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Lower Bound of the Upper Topological Limit of a Sequence of Subspaces

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Dedicated to Alexander Ioffe on the occasion of his 80th birthday.

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A new short proof of the following assertion is presented. Given an integer k and a sequence of closed linear subspaces of a Banach space, assume that the codimension of each subspace does not exceed k. Then the upper topological limit of this sequence contains a closed linear subspace with codimension not exceeding k.

Keywords: Sequence of subspaces, upper topological limit, lower bound.

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1. Main result

Let X be a normed space, X^* be its topologically dual space. As usual, for a closed linear space $\Pi \subset X$, its codimension codimΠ equals k if

$$\Pi = \{x : \langle x_i^*, x\rangle = 0,\ i = 1, \dots, k\}$$

for some linearly independent $x_i^* \in X^*$. Denote the upper topological limit of a sequence $\{C_n\}$ of subsets of X by $\mathrm{Ls}\{C_n\}$. Recall that it consists of all limit points of all sequences $\{x_n\} : x_n \in C_n$ for all n.

Let us state an assertion that plays an important role in applications, in particular, in optimization theory, theory of quadratic forms (see [3]), etc.

Theorem 1.1. *Let Y be a k-dimensional linear space, $\{A_n : X \to Y\}$ be a sequence of linear continuous operators that converges in norm to an operator A. Then there exists a closed linear subspace $\Pi \subset X$ such that*

$$\mathrm{codim}\Pi \le k, \quad \Pi \subset \ker A, \quad \Pi \subset \mathrm{Ls}\{\ker A_n\}.$$

This theorem is equivalent to the following assertion.

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Theorem 1.2. *Let $\{\Pi_n\}$ be a sequence of closed linear subspaces of X, and assume that there exists a positive integer k such that*

$$\mathrm{codim}\Pi_n \leq k \,\forall\, n.$$

Then there exists a closed linear subspace $\Pi \subset X$ such that

$$\mathrm{codim}\Pi \leq k, \quad \Pi \subset \mathrm{Ls}\{\Pi_n\}.$$

If X is a Hilbert space, then the proof of these theorems isn't difficult (see e.g. [3]). For Banach spaces, these theorems were proved in [1]. The proof was based on the theory of filters and required the usage of the so-called Dubovitskii operator generated by a linear continuous functional on l_∞ of a special type (see [1], [2]). In [6], Theorem 1.1 was generalized to the case when X is a linear topological space. But the proof in [6] is also based on the existence of uniform ultrafilters.

The aim of this paper is to give a direct, simple and short proof of Theorem 1.2 that uses only the standard methods of functional analysis, namely the Tikhonov theorem on the product of compact spaces and the Banach-Alaoglu theorem, and does not use the theory of filters (see [5]). First we give the following simple assertion.

Proposition 1.3. *Let $\Pi \subset X$ be a closed linear subspace of finite codimension k. Then for any $\varepsilon > 0$ there exists a biorthogonal system $x_i \in X$, $x_i^* \in X^*, i = 1, \ldots, k$, such that*

$$\Pi = \{x : \langle x_i^*, x\rangle = 0,\, i = 1, \ldots, k\}, \;\; \langle x_i^*, x_j\rangle = \delta_j^i,\; i, j = 1, \ldots, k, \tag{1}$$

$$1 \leq \|x_i\| \leq 1 + \varepsilon, \qquad 1 - \varepsilon \leq \|x_i^*\| \leq 1, \quad i = 1, \ldots, k. \tag{2}$$

Here δ_j^i are the Kronecker symbols.

Proof. Consider the quotient space $\Xi = X/\Pi$ with the norm $\|\xi\| = \inf_{x\in\xi} \|x\|$ for $\xi \in \Xi$. In the k -dimensional normed space Ξ, there exists a biorthogonal Auerbach system (see, for example, [4], Proposition 6.10.45), i.e. $\xi_i \in \Xi$, $\xi_i^* \in \Xi^*$, such that

$$\langle \xi_i^*, \xi_j\rangle = \delta_j^i,\; \|\xi_i\| = \|\xi_i^*\| = 1, \qquad i, j = 1, \ldots, k. \tag{3}$$

For each equivalence class $\xi_i \in \Xi$ there exists an element $x_i \in X$ such that $x_i \in \xi_i$ and $\|x_i\| \leq 1 + \varepsilon$. We define linear functionals x_i^* on X by the relations $\langle x_i^*, x\rangle = \langle \xi_i^*, \xi(x)\rangle$, where $\xi(x)$ is the equivalence class generated by $x \in X$. Then $\langle x_i^*, x\rangle = 0 \,\forall\, x \in \Pi$ and for any $x \in X$ in virtue of (3) we have

$$|\langle x_i^*, x\rangle| = |\langle \xi_i^*, \xi(x)\rangle| \leq \|\xi(x)\| \leq \|x\| \Rightarrow \|x_i^*\| \leq 1,$$

$$\langle x_i^*, x_j\rangle = \delta_j^i \Rightarrow \langle x_i^*, x_i/\|x_i\|\rangle = \|x_i\|^{-1} \geq (1+\varepsilon)^{-1} \geq 1 - \varepsilon.$$

Thus, the constructed x_i, x_i^*, $i = 1, \ldots, k$ are the desired ones. Thus, the proposition is proved. □

Proof of Theorem 1.2. Reducing, if necessary, the subspaces Π_n, without loss of generality we assume $\mathrm{codim}\Pi_n = k$ for all n. For each number n we take a biorthogonal system $x_{n,i} \in X$, $x^*_{n,i} \in X^*, i = 1, \ldots, k$ corresponding to the subspace Π_n by virtue of proposition 1.3 for $\varepsilon = 1/2$.

We denote by B the unit ball in X^* endowed with the weak* topology. By the Banach-Alaoglu theorem, the topological space B is compact. We denote by $\mathcal{B}$ the k -fold product of the topological space B by itself. The space $\mathcal{B}$ is compact by Tikhonov's theorem on the product of compact spaces. We denote the elements of $\mathcal{B}$ by $z = (z^1, ..., z^k)$, where $z_i \in B$.

Consider the sequence $z_n = (x^*_{n,1}, ..., x^*_{n,k})$ lying in $\mathcal{B}$, $n = 1, 2, ...$ It has at least one limit point $z = (x^*_1, ..., x^*_k)$. Consider the subspace $\Pi = \{x \in X : \langle x^*_i, x\rangle = 0,\ i = 1, \ldots, k\}$. Obviously, $\mathrm{codim}\Pi \leq k$.

Let us prove that $\Pi \subset \mathrm{Ls}\{\Pi_n\}$. Indeed, let $x \in \Pi$. Then $\langle x^*_i, x\rangle = 0\, \forall i$. Consider an arbitrary $\delta > 0$. By the definition of the weak* topology and since z is the limit point of the sequence $\{z_n\}$ there exists an infinite set of numbers n such that $|\langle x^*_{n,i}, x\rangle| < \delta\, \forall i$. For each such a number n, we set

$$x_n = x - \sum_{i=1}^{k} \langle x^*_{n,i}, x\rangle x_{n,i}.$$

By virtue of (1) $x_n \in \Pi_n$ and by virtue of (2) $\|x - x_n\| < \frac{3}{2}k\delta$. Hence, because of the arbitrariness of δ we have $x \in \mathrm{Ls}\{\Pi_n\}$. The theorem is proved. $\square$

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