represented in terms of the 2-dimensional circumscribed cones as

$$C_\varphi = \big(\cap_{0\le t\le 1} K_t(\varphi)\big) \cap \mathrm{Conv}\big(C_\varphi \cup \{O\}\big).$$

Using the definitions of the functions α, z, t, σ, one readily obtains the formulas

$$z(r,\varphi) = \sigma(t(r,\varphi),\varphi)\,r + \alpha(t(r,\varphi)) \quad \text{and} \quad \frac{\partial z}{\partial r}(r,\varphi) = \sigma(t(r,\varphi),\varphi). \qquad (6)$$

In terms of the inverse function $t \mapsto r(t,\varphi)$ we have

$$z(r(t,\varphi),\varphi) = \sigma(t,\varphi)\,r(t,\varphi) + \alpha(t), \qquad \frac{\partial z}{\partial r}(r(t,\varphi),\varphi) = \sigma(t,\varphi). \qquad (7)$$

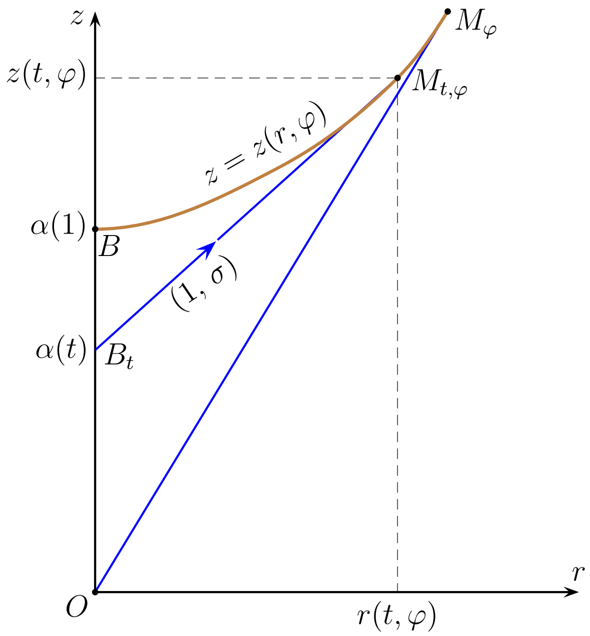


Figure 4.3: The cone $K_t(\varphi)$ is bounded by the z-axis and by the ray $B_tM_{t,\varphi}$.

If all functions in (7) are continuously differentiable, one differentiates by φ both parts in the former formula in (7) to obtain $\frac{\partial z}{\partial r}\frac{\partial r}{\partial \varphi} + \frac{\partial z}{\partial \varphi} = \sigma \frac{\partial r}{\partial \varphi} + r\frac{\partial \sigma}{\partial \varphi}$, and using the latter formula in (7) one gets

$$\frac{1}{r(t,\varphi)}\frac{\partial z}{\partial \varphi}(r(t,\varphi),\varphi) = \frac{\partial \sigma}{\partial \varphi}(t,\varphi). \tag{8}$$

Since differentiability is not assumed a priori, we need some additional effort to justify this formula. We shall prove that (8) holds for almost all $(t,\varphi) \in [0,\,1] \times [0,\,2\pi)$.

For the sake of brevity, in the next paragraph we introduce the shorthand notation $r = r(t,\varphi)$, $z = z(r,\varphi)$, $\Delta z = z(r,\varphi+\Delta\varphi) - z(r,\varphi)$, and $\Delta\sigma = \sigma(r,\varphi+\Delta\varphi) - \sigma(r,\varphi)$ for $\Delta\varphi \in \mathbb{R}$. We have

$$\sigma(t,\varphi) = \frac{z-\alpha(t)}{r}. \tag{9}$$

On the other hand, the tangent line from the point $(0,\alpha(t))$ to the graph of the function $r \mapsto z(r,\varphi+\Delta\varphi)$ (which is a concave curve) lies below the ray from $(0,\alpha(t))$ through $(r,\,z+\Delta z)$; see Fig. 4.4.

It follows that
$$\sigma(t,\varphi+\Delta\varphi) \le \frac{z+\Delta z-\alpha(t)}{r}. \tag{10}$$

From (9) and (10) one derives that

$$\frac{\Delta\sigma}{\Delta\varphi} \le \frac{1}{r}\frac{\Delta z}{\Delta\varphi} \quad \text{for} \quad \Delta\varphi > 0 \qquad \text{and} \qquad \frac{\Delta\sigma}{\Delta\varphi} \ge \frac{1}{r}\frac{\Delta z}{\Delta\varphi} \quad \text{for} \quad \Delta\varphi < 0.$$

Going to the limit $\Delta\varphi \to 0$, one obtains formula (8), provided that the partial derivatives in both sides of (8) exist.

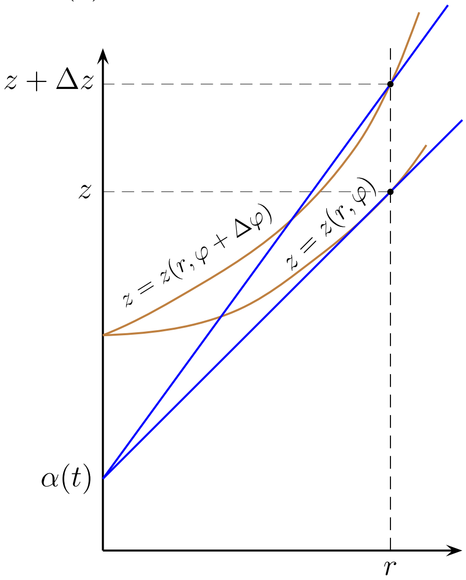


Figure 4.4: Auxiliary construction to the proof of formula (8).

The statement (b) of Lemma 4.2 guarantees that the right hand side of (8) exists for almost all $(t, \varphi) \in [0,\, 1] \times [0,\, 2\pi)$.

Note that ∂_C is the graph of a convex function z of variables x, y. In polar coordinates $x = r\cos\varphi$, $y = r\sin\varphi$ this function takes the form $(r,\, \varphi) \mapsto z(r, \varphi)$. For almost all (r, φ) the corresponding point of ∂_C is regular, and therefore, the partial derivative $\frac{\partial z}{\partial\varphi}(r, \varphi)$ exists. In other words, the set $A = \{(r, \varphi) : \frac{\partial z}{\partial\varphi}(r, \varphi) \text{ does not exist}\}$ has Lebesgue measure 0, and using Corollary 4.4 one concludes that the set $g^{-1}(A) = \{(t, \varphi) : \frac{\partial z}{\partial\varphi}(r(t, \varphi), \varphi) \text{ does not exist}\}$ also has Lebesgue measure 0. Thus, the left hand side of (8) also exists for almost all (t, φ).

Lemma 4.5. *For all φ and all $t_1,\ t_2$,*

$$r(t_2, \varphi)\big[\sigma(t_1, \varphi) - \sigma(t_2, \varphi)\big] \le \alpha(t_2) - \alpha(t_1) \le r(t_1, \varphi)\big[\sigma(t_1, \varphi) - \sigma(t_2, \varphi)\big]. \quad (11)$$

Proof. The curve ∂_C_φ lies above all its lines of support. Denote $r_1 := r(t_1, \varphi)$ and $r_2 := r(t_2, \varphi)$. The lines of support through the points $M_{t_1,\varphi} = (r_1, z(r_1, \varphi))$ and $M_{t_2,\varphi} = (r_2, z(r_2, \varphi))$ are, respectively, given by the equations $z = \sigma(t_1, \varphi)\, r + \alpha(t_1)$ and $z = \sigma(t_2, \varphi)\, r + \alpha(t_2)$. The point $M_{t_1,\varphi}$ lies above the support line through $M_{t_2,\varphi}$, hence

$$z(r_1, \varphi) \ge \sigma(t_2, \varphi)\, r_1 + \alpha(t_2),$$

and using the first formula in (7) we have

$$\sigma(t_1, \varphi)\, r(t_1, \varphi) + \alpha(t_1) \ge \sigma(t_2, \varphi)\, r(t_1, \varphi) + \alpha(t_2). \quad (12)$$

Similarly, the point $M_{t_2,\varphi}$ lies above the support line through $M_{t_1,\varphi}$, hence

$$z(r_2, \varphi) \ge \sigma(t_1, \varphi)\, r_2 + \alpha(t_1),$$

and therefore,

$$\sigma(t_2,\varphi)\,r(t_2,\varphi)+\alpha(t_2)\geq \sigma(t_1,\varphi)\,r(t_2,\varphi)+\alpha(t_1). \tag{13}$$

From (12) and (13) one obtains the statement of Lemma 4.5. $\square$

A plane of support to C through a point $\xi\in\partial C$ divides the space into the closed half-space containing C and the open half-space disjoint with C. If $\xi\in\partial_- C$ then O is contained in the half-space disjoint with C, and therefore, for a regular point ξ, $\langle n_\xi,\, \overrightarrow{OP}\rangle<0$ for any point $P\in C$. This inequality remains valid for all points $P\in K\setminus\{O\}$.

On the other hand, if $\xi\in\partial_+C$ then O is contained in the half-space containing C, and therefore, for a regular point ξ, $\langle n_\xi,\, \overrightarrow{O\xi}\rangle\geq 0$. Taking $P=\xi$, one concludes that the inequality $\langle n_\xi,\, \overrightarrow{OP}\rangle\geq 0$ is valid for at least one point $P\in K\setminus\{O\}$. Define

$$\Omega_-=\Omega_-(K)=\big\{n\in S^2:\langle n,\, \overrightarrow{OP}\rangle<0 \text{ for all } P\in K\setminus\{O\}\big\} \text{ and} \tag{14}$$

$$\Omega_+=\Omega_+(K)=\big\{n\in S^2:\langle n,\, \overrightarrow{OP}\rangle\geq 0 \text{ for some } P\in K\setminus\{O\}\big\}. \tag{15}$$

We have $S^2=\Omega_-\cup\Omega_+$. It follows from the aforementioned argument that

$$\mathcal{N}_C^{-1}(\Omega_-)=\partial'_-C \qquad\text{and}\qquad \mathcal{N}_C^{-1}(\Omega_+)=\partial'_+C.$$

Denote by $[0,\,1]\times[0,\,2\pi)'$ the set of values $(t,\varphi)\in[0,\,1]\times[0,\,2\pi)$ that correspond to regular points of ∂C; this means that for $r=r(t,\varphi)$, the point $(r\cos\varphi, r\sin\varphi, z(r,\varphi))\in\partial C$ is regular. The set of pairs (r,φ) corresponding to irregular points of ∂C has measure zero; using Corollary 4.4 we conclude that $[0,\,1]\times[0,\,2\pi)'$ is a full measure subset of $[0,\,1]\times[0,\,2\pi)$.

Define the map $H_C\colon[0,\,1]\times[0,\,2\pi)'\to\Omega_-$ by $H_C(t,\varphi)=n_{M_{t,\varphi}}$. H_C is measurable, since it is the composition of two measurable maps $(t,\varphi)\mapsto M_{t,\varphi}$ and $M\overset{\mathcal{N}_C}{\mapsto} n_M$.

The intersection of the curve ∂_-C_φ with the ray of support from B_t is a line segment with an endpoint at $M_{t,\varphi}$. A plane of support to C at a point of this segment contains the ray of support, and therefore, is also a plane of support to any other point of the segment. It follows that the points of the segment are either all regular, or singular. In the case of regularity, the normal vectors at the points of the segment coincide with $n_{M_{t,\varphi}}$.

Remark 4.6. Note that

(a) $(t,\varphi)\in[0,\,1]\times[0,\,2\pi)'$ iff the points on the ray $\partial K_t\cap\Pi_\varphi$ are regular points of the cone K_t, and

(b) the vector $H_C(t,\varphi)$ coincides with the outer normal to the cone K_t at any point of the ray $\partial K_t\cap\Pi_\varphi$.

Lemma 4.7. *For any Borel set $D \subset \Omega_-$,*

$$\nu_C(D) = \iint_{H_C^{-1}(D)} \sqrt{1 + \sigma^2(t,\varphi) + \Big(\frac{\partial\sigma}{\partial\varphi}(t,\varphi)\Big)^2}\, \frac{1}{2}\, d\big[-r^2(t,\varphi)\big]\, d\varphi. \tag{16}$$

The expression in the right hand side of (16) should be understood as a repeated integral, where the interior integral is a Stieltjes integral with the (monotone increasing) generating function $t \mapsto -r^2(t,\varphi)$.

Proof. Notice that in general, if a measurable subset $\mathcal{O} \subset \partial C$ of the surface of a convex body is the graph of a function $z = z(x,y)$ defined on $\mathrm{pr}_{xy}\mathcal{O}$, then the gradient of this function $\nabla z(x,y)$ is a measurable function defined almost everywhere in $\mathrm{pr}_{xy}\mathcal{O}$ and the area of $\mathcal{O}$ equals

$$|\mathcal{O}| = \iint_{\mathrm{pr}_{xy}\mathcal{O}} \sqrt{1 + |\nabla z(x,y)|^2}\, dx\, dy.$$

Denoting by A the corresponding domain of integration in the polar coordinates $x = r\cos\varphi,\ y = r\sin\varphi$, one comes to the formula

$$|\mathcal{O}| = \iint_A \sqrt{1 + \Big(\frac{\partial z}{\partial r}\Big)^2 + \frac{1}{r^2}\Big(\frac{\partial z}{\partial\varphi}\Big)^2}\, r\, dr\, d\varphi. \tag{17}$$

By the second formula in (6) we have $\frac{\partial z}{\partial r} = \sigma$, and by formula (8), almost everywhere we have $\frac{1}{r}\frac{\partial z}{\partial\varphi} = \frac{\partial\sigma}{\partial\varphi}$, therefore the integrand in (17) equals $\sqrt{1 + \sigma^2 + (\frac{\partial\sigma}{\partial\varphi})^2}$. Thus, we obtain

$$\nu_C(D) = \iint \sqrt{1 + \sigma^2(t(r,\varphi),\varphi) + \Big(\frac{\partial\sigma}{\partial\varphi}(t(r,\varphi),\varphi)\Big)^2}\, r\, dr\, d\varphi,$$

where the integration is taken over the set $\{(r,\varphi) :\ H_C(t(r,\varphi),\varphi) \in D\}$.

Making the change of variables induced by the map $g(t,\varphi) = (r(t,\varphi),\varphi)$, one obtains formula (16). The sign "$-$" appearing in the differential in the right hand side of (16) is due to the fact that the map g reverses the orientation. $\square$

Take a positive C^1 function $\eta(t),\ t \in [0,1]$ with $\eta(0) = 1$, and introduce the auxiliary functions

$$\tilde r(t,\varphi) = \eta(t) r(t,\varphi) \quad \text{and} \quad \tilde\alpha(t) = \int_0^t \eta(\xi)\, d\alpha(\xi). \tag{18}$$

One obviously has $\tilde r(0,\varphi) = r(0,\varphi)$ and $\tilde r(1,\varphi) = 0$.

Definition 4.8. The function η is called *admissible*, if

(a) for any $0 < \tau < 1$ there exists $\tilde\gamma = \tilde\gamma(\tau) > 0$ such that

$$\tilde r(t_2,\varphi) - \tilde r(t_1,\varphi) \le -\tilde\gamma\,(t_2 - t_1) \tag{19}$$

for all $0 \le t_1 < t_2 \le \tau$ and for all φ;

(b) $\tilde\alpha(1) < |OB'|$.

The following lemma guarantees that there is a huge variety of admissible functions. Fix a value $0 < T < 1$.

Lemma 4.9. *There exists $c = c(T) > 0$ such that any function of the form $\eta(t) = e^{\chi(t)}$ with χ of class C^1, $\chi(0) = 0$, $|\chi'(t)| \le c$ for $0 \le t \le T$ and $\chi(t) = \chi(T)$ for $T < t \le 1$ is admissible.*

Proof. Recall that by Lemma 4.3, for any $0 < \tau < 1$ there exists $\gamma = \gamma(\tau) > 0$ such that for all φ and $0 \le t_1 < t_2 \le \tau$, (5) is satisfied.

Take a positive value c satisfying the two inequalities $c \le \gamma(T)/(2\,\mathrm{diam}(C))$ and $e^{cT} < |OB'|/|OB|$.

For each φ the function $t \mapsto r(t,\varphi)$ is monotone decreasing and therefore is differentiable for almost all t. At each point of differentiability $t \in [0,\, T]$ we have $\frac{\partial r}{\partial t}(t,\varphi) \le -\gamma(T)$.

Let η and χ satisfy the hypotheses of the lemma. If r is differentiable in t at a certain t, then the product ηr also is, and we use the estimates $\eta'(t)/\eta(t) = \chi'(t) \le c$ and $\frac{\partial r}{\partial t}(t,\varphi)/r(t,\varphi) \le -\gamma(T)/\mathrm{diam}(C) \le -2c$ to obtain

$$\frac{\frac{\partial}{\partial t}(\eta(t) r(t,\varphi))}{\eta(t) r(t,\varphi)} = \frac{\eta'(t)}{\eta(t)} + \frac{\frac{\partial r}{\partial t}(t,\varphi)}{r(t,\varphi)} \le -c.$$

Of course the function $t \mapsto \eta(t) r(t,\varphi)$ has bounded variation. Additionally, we have just obtained that at each point of differentiability of this function, for $t \in [0,\, T]$, $\frac{\partial}{\partial t}(\eta(t) r(t,\varphi))/(\eta(t) r(t,\varphi)) \le -c$.

Take $0 \le t_1 < t_2 \le T$. We use Lebesgue's decomposition to represent ηr as the sum of an absolutely continuous function, a singular function, and a jump function. The set of singular values t, that is, values at which the derivative of the singular function is not defined and values corresponding to jumps, has measure zero. We first take t_1 and t_2 not in this set.

Take $\varepsilon > 0$ and consider a covering of the set of singular values t by the union of countably many disjoint open intervals, $J = \cup_i (a_i,\, b_i)$, with the total sum of lengths being smaller than ε, that is, $|J| = \sum_i (b_i - a_i) < \varepsilon$. One can choose J in such a way that it does not contain t_1 and t_2.

Substitute the function $\eta(\cdot) r(\cdot,\varphi)$ with the auxiliary continuous function $\hat\eta(\cdot)$ which coincides with $\eta(\cdot) r(\cdot,\varphi)$ on $[0,\, 1] \setminus J$ and is affine on each interval $(a_i,\, b_i)$. One easily checks that this function is absolutely continuous and its derivative on $[0,\, 1] \setminus J$ coincides with the derivative of $\eta(\cdot) r(\cdot,\varphi)$. Hence,

$$\ln\big(\eta(t_2) r(t_2,\varphi)\big) - \ln\big(\eta(t_1) r(t_1,\varphi)\big) = \ln\big(\hat\eta(t_2)\big) - \ln\big(\hat\eta(t_1)\big) = \int_{t_1}^{t_2} \frac{\hat\eta'(t)}{\hat\eta(t)}\, dt$$

$$= \int_{[t_1,t_2]\setminus J} \frac{\frac{\partial}{\partial t}(\eta(t) r(t,\varphi))}{\eta(t) r(t,\varphi)}\, dt + \sum_i \big[\ln(\eta(b_i) r(b_i,\varphi)) - \ln(\eta(a_i) r(a_i,\varphi))\big]. \quad (20)$$

The integral in (20) is less than $-c(t_2 - t_1 - \varepsilon)$, and we use the inequalities $r(b_i, \varphi)) < r(a_i, \varphi))$ to estimate the sum in the right hand side of (20) as follows:

$$\sum_i \big[\ln(\eta(b_i) r(b_i, \varphi)) - \ln(\eta(a_i) r(a_i, \varphi)) \big]$$
$$< \sum_i \big[\ln(\eta(b_i) - \ln(\eta(a_i) \big] \le \max_t \frac{\eta'(t)}{\eta(t)} \sum_i (b_i - a_i) \le \varepsilon\, c.$$

Thus, one has

$$\ln \tilde r(t_2, \varphi) - \ln \tilde r(t_1, \varphi)) = \ln \big(\eta(t_2) r(t_2, \varphi)\big) - \ln \big(\eta(t_1) r(t_1, \varphi)\big) \le -c(t_2 - t_1) + 2\varepsilon c,$$

and taking into account that $\varepsilon > 0$ is arbitrary, one obtains

$$\ln \tilde r(t_2, \varphi) - \ln \tilde r(t_1, \varphi)) \le -c(t_2 - t_1), \text{ and so,}$$
$$\tilde r(t_2, \varphi) - \tilde r(t_1, \varphi) \le -\big(e^{c(t_2 - t_1)} - 1\big) \tilde r(t_2, \varphi) \le -c(t_2 - t_1)\, e^{-cT} \min_\varphi r(T, \varphi).$$

Now assume that one or both values $t_1,\ t_2$ are singular. Since each singular value is a limiting point of nonsingular ones, and using continuity in t of the function $\eta(t) r(t, \varphi)$, by the limiting process we come to the same inequality

$$\tilde r(t_2, \varphi) - \tilde r(t_1, \varphi) \le -[c\, e^{-cT} \min_\varphi r(T, \varphi)]\, (t_2 - t_1).$$

Hence, in the case $\tau \le T$, inequality (19) is satisfied with $\tilde\gamma(\tau) = c\, e^{-cT} \min_\varphi r(T, \varphi)$. Further, for two values $T \le t_1 < t_2 \le \tau$ we have

$$\tilde r(t_2, \varphi) - \tilde r(t_1, \varphi) = \eta(T) \big(r(t_2, \varphi) - r(t_1, \varphi) \big) \le -e^{-cT} \gamma(\tau)\, (t_2 - t_1).$$

It follows that inequality (19) in the case $\tau > T$ is satisfied with the constant $\tilde\gamma(\tau) = \min \big\{ c\, e^{-cT} \min_\varphi r(T, \varphi),\ e^{-cT} \gamma(\tau) \big\}$. Further, using that $\alpha(1) = |OB|$ and $\max_\xi \eta(\xi) \le e^{cT}$, one has

$$\tilde\alpha(1) = \int_0^1 \eta(\xi)\, d\alpha(\xi) \le \max_\xi \eta(\xi) \int_0^1 d\alpha(\xi) \le e^{cT} \cdot \alpha(1) = e^{cT} |OB| < |OB'|.$$

Thus, condition (b) in the definition of admissibility is also satisfied. □

In what follows we assume that the function η is admissible.

Lemma 4.10. *For any t_1 and t_2 the following formula analogous to* (11) *holds*

$$\tilde r(t_2, \varphi) \big(\sigma(t_1, \varphi) - \sigma(t_2, \varphi) \big) \le \tilde\alpha(t_2) - \tilde\alpha(t_1) \le \tilde r(t_1, \varphi) \big(\sigma(t_1, \varphi) - \sigma(t_2, \varphi) \big).$$

Proof. We shall give the proof for the case $t_1 < t_2$; for $t_2 < t_1$ the argument is completely similar.

After suitable substitutions the above formula is transformed into

$$\eta(t_2) r(t_2, \varphi) \big(\sigma(t_1, \varphi) - \sigma(t_2, \varphi) \big)$$
$$\le \int_{t_1}^{t_2} \eta(\xi)\, d\alpha(\xi) \le \eta(t_1) r(t_1, \varphi) \big(\sigma(t_1, \varphi) - \sigma(t_2, \varphi) \big). \tag{21}$$

Consider partitions of $[t_1,\, t_2]$ into subintervals, $\mathcal{P} = \{t_1 = t^{(0)} < t^{(1)} < \ldots < t^{(n-1)} < t^{(n)} = t_2\}$, and take the limit as $\delta(\mathcal{P}) := \max_{1\le i\le n}(t^{(i)} - t^{(i-1)}) \to 0$. In the formulas below we make use of the left inequality in (11) and take into account that the function $\eta(t)r(t,\varphi)$ is monotone decreasing in t.

$$\int_{t_1}^{t_2} \eta(\xi)\, d\alpha(\xi) = \lim_{\delta(\mathcal{P})\to 0} \sum_{i=1}^{n} \eta(t^{(i)})\big(\alpha(t^{(i)}) - \alpha(t^{(i-1)})\big)$$
$$\ge \lim_{\delta(\mathcal{P})\to 0} \sum_{i=1}^{n} \eta(t^{(i)}) r(t^{(i)},\varphi)\big(\sigma(t^{(i-1)},\varphi) - \sigma(t^{(i)},\varphi)\big)$$
$$\ge \eta(t_2)r(t_2,\varphi) \sum_{i=1}^{n} \big(\sigma(t^{(i-1)},\varphi) - \sigma(t^{(i)},\varphi)\big) = \eta(t_2)r(t_2,\varphi)\big(\sigma(t_1,\varphi) - \sigma(t_2,\varphi)\big).$$

The left inequality in (21) is proved.

Now we make use of the right inequality in (11).

$$\int_{t_1}^{t_2} \eta(\xi)\, d\alpha(\xi) = \lim_{\delta(\mathcal{P})\to 0} \sum_{i=1}^{n} \eta(t^{(i-1)})\big(\alpha(t^{(i)}) - \alpha(t^{(i-1)})\big)$$
$$\le \lim_{\delta(\mathcal{P})\to 0} \sum_{i=1}^{n} \eta(t^{(i-1)}) r(t^{(i-1)},\varphi)\big(\sigma(t^{(i-1)},\varphi) - \sigma(t^{(i)},\varphi)\big)$$
$$\le \eta(t_1)r(t_1,\varphi) \sum_{i=1}^{n} \big(\sigma(t^{(i-1)},\varphi) - \sigma(t^{(i)},\varphi)\big) = \eta(t_1)r(t_1,\varphi)\big(\sigma(t_1,\varphi) - \sigma(t_2,\varphi)\big).$$

The right inequality in (21) is proved. □

Given an admissible function η, one then defines a convex body $\tilde{C} \subset \mathbb{R}^3$. This is done in the following remark.

Remark 4.11. (Construction of $\tilde{C}$) Fix η, for any φ take the function $t \mapsto \tilde{r}(t,\varphi) = \eta(t)r(t,\varphi)$, and define the generalized inverse function $r \mapsto \tilde{t}(r,\varphi)$, $r \in [0,\, r(1,\varphi)]$ by the relation $\tilde{t}(r,\varphi) := \inf\{t : \tilde{r}(t,\varphi) \le r\}$. The generalized inverse function $\tilde{t}(\cdot,\varphi)$ is continuous and monotone non-increasing. The intervals where it is constant correspond to jumps of the function $\eta(\cdot)r(\cdot,\varphi)$.

For each φ and $t \in [0,\, 1]$ consider the planar cone $\tilde{K}_t(\varphi)$ in the half-plane Π_φ with the coordinates (r, z) given by the inequalities

$$r \ge 0, \quad z \ge \sigma(t,\varphi)\, r + \tilde{\alpha}(t). \tag{22}$$

The cone $\tilde{K}_t(\varphi)$ is bounded by the z-axis and by the ray with the origin $\tilde{B}_t := (0,\, \tilde{\alpha}(t))$ through the point $\tilde{M}_{t,\varphi} := (\tilde{r}(t,\varphi),\ \sigma(t,\varphi)\,\tilde{r}(t,\varphi) + \tilde{\alpha}(t))$, and its vertex is at $\tilde{B}_t$. Note that for $t = 0$ one has $\tilde{M}_{0,\varphi} = (r(0,\varphi),\ \sigma(0,\varphi)\, r(0,\varphi)) = M_\varphi$.

The union over all φ of these planar cones is the 3-dimensional convex cone $\tilde{K}_t := \cup_\varphi \tilde{K}_t(\varphi)$ with the vertex at $\tilde{B}_t$, which is actually the translation of K_t along the z-axis by the distance $\tilde{\alpha}(t) - \alpha(t) = \int_0^t (\eta(\tau) - 1)\, d\alpha(\tau)$. One has, in particular,

$\tilde{B}_0 = O$, $\tilde{K} := \tilde{K}_0 = K$, and $\tilde{K}_0(\varphi) = K_0(\varphi)$. Denote $\tilde{B} := \tilde{B}_1$. Condition (b) in Definition 4.8 means that $\tilde{B}$, and therefore all points $\tilde{B}_t$, $0 \le t \le 1$, lie below B'.

Define the new planar convex body $\tilde{C}_\varphi$ by

$$\tilde{C}_\varphi = (\cap_{0 \le t \le 1} \tilde{K}_t(\varphi)) \cap \mathrm{Conv}(\mathrm{C}_\varphi, \{\mathrm{O}\})$$

and the corresponding 3-dimensional convex body $\tilde{C}$ by

$$\tilde{C} = \cup_\varphi \tilde{C}_\varphi = \Big(\cap_{0 \le t \le 1} \tilde{K}_t \Big) \cap \mathrm{Conv}(C \cup \{O\}). \tag{23}$$

This completes the construction. □

Note that it does not follow automatically from these definitions that all cones $\tilde{K}_t(\varphi)$ are circumscribed about $\tilde{C}_\varphi$ or that the 3-dimensional cones $\tilde{K}_t$ are circumscribed about $\tilde{C}$. This is justified in the following Lemma 4.12.

$\tilde{C}_\varphi$ is bounded on the left by a segment of the positive z-semiaxis and on the right by a concave curve with the endpoints at $\tilde{B}$ and B'. This curve can also be defined as $\partial \tilde{C} \cap \Pi_\varphi$.

Similarly to what was done above for C_φ, we draw the ray of support from O in the half-plane Π_φ to that curve, denote by $\partial_- \tilde{C}_\varphi$ the part of the curve between $\tilde{B}$ and the closest to O point of intersection of the ray with the curve (excluding this point), and denote by $\partial_+ \tilde{C}_\varphi$ the complementary part of the curve between this point of intersection and B'. From each point $\tilde{B}_t$, $0 \le t \le 1$, draw the ray of support to that curve and denote by $\tilde{\sigma}_{t,\varphi}$ the inverse slope of the ray and by $H_{\tilde{C}}(t, \varphi)$ the outer normal to $\tilde{C}$ at the closest to $\tilde{B}_t$ point of intersection of this ray with the curve.

Lemma 4.12. (a) *The ray of support in Π_φ from O to $\tilde{C}_\varphi$ is $z = \sigma(0, \varphi)\, r$, $r \ge 0$. The closest to O point of intersection of the ray with the curve is M_φ, and*

$$\partial_+ \tilde{C}_\varphi = \partial_+ C_\varphi.$$

(b) *For any t the ray of support in Π_φ from $\tilde{B}_t$ to the curve is $z = \sigma(t, \varphi)\, r + \tilde{\alpha}(t)$, $r \ge 0$, and therefore,*

$$\tilde{\sigma}_{t,\varphi} = \sigma_{t,\varphi}. \tag{24}$$

Additionally, $\tilde{M}_{t,\varphi}$ is the closest to $\tilde{B}_t$ point of intersection of the ray with the curve.

(c) *For all $(t, \varphi) \in [0,\, 1] \times [0,\, 2\pi)'$, $H_{\tilde{C}}(t, \varphi)$ exists and $H_{\tilde{C}}(t, \varphi) = H_C(t, \varphi)$.*

(d) $\mathrm{Conv}(\tilde{C}_\varphi \cup \{O\}) = \mathrm{Conv}(C_\varphi \cup \{O\})$.

Proof. Fix φ. One easily derives from Lemma 4.10 that each point $\tilde{M}_{t,\varphi} = (\tilde{r}(t, \varphi),\ \sigma(t, \varphi)\, \tilde{r}(t, \varphi) + \tilde{\alpha}(t))$ lies above all rays of the kind $z = \sigma(t', \varphi)\, r + \tilde{\alpha}(t')$, $r \ge 0$ $(0 \le t' \le 1)$, and therefore belongs to the intersection of cones $\cap_{0 \le t \le 1} \tilde{K}_t(\varphi)$.

Further, for any $0 \le t \le 1$, both the points B' and M_φ lie above the ray $z = \sigma(t,\varphi)\, r + \tilde\alpha(t),\ r \ge 0$ containing $\tilde M_{t,\varphi}$, and $\tilde M_{t,\varphi}$ lies above the line OM_φ, and therefore, $\tilde M_{t,\varphi}$ belongs to the triangle $OM_\varphi B'$. Since this triangle is contained in $\mathrm{Conv}(C_\varphi \cup \{O\})$, we have

$$\tilde M_{t,\varphi} \in \mathrm{Conv}(C_\varphi \cup \{O\}).$$

It follows that the point $\tilde M_{t,\varphi}$ lies in $\tilde C_\varphi = \big(\cap_{0\le t'\le 1} \tilde K_{t'}(\varphi)\big) \cap \big(\mathrm{Conv}(C_\varphi \cup \{O\})\big)$. Since it also lays on the ray $z = \sigma(t,\varphi)\, r + \tilde\alpha(t),\ r \ge 0$ bounding $\tilde K_t(\varphi)$ on the right, we conclude that this ray is the ray of support from $\tilde B_t$ to $\tilde C_\varphi$, and therefore, $\tilde\sigma_{t,\varphi} = \sigma_{t,\varphi}$. Thus, formula (24) in (b) is proved.

In particular, the ray $z = \sigma(0,\varphi)\, r,\ r \ge 0$ is the ray of support from O to $\tilde C_\varphi$, and the point $\tilde M_{0,\varphi} = M_\varphi$ lies in the intersection of this ray with the curve. Further, the r-coordinate of the closest to O point of the intersection is smaller than or equal to the r-coordinate of M_φ and greater than or equal to the r-coordinate of $\tilde M_{t,\varphi}$ for all $t > 0$ (these coordinates are equal, correspondingly, to $r(0,\varphi)$ and $\tilde r(t,\varphi)$). Since the function $t \mapsto \tilde r(t,\varphi)$ is right semicontinuous, we have $\lim_{t\to 0^+} \tilde r(t,\varphi) = \tilde r(0,\varphi) = r(0,\varphi)$, and therefore, the closest to O point of the intersection is $M_\varphi = \tilde M_{0,\varphi}$.

Exactly the same argument can be used to prove that $\tilde M_{t,\varphi}$ is the closest to $\tilde B_t$ point of intersection of the ray $z = \sigma(t,\varphi)\, r + \tilde\alpha(t),\ r \ge 0$ with the curve. Thus, the proof of (b) is complete.

The curve $\partial_+ C_\varphi$ lies in the intersection of cones $\cap_{0\le t\le 1} \tilde K_t(\varphi)$, and therefore, in $\tilde C_\varphi = (\cap_{0\le t\le 1} \tilde K_t(\varphi)) \cap \mathrm{Conv}(\mathrm{C}_\varphi \cup \{\mathrm{O}\})$. Since $\partial_+ C_\varphi$ lies in the boundary of $\mathrm{Conv}(\mathrm{C}_\varphi \cup \{\mathrm{O}\})$, it also lies in the boundary of $\tilde C_\varphi$. Thus, both curves $\partial_+ \tilde C_\varphi$ and $\partial_+ C_\varphi$ lie in $\partial \tilde C_\varphi$ and are bounded by the points M_φ and B' (and include them). It follows that they coincide, and (a) is proved.

Remark 4.6 is applicable also to $\tilde C$, that is, (a) $H_{\tilde C}(t,\varphi)$ exists *iff* the points of the ray $\partial \tilde K_t \cap \Pi_\varphi$ are regular points of the cone $\tilde K_t$, and (b) $H_{\tilde C}(t,\varphi)$ coincides with the outer normal to $\tilde K_t$ at any point of the ray $\partial \tilde K_t \cap \Pi_\varphi$. Since $\tilde K_t$ is a translation of K_t along the z-ray and Π_φ is invariant with respect to this translation, the points of the ray $\partial \tilde K_t \cap \Pi_\varphi$ are regular points of $\tilde K_t$ *iff* the points of the ray $\partial K_t \cap \Pi_\varphi$ are regular points of K_t, and in the case of regularity, the outer normal to $\tilde K_t$ at a point of $\partial \tilde K_t \cap \Pi_\varphi$ coincides with the outer normal to K_t at a point of $\partial K_t \cap \Pi_\varphi$. It follows that $H_{\tilde C}(t,\varphi)$ is defined on $[0,\,1] \times [0,\,2\pi)'$ and coincides with $H_C(t,\varphi)$, and (c) is proved.

Note that the set $\mathrm{Conv}(C_\varphi \cup \{O\})$ is bounded by the line segments OB' and OM_φ and by the curve $\partial_+ C_\varphi$, and therefore, coincides with $\mathrm{Conv}(\partial_+ C_\varphi \cup \{O\})$. Using that $\partial_+ C_\varphi \subset \tilde C_\varphi$, we obtain

$$\mathrm{Conv}(C_\varphi \cup \{O\}) = \mathrm{Conv}(\partial_+ C_\varphi \cup \{O\}) \subset \mathrm{Conv}(\tilde C_\varphi \cup \{O\}).$$

On the other hand, since $\tilde C_\varphi \subset \mathrm{Conv}(\mathrm{C}_\varphi \cup \{\mathrm{O}\})$, we have

$$\mathrm{Conv}(\tilde C_\varphi \cup \{O\}) \subset \mathrm{Conv}\big(\mathrm{Conv}(C_\varphi \cup \{O\}) \cup \{O\}\big) = \mathrm{Conv}(C_\varphi \cup \{O\}).$$

Thus, (d) is proved. $\square$

The statement (b) of the following lemma is, in a sense, inverse to Lemma 4.3.

Lemma 4.13. (a) *The ray* $z = \sigma(0, \varphi)\, r,\ r \geq 0$ *is tangent to* $\tilde{C}_\varphi$.

(b) *For any* $0 < \tau < 1$ *there exists* $\tilde{k} = \tilde{k}(\tau)$ *such that for any* φ, *the part of the curve* $\partial_- \tilde{C}_\varphi$ *between the points* $\tilde{M}_{\tau,\varphi}$ *and* M_φ *is of class* C^1 *and has curvature* $\leq \tilde{k}$.

Proof. The proof of (b) is essentially an inversion of the argument of Lemma 4.3. We use the same Fig. 4.2 with a slight correction of notation: $M_{t_i,\varphi}$ and $\alpha(t_i)$, $i = 1, 2$ in the figure should be replaced, respectively, by $\tilde{M}_{t_i,\varphi}$ and $\tilde{\alpha}(t_i)$.

Fix $0 < \tau < 1$ and take $0 \leq t_1 < t_2 \leq \tau$. We use the shorthand notation $r_1 := \tilde{r}(t_1, \varphi)$, $r_2 := \tilde{r}(t_2, \varphi)$, $\psi_1 := \arctan(\sigma(t_1, \varphi))$, $\psi_2 := \arctan(\sigma(t_2, \varphi))$. One has $t_1 < t_2$, therefore $r_1 > r_2$ and $\psi_1 > \psi_2$.

The length of the part of the curve between the points $\tilde{M}_{t_1,\varphi}$ and $\tilde{M}_{t_2,\varphi}$ (to be referred to as *the curve* later on in the proof) is greater than or equal to $r_1 - r_2$, and since the function $\tilde{r}$ is admissible, for $\tilde{\gamma} = \tilde{\gamma}(\tau)$ one has $r_1 - r_2 \geq \tilde{\gamma}(t_2 - t_1)$.

Draw the line through the point $\tilde{M}_{t_2,\varphi}$ parallel to the ray of support through $\tilde{M}_{t_1,\varphi}$, and let $\hat{\alpha}$ be the z-coordinate of intersection of this line with the z-axis. Since the curve is convex, we have $\hat{\alpha} \geq \tilde{\alpha}(t_1)$. Thus, we have

$$\tilde{\alpha}(t_2) - \tilde{\alpha}(t_1) \geq \tilde{\alpha}(t_2) - \hat{\alpha} = r_2 \big(\sigma(t_1, \varphi) - \sigma(t_2, \varphi)\big).$$

Note that $\sigma(t_1, \varphi) - \sigma(t_2, \varphi) = \tan \psi_1 - \tan \psi_2 > \psi_1 - \psi_2$. Introducing the value $\tilde{R}_\tau := \min_\varphi \tilde{r}(\tau, \varphi) \geq \tilde{\gamma}\tau > 0$ and using that $r_2 \geq \tilde{R}_\tau$, we get

$$\big(\text{length of the curve}\big) \geq r_1 - r_2 \geq \tilde{\gamma}(t_2 - t_1) \geq \tilde{\gamma}\, \frac{\tilde{\alpha}(t_2) - \tilde{\alpha}(t_1)}{\max_{t\in[0,1]} \alpha'(t)} \geq \frac{1}{\tilde{k}}\, (\psi_1 - \psi_2),$$

where
$$\tilde{k} = \tilde{k}(\tau) = \frac{\max_{t\in[0,1]} \alpha'(t)}{\tilde{\gamma}\, \tilde{R}_\tau}.$$

This implies that the part of the curve between $\tilde{M}_{\tau,\varphi}$ and M_φ is of class C^1 and its curvature is $\leq \tilde{k}$. Claim (b) of the lemma is proved.

Thus, we conclude that the slope of the curve at a point converges to $\sigma(0, \varphi)$ when the point approaches M_φ. It follows that the ray $z = \sigma(0, \varphi)\, r,\ r \geq 0$ is tangent to the curve $\partial_- \tilde{C}_\varphi$ at the endpoint M_φ. On the other hand, according to condition (i) of Theorem 3.1′, this ray is also tangent to $\partial_+ C_\varphi$ at the endpoint M_φ. Taking into account that $\partial_+ C_\varphi = \partial_+ \tilde{C}_\varphi$, one concludes that the ray is tangent to $\tilde{C}_\varphi$. Thus, claim (a) of the lemma is also proved. □

Denote $\partial_+ \tilde{C} = \cup_\varphi \partial_+ \tilde{C}_\varphi = \partial \tilde{C} \cap \partial\big(\mathrm{Conv}(C \cup \{O\})\big)$. It follows from claim (d) of Lemma 4.12 that $\partial_+ \tilde{C} = \partial_+ C$, and therefore,

$$\partial_+ C \subset \partial \tilde{C}. \tag{25}$$

Since $\tilde{K}$ coincides with K, the sets $\Omega_-(K)$ and $\Omega_+(K)$, defined by (14) and (15), coincide with $\Omega_-(\tilde{K})$ and $\Omega_+(\tilde{K})$, respectively. It follows that $\mathcal{N}_{\tilde{C}}^{-1}(\Omega_+) = \partial'_+ \tilde{C} = \partial'_+ C$ and $\mathcal{N}_{\tilde{C}}\rfloor_{\partial'_+ \tilde{C}} = \mathcal{N}_C\rfloor_{\partial'_+ C}$.

Lemma 4.13 and claims (b), (c) of Lemma 4.12 mean that the body $\tilde{C}$ satisfies conditions (i) and (ii)$'$ of Theorem 3.1$'$, the corresponding auxiliary functions $\tilde{\sigma}(t,\varphi)$ and $H_{\tilde{C}}(t,\varphi)$ coincide with $\sigma(t,\varphi)$ and $H_C(t,\varphi)$, respectively, and the function $\tilde{r}(t,\varphi) = \eta(t)r(t,\varphi)$ plays the role of $r(t,\varphi)$ as applied to $\tilde{C}$. Therefore Lemma 4.7 is also applicable to the body $\tilde{C}$.

For any Borel set $D \subset \Omega_+$ we have

$$\nu_{\tilde{C}}(D) = |\mathcal{N}_{\tilde{C}}^{-1}(D)| = |\mathcal{N}_C^{-1}(D)| = \nu_C(D). \tag{26}$$

For a Borel set $D \subset \Omega_-$ we get

$$\nu_{\tilde{C}}(D) = \iint_{H_C^{-1}(D)} \sqrt{1+\sigma^2(t,\varphi) + \Big(\frac{\partial\sigma}{\partial\varphi}(t,\varphi)\Big)^2}\, \frac{1}{2}\, d\big[-\eta^2(t)r^2(t,\varphi)\big]\, d\varphi. \tag{27}$$

Now prove Theorem 3.1$'$. Fix $0 < T < 1$ and take $c = c(T)$ as explained in Lemma 4.9. Choose a C^1 function $\theta\colon [0,\,1] \to \mathbb{R}$ such that

(i) $\theta(0) = 0$ and $\theta(t) = 0$ for $t \in [T,\,1]$;

(ii) $|\theta(t)| < 1$ for all t;

(iii) $\big|\frac{d}{dt}\big[\frac{1}{2}\ln(1 \pm \theta(t))\big]\big| \le c(T)$ for all $0 \le t \le T$.

It is straightforward to see that for $s \in [-1,\,1]$ the function $\eta(\cdot,s) = \sqrt{1+s\theta(\cdot)}$ satisfies the conditions of Lemma 4.9, and therefore is admissible. For each $s \in [-1,\,1]$ take $\eta(t) := \eta(t,s)$ and construct the corresponding convex body $\tilde{C}$ according to the procedure described on page 1396 ending with formula (23). Thus, we have a vast family of convex bodies depending on the function θ and on the parameter s; we will use the notation $\tilde{C} = C(s) = C^{(\theta)}(s)$.

One obviously has $C(0) = C$; thus, statement (a) of the theorem holds true for these bodies.

Formula (25) remains valid when $\tilde{C}$ is replaced with $C(s)$; thus, statement (b) of the theorem is also true.

According to formulas (26) and (27), for a Borel set $D \subset \Omega_+$ we have $\nu_{C(s)}(D) = \nu_C(D)$, and for a Borel set $D \subset \Omega_-$,

$$\nu_{C(s)}(D) = \iint_{H_C^{-1}(D)} \sqrt{1+\sigma^2(t,\varphi) + \Big(\frac{\partial\sigma}{\partial\varphi}(t,\varphi)\Big)^2}\, \frac{1}{2}\, d\big[-(1+s\,\theta(t))r^2(t,\varphi)\big]\, d\varphi.$$

Using these relations, one easily derives that for any Borel set D,

$$\nu_{C(s)}(D) = \nu_C(D) + s\,\big(\nu_{C(1)}(D) - \nu_C(D)\big),$$

and so, statement (c) of the theorem is true. Further, the signed measure that serves as a director vector, $\nu_{C^{(\theta)}(1)} - \nu_C$, satisfies the relations $(\nu_{C^{(\theta)}(1)} - \nu_C)(D) = 0$, if $D \subset \Omega_+$, and, if $D \subset \Omega_-$,

$$(\nu_{C^{(\theta)}(1)} - \nu_C)(D) = \iint_{H_C^{-1}(D)} \sqrt{1+\sigma^2(t,\varphi)+\Big(\frac{\partial\sigma}{\partial\varphi}(t,\varphi)\Big)^2}\, \frac{1}{2}\, d\big[-\theta(t)\,r^2(t,\varphi)\big]\, d\varphi.$$

Let Θ be the set of C^1 functions θ satisfying (i), (ii), and (iii). The union of the 1-dimensional subspaces spanned by all the director vectors corresponding to $\theta \in \Theta$ contains all signed measures $\nu = \nu^{(\theta)}$ induced by C^1 functions θ such that $\theta(0) = 0$ and $\theta(t) = 0$ for $t \in [T, 1]$. These measures are defined by

$$\nu^{(\theta)}(D) = \begin{cases} 0, & \text{if } D \subset \Omega_+; \\ \iint_{H_C^{-1}(D)} \frac{1}{2}\sqrt{1 + \sigma^2(t,\varphi) + \left(\frac{\partial\sigma}{\partial\varphi}(t,\varphi)\right)^2} \times & \\ \times d\Big[-\theta(t)\, r^2(t,\varphi)\Big] d\varphi, & \text{if } D \subset \Omega_-. \end{cases}$$

All such measures form an infinite-dimensional linear space. Thus, statement (d) of the theorem is proved.

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References

[1] A. D. Alexandrov: *Selected Works. I: Selected Scientific Papers*, Chapter V, §3, Yu. G. Reshetnyak and S. S. Kutateladze (eds.), Gordon & Breach, Philadelphia et al. (1996).

[2] V. Alexandrov, N. Kopteva, S. S. Kutateladze, *Blaschke addition and convex polyhedra*, arXiv: 0502345 (2005).

[3] F. Brock, V. Ferone, B. Kawohl: *A symmetry problem in the calculus of variations*, Calc. Var. 4 (1996) 593–599.

[4] G. Buttazzo, V. Ferone, B. Kawohl: *Minimum problems over sets of concave functions and related questions*, Math. Nachrichten 173 (1995) 71–89.

[5] G. Buttazzo, P. Guasoni: *Shape optimization problems over classes of convex domains*, J. Convex Analysis 4(2) (1997) 343–351.

[6] G. Buttazzo, B. Kawohl: *On Newton's problem of minimal resistance*, Math. Intell. 15 (1993) 7–12.

[7] G. Carlier, T. Lachand-Robert: *Convex bodies of optimal shape*, J. Convex Analysis 10(1) (2003) 265–273.

[8] M. Comte, T. Lachand-Robert: *Newton's problem of the body of minimal resistance under a single-impact assumption*, Calc. Var. Partial Differ. Equ. 12 (2001) 173–211.

[9] M. Comte, T. Lachand-Robert: *Existence of minimizers for Newton's problem of the body of minimal resistance under a single-impact assumption*, J. Anal. Math. 83 (2001) 313–335.

[10] T. Lachand-Robert, E. Oudet: *Minimizing within convex bodies using a convex hull method*, SIAM J. Optim. 16 (2006) 368–379.

[11] T. Lachand-Robert, M. A. Peletier: *Newton's problem of the body of minimal resistance in the class of convex developable functions*, Math. Nachrichten 226 (2001) 153–176.

[12] T. Lachand-Robert, M. A. Peletier: *An example of non-convex minimization and an application to Newton's problem of the body of least resistance*, Ann. Inst. H. Poincaré, Anal. Non Lin. 18 (2001) 179–198.

[13] A. Plakhov: *Exterior Billiards. Systems with Impacts Outside Bounded Domains*, Springer, New York (2012).

[14] A. Plakhov, D. Torres: *Newton's aerodynamic problem in media of chaotically moving particles*, Sbornik: Math. 196 (2005) 885–933.

[15] G. Wachsmuth: *The numerical solution of Newton's problem of least resistance*, Math. Program. A 147 (2014) 331–350.

[16] H. Zouaki: *Convex set symmetry measurement using Blaschke addition*, Pattern Recognition 36 (2003) 753–763.