Journal of Convex Analysis
Volume 27 (2020), No. 1, 19–30

Sufficient Condition for Tangential Transversality*

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Dedicated to Alexander Ioffe on the occasion of his 80th birthday.

Received: February 14, 2019
Accepted: October 8, 2019

A general sufficient condition for tangential transversality is proved. The underlying idea is that in many cases the uniformness of the local approximation of a closed set can be used instead of some suitable compactness assumption.

Keywords: Subtransversality, Aubin property, tangential transversality.

2010 Mathematics Subject Classification: 49K27, 35F25, 46N10.

1. Introduction

Transversality is a classical concept of mathematical analysis and differential topology. Recently, it has proven to be useful in variational analysis as well. As A. Ioffe stated in [12], the transversality-oriented language is extremely natural and convenient in some parts of variational analysis, including subdifferential calculus and nonsmooth optimization.

The classical definition of transversality at an intersection point of two smooth manifolds in a Euclidean space is that the sum of the corresponding tangent spaces at the intersection point is the whole space (cf. [8], [9]).

*This work was partially supported by the Sofia University “St. Kliment Ohridski” fund “Research and Development” under contract 80-10-20/09.04.2019, by the Bulgarian National Scientific Fund under Grant KP-06-H22/4/04.12.2018 and by the Center of Excellence in Informatics and ICT, Grant No. BG05M2OP001-1.001-0003 (financed by the Science and Education for Smart Growth Operational Program (2014-2020) and co-financed by the European Union through the European structural and Investment funds).

ISSN 0944-6532 / $ 2.50

In order to prove the Pontryagin maximum principle (cf., for example, the bibliography of [24]), Hector Sussmann generalizes the definition of transversality for closed convex cones in $\mathbb{R}^n$: the cones C^A and C^B are transversal if and only if

$$C^A - C^B = \mathbb{R}^n$$

and strongly transversal, if they are transversal and $C^A \cap C^B \neq \{0\}$ (cf. Definitions 3.1 and 3.2 from [24]). In the finite-dimensional case, strong transversality of the approximating cones of the same type (either Clarke or Boltyanski) is a sufficient condition for local nonseparation of sets. The sets A, B containing a point x_0 are said to be locally separated at x_0, if there exists a neighborhood Ω of x_0 so that $\Omega \cap A \cap B = \{x_0\}$.

In the infinite-dimensional case, strong transversality of the approximating cones of the same type does not imply local nonseparation of sets – take for example a Hilbert cube $A := \{(x_n) \in l_2 : \ |x_n| \leq 1/n\} \subset l_2$ and a ray $B := \{\lambda y : \ \lambda \geq 0\}$, where $y := (1/n^{3/4})_{n=1}^{\infty}$. We have that the corresponding Clarke tangent cones $\widehat{T}_A(\mathbf{0}) = l_2$ and $\widehat{T}_B(\mathbf{0}) = B$ are strongly transversal, while the sets A and B are locally separated at 0.

In the literature there exist many notions generalizing the classical transversality as well as transversality of cones. Some of them are introduced under different names by different authors, but actually coincide. We refer to [18] for a survey of terminology and comparison of the available concepts. The central ones among them are *transversality* and *subtransversality*. They are also objects of study in the recent book [13].

The term subtransversality is recently introduced in [7] in relation to proving linear convergence of the alternating projections algorithm. However, this property has been around for more than 20 years, but under different names – see Remark 4 in [18] and the references therein. It is a key assumption for two types of results: linear convergence of sequences generated by projection algorithms and a qualification condition for normal intersection property with respect to the limiting normal cone and a sum rule for the limiting subdifferential. The precise definition is

Definition 1.1. Let A and B be closed subsets of the Banach space X. A and B are said to be *subtransversal* at $x_0 \in A \cap B$, if there exists $K > 0$ such that

$$d(x, A \cap B) \leq K(d(x, A) + d(x, B))$$

for all x in a fixed neighborhood of x_0. □

From our point of view the most remarkable thing about subtransversality is the fact that it implies a rather general nonseparation result which is crucial for obtaining necessary optimality conditions of Pontryagin maximum principle type (including optimal control problems with infinite-dimensional state space). Moreover, subtransversality is a natural assumption for proving abstract Lagrange multiplier rule.

Proposition 1.2. (Nonseparation result) *Let A and B be closed subsets of the Banach space X. Let A and B be subtransversal at $x_0 \in A \cap B$ with constants $\delta > 0$ and $K > 0$. Assume that there exist v^A with unit norm which belongs to the Bouligand tangent cone to A at x_0 and v^B with unit norm which belongs to the derivable tangent cone to B at x_0, such that $\|v^A - v^B\| < \frac{1}{K}$. Then A and B cannot be locally separated at x_0.*

Theorem 1.3. (Lagrange multiplier rule) *We consider the optimization problem*

$$f(x) \to \min \quad \textit{subject to} \quad x \in S\ ,$$

where $f\colon X \longrightarrow \mathbb{R} \cup \{+\infty\}$ is lower semicontinuous and proper and S is a closed subset of the Banach space X. Let x_0 be a solution of the above problem. Let $\tilde{C}_{epif}(x_0, f(x_0))$ and $C_S(x_0)$ be closed convex cones, contained in the corresponding Bouligand approximating cones $T_{epif}(x_0, f(x_0))$ and $T_S(x_0)$. Let at least one of them consist of derivable tangent vectors.

(a) *If $\tilde{C}_{epif}(x_0, f(x_0)) - C_S(x_0) \times (-\infty, 0]$ is not dense in $X \times \mathbb{R}$, then there exists a pair $(\xi, \eta) \in X^* \times \mathbb{R}$ such that*
   - (i) $(\xi, \eta) \neq (\mathbf{0}, 0)$;
   - (ii) $\eta \in \{0, 1\}$;
   - (iii) $\langle \xi, v\rangle \ \leq 0$ *for every* $v \in C_S(x_0)$;
   - (iv) $\langle \xi, w\rangle + \eta s \geq 0$ *for every* $(w, s) \in \tilde{C}_{epif}(x_0, f(x_0))$.

(b) *If $\tilde{C}_{epif}(x_0, f(x_0)) - C_S(x_0) \times (-\infty, 0]$ is dense in $X \times \mathbb{R}$, then epif and $S \times (-\infty, f(x_0)]$ are not subtransversal at $(x_0, f(x_0))$.*

The precise definitions of the concepts used in the assertions formulated above are given in the second section. These results are taken from [1], because we couldn't find them clearly stated in the literature elsewhere. Nevertheless, we are pretty sure that Prof. Ioffe is well aware that something like that is true. The intriguing thing here is to verify the subtransversality assumption in nontrivial cases. The aim of this note is to find some conditions which are sufficient for subtransversality of two sets.

Our approach is proving tangential transversality instead of subtransversality.

Definition 1.4. Let A and B be closed subsets of the Banach space X. We say that A and B are *tangentially transversal* at $x_0 \in A \cap B$, if there exist $M > 0$, $\delta > 0$ and $\eta > 0$ such that for any two different points $x^A \in (x_0 + \delta\overline{\mathbf{B}}) \cap A$ and $x^B \in (x_0 + \delta\overline{\mathbf{B}}) \cap B$, there exists a sequence $\{t_m\}$, $t_m \searrow 0$, such that for every $m \in \mathbb{N}$ there exist $w_m^A \in X$ with $\|w_m^A\| \leq M$ and $x^A + t_m w_m^A \in A$, and $w_m^B \in X$ with $\|w_m^B\| \leq M$, $x^B + t_m w_m^B \in B$, and the following inequality holds true

$$\|x^A - x^B + t_m(w_m^A - w_m^B)\| \leq \|x^A - x^B\| - t_m\eta\,. \qquad \square$$

The above condition is a stronger condition than subtransversality (c.f. [1], Proposition 2.8), but for the time being we do not know whether it is strictly stronger or the class of tangentially transversal pairs of sets coincides with the class of subtransversal pairs of sets. For a discussion on this notion the reader is referred to [1]. It happens that usually tangential transversality is easier to verify than subtransversality when the information known concerns the tangential structure of the sets.

We are going to present a general sufficient condition for tangential transversality (cf. Theorem 3.1 and Theorem 3.2). The underlying idea is that in many cases the uniformness of the local approximation of a closed set can be used instead of some suitable compactness assumption. This is especially important in the infinite-dimensional case. The concept of uniform tangent set is introduced in [15] and its study is continued in [3]. For the definition of a uniform tangent set and some of its properties see section two.

The starting point of this research was the famous Aubin property used by Clarke in [6] for the basic problem of the calculus of variations. We formulate here an abstract (infinite-dimensional) version of this condition. This abstract version inspired the rest of the results in this note. We show that if a function (actually its epigraph) satisfies this assumption and the constraint has a specific form (tailored after the constraint in the basic problem of the calculus of variations as an infinite dimensional optimization problem), one can find a Lagrange multiplier. Sure, the proof makes use of our main result. It is worth noting that our abstract version of the Aubin condition is essentially weaker than the original pointwise version – the "correction" set in our version is a ball while it is a weak compact set in the original version (cf. [6]). We think that such more restrictive version is related to the regularity of the obtained Lagrange multiplier.

To further motivate our main result, we show that some known sufficient conditions for tangential transversality can be obtained as its particular cases. Namely, we obtain Theorem 2.6 (taken from [1]) and Proposition 2.7 (taken from [2]) as corollaries of our main result Theorem 3.2. Moreover, the well known notion of compactly epi-Lipschitz set is extended for a pair of closed sets (cf. Definition 4.4).

2. Preliminaries

Throughout the paper if Y is a Banach space, we will denote by $\mathbf{B}_Y$ [$\overline{\mathbf{B}}_Y$] its open [closed] unit ball, centered at the origin. The index could be omitted if there is no ambiguity about the space. If S is a closed subset of Y and $y \in S$, we will denote by $T_S(y)$ the *Bouligand tangent cone* to S at y, i.e.

$$T_S(y) := \left\{ v \in Y : \frac{y_k - y}{\tau_k} \to v \quad \begin{array}{l} \text{for some sequences } y_k \in S, y_k \to y \\ \text{and } \tau_k > 0, \tau_k \to 0 \end{array} \right\};$$

by $G_S(y)$ the *derivable tangent cone* to S at y, i.e.

$$G_S(y) := \left\{ v \in Y : \frac{\xi(\tau_k) - y}{\tau_k} \to v \quad \begin{array}{l} \text{for some vector-valued function} \\ \xi : [0,\varepsilon] \to S, \xi(0) = y \text{ and for every} \\ \text{choice of a sequence } \tau_k > 0, \tau_k \to 0 \end{array} \right\};$$

and by $\widehat{T}_S(y)$ the *Clarke tangent cone* to S at y, i.e.

$$\widehat{T}_S(y) := \left\{ v \in Y : \begin{array}{l} \text{for every } \varepsilon > 0 \text{ there exists } \delta > 0 \\ \text{such that for every } t \in [0, \delta] \text{ it holds true that} \\ S \cap (y + \delta \overline{\mathbf{B}}) + tv \subset S + t\varepsilon \overline{\mathbf{B}} \end{array} \right\}.$$

The following definitions are from [15]:

Definition 2.1. Let S be a closed subset of X and x_0 belong to S. We say that the bounded set $D_S(x_0)$ is a *uniform tangent set* to S at the point x_0 if for each $\varepsilon > 0$ there exists $\delta > 0$ such that for each $v \in D_S(x_0)$ and for each point $x \in S \cap (x_0 + \delta\overline{\mathbf{B}})$ one can find $\lambda > 0$ for which $S \cap (x + t(v + \varepsilon\overline{\mathbf{B}}))$ is non empty for each $t \in [0, \lambda]$. □

Definition 2.2. Let S be a closed subset of X and x_0 belong to S. We say that the bounded set $D_S(x_0)$ is a *sequence uniform tangent set* to S at the point x_0 if for each $\varepsilon > 0$ there exists $\delta > 0$ such that for each $v \in D_S(x_0)$ and for each point $x \in S \cap (x_0 + \delta\overline{\mathbf{B}})$ one can find a sequence of positive reals $t_m \to 0$ for which $S \cap (x + t_m(v + \varepsilon\overline{\mathbf{B}}))$ is non empty for each positive integer m. □

The next theorem is the main result from [3].

Theorem 2.3. *Let S be a closed subset of X and x_0 belong to S. The following conditions are equivalent*

(1) *$D_S(x_0)$ is a uniform tangent set to S at the point x_0;*

(2) *$D_S(x_0)$ is a sequence uniform tangent set to S at the point x_0;*

(3) *for each $\varepsilon > 0$ there exist $\delta > 0$ and $\lambda > 0$ such that for each $v \in D_S(x_0)$ and for each point $x \in S \cap (x_0 + \delta\overline{\mathbf{B}})$ the set $S \cap (x + t(v + \varepsilon\overline{\mathbf{B}}))$ is non empty for each $t \in [0, \lambda]$.*

The basic properties of uniform tangent sets are gathered in the next proposition taken from [2]:

Proposition 2.4. *Let S be a closed subset of X and let $x_0 \in S$. Let $D_S(x_0)$ be a uniform tangent set to S at the point x_0. Then, the following conditions hold true:*

(1) *the set $c\, D_S(x_0)$ is a uniform tangent set to S at x_0 for each fixed constant $c > 0$;*

(2) *if $D'_S(x_0) \subset D_S(x_0)$, then $D'_S(x_0)$ is a uniform tangent set to S at x_0;*

(3) *if $D'_S(x_0)$ is another uniform tangent set to S at x_0, then $D_S(x_0) \cup D'_S(x_0)$ is a uniform tangent set to S at x_0;*

(4) *the convex closed closure $\overline{\text{co}}\, D_S(x_0)$ of $D_S(x_0)$ is a uniform tangent set to S at x_0;*

(5) *if S is convex, then $(S - x_0) \cap M\overline{\mathbf{B}}$ is a uniform tangent set to S at x_0 for every $M > 0$.*

Next we remind the classical concept of compactly epi-Lipschitz sets in Banach spaces. It was introduced by J.M. Borwein and H.M. Strojwas in 1985 in [4] and it includes all finite-dimensional and all epi-Lipschitsian sets in Banach spaces. Since then, it has been an important notion in nonsmooth analysis and has been frequently used in qualification conditions for obtaining normal intersection properties and calculus rules concerning limiting Fréchet cones and subdifferentials (in Asplund spaces, cf. [20] and [21]) and G-cones and G-subdifferentials (in general Banach spaces, cf. [14] and [12]). Compactly epi-Lipschitz sets are called *massive* in [13]. Here is the corresponding

Definition 2.5. Let A be a closed subset of the Banach space X and $x_0 \in A$. We say that A is *compactly epi-Lipschitz (massive)* at x_0, if there exist $\varepsilon > 0$, $\delta > 0$ and a compact set $K \subset X$, such that for all $t \in [0, \delta]$ the following inclusion holds true: $A \cap (x_0 + \delta\overline{\mathbf{B}}) + t\varepsilon\overline{\mathbf{B}} \subset A + tK$. □

Next we formulate two sufficient conditions for tangential transversality which can be obtained from the main result of this paper in a uniform manner. Theorem 2.6 is taken from [1] and Proposition 2.7 can be found in [2].

Theorem 2.6. *Let A and B be closed subsets of the Banach space X and let $x_0 \in A \cap B$. Let A be massive (or compactly epi-Lipschitz) and $\widehat{T}_A(x_0) - \widehat{T}_B(x_0)$ be dense in X. Then A and B are tangentially transversal at x_0.*

Proposition 2.7. *Let A and B be closed subsets of the Banach space X and let $x_0 \in A \cap B$. If A and B are strongly tangentially transversal at x_0 (that is there exist $D_A(x_0)$ – uniform tangent set to A at the point x_0, $D_B(x_0)$ – uniform tangent set to the set B at the point x_0 and $\rho > 0$ such that $\rho\overline{\mathbf{B}} \subset \overline{co}\,(D_A(x_0) - D_B(x_0))$), then A and B are tangentially transversal at x_0.*

3. Main result

We first formulate a simpler version of our main result:

Theorem 3.1. *Let A and B be closed subsets of the Banach space X and let $x_0 \in A \cap B$. Assume that there exist $\varepsilon > 0$, $\delta > 0$ and:*

(i) *there exist bounded "ball covering" sets M_A, M_B such that $M_A - M_B$ is dense in $\varepsilon\overline{\boldsymbol{B}}$ and "correcting" sets U_A, U_B such that, whenever $t \in [0, \delta]$,*

$$A \cap (x_0 + \delta\overline{\boldsymbol{B}}) + tM_A \subset A + tU_A \text{ and } B \cap (x_0 + \delta\overline{\boldsymbol{B}}) + tM_B \subset B + tU_B;$$

(ii) *there exist uniform tangent sets D_A (to A at x_0) and D_B (to B at x_0) such that $D_A - D_B$ is dense in $U_A - U_B$.*

Then A and B are tangentially transversal at x_0.

If, in the above theorem, $M_B = U_B = \{\mathbf{0}\}$, our assumption would be reduced to the definition of A being compactly epi-Lipschitz, but with the difference that the compact set is replaced by a difference of two uniform tangent sets. Moreover, the "massiveness-like" property is split between the sets. The above theorem is a direct consequence of its "quantified" version below. For the statement of the next result we will need the notion of ε-density: we say that a set A is ε-dense in the set B, if for all $v \in B$ there is $u \in A$ such that $\|v - u\| < \varepsilon$.

Theorem 3.2. *Let A and B be closed subsets of the Banach space X and let $x_0 \in A \cap B$. Assume that there exist $\varepsilon > 0$, $\delta > 0$, $q_1 > 0$, $q_2 > 0$, such that $q_1 + q_2 < 1$ and:*

(i) *there exist bounded "ball covering" sets M_A and M_B such that $M_A - M_B$ is εq_1-dense in $\varepsilon\overline{\boldsymbol{B}}$ and "correcting" sets U_A, U_B such that, whenever $t \in [0, \delta]$,*

$$A \cap (x_0 + \delta\overline{\boldsymbol{B}}) + tM_A \subset A + tU_A \text{ and } B \cap (x_0 + \delta\overline{\boldsymbol{B}}) + tM_B \subset B + tU_B;$$

(ii) *there exist two bounded sets D_A and D_B such that $D_A - D_B$ is εq_2-dense in $U_A - U_B$ and they are "η-uniform" with $\eta := \varepsilon(1 - q_1 - q_2)/3$, i.e. for each $t \in [0, \delta]$*

$$A \cap (x_0 + \delta\overline{\boldsymbol{B}}) + tD_A \subset A + t\eta\overline{\boldsymbol{B}} \quad \text{and} \quad B \cap (x_0 + \delta\overline{\boldsymbol{B}}) + tD_B \subset B + t\eta\overline{\boldsymbol{B}}.$$

Then A and B are tangentially transversal at x_0.

Proof. From (ii) we see that $U_A - U_B$ is bounded, and hence each of the sets U_A and U_B is bounded as well. We set

$$N := \sup\{\|u\| \mid u \in U_A \cup U_B \cup D_A \cup D_B \cup M_A \cup M_B\} \quad \text{and} \quad \bar{\delta} := \frac{\delta}{1 + 2N}.$$

Let $x^A \in (x_0 + \bar{\delta}\overline{\mathbf{B}}) \cap A$ and $x^B \in (x_0 + \bar{\delta}\overline{\mathbf{B}}) \cap B$, $x_A \neq x_B$ be arbitrary. Let us fix $t \in \left(0, \min\left\{\bar{\delta}, \|x^A - x^B\|/\varepsilon\right\}\right)$ and let us set

$$v := -\frac{x^A - x^B}{\|x^A - x^B\|}.$$

Then $\|\varepsilon v\| = \varepsilon$. According to (i), there exist $m_A \in M_A$ and $m_B \in M_B$, such that $\|\varepsilon v - (m_A - m_B)\| \leq \varepsilon q_1$. Since $0 < t < \bar{\delta} \leq \delta$ and

$$x^A \in (x_0 + \bar{\delta}\overline{\mathbf{B}}) \cap A \subset (x_0 + \delta\overline{\mathbf{B}}) \cap A, \; x^B \in (x_0 + \bar{\delta}\overline{\mathbf{B}}) \cap B \subset (x_0 + \delta\overline{\mathbf{B}}) \cap B,$$

there exist $u_A \in U_A$ and $u_B \in U_B$ such that

$$\widetilde{x}^A := x^A + t(m_A - u_A) \in A \text{ and } \widetilde{x}^B := x^B + t(m_B - u_B) \in B.$$

Because $u_A \in U_A$, $u_B \in U_B$ and $D_A - D_B$ is εq_2-dense in $U_A - U_B$, we obtain that $\|(d^A - d^B) - (u_A - u_B)\| \leq \varepsilon q_2$ for some $d^A \in D_A$ and $d^B \in D_B$. We estimate

$$\|\widetilde{x}^A - x_0\| \leq \|x^A - x_0\| + t\|m_A - u_A\| \leq \bar{\delta} + \bar{\delta}(\|m_A\| + \|u_A\|\|) \leq \bar{\delta}(1 + 2N) = \delta.$$

Therefore there exists $w_A \in \eta\overline{\mathbf{B}}$ such that $\widetilde{x}^A + t(d^A - w_A) \in A$ and we obtain

$$\widetilde{x}^A + t(d^A - w_A) = x^A + t(m_A - u_A + d^A - w_A) \in A$$

and

$$\|m_A - u_A + d^A - w_A\| \leq \|m_A\| + \|u_A\| + \|d^A\| + \|w_A\| \leq 3N + \eta.$$

Analogously we obtain $w_B \in \eta\overline{\mathbf{B}}$ such that $x^B + t(m_B - u_B + d^B - w_B) \in B$ and $\|m_B - u_B + d^B - w_B\| \leq 3N + \eta$. Hence

$$\begin{aligned}
&\|(x^A + t(m_A - u_A + d^A - w_A)) - (x^B + t(m_B - u_B + d^B - w_B))\| \\
=&\left\| x^A - x^B - t\varepsilon \frac{x^A - x^B}{\|x^A - x^B\|} - t(\varepsilon v - (m_A - m_B)) - tw_A + tw_B + \right. \\
&\qquad\qquad\qquad\qquad \left. + t((d^A - d^B) - (u_A - u_B)) \right\| \\
\leq&\|x^A - x^B\| - t\varepsilon + t(\varepsilon q_1 + \eta + \eta + \varepsilon q_2) = \|x^A - x^B\| - t\eta.
\end{aligned}$$

This verifies the tangential transversality of the sets A and B at the point x_0 with constants $3N + \eta > 0$, $\bar{\delta} > 0$ and $\eta > 0$. $\square$

Remark 3.3. We would like to note that Proposition 2.7 can be considered as a particular case of Theorem 3.1. Indeed, let us set $M_A = U_A = D_A := D_A(x_0)$ and $M_B = U_B = D_B := D_B(x_0)$ and $\varepsilon := \rho$. Without loss of generality we may think that the sets $D_A(x_0)$ and $D_B(x_0)$ are convex. Then the conclusion follows trivially.

4. Applications of the main result

Below we formulate an abstract (infinite-dimensional) version of the well-known Aubin condition from [6] for the basic problem of the calculus of variations:

Definition 4.1. Let X and Y be Banach spaces and $f\colon X \times Y \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous function which has finite value at $(\overline{x}, \overline{y}) \in X \times Y$. It is said that f satisfies the *Aubin condition* at $(\overline{x}, \overline{y}, f(\overline{x}, \overline{y}))$ iff there exist positive reals $\overline{\delta} > 0$ and $K > 0$ such that for every $t \in [0, \overline{\delta}]$ we have the following inclusion:

$$\operatorname{epi} f \cap \left((\overline{x}, \overline{y}, f(\overline{x}, \overline{y})) + \overline{\delta} \cdot \overline{\mathbf{B}}_{X \times Y \times R}\right) + t\left(\overline{\mathbf{B}}_X, \mathbf{0}, 0\right) \subset$$

$$\subset \operatorname{epi} f + t\left(\mathbf{0}, K \cdot \overline{\mathbf{B}}_Y, K[-1, 1]\right).$$

The next theorem is the main motivation of our research:

Theorem 4.2. *Let X and Y be Banach spaces and $f\colon X \times Y \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous function which satisfies the Aubin condition at $(\overline{x}, \overline{y}, f(\overline{x}, \overline{y}))$. Let $L\colon Y \longrightarrow X$ be a compact linear operator and $S := \{(Ly, y) : y \in Y\}$. We assume that $\widehat{T}_{\operatorname{epi} f}(\overline{x}, \overline{y}, f(\overline{x}, \overline{y})) - S \times (-\infty, 0]$ is dense in $X \times Y \times R$. Then* $\operatorname{epi} f$ *and $S \times (-\infty, f(\overline{x}, \overline{y})]$ are tangentially transversal at $(\overline{x}, \overline{y}, f(\overline{x}, \overline{y}))$.*

Proof. We apply the main result to the sets $A := \operatorname{epi} f$ and $B := S \times (-\infty, f(\overline{x}, \overline{y})]$. The ball covering sets are

$$M_A := \left(\overline{\mathbf{B}}_X, \mathbf{0}, [-1, 1]\right) \text{ and } M_B := \{(Ly, y, 0) : \|y\| \leq 1\},$$

i.e. $M_A - M_B \supset \varepsilon \overline{\mathbf{B}}_{X\times Y\times R}$, where $\varepsilon = \dfrac{1}{1+\|L\|}$. Indeed, let $(x,y,r) \in \varepsilon \overline{\mathbf{B}}_{X\times Y\times R}$ be arbitrary. Then

$$(x,y,r) = (x - Ly, \mathbf{0}, r) - (-Ly, -y, 0)$$

and $(L(-y), -y, 0) \in M_B$, because $\|y\| \le \varepsilon \le 1$. Moreover, $(x - Ly, \mathbf{0}, r) \in M_A$ because $|r| \le \varepsilon \le 1$ and

$$\|x - Ly\| \le \|x\| + \|L\| \cdot \|y\| \le (1 + \|L\|)\varepsilon = 1.$$

The correcting sets are

$$U_A := \left(\mathbf{0}, K \cdot \overline{\mathbf{B}}_Y, (1+K)[-1,1]\right) \text{ and } U_B := \{(\mathbf{0}, \mathbf{0}, 0)\}.$$

The Aubin condition implies

$$A \cap \left((\overline{x}, \overline{y}, f(\overline{x}, \overline{y})) + \overline{\delta} \cdot \overline{\mathbf{B}}_{X\times Y\times R}\right) + tM_A \subset A + tU_A \text{ for every } t \in [0, \overline{\delta}].$$

The fact that S is a linear space implies that

$$B \cap \left((\overline{x}, \overline{y}, f(\overline{x}, \overline{y})) + \overline{\delta} \cdot \overline{\mathbf{B}}_{X\times Y\times R}\right) + tM_B \subset B + tU_B \text{ for every } t \in [0, \overline{\delta}].$$

Now we have to cover $U_A - U_B = U_A$ (with some accuracy $\eta > 0$) by the difference of two uniform tangent sets to A and B, respectively. To this end, we fix an arbitrary $\eta > 0$ (sufficiently small) and we consider the set

$$C := \left(K \cdot L\left(\overline{\mathbf{B}}_Y\right), \mathbf{0}, (1+K)[-1,1]\right) .$$

The fact that L is a compact linear operator implies that C is totally bounded. Now the density of $\widehat{T}_{\operatorname{epi} f}(\overline{x}, \overline{y}, f(\overline{x}, \overline{y})) - S \times (-\infty, 0]$ implies the existence of a finite η-net for C consisting of elements of this dense difference. Therefore there exist finite sets $F \subset \widehat{T}_{\operatorname{epi} f}(\overline{x}, \overline{y}, f(\overline{x}, \overline{y}))$ and $G \subset S \times (-\infty, 0]$ such that

$$(F - G) + \eta \overline{\mathbf{B}}_{X\times Y\times R} \supset C .$$

Let $(\mathbf{0}, y, r)$ be an arbitrary element of U_A. Then

$$(\mathbf{0}, y, r) = (Ly, y, 0) + (-Ly, \mathbf{0}, r) = (-Ly, \mathbf{0}, r) - (-Ly, -y, 0) .$$

Therefore

$$U_A \subset C - \left(S \cap \left(K \cdot \max\{1, \|L\|\} \cdot \overline{\mathbf{B}}_{X\times Y}\right), 0\right) .$$

Using above inclusion and the η-net for C we obtain that

$$\begin{aligned} U_A &\subset (F - G) + \eta \overline{\mathbf{B}}_{X\times Y\times R} - \left(S \cap \left(K \cdot \max\{1, \|L\|\} \cdot \overline{\mathbf{B}}_{X\times Y}\right), 0\right) \\ &= F - \left(G + \left(S \cap \left(K \cdot \max\{1, \|L\|\} \cdot \overline{\mathbf{B}}_{X\times Y}\right), 0\right)\right) + \eta \overline{\mathbf{B}}_{X\times Y\times R} . \end{aligned}$$

It remains to notice that $D_A := F$ is a uniform tangent set to A at $(\overline{x}, \overline{y}, f(\overline{x}, \overline{y}))$ (it is a finite subset of the respective Clarke tangent cone) and $D_B := G + \left(S \cap \left(K \cdot \max\{1, \|L\|\} \cdot \overline{\mathbf{B}}_{X\times Y}\right), 0\right)$ is a uniform tangent set to B at $(\overline{x}, \overline{y}, f(\overline{x}, \overline{y}))$. Therefore, the assumptions of Theorem 3.2 hold true, and hence the sets A and B are tangentially transversal. □

The next corollary follows directly from Theorem 1.3 and Theorem 4.2:

Corollary 4.3. *Let X and Y be Banach spaces. We consider the optimization problem*

$$f(x,y) \to \min \quad \textit{subject to} \quad (x,y) \in S\ ,$$

where $f\colon X \times Y \longrightarrow \mathbb{R} \cup \{+\infty\}$ is lower semicontinuous, proper and satisfies the Aubin condition at $(\overline{x}, \overline{y}, f(\overline{x}, \overline{y}))$ and $S := \{(Ly, y) : y \in Y\}$, where $L\colon Y \longrightarrow X$ is a compact linear operator. Let $(\overline{x}, \overline{y})$ be a solution of the above problem.

Then there exists a triple $(\xi, \eta, \zeta) \in X^* \times Y^* \times \mathbb{R}$ such that

(i) $(\xi, \eta, \zeta) \neq (\mathbf{0}, \mathbf{0}, 0)$;

(ii) $\zeta \in \{0, 1\}$;

(iii) $\langle \xi, Ly \rangle + \langle \eta, y \rangle = 0$ *for every* $y \in Y$;

(iv) $\langle \xi, u \rangle + \langle \eta, v \rangle + \zeta w \geq 0$ *for every* $(u, v, w) \in \widehat{T}_{epif}(\overline{x}, \overline{y}, f(\overline{x}, \overline{y}))$.

Now we turn to an extension of the notion of compactly epi-Lipschitz (massive) sets (cf. Definition 2.5). The main difference with respect to the classical one is that we speak about "massiveness" of two sets as a pair.

Definition 4.4. Let A and B be closed subsets of the Banach space X and $x_0 \in A \cap B$. We say that A and B are *jointly massive* at x_0 if there exist $\varepsilon > 0$, $\bar{\delta} > 0$, bounded sets $M_A \subset X$, $M_B \subset X$ and a compact set $K \subset X$ such that:

(i) $\varepsilon \overline{\mathbf{B}}_X \subset \overline{M_A - M_B}$;

(ii) $A \cap (x_0 + \bar{\delta}\overline{\mathbf{B}}) + tM_A \subset A + tK$ and $B \cap (x_0 + \bar{\delta}\overline{\mathbf{B}}) + tM_B \subset B + tK$ whenever $t \in [0, \bar{\delta}]$. □

Clearly if the sets A and B are closed, $x_0 \in A \cap B$ and A is massive at x_0, then A and B are jointly massive at x_0. Indeed, let $A \cap (x_0 + \bar{\delta}\overline{\mathbf{B}}) + t\varepsilon\overline{\mathbf{B}}_X \subset A + tK$ for some compact set K. Setting $M_A := \varepsilon\overline{\mathbf{B}}_X$, $M_B := \{\mathbf{0}\}$ and $\tilde{K} := K \cup \{\mathbf{0}\}$, the above written definition holds true because $\tilde{K}$ is compact as well.

The next assertion is a direct generalization of Theorem 4.3 of [1].

Proposition 4.5. *Let A and B be jointly massive at x_0 and $\widehat{T}_A(x_0) - \widehat{T}_B(x_0)$ be dense in X. Then A and B are tangentially transversal at x_0.*

Proof. Let ε and $\bar{\delta}$ are the constants from (i) in Definition 4.4. We put $U_A = U_B = K$. From (i) we see that for each $q_1 > 0$ the assumption (i) of Theorem 3.2 holds true. We fix arbitrary $q_1 > 0$ and $q_2 > 0$ with $q_1 + q_2 < 1$. Because $K - K$ is compact, there is a finite $\varepsilon q_2/2$-net for $K - K$, i.e. a set $F = \{z_1, \dots, z_n\}$, such that for any $z \in K - K$ there is $i \in \{1, \dots, n\}$ with $\|z_i - z\| \leq \varepsilon q_2/2$. Since $\widehat{T}_A(x_0) - \widehat{T}_B(x_0)$ is dense in X, it is dense in F. Thus for any $i \in \{1, \dots, n\}$, one finds $d_i^A \in \widehat{T}_A(x_0)$ and $d_i^B \in \widehat{T}_B(x_0)$, such that $\|d_i^A - d_i^B - z_i\| \leq \varepsilon q_2/2$. So for any $z \in K - K$, there is $i \in \{1, \dots, n\}$, such that

$$\|d_i^A - d_i^B - z\| \leq \|d_i^A - d_i^B - z_i\| + \|z_i - z\| < \varepsilon q_2/2 + \varepsilon q_2/2 = \varepsilon q_2.$$

Therefore, if $D_A := \{d_i^A\}_{i=1}^n$ and $D_B := \{d_i^B\}_{i=1}^n$, then $D_A - D_B$ is εq_2-dense in $U_A - U_B$. As the set D_A (D_B, respectively) is a finite subset of $\widehat{T}_A(x_0)$ (of $\widehat{T}_B(x_0)$, respectively), it is a uniform tangent set to A (to B, respectively) at x_0. In particular, for $\eta = \varepsilon(1-q_1-q_2)/3 > 0$ there exits $\widetilde{\delta}$ such that for each $t \in [0, \widetilde{\delta}]$

$$A \cap (x_0 + \widetilde{\delta}\overline{\mathbf{B}}) + tD_A \subset A + t\eta\overline{\mathbf{B}} \text{ and } B \cap (x_0 + \widetilde{\delta}\overline{\mathbf{B}}) + tD_B \subset B + t\eta\overline{\mathbf{B}}.$$

Now (i) and (ii) from Theorem 3.2 are fulfilled with ε and $\delta = \min\{\bar{\delta}, \widetilde{\delta}\}$. □

The next corrollary follows directly from Theorem 1.3 and Proposition 4.5:

Corollary 4.6. *Let us consider the optimization problem*

$$f(x) \to \min \quad \textit{subject to} \quad x \in S \ ,$$

where $f\colon X \longrightarrow \mathbb{R} \cup \{+\infty\}$ *is lower semicontinuous and proper, and* S *is a closed subset of the Banach space* X*. Let* $epi f$ *and* $S \times (-\infty, f(x_0)]$ *be jointly massive at the point* $(x_0, f(x_0))$*. Then there exists a pair* $(\xi, \eta) \in X^* \times \mathbb{R}$ *such that*

(i) $(\xi, \eta) \neq (\mathbf{0}, 0)$;

(ii) $\eta \in \{0, 1\}$;

(iii) $\langle \xi, v \rangle \ \leq 0$ *for every* $v \in \widehat{T}_S(x_0)$;

(iv) $\langle \xi, w \rangle + \eta s \geq 0$ *for every* $(w, s) \in \widehat{T}_{epif}(x_0, f(x_0))$.

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