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Interval-Decomposability of Continuous Functions, and Convexity

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Dedicated to Alexander Ioffe on the occasion of his 80th birthday.

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A generalization of the notion of a decomposable set of functions is offered. It applies, in particular, to continuous functions which are not differentiable. Following the classical analysis of decomposable families of integrable functions, it is shown that many convex analysis features hold also in the general framework.

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1. Introduction

Decomposability of a family of functions has been recognized a long time ago as a substitute for convexity in the space of integrable functions. Indeed, many properties of convex sets, and features of convex analysis, are shared by decomposable sets of functions, see Olech [15]. The notion of a decomposable family of functions is rooted in the fundamental work of Rockafellar [17], and the seminal analysis of Antosiewicz and Cellina [1]. These contributions were followed by many authors, developing the theoretical scope and enriching the arsenal of applications of the notion. See, e.g., Anza Hafza and Mandallena [2] and Bouchitté and Valadier [8], for applications of Rockafellar's findings to convex functionals. An account of the theory that emerged from the Antosiewicz and Cellina's paper, its history and its applications, in particular in respect to fixed point properties, selections, and approximations of set-valued maps with decomposable values, is offered in Fryszkowski [9].

Recall that a family $\mathcal{F}$ of measurable functions from a measure space $(S, \mathcal{B}, \mu)$ to, say, the n-dimensional space R^n, is *decomposable*, if whenever $f_1, ..., f_k$ are in $\mathcal{F}$, and $(B_1, ..., B_k)$ is a partition of S with B_i in $\mathcal{B}$, then

$$I_{B_1} f_1 + \cdots + I_{B_k} f_k \tag{1}$$

is also in $\mathcal{F}$. Here I_B is the indicator function of B, namely, $I_B(x) = 1$ when $x \in B$ and $I_B(x) = 0$ otherwise.

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While the summation in (1) resembles the operation of taking a convex combination of the functions involved, it is different. It would be a convex combination of the functions if the coefficients, here I_{B_i}, were non-negative constants, summing up to 1. The expression (1), however, is related to swapping, measurably, among the functions.

Recently, a generalization of the decomposability notion, motivated by an application, has been employed by the author in [4]. The mathematical framework of the application is an optimization of integrals of a continuous map, with respect to a family of probability measures. The decomposability then applies to a family of measures. It emerged in a natural way in the modeling of optimal allocation problems in mathematical economics, and was employed in [4] in generalizing models used in, e.g., Aumann and Perles [6], Berliocchi and Lasry [7], Ioffe [11], and many others.

In the present paper we examine an abstract version of the decomposability suggested in [4]. The sets in question are sets of continuous maps on the unit interval, reflecting, roughly, the integrals of the functions in the classical decomposable sets. The definition, however, applies to general continuous functions, which may not be represented as integrals. In turn, we have to restrict the measurable sets in (1) to be unions of intervals, hence the term interval-decomposability. The purpose of the abstraction is twofold: First, to unravel the role played in the available theory by properties of continuously differentiable functions. Indeed, we find that many of the convexity consequences hold for merely continuous functions. Second, to allow for new applications in a wider scope.

The interval-decomposable sets are introduced in the next section, along with some examples and comments, including comments on the attainable set, and the structure of decomposed combinations and the decomposed hull. In Section 3 we verify a Lyapunov Convexity Theorem type result, namely, the convexity of the interval-range of a continuous function. Convexity and closure properties, including a generalization of the convexity of the Aumann integral (for the latter see Aumann [5]) are examined in Section 4.

The extremal functions of an interval-decomposable family, and the manner they generate the points in the attainable set of the family, are presented in Section 5. In the closing section we examine two approximation problems. First we examine how the convex hull of the set of functions is approximated by the interval-decomposed hull. Then we analyze the process of the construction of the decomposed hull, and show that it establishes approximations to the attainable set. We conclude with a remark on why the analysis offered in the paper is not of much help (yet?) when fixed-point properties are sought.

2. Interval-decomposable sets

A generalization of the notion of a decomposable set displayed in (1), is introduced now. First we need some terminology.

Definition 2.1. Let X be a Banach space. Let $\mathcal{H}$ be a family of functions

$$h(t)\colon [0,1] \to X, \tag{2}$$

in $C([0,1],X)$ (i.e. the continuous mappings from $[0,1]$ to X), each satisfying $h(0) = 0$. We call such a set an *integrated set*. The set $A = \{h(1) : h(\cdot) \in \mathcal{H}\}$ in X will be referred to as the *attainable set* of $\mathcal{H}$. At times we shall be interested in $A(s) = \{h(s) : h(\cdot) \in \mathcal{H}\}$, namely the points attained at the time s (thus $A = A(1)$).

An example of an integrated set is the set of integrals $h(\cdot)$ determined by $h(t) = \int_0^t f(s)ds$, when $f(\cdot)$ in a set of integrable functions on $[0,1]$, as described in the Introduction, for the measure space $[0,1]$. Hence the term 'integrated set'. The term 'attainable set' is chosen since it arises in many applications related to Control Theory and in Mathematical Economics, as explained later. The attainable set A is also the analog of the Aumann integral, see Example 2.4 below.

We shall use the terminology and notations as follows. Given a function $h(\cdot)$ in an integrated set, and a sub-interval I in $[0,1]$, say $[t_1,t_2]$ (but it could also be open or half open), we denote by $h_I(\cdot)$ the restriction of $h(\cdot)$ to I. If E is a disjoint union of intervals in $[0,1]$, we denote by $h_E(\cdot)$ the restriction of $h(\cdot)$ to E.

Let $h_1(\cdot)$ and $h_2(\cdot)$ be two functions in an integrated set, and let $I = [t_1,t_2]$ be an interval in $[0,1]$. The result of *swapping* $h_{1,I}(\cdot)$ in $h_1(\cdot)$, for $h_{2,I}(\cdot)$, is the mapping, say $h(\cdot)$, obtained by replacing $h_1(t)$ with $h_2(t)$ on I, shifted, however, on I and on $[t_2,1]$ so to make the outcome continuous. Formally, $h(t)$ is defined by $h(t) = h_1(t)$ when $t \le t_1$, by $h(t) = h_2(t) + (h_1(t_1) - h_2(t_1))$ when $t_1 < t \le t_2$, and by $h(t) = h_1(t) + (h_1(t_1) - h_2(t_1)) + (h_2(t_2) - h_1(t_2))$ when $t_2 < t$.

Similarly, if E is a finite union of disjoint intervals (closed, half closed or open) in $[0,1]$, and $h_{1,E}(\cdot)$ is the restriction of $h_1(\cdot)$ to E, then swapping $h_{1,E}(\cdot)$ for $h_{2,E}(\cdot)$ is the continuous mapping, say $h(\cdot)$, obtained by replacing $h_1(t)$ by $h_2(t)$ on E, shifted, however, so to make the outcome continuous (namely, applying the swapping inductively each time on one interval).

Now, let $h_1(\cdot), ..., h_k(\cdot)$ be a finite sequence of functions in an integrated set. Let $E_1, ..., E_k$ be a finite partition of $[0,1]$ such that each E_i is a finite union of disjoint intervals (each closed or half closed or open). The *interval-decomposed combination* of the two sequences is the outcome in $C([0,1],X)$ obtained by swapping $h_{1,E_j}(\cdot)$ for $h_{j,E_j}(\cdot)$, say inductively, for $j = 2, ..., k$.

Definition 2.2. Let $\mathcal{H}$ be an integrated set. The *interval-decomposed hull* of $\mathcal{H}$ is the family of continuous functions obtained as interval-decomposed combinations of elements in $\mathcal{H}$.

In the sequel, when it is clear that we refer to an interval operation (e.g, in interval-decomposition, etc.), we may suppress the prefix 'interval'.

With the terminology and definitions introduced earlier, we are ready to display the promised generalization.

Definition 2.3. Let $\mathcal{H}$ be an integrated set. We say that it is *interval-decomposable* if whenever $h_1(\cdot), ..., h_k(\cdot)$ are in $\mathcal{H}$, and $E_1, ..., E_k$ is a partition of $[0,1]$ such that each E_i is a finite union of disjoint intervals (each closed or half closed or open), then the decomposed combination of the sequence is in $\mathcal{H}$.

We offer now some comments on, and examples of, interval-decomposable families. The first example draws the similarity between the definition in this section and the classical one recalled in the Introduction.

Example 2.4. Let $\mathcal{F}$ be a family of L_1 functions from the unit interval $[0,1]$ with the Lebesgue measure ds, to R^n. Consider the collection $\mathcal{H}$ of continuous functions from $[0,1]$ into R^n given by $h(t) = \int_0^t f(s)ds$, for $f(\cdot)$ in $\mathcal{F}$. Then, clearly, $\mathcal{H}$ is an integrated set according to Definition 2.1. If $\mathcal{F}$ is decomposable in the sense described in the Introduction, then, clearly, $\mathcal{H}$ is interval-decomposable according to Definition 2.3.

The converse statement, however, holds only under an additional condition. Specifically, let $\mathcal{H}$ be an integrated set according to Definition 2.1, where the values of the functions are in R^n. Suppose that each $h(\cdot)$ in $\mathcal{H}$ is absolutely continuous, in particular, it is the integral of its almost everywhere derivative. Even if $\mathcal{H}$ is interval-decomposable according to Definition 2.3, the family $\mathcal{F}$ of a.e. derivatives may not be decomposable according to the classical definition. Indeed, the decomposability in Definition 2.3 only requires swapping on unions of intervals, while (1) allows any measurable partition. If, however, $\mathcal{H}$ is known to be closed under the $W^{1,1}$-topology (in particular, the family of derivatives is closed in L_1), then the decomposability of $\mathcal{F}$ is equivalent to the interval-decomposability of $\mathcal{H}$. Indeed, the decomposed combinations in (1) can be approximated in L_1 by a sequence of decomposed combinations of interval-decompositions.

In case where the integrated set $\mathcal{H}$ is the set of integrals of a family $\mathcal{F}$, and the latter is the family of selections of a set-valued map, then the attainable set of $\mathcal{H}$ coincides with the *Aumann integral* of $\mathcal{F}$. □

Remark 2.5. The unit interval is used in the previous example, while a general measure space was allowed in (1). This is not a major restriction as any separable probability space which is atom-less, is isomorphic to the unit interval. The specific isomorphism which is used, may affect the family of unions of intervals which takes part in the swapping. Given, however, a partition as in (1), an isomorphism can be found that sends each measurable set B_i in the partition into an interval in $[0,1]$.

Example 2.6. As a simple demonstration of interval-decomposability take a set C in the Banach space X and consider $\mathcal{H}$ to be the set of functions from the unit interval into X given by $h(t) = tx$, this for $x \in C$. This is clearly an integrated set. It is not interval-decomposable. Enlarge now the set by including the piecewise continuous functions obtained by having a partition of the unit interval into a number of sub-intervals, say $[t_i, t_{i+1})$, and on each such sub-interval let the function take the form $h(s) = h(t_i) + (s - t_i)x$, for some $x \in C$. Namely, consider the decomposed combination of the linear functions, determined by the partition. The resulting set is interval-decomposable. The attainable set of the latter family is the set of all finite convex combinations of vectors in C. If C is a compact set in R^n, the aforementioned attainable set is the convex hull of C. Note, however, that the family of decomposed combinations in this example is not convex in the space of continuous functions. It is easy to see that the closure of this family is convex, namely, the set of interval-decomposed combinations in this example forms a dense subset of a convex closed set. A general result in this respect is offered in the closing section. □

Example 2.7. We describe now the example (worked out in [4]) that motivated the present study. A measurable space (say a separable metric space S with its Borel field) is given. It is the set of producers. The integrated set consists of mappings $\mu(t)$ defined on $[0,1]$ where for each t the value $\mu(t)$ is a non-negative measure on S, such that $\mu(t)(S) = t$, and such that $\mu(t_1) < \mu(t_2)$ when $t_1 < t_2$ (here $<$ means that $\mu(t_2) - \mu(t_1)$ is a positive measure). The attainable set consists then of probability measures, the interpretation of which is that a decision maker can use these measures (on the producers' space) for the allocation of prescribed resources. Swapping among the measures means that the decision maker may use a mixture of portions of the probability measures, restricted, however, to segments of the continuous functions in the integrated set. Thus, the structure of the integrated set reflects restrictions imposed on the decision maker. In this model each element in the integrated set is absolutely continuous with respect to the Lebesgue measure on the unit interval, hence can be disintegrated in the space of measure-valued maps, resembling the case described in Example 2.4, with 'derivatives' being probability measure-valued mappings. □

For the next example recall the Cantor function, namely, the continuous and non-decreasing function $c(t)\colon [0,1] \to [0,1]$, with $c(0) = 0$ and $c(1) = 1$, which is constant on each interval in a sequence of intervals with total length equal to 1. In particular, $c(\cdot)$ is differentiable almost everywhere, but its derivative is equal to 0 almost everywhere. In particular, $c(\cdot)$ is not the integral of its a.e. derivative.

Example 2.8. Let $\mathcal{H}$ consist of the interval-decomposable hull of two real-valued functions: The Cantor function $c(\cdot)$ and the function which is identically equal to 0. For every t, the attainable set $A(t)$ is the interval $[0, c(t)]$. Indeed, any point $r \in [0, c(t)]$ is attained by identifying a point s where $c(s) = r$ (it exists by continuity), then swapping $c(\cdot)$ on $[s,1]$, for the 0-function on $[s,1]$. The integrated set in this example cannot be presented as a result of integration of L_1 functions. □

Example 2.9. Consider two real-valued functions as follows. The first is the mapping $h_1(t) = 0$. The second is defined by $r(0) = 0$, $r(1) = 1$, $r(\frac{1}{2k}) = \frac{1}{2k}$ and $r(\frac{1}{2k+1}) = \frac{-1}{2k+1}$, this for $k = 1, 2, ...$, and on the rest of the interval the function $r(\cdot)$ connects the values defined so far with linear segments. The construction is such that the graph of $r(\cdot)$ has an infinite length. This implies that although $A(0)$ is the origin, the attainable set $A(t)$, for any $t > 0$, of the interval-decomposable hull of the two functions, is the entire real line. Indeed, arbitrarily large values can be attained arbitrarily close to the origin, by swapping $r(\cdot)$ on sub-intervals of $[\frac{1}{2k}, \frac{1}{2k-1}]$ where $r(t) < 0$, with the constant-valued $h_1(\cdot)$, and likewise for arbitrarily negative values. □

The previous example indicates that the attainable set $A(t)$ of the decomposed hull of, even, two continuous mappings, may not vary continuously, as a set-valued map. Indeed, in the example, the attainable set becomes unbounded right after $t = 0$, while at $t = 0$ it is bounded. Changing $r(\cdot)$ to $r_1(\cdot)$ given by $r_1(t) = r(t-t_0)$ for $t > t_0$, and $r_1(t) = 0$ otherwise, will result in a bounded attainable set for $t \le t_0$ and an unbounded attainable set for $t > t_0$. It is also easy to come up with an example where $A(t)$ is bounded for, say, $t < t_0$ and unbounded for $t \ge t_0$. Just change $r(t)$ to $r_2(t) = r(t_0 - t)$, for $t_1 \le t \le t_0$, with $r_2(t) = 0$ for $t \ge t_0$ and $t \le t_1$, where t_1 chosen such that $r_2(t_1)$ previously defined is equal to 0. The next result verifies that these are the only two ways of discontinuity. To this end recall the notion of *equi-uniform continuity*, that is, a family $\mathcal{H}$ of continuous functions, say, from $[0, 1]$ to a Banach space X, is equi-uniformly continuous if for every $\varepsilon > 0$ there is a $\delta > 0$ such that $|t - s| \le \delta$ in $[0, 1]$, implies that $\|h(t) - h(s)\| \le \varepsilon$ in X for each $h(\cdot)$ in $\mathcal{H}$.

Proposition 2.10. *Let $\mathcal{H}$ be an integrated set of mappings from $[0, 1]$ into R^n which is interval-decomposable. Suppose that the attainable set $A = A(1)$ is bounded. Then $A(t)$ is bounded for every $t \in [0, 1]$. Furthermore, the functions $h(\cdot)$ in $\mathcal{H}$ are then equi-uniformly continuous. Consequently, the set-valued mapping $A(t)$ is then continuous in the Hausdorff distance.*

Proof. Suppose that $A(t_0)$, for some $t_0 < 1$, is unbounded. Let $h_0(\cdot)$ be a fixed function in $\mathcal{H}$. Swapping $h(\cdot)$ for $h_0(\cdot)$ on $[t_0, 1]$, this for all elements in $\mathcal{H}$, results in an unbounded $A(1)$. Thus, if $A(1)$ is bounded, them $A(t)$ is bounded for all t.

Suppose now that $\mathcal{H}$ is not equi-uniformly continuous on $[0, 1]$. Then an $\varepsilon > 0$ exists, and a sequence of functions $h_i(\cdot)$ in $\mathcal{H}$, and a sequence of intervals $[a_i, b_i]$ in the unit interval, such that $b_i - a_i$ tends to 0, while $\|h_i(b_i) - h_i(a_i))\| \ge \varepsilon$. We may assume, by possibly taking a sub-sequence, that a_i converges, say to t_0, and it is either increasing or decreasing toward t_0, and that the unit vectors $(h_i(b_i) - h_i(a_i))\|h_i(b_i) - h_i(a_i)\|^{-1}$ converge, say to v. We fix now a function $h_0(\cdot)$ in $\mathcal{H}$. By successively swapping $h_0(\cdot)$ on $[a_i, b_i]$ for $h_i(\cdot)$, with $b_i - a_i$ small enough, we get points in the attainable set $A(t)$ for t arbitrarily close to t_0, with absolute value arbitrarily large. This implies that either $A(t_0)$ is unbounded (in case a_i is increasing toward t_0), or $A(t)$ is unbounded for t arbitrarily close to t_0 (in case a_i is decreasing toward t_0). This, in view of the first assertion, contradicts the assumption that $A(1)$ is bounded. The last claim is a direct consequence of the equi-uniform continuity. □

It is worth pointing out, for the sake of possible applications, that if $\mathcal{H}$ is an integrated set of functions with values in a space X, and $L\colon X \to Y$, say $Y = R^n$, is a linear continuous mapping, then the family $\mathcal{G}$ formed by $g(t) = Lh(t)$ for $h(\cdot) \in \mathcal{H}$, is an integrated set of functions into Y, and the interval-decomposed hull of $\mathcal{G}$ is obtained by applying the linear mapping L. This becomes useful (it served well in [4]) as many of our results below apply when the range is R^n.

3. An interval-Lyapunov Convexity Theorem

Our result concerning the convexity of the attainable set is presented in the next section. It is a straightforward implication of a generalized Lyapunov Convexity Theorem (rooted in Halkin [10]), which we state and verify in this section. On Lyapunov Convexity Theorem and its generalizations see Olech [16].

We introduce the following notation. Let $h(\cdot)$ be a continuous function on $[0,1]$ with values in, say, R^n, such that $h(0) = 0$. For an interval $I = [\alpha, \beta]$ we write $h(I) = h(\beta) - h(\alpha)$. For a set E which is a disjoint union of intervals I_j, we write $h(E) = h(I_1) + ... + h(I_k)$. The continuity assures that it does not matter if the intervals in question are closed, open, or half open, and in the sequel we refer to them simply as intervals. Also, dividing an interval I into sub-intervals will not change the h-value of the union. The set of values $h(E)$ where E runs over all finite union of disjoint sub-intervals of $[0,1]$, is referred to as the *interval-range* of $h(\cdot)$.

Theorem 3.1. *Let $h(t)\colon [0,1] \to R^n$ be continuous. The interval-range of $h(\cdot)$ is a convex set.*

The result is a generalization of the Lyapunov Convexity Theorem in two respects. First, it refers to continuous functions, rather than absolutely continuous functions. Second, the result addresses the interval-range, rather than the range, namely, (in the case when $h(\cdot)$ is the integral of its a.e. derivative) the collection of integrals over all measurable subsets. The convexity of the interval-range of an atom-less vector measure on $[0,1]$ was established by Halkin [10]. Our proof follows Halkin's approach, adapted, however, to the present framework, namely, continuous functions (in previous publications I announced, without proof, that Halkin's proof can be modified so to cover the continuous case).

We start with two auxiliary observations.

Lemma 3.2. *Let $v_1, ..., v_m$ be a sequence of vectors in R^n and let $v = v_1 + ... + v_m$ be their sum. Suppose that $\|v_j\| \le 1$, for $j = 1, ..., m$. Then the vectors v_j can be rearranged, i.e., there is a permutation $i(j)$ of the indices, such that for each k*

$$\|(v_{i_1} + ... + v_{i_k}) - \frac{k}{m} v\| \le C, \tag{3}$$

(here i_j is the $j-th$ term after the permutation) where C is a universal constant, that depends only on the dimension n of R^n.

Proof. When $v = 0$ the result goes back to Steinitz [19] and is known in the literature as the Steinitz Lemma (not to be confused with many other key lemmas of Steinitz), or as the Polygonal Confinement Lemma. A nice proof can be found in Rosenthal [18]. The inequality in (3) follows then by applying the Steinitz Lemma to the vectors $v_i - \frac{1}{m} v$, whose sum is indeed equal to 0. □

The homogeneity of the terms in the previous lemma implies that if $\|v_i\| \leq \theta$, for each i, then in (3) the estimate becomes θC.

Lemma 3.3. *Let $h(t)\colon [0,1] \to R^n$ be continuous and let E be a finite union of disjoint intervals in $[0,1]$. Let $\varepsilon > 0$ be given. Then the intervals forming E can be divided into, say m, sub-intervals, then rearranged, resulting in, say, the sequence $I_1, ..., I_m$, such that for every k*

$$\|(h(I_1) + ... + h(I_k)) - \frac{k}{m} h(E)\| \leq \varepsilon. \tag{4}$$

Furthermore, once m is determined, each of the m sub-intervals $I_j = [\alpha_j, \beta_j]$, can be further cut into tiny intervals, say $I_{j,l}$ for $l = 1, ..., m(j)$, that can be rearranged (the boundaries of these sub-intervals can be ignored) such that the following estimate holds. Denote by E_j the interval I_j after the rearrangement. An interval $[s,t]$ in E_j is then a union of sub-interval $I_{j,i}$ contained in the interval $[s,t]$ (in E_j), and, possibly, sub-intervals of the intervals $I_{j,l}$ and $I_{j,k}$ to which s and t belong (in E_j). Then, for every interval $[s,t]$ in E_j

$$\|h([s,t]) - \frac{t-s}{\beta_j - \alpha_j} h(I_j)\| \leq \frac{\varepsilon}{m} \tag{5}$$

(notice that $h([s,t])$ is not equal to $h(t) - h(s)$, but it is the sum of the h-value of the intervals that form $[s,t]$ in E_j).

Proof. The first claim follows directly from Lemma 3.2. Indeed, when each of the intervals I_j is small, specifically, when $\|h(I_j)\| < \frac{\varepsilon}{C}$, they can be arranged as desired. The second claim also follows from the same lemma, together with the observation that the continuous function $h(\cdot)$ is uniformly continuous. For completeness, we display the estimates involved.

Once m is determined, each I_j can be cut into, say $m(j)$, tiny sub-intervals, say $I_{j,k}$, such that $\|h(I_{j,k})\| < \frac{\varepsilon}{4mC}$ for each k. By Lemma 3.2, these sub-intervals can be rearranged such that for each k

$$\|(h(I_{j,1}) + ... + h(I_{j,k})) - \frac{k}{m(j)} h(I_j)\| \leq \frac{\varepsilon}{4m}. \tag{6}$$

The latter inequality implies that

$$\|h([\alpha_{j,k}, \beta_{j,i}]) - \frac{\beta_{j.i} - \alpha_{j,k}}{\beta_j - \alpha_j} h(I_j)\| \leq \frac{\varepsilon}{2m}, \tag{7}$$

where $\alpha_{j,k}$ and $\beta_{j.i}$ are the beginning and the end of sub-intervals in the rearrangement E_j of I_j (and, again, recall that $h([\alpha_{j,k}, \beta_{j,i}])$ is the h-value of the intervals, in E_j, between $\alpha_{j,k}$ and $\beta_{j,i}$).

Now we invoke the uniform continuity. It implies that a $\delta > 0$ exists such that $|\beta - \alpha| < \delta$ implies that $\|h(\beta) - h(\alpha)\| < \frac{\varepsilon}{8m\|h(I_j)\|}$. Now, if the length of each of the sub-intervals $I_{j,k}$, in addition to being less than $\frac{\varepsilon}{4mC}$, is also less than δ, the desired inequality (5) follows easily from (7). This completes the proof. □

We now prove the aforementioned generalization of the Lyapunov Convexity Theorem.

Proof of Theorem 3.1. Let $\mathcal{R}$ denote the interval range of $h(\cdot)$. Denote by $co\mathcal{R}$ the convex hull of $\mathcal{R}$, and recall the Carathéodory Theorem, assuring that any point v in $co\mathcal{R}$ is a convex combination of at most $n+1$ points in $\mathcal{R}$. We split the discussion into two cases.

Case 1: Let v be in the interior of $co\mathcal{R}$. Then v is in the interior of the convex hull of $n+1$ points, say $V = \{v_1, ..., v_{n+1}\}$, each in $\mathcal{R}$. In particular, there is an $\varepsilon > 0$, such that the ball of radius, say, $4(n+1)\varepsilon$, around v, is in the interior of the convex hull coV of V.

Being in $\mathcal{R}$, each v_j is of the form $h(E_j)$ with E_j being a union of disjoint intervals in $[0,1]$. By Lemma 3.2, each E_j is the union of sub-intervals $I_{j,i}$, for $i = 1, ..., m(j)$, satisfying (4). Each of the $I_{j,i}$ can be further divided to sub-intervals that after rearrangement satisfy (5) (again, with $m = m(j)$). Clearly, the divisions of all E_j can be arranged (possibly by a further division) such that any two intervals in the entire collection are either identical or do not intersect.

The convex hull coV of V is represented as the collection of vectors

$$\lambda_1 v_1 + ... + \lambda_{n+1} v_{n+1}, \tag{8}$$

with $0 \le \lambda_j \le 1$ and $\lambda_1 + ... + \lambda_{n+1} = 1$.

We construct now a mapping from coV to itself, as follows. For each sub-interval $I_{j,i}$, say $[a,b]$, (here $j = 1, ..., n+1$ and $i = 1, ..., m(j)$), and each $(n+1)$-tuple $(\lambda_1, ..., \lambda_{n+1})$, we consider the partition of the interval determined by $a = \alpha_1 \le \alpha_2 \le ... \le \alpha_{n+1} = b$, such that $\alpha_{i+1} - \alpha_i = \lambda_{i+1}(b-a)$. Now define $E(\lambda_1, ..., \lambda_{n+1})$ to be the union of all intervals of the form $[\alpha_i, \alpha_{i+1}]$ of sub-intervals of E_j, this for $j = 1, ..., n+1$. (That is, E takes from E_j all the sub-intervals of intervals in E_j, of the form $[\alpha_j - \alpha_{j+1}]$.) In particular, since each E_j is a union of intervals $I_{j,k}$, the value $h(E(\lambda_1, ..., \lambda_{n+1}))$ is in the interval-range of $h(\cdot)$.

Clearly, the mapping $h(E(\lambda_1, ..., \lambda_{n+1}))$ is continuous in $(\lambda_1, ..., \lambda_{n+1})$, hence continuous in $\lambda_1 v_1 + ... + \lambda_{n+1} v_{n+1}$, which is the said convex hull coV. The estimates above imply that

$$\|h(E(\lambda_1, ..., \lambda_{n+1})) - (\lambda_1 v_1 + ... + \lambda_{n+1} v_{n+1})\| \le 2(n+1)\varepsilon. \tag{9}$$

Consider now the mapping

$$\Gamma(\lambda_1 v_1 + ... + \lambda_{n+1} v_{n+1}) = v - h(E(\lambda_1, ..., \lambda_{n+1})) + (\lambda_1 v_1 + ... + \lambda_{n+1} v_{n+1}). \tag{10}$$

The choice of ε implies that the continuous mapping Γ is from coV to itself. Hence it has a fixed point. This fixed point satisfies $v = h(E(\lambda_1, ..., \lambda_{n+1}))$, and completes the proof in the first case.

Case 2: Let v be on the boundary of $co\mathcal{R}$. Then a vector p exists such that $p \cdot v \ge p \cdot x$ for all $x \in co\mathcal{R}$ (the dot signifies scalar product). Let v_1 and v_2 be two vectors in $\mathcal{R}$ such that v is a convex combination of them. Let E_1 and E_2 be unions of disjoint intervals in $[0,1]$ such that $h(E_1) = v_1$ and $h(E_2) = v_2$. Then $p \cdot v_i = p \cdot v$ for $i = 1, 2$. In particular, whenever I is an interval in $E_1 \backslash E_2$, then $p \cdot h(I) = 0$ (otherwise we can add it to E_2 and get a higher p-value, or subtract it from E_1 and get a higher p-value). Consider now the family of all $h(E)$ such that E is a union of intervals within the union of $E_1 \backslash E_2$ and $E_2 \backslash E_1$. Denote the set of h-values of all these union of intervals by $\mathcal{R}_{n-1}$. It is included in the subspace of R^n determined by $p \cdot x = 0$. An induction argument implies that it is a convex set (the case $n = 1$ is trivial, while passing from $n-1$ to n is covered by Case 1). Then $h(E_1 \cup E_2) + \mathcal{R}_{n-1}$ is also convex, and it includes v_1 and v_2. This completes the proof. □

Remark 3.4. Unlike the case of the original Lyapunov Convexity Theorem, where the range is determined by a bounded vector measure, the range in the preceding result can be unbounded, and even when bounded, it may not be compact. The unboundedness is demonstrated in Example 2.9 (with respect to the interval-range of $r(\cdot)$). A simple example of a bounded, yet not compact, interval-range is the function $h(x,y) = (x, c(x))$ defined on $[0,1]$ with values in R^2, where $c(x)$ is the Cantor function (see also Example 2.8). Then the point $(0,1)$ is an extreme point of the closure of the interval-range of $h(\cdot)$, yet, it is not in the interval-range itself. Likewise for the point $(1,0)$. A similar example can be provided in the differentiable case. For instance, let $h(x,y) = (x, d(x))$ with $d(x)$ being differentiable and strictly increasing except for an infinite sequence of disjoint closed intervals on each of which it is constant.

4. The convexity and closure of the attainable set

The passage from the Lyapunov Theorem to the case of the convexity of the attainable set is standard, as follows.

Theorem 4.1. *Let $\mathcal{H}$ be an interval-decomposable integrated set of mappings from $[0,1]$ into R^n. Then its attainable set, namely $A = \{h(1) : h(\cdot) \in \mathcal{H}\}$, is a convex set.*

Proof. Let $h_1(\cdot)$ and $h_2(\cdot)$ be in $\mathcal{H}$. Consider the mapping into R^{2n} given by $h(t) = (h_1(t), h_2(t))$. It is continuous, hence its interval-range is convex. In particular, Theorem 3.1 implies that given $\alpha \in [0,1]$, there exists a union of disjoint intervals E, such that $h(E) = \alpha h(1)$. That is, $h_1(E) + h_2([0,1] \backslash E) = \alpha h_1(1) + (1-\alpha) h_2(1)$. Hence, if we denote by $h_3(\cdot)$ the result of swapping $h_1(\cdot)$ on $[0,1] \backslash E$ for $h_2(\cdot)$, then $h_3(1) = \alpha h_1(1) + (1-\alpha) h_2(1)$. This verifies the desired convexity of the attainable set. □

Remark 4.2. The preceding proof relies on the interval-Lyapunov Convexity Theorem 3.1, which in turn is a particular case of Theorem 4.1. Indeed, if along with the mapping $h(\cdot)$ in Theorem 3.1 we consider the mapping $h_0(\cdot)$ which is identically zero, and take the interval-decomposed hull of the two, then the resulting attainable set is identical to the interval-range of $h(\cdot)$.

The preceding result generalizes the convexity of the Aumann integral (see Example 2.4) in two ways. First, the continuous functions in $\mathcal{H}$ may not be differentiable, let alone they may not be the integral of their derivatives. Second, the swapping we require is only of a union of intervals in $[0, 1]$, and not necessarily general measurable sets. Also notice that to get the function $h_3(\cdot)$ with the desired equality $h_3(1) = \alpha h_1(1) + (1 - \alpha)h_2(1)$, it suffices to carry out swapping on a union of intervals, invoking only $h_1(\cdot)$ and $h_2(\cdot)$.

It is clear that the convexity established for the attainable set $A(1)$ is valid also for $A(t)$, for all $t \in [0, 1]$. This does not imply that the family $\mathcal{H}$ itself is convex. We deal with the possible convexity of $\mathcal{H}$ in the closing section.

As noted already in Remark 3.4, in the general case the attainable set may not be closed, even if bounded (unlike the case of the Aumann integral of a set-valued map with closed values in R^n). An interesting observation related to closedness is the following.

Theorem 4.3. *Let $\mathcal{H}$ be an interval-decomposable set of mappings from $[0, 1]$ into R^n. Let e in R^n be an extreme point of the closure of the attainable set $A(1)$. Then for every $\varepsilon > 0$ there is a $\delta > 0$ such that if $h_1(\cdot)$ and $h_2(\cdot)$ are in $\mathcal{H}$, and both $h_1(1)$ and $h_2(1)$ are in a δ-neighborhood of e, then $\|h_1(\cdot) - h_2(\cdot)\| < \varepsilon$ in the $C([0, 1], R^n)$ sup-norm. In particular, if $h_j(1)$ converges to e for a sequence $h_j(\cdot)$ in $\mathcal{H}$, then the sequence of functions is a Cauchy sequence in the sup-norm.*

Proof. To verify the claim suppose the contrary. Then an $\varepsilon > 0$ exists, and sequences $h_{1,i}(\cdot)$ and $h_{2,i}(\cdot)$ such that $h_{1,i}(1)$ and $h_{2,i}(1)$ both converge to e, yet $\|h_{1,i}(\cdot) - h_{2,i}(\cdot)\| \geq \varepsilon$. Without loss of generality, the latter inequality applies to one coordinate, say the first coordinate, of the sequences $h_{1,i}(\cdot)$ and $h_{2,i}(\cdot)$. Denote the first coordinate by the superscript 1. Let t_i be a point in $[0, 1]$ such that $|h^1_{1,i}(t_i) - h^1_{2,i}(t_i)| = \varepsilon$. Such a point exists by continuity. Let $g_{1,i}(\cdot)$ and $g_{2,i}(\cdot)$ be obtained by swapping $h_{1,i}(\cdot)$ on $[0, t_i]$ for $h_{2,i}(\cdot)$, and likewise, swapping $h_{2,i}(\cdot)$ on $[0, t_i]$ for $h_{1,i}(\cdot)$. The interval-decomposability implies that $g_{1,i}(\cdot)$ and $g_{2,i}(\cdot)$ are in $\mathcal{H}$. The definition of swapping (see Definition 2.2) implies that $\|g_{1,i}(1) - g_{2,i}(1)\| \geq \varepsilon$. On the other hand, $\frac{1}{2}(g_{1,i}(1) + g_{2,i}(1))$ converge to e. This, together with the convexity of the interval-range, imply that e is not an extreme point of the closure of $A(1)$. Thus the claim is verified. □

Theorem 4.4. *Let $\mathcal{H}$ be an interval-decomposable integrated set of mappings from $[0, 1]$ into R^n. Suppose that $\mathcal{H}$ is closed in the sup-norm. If e is an extreme point of the closure of the attainable set, then e is in the attainable set itself.*

Proof. By Theorem 4.3, if $h_i(1)$ converges to e for a sequence $h_i(\cdot)$ in $\mathcal{H}$, then the sequence is a Cauchy sequence in the sup-norm. Hence it has a limit, say $h_0(\cdot)$, and since $\mathcal{H}$ is assumed to be closed, the limit is also in $\mathcal{H}$. Clearly $h_0(1) = e$. This completes the proof. □

Remark 4.5. The former two results are adaptations to the present framework of similar arguments in respect to the Aumann integral, or the analogous integral for the classical decomposability property. There the closedness is assumed in the L_1-norm, on the derivative level. See Theorem 1 in Olech [15] and references therein, see also the Appendix in Artstein [3]. Assuming closedness of $\mathcal{H}$ in the sup-norm is a strong assumption; in the next section we offer refined conditions for the compactness of the attainable set.

Furthermore, a consequence of Theorem 4.3 is that if e is an extreme point of the closure of the attainable set, and e is in the attainable set, then there is a unique function $h_e(\cdot)$ in $\mathcal{H}$ with $h_e(1) = e$. This uniqueness property will be verified below (see Corollary 5.2) under weaker conditions.

The inclusion of the extreme points in the attainable set implies, in a standard way, closedness of the attainable set under some conditions. For completeness, and further reference, We display this result.

Corollary 4.6. *Under the conditions of Theorem* 4.4*, if the attainable set is bounded, then it is compact. If the boundary of the attainable set does not contain an unbounded ray, then the attainable set is closed.*

Proof. The attainable set is convex, by Theorem 4.1. A closed convex set with a boundary that does not contain a ray is the convex hull of the extreme points. Since the extreme points of the closure of the attainable set are included in the set, the result follows. □

Remark 4.7. As was pointed in Remark 4.2, the interval-Lyapunov Convexity Theorem is a particular case of Theorem 4.1. Earlier, in Remark 3.4, we provided an example where the point $(0, 1)$ is not in the interval-range. This point is an extreme point of the closure of the interval-range. This does not contradict Theorem 4.4, since the latter requires closedness of the family of continuous functions, a property not satisfied when taking all the interval-swapping.

5. Extremal trajectories and a Carathéodory type result

The celebrated Carathéodory Theorem asserts that any point in a convex compact set in R^n is a convex combination of $n+1$ extreme points of the set. In several contributions, mainly within linear control theory, Olech established a similar result concerning the generation of points in the attainable set by extremal trajectories. Namely, any point in the attainable set is generated by switching among $n+1$ extreme trajectories on at most $n+1$ intervals. See, e.g., Olech [13]. The analogous result for decomposable sets in L_1 was established by Olech, see [15] and references therein. We verify here that the result holds also for interval-decomposable continuous mappings. On the general theory of extremals in convex analysis and their role in optimization see Ioffe and Tihomirov [12]

Recall that a point e is an extreme point of a convex set C in R^n, if it cannot be presented as $e = \alpha v_1 + (1-\alpha)v_2$ with v_1 and v_2 not equal to e, in C and $0 < \alpha < 1$. Here e may not be an extreme point of the closure of C. Also, when a point v is on the boundary of a convex set C, then a vector p exists such that $p \cdot v \geq p \cdot x$ for every $x \in C$. Here v may not be in C. Given p, we refer to the set of all $v \in C$ with this property as the p-boundary of C, denoted C_p. The

p-boundary of a convex set C is convex, and may have its own q-boundary (it could be empty), which we denote $C_{p,q}$, etc. It is easy to see that e is an extreme point of C if and only if it is the *lexicographic maximum* in C with respect to a sequence of (at most) n linearly independent vectors $p_1, p_2, ..., p_n$, in the sense that e is in $C_{p_1,p_1,...,p_n}$. The latter is, in fact, a singleton, so, abusing rigor, we may write $e = C_{p_1,p_1,...,p_n}$. Notice that at no point we assume that C or any of its p-boundaries, is closed. Olech exploited this analysis in his study of linear control systems, see. e.g., [14]. We show now that the relevant analysis can be applied in the interval-decomposable setting of the present paper.

Lemma 5.1. *Let $\mathcal{H}$ be an interval-decomposable set in $C([0,1], R^n)$. Let v be in the p-boundary of its attainable set $A(1)$. Let $h_0(\cdot) \in \mathcal{H}$ be such that $h_0(1) = v$. Then $h_0(s)$ is on the p-boundary of the attainable set $A(s)$ for every $s \in [0,1]$.*

Proof. If for some s_0 in the interior of $[0,1]$ and some $h_1(\cdot) \in \mathcal{H}$, the inequality $p \cdot h_1(s_0) > p \cdot h_0(s_0)$ holds, then swapping $h_1(\cdot)$ on $[s_0, 1]$ for $h_0(\cdot)$ will result in a function in $\mathcal{H}$, say $h_2(\cdot)$ such that $p \cdot h_2(1) > p \cdot h_0(1)$, a contradiction. □

Corollary 5.2. *Let $\mathcal{H}$ be an interval-decomposable set in $C([0,1], R^n)$. Let e be an extreme point of its attainable set. Then there is a unique function $h_e(\cdot) \in \mathcal{H}$ such that $h_e(1) = e$. If e is the lexicographic maximum in $A(1)$ with respect to the linearly independent vectors $p_1, p_2, ..., p_n$, then for every s the point $h_e(s)$ is the lexicographic maximum in $A(s)$ with respect to the same linearly independent vectors. In particular, there is a unique function $h(\cdot) \in \mathcal{H}$ such that $h(1) = e$.*

Proof. Apply the previous lemma, successively, to $A_{p_1,p_2,...,p_k}$, for $k = 1, ..., n$. □

Remark 5.3. The uniqueness established in the preceding corollary is stronger than the one referred to in Remark 4.5. Indeed, here it is not required that e be an extreme point of the closure of C. In turn, in the framework of the preceding corollary it may not be true that a sequence $h_j(\cdot)$ in $\mathcal{H}$ such that $h_j(1)$ converges to e, is a Cauchy sequence. To guarantee this property, e needs to be an extreme point of the closure.

The preceding derivations allow us to characterize closedness of the attainable set in terms of generation of the extreme points. Indeed, the attainable set may be compact without the integrated set $\mathcal{H}$ being compact. Thus, assuming the latter property (as was done in Theorem 4.4), is too strong.

Theorem 5.4. *Let $\mathcal{H}$ be an interval-decomposable integrated set in $C([0,1], R^n)$ such that its attainable set $A(1)$ is bounded. Denote by $C(t)$ the closure of the attainable set $A(t)$. For every sequence $p_1, p_2, ..., p_n$ of linearly independent vectors in R^n, and every t, let $p_1(C(t)) = \max\{p_1 \cdot x : x \in C(t)\}$, and, successively, $p_j(C(t)) = \max\{p_j \cdot x : x \in C(t)_{p_1,...,p_{j-1}}\}$, for $j = 2, ..., n$. The attainable set $A(1)$ is compact if and only if, for every such sequence of linearly independent vectors, the equalities $p_j \cdot x(t) = p_j(C(t))$ determine a (necessarily unique) function in $\mathcal{H}$.*

Proof. The 'only if' part is guaranteed by Corollary 5.2. The 'if' direction follows since the function is $\mathcal{H}$ that satisfies the equalities has the corresponding extreme point as a value in $A(1)$. The convexity of the latter assures that it is compact. □

The following result is the one promised at the beginning of the section. A function $h(\cdot)$ in $\mathcal{H}$ such that $h(1)$ is an extreme point of the attainable set is called an *extremal function.*

Theorem 5.5. *Let $\mathcal{H}$ be an interval-decomposable integrated set in $C([0,1],R^n)$ such that its attainable set $A(1)$ is compact. Then for every $v \in A(1)$ there is a function $h(\cdot)$ in $\mathcal{H}$ with $h(1) = v$, and such that $h(\cdot)$ is a result of swapping among at most $n+1$ extremal functions, over at most $n+1$ intervals in $[0,1]$.*

Proof. Assume first that v is in the interior of $A(1)$. Let $h_1(\cdot)$ be an arbitrary extremal function, say $h_1(1) = e$. Consider the mapping $g_1(\cdot)$ given by $g_1(t) = h_1(t) - e + v$. Then $g_1(1) = v$ and, by continuity, on some interval $[t_1, 1]$ the values $g_1(s) \in A(s)$. By Proposition 2.10, the mapping $A(s)$ is continuous in the Hausdorff distance. It is also convex-valued (by Theorem 4.1), and compact-valued (since, by Corollary 5.2, it contains all the extreme points of its closure). Since $g_1(0)$ is not the origin, it follows that a point $s_1 > 0$ exists such that $g_1(s)$ is in the interior of $A(s)$ for all $s_1 < s \leq 1$, and $g_1(s_1)$ is on the boundary of $A(s_1)$. An induction argument would complete now the proof. Indeed, the p-boundary of the attainable set $A(s_1)$ is attained by the functions $h(\cdot)$ in $\mathcal{H}$ with $h(s)$ in the p-boundary of $A(s)$ for $s \leq s_1$, and the case of a zero-dimensional p-boundary is covered by the definition of an extremal function. □

6. Approximations

In this section we deal with two approximation issues. First we ask: Is the convex hull in $C([0,1],R^n)$ of an integrated set approximated by the interval-decomposed hull of the set? Second, we examine the approximation of the interval-decomposed hull of a set of functions, when the family of intervals on which swapping is allowed is restricted.

Proposition 6.1. *Let $\mathcal{H}$ be an interval-decomposable integrated set of mappings from $[0,1]$ into R^n. Let $h_1(\cdot)$ and $h_2(\cdot)$ be in $\mathcal{H}$. Let $t_1 < t_2 < \ldots < t_k$ be points in $[0,1]$. Then a function $h_3(\cdot)$ exists in $\mathcal{H}$, which is formed by swapping $h_1(\cdot)$ for $h_2(\cdot)$, on an appropriate union of intervals, such that $h_3(t_i) = \frac{1}{2}(h_1(t_i) + h_2(t_i))$, for $i = 1, \ldots, k$.*

Proof. Apply Theorem 4.1 on $[0, t_1]$, namely, in regard to the attainable set $A(t_1)$. Then, inductively, apply the theorem on $[t_i, t_{i+1}]$, each time with $h_1(\cdot)$ and $h_2(\cdot)$ shifted appropriately so to be equal at t_i. □

Proposition 6.2. *Let $\mathcal{H}$ be an interval-decomposable integrated set of mappings from $[0,1]$ into R^n, such that its attainable set $A(1)$ is bounded. Given $\varepsilon > 0$, there is a $\delta > 0$ such that if $0 = t_1 < t_2 < \ldots < t_k = 1$ are points in the unit interval, and $t_j - t_{j-1} < \delta$ for $j = 2, \ldots, k$, and $h_1(\cdot)$ and $h_2(\cdot)$ are in $\mathcal{H}$, and $h_1(t_j) = h_2(t_j)$ for $j = 1, \ldots, k$, then $\|h_1(\cdot) - h_2(\cdot)\| < \varepsilon$, in $C(([0,1],R^n)$.*

Proof. Since A is bounded, the result follows from Proposition 6.1 and the equi-uniform continuity of the functions in $\mathcal{H}$, establish in Proposition 2.10. □

The preceding two results provide a recipe for the approximation of a convex combination of functions in $C(([0,1],R^n)$, say the average $\frac{1}{2}(h_1(\cdot)+h_2(\cdot))$, by swapping part of the first function for the other. Simply determine the partition $0=t_1<t_2<...<t_k=1$ such that the intervals $[t_i,t_{i+1}]$ are small enough (less than the δ in Proposition 6.2), then use Proposition 6.1 and find a function $h_3(\cdot)$ satisfying $h_3(t_i)=\frac{1}{2}(h_1(t_i)+h_2(t_i))$, for $i=1,...,k$. This function constitutes the approximation. However, this recipe works only in the case where the attainable set is bounded. It may not be valid in the case of an unbounded attainable set, even if $\mathcal{H}$ is the interval-decomposable hull of two mappings, and the average of these two mappings is sought, as the following variation of Example 2.9 shows.

Example 6.3. Consider two real-valued functions. The first is the mapping $h_1(t)=0$. The second is defined as follows. First consider the partition of the unit interval into an infinite number of sub-intervals, given by $I_j=[2^{-(j+1)},2^{-j}]$, this for $j=1,2,....$ In each I_j consider the partition to sub-intervals of equal length, determined, say, by points $s_1<s_2<...<s_{m(j)}$, with $m(j)$ to be determined shortly.

Now define $r(s_i)=s_i$ if i is even and $r(s_i)=-s_i$ if i is odd, and on the rest of the interval let the function $r(\cdot)$ connect the values defined so far with linear segments, and continuous on the entire interval $[0,1]$. The number $m(i)$ is chosen such that the graph of $r(\cdot)$ on I_j has length tending to ∞ as $j\to\infty$. Now, when the unit interval is partitioned into tiny intervals, say I_j for a large j is one of them, it could happen that a function $h_3(\cdot)$, in the interval-decomposed hull of the two functions, satisfies $h_3(s)=\frac{1}{2}(h_1(s)+r(s))$ for $s=2^{-j}$ and $s=2^{-(j+1)}$, namely the beginning and end of the interval I_j, while in the middle of the interval $\|h_3(s)-\frac{1}{2}(h_1(s)+r(s))\|$ gets arbitrarily large. □

The previous example does not preclude the possibility to approximate the convex hull of $\mathcal{H}$, in $C([0,1],R^n)$, by elements in the interval-decomposed hull of the set. It just says that the intuition (valid in many approximation procedures) that taking tiny sub-intervals yields good approximations, is wrong when the functions in the decomposed hull are not equi-uniformly continuous. As we see in the following considerations, in such a case the approximation can be carried out when the swapping is on intervals with length bounded away from zero.

Theorem 6.4. *Let $\mathcal{H}$ be an integrated set in $C([0,1],R^n)$. Each convex combination of mappings in $\mathcal{H}$ can be approximated, arbitrarily close in the sup-norm of $C([0,1],R^n)$, by an interval-decomposed combination of mappings in $\mathcal{H}$.*

Proof. Clearly, it is enough to show that given $h_1(\cdot)$ and $h_2(\cdot)$ in $\mathcal{H}$, their average $\frac{1}{2}(h_1(\cdot)+h_2(\cdot))$ can be approximated by a function $h_3(\cdot)$ obtained by swapping $h_1(\cdot)$ on a union of intervals, for $h_2(\cdot)$. Also, we may assume that $h_1(t)=-h_2(t)$ for all t, namely, the average of the two mappings is identically equal to 0, this by subtracting the average function from both functions.

The uniform continuity of the two functions implies that for every $\varepsilon>0$, a $\delta>0$ exists, such that $\|h_1(s)-h_1(t)\|\geq\varepsilon$, then $|s-t|\geq\delta$, and likewise for $h_2(\cdot)$.

Consider now the interval $[0, t_1]$ such that t_1 is the first point where $\|h_1(t_1)\| = 2\varepsilon$. If such a point does not exist, the mapping $h_1(\cdot)$ is a 2ε-approximation of the average. If $t_1 < 1$, consider the last point s_1 in $[0, t_1]$, such that $\|h_1(s_1)\| = \varepsilon$. Such a point exits, by continuity, and on $[s_1, t_1]$ the values of $\|h_1(t)\|$ are between ε and 2ε. Swapping $h_1(\cdot)$ for $h_2(\cdot)$ on $[s_1, t_1]$ results in a function $h_3(\cdot)$, satisfying $\|h_3(t)\| \leq 2\varepsilon$ on the interval, and $h_3(t_1) = 0$. Also, the length of the interval $[0, t_1]$ is greater than or equal to 2δ. Iterating the procedure a finite number of times, specifically, less than or equal to $\frac{1}{2\delta}$ iterates, results in a 2ε-approximation of the average. This completes the proof. □

Corollary 6.5. *Let $\mathcal{H}$ be an intervals-decomposable set in $C([0, 1], R^n)$. If $\mathcal{H}$ is closed in the sup-norm in $C([0, 1], R^n)$, then it is a convex set in $C([0, 1], R^n)$.*

Proof. The claim follows from Theorem 6.4 and the observation that the closure of a convex set is convex. □

We turn now to the second approximation problem, Given an integrated family $\mathcal{H}$ of continuous functions, its interval-decomposed hull is the family of functions obtained by swapping among functions in $\mathcal{H}$. A natural question then is to what extent an iterated swapping process results in approximations to the interval-decomposed hull. Likewise, if swapping is allowed on a restricted family of intervals, how does the attainable set of the restricted family, approximate the attainable set of the interval-decomposed hull.

Consider an integrated family $\mathcal{H}$ of continuous functions in $C([0, 1], R^n)$. Let $\mathcal{P}$ be a finite partition of the unit interval, determined by $0 = t_0 < t_1 < ... < t_k = 1$. The *$\mathcal{P}$-decomposed hull* of $\mathcal{H}$ is the family of functions obtained when swapping among functions in $\mathcal{H}$ is allowed only on intervals in $\mathcal{P}$. We denote this family by $\mathcal{H}_{\mathcal{P}}$. The complete interval-decomposed hull of $\mathcal{H}$ will be denoted by $\mathcal{H}_{\mathcal{W}}$ (here $\mathcal{W}$ stands for whole). Likewise, we denote by $A_{\mathcal{P}}(1)$ and $A_{\mathcal{W}}(1)$ the attainable sets of $\mathcal{H}_{\mathcal{P}}$ and $\mathcal{H}_{\mathcal{W}}$, respectively.

Theorem 6.6. *Let $\mathcal{H}$ be an integrated set of mappings from $[0, 1]$ into R^n. Assume that the attainable set $A_{\mathcal{W}}(1)$ is bounded. The Hausdorff distance in $C([0, 1], R^n)$ between $\mathcal{H}_{\mathcal{P}}$ and $\mathcal{H}_{\mathcal{W}}$ (in particular, the distance between $A_{\mathcal{P}}(1)$ and $A_{\mathcal{W}}(1)$) converges to zero when the maximal length of the intervals which determine $\mathcal{P}$, tends to zero.*

Proof. In view of Proposition 2.10, the functions $h(\cdot)$ in $\mathcal{H}_{\mathcal{W}}$ are equi-uniformly continuous. Let $h_0(\cdot)$ be in $\mathcal{H}_{\mathcal{W}}$. It is the result of swapping among a finite number of elements in $\mathcal{H}$, on intervals among the intervals determined by a partition, say $\mathcal{P}_0$, given by, say, $0 = s_0 < s_1 < ... < s_m = 1$. If the partition $\mathcal{P}$ consists of small intervals, a sub-partition of $\mathcal{P}$ can be identified, say $0 = \tau_0 < \tau_1 < ... < \tau_m = 1$, such that $|s_i - \tau_{j(i)}|$ is small, for an appropriate index $j(i)$, for every i. In view of the equi-uniform continuity, carrying the swapping on the τ_i-intervals, rather than on the s_j-intervals, will result in a small deviation from $h_0(\cdot)$. The boundedness and equi-uniform continuity imply that the closure of $\mathcal{H}_{\mathcal{W}}$ in $C([0, 1], R^n)$ is compact. Hence, approximating a finite number of elements in

$C([0,1], R^n)$ suffices to guarantee a small Hausdorff distance. This completes the proof. $\square$

Without the boundedness assumed in the previous theorem, the result does not hold. In view of Theorem 6.4, however, each fixed convex combination of mappings in $\mathcal{H}$ will be approximated by mappings in $\mathcal{H}_{\mathcal{P}}$, when the intervals in $\mathcal{P}$ are small enough.

Conclusion. We have demonstrated in this paper that many convex analysis features of decomposable families of L_1-functions can be verified on the level of the integrated sets, in particular continuous functions, allowing, for instance, functions that are not absolutely continuous, even nowhere differentiable. Among these features are the convexity of the attainable set, closure properties and the structure of extremal and extreme points. We note, however, that a prime achievement of the theory of decomposable sets, is the fixed point theory, and the associated approximation results, starting with the Antosiewicz and Cellina theorem [1], which was followed by many theoretical developments and applications, see Fryszkowski [9] and references therein. While the fixed point results hold also in the framework of the present paper, our approach, so it seems, does not add much to this facet of the theory. Indeed, the fixed point theory makes use of compactness. Compact decomposable sets in L_1 may not be convex. Compact interval-decomposable sets of integrated sets are convex (see above). Then the classical fixed point theory can be invoked.

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