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Lipschitz Stability of Extremal Problems with a Strongly Convex Set

Maxim V. Balashov*
V. A. Trapeznikov Institute of Control Sciences of the Russian Academy of Sciences,
65 Profsoyuznaya street, Moscow 117997, Russia
balashov73@mail.ru

Dedicated to Alexander Ioffe on the occasion of his 80th birthday.

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We prove that in a real Hilbert space some extremal problems are Lipschitz stable with respect to the set in some special metric (Pliś metric). We also consider the Lipschitz stability of such problems in the Hausdorff metric and characterize metrics on the space of closed bounded convex sets with uniformly continuous metric projection as function of the set.

Keywords: Hilbert space, metric projection, summand of a convex set, Pliś metric, Hausdorff metric, uniform continuity, integral of set-valued mapping.

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1. Introduction and main notations

The Hausdorff metric is one of the most important metrics in the space of closed bounded subsets. It has a very natural geometrical meaning and also can be specified (with the help of the supporting function) for the case of closed convex bounded subsets. It has very useful topological properties as well.

Nevertheless, the Hausdorff metric has certain disadvantages. Suppose that $\mathcal{F}$ is the space of closed convex bounded subsets of a real Hilbert space $\mathcal{H}$.

For example, the unique solution of the minimization problem

$$\min_{x\in A} \|x\|^2$$

$a = a(A) \in A \subset B_R(0) \cap \mathcal{F}$, is Holder continuous (with the power $\frac{1}{2}$) with respect to the set A in the Hausdorff metric [11]. The function $A \to a(A)$, $A \in \mathcal{F}$, is not uniformly continuous in the Hausdorff metric with respect to the set. Moreover, there is no uniformly continuous (in the Hausdorff metric) function $f\colon \mathcal{F} \to \mathcal{H}$ with the property $f(A) \in A$ for any $A \in \mathcal{F}$ [19, Theorem 5.86 (1)].

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In the paper [4] we considered some special metric ρ, introduced by A. Pliś [17]. We proved that for any $A, B \in \mathcal{F}$ and for any point $x \in \mathcal{H}$ we have $\|P_A x - P_B x\| \le \rho(A, B)$. Here $P_A x$ is the metric projection of the point x on the set A.

We also proved that the solution $a(A)$ of the minimization problem

$$\min_{x \in A} f(x)$$

for a strongly convex (with constant of strong convexity $\varkappa$) function with the Lipschitz continuous gradient (with Lipschitz constant L_1) satisfies the estimate

$$\|a(A) - a(B)\| \le \frac{1}{1 - \sqrt{1 - \frac{\varkappa^2}{L_1^2}}}\,\rho(A, B) \qquad \forall A, B \in \mathcal{F} \tag{1}$$

(note that $\varkappa \le L_1$). We also obtained some estimates for quasiconvex functions with strongly convex level sets and for some maximization problems.

In the present paper we prove some results about the Lipschitz stability for strongly convex sets and (noncovex in general) functions with the Lipschits continuous gradient and weakly convex level sets. We also characterize metrics on the space $\mathcal{F}$ with globally uniformly continuous metric projection as function of the set.

Let $\mathcal{H}$ be a real Hilbert space with the inner product $(\cdot, \cdot)$ and $B_r(a) = \{x \in \mathcal{H} \mid \|x - a\| \le r\}$ the ball around a of radius r. We denote by $\mathcal{F}$ the set of all closed convex bounded subsets from $\mathcal{H}$. The distance from the point $x \in \mathcal{H}$ to the set $A \subset \mathcal{H}$ is defined by the formula $\varrho(x, A) = \inf_{a \in A} \|x - a\|$. Denote the *metric projection* of the point $x \in E$ on the set $A \subset E$ as

$$P_A(x) = \{a \in A : \ \|x - a\| = \varrho(x, A)\}.$$

Recall that the set $A_1 + A_2 = \{a_1 + a_2 \mid a_i \in A_i, i = 1, 2\}$ is called the *Minkowski sum* of the sets $A_1, A_2 \subset \mathcal{H}$ and the set

$$A_1 \overset{*}{-} A_2 = \{x \in \mathcal{H} \mid x + A_2 \subset A_1\} = \bigcap_{a \in A_2} (A_1 - a)$$

is called the *geometric* (or *Minkowski-Pontryagin*) *difference* of the sets. If the nonempty sets $A_i \in \mathcal{F}$, $i = 1, 2, 3$, satisfy the equality $A_1 = A_2 + A_3$, one easily sees by using supporting functions that $A_3 = A_1 \overset{*}{-} A_2$. For a function $f : \mathcal{H} \to \mathbb{R}$ define the *lower level set* as

$$\mathcal{L}(\mu) = \{x \in \mathcal{H} \mid f(x) \le \mu\}.$$

A function $f : \mathcal{H} \to \mathbb{R}$ is called *quasiconvex*, if its lower level sets $\mathcal{L}(\mu)$ are convex.

We shall denote by $f'(x)$ the *Fréchet derivative* of the function f at the point x.

The Hausdorff distance (*metric*) for the sets $A_1, A_2 \subset \mathcal{H}$ is defined by the formula

$$h(A_1, A_2) = \max\left\{ \sup_{a \in A_1} \varrho(a, A_2), \quad \sup_{b \in A_2} \varrho(b, A_1) \right\} =$$
$$= \inf\{r > 0 \mid A_1 \subset A_2 + B_r(0), \ A_2 \subset A_1 + B_r(0)\}.$$

By the separation theorem for closed convex bounded subsets $A_1, A_2 \subset \mathcal{H}$ we get

$$h(A_1, A_2) = \sup_{\|p\|=1} |s(p, A_1) - s(p, A_2)|, \tag{2}$$

where $s(p, A) = \sup\limits_{x\in A}(p, x)$ is the supporting function of the set A.

For any vector $p \in \mathcal{H}$ and set $A \in \mathcal{F}$ the *subdifferential* of the supporting function $s(\cdot, A)$ at the point p is denoted by $A(p) = \{x \in A \mid (p, x) = s(p, A)\}$.

For a set $A \in \mathcal{F}$ and a point $a \in \partial A$ (the boundary of A) define the *normal cone*

$$N(A, a) = \{p \in \mathcal{H} \mid (p, a) \geq s(p, A)\}.$$

Definition 1.1. For the sets $A_1, A_2 \in \mathcal{F}$ define the *Pliś distance* (*metric*) between A_1 and A_2 as

$$\rho(A_1, A_2) = \sup_{\|p\|=1} h(A_1(p), A_2(p)). \qquad \square$$

Some topological properties of the Pliś metric were described in [8], in particular, for any sets $A_1, A_2 \in \mathcal{F}$ we have $h(A_1, A_2) \leq \rho(A_1, A_2)$.

The set $A \in \mathcal{F}$ is called *strongly convex of radius* $R > 0$, if it can be represented as intersection of closed balls of radius R: $A = \bigcap_{x\in X} B_R(x) = B_R(0) \overset{*}{-} (-X)$, where $X \subset \mathcal{H}$ is an arbitrary set of centers.

For $r > 0$ *the open r-neighborhood* of the set $A \subset \mathcal{H}$ is defined as

$$U_r(A) = \{x \in \mathcal{H} \mid \varrho(x, A) < r\}.$$

We say that a subset $A \subset \mathcal{H}$ satisfies the *supporting condition with constant* $r > 0$, if for any point $x \in U_r(A)\backslash A$ such that $a \in P_A x$, we have the equality

$$A \bigcap \operatorname{int} B_R\left(a + R\,\frac{x-a}{\|x-a\|}\right) = \emptyset. \tag{3}$$

A closed subset A of a Banach space E is called *proximally smooth (or prox-regular)* with constant $r > 0$ [10] if the distance function $\varrho(x, A)$ is continuously differentiable on the set $U_r(A)\backslash A$. Proximally smooth sets in a Hilbert space are also called *weakly convex,* their properties can be found in [6, 10, 13, 15, 18, 20].

Proposition 1.2. ([10, Theorem 4.1]) *Let $A \subset \mathcal{H}$ be a closed subset, $r > 0$. The following conditions are equivalent:*

(1) *The set A satisfies the supporting condition with constant r.*

(2) *The set A is proximally smooth = weakly convex with constant r.*

(3) *The metric projection $P_A x$ is singleton and continuous for all $x \in U_r(A)\backslash A$.*

2. Metric projection, sums and integrals

Lemma 2.1. *Assume $A_1, A_2 \in \mathcal{F}$ and $d > 0$. Suppose that A_2 is strongly convex of radius $R>0$. Then for any $x_1, x_2 \in \mathcal{H}\backslash U_d(A_2)$ and $a_i = P_{A_i} x_i$, $i=1,2$, we have*

$$\|a_1 - a_2\| = \|P_{A_1}x_1 - P_{A_2}x_2\| \leq \frac{R}{R+d}\|x_1 - x_2\| + \rho(A_1, A_2).$$

Proof. First we prove that for a point $x \in \mathcal{H}$ and $a_i = P_{A_i}x$, $i = 1, 2$,

$$\|a_1 - a_2\| \le \rho(A_1, A_2). \tag{4}$$

We can assume that $x = 0$. Suppose that $a_i \ne 0$, $i = 1, 2$. Let $p_i = -\frac{a_i}{\|a_i\|}$ be the normal vector to the set A_i at the point a_i. Define $r_1 = \varrho(0, A_1)$, $r_2 = \varrho(0, A_2)$. Note that $a_i = -r_i p_i$ and $a_i \in A_i(p_i)$, $i = 1, 2$. Fix $t > 1$.

Let $y \in A_2(p_1)$ be a point such that $\|a_1 - y\| \le t\rho(A_1, A_2)$. It is easy to see, that $(p_2, y) \le (p_2, a_2)$ and $(p_1, y) \ge (p_1, a_2)$. We have

$$(a_1 - a_2, y - a_2) = (-r_1 p_1 + r_2 p_2, y - a_2) = \\ = r_1 \left(-(p_1, y) + (p_1, a_2)\right) + r_2 \left((p_2, y) - (p_2, a_2)\right) \le 0.$$

Thus, the inner product $(a_1 - a_2,\, y - a_2)$ is nonpositive, and hence the angle between vectors $a_1 - a_2$ and $y - a_2$ is at least $\frac{\pi}{2}$. Therefore, the length of the side $a_1 y$ of the triangle $a_1 a_2 y$ is not less than the length $a_1 a_2$, i.e. $\|a_1 - a_2\| \le \|a_1 - y\| \le t\rho(A_1, A_2)$ for all $t > 1$. Passing to the limit $t \to 1 + 0$, we prove (4).

If $a_2 = 0$, $a_1 \ne 0$, then (taking $t > 1$) put $p_1 = -\frac{a_1}{\|a_1\|}$, and choose $y \in A_2(p_1)$ such that $\|a_1 - y\| \le t\rho(A_1, A_2)$. As above it's easy to see that $\|0 - a_1\| \le \|y - a_1\| \le t\rho(A_1, A_2)$. It remains to pass to the limit as $t \to 1 + 0$.

Now let $b = P_{A_2}a_1$. We get

$$\|a_1 - a_2\| \le \|a_1 - b\| + \|b - a_2\|, \quad \text{and} \quad \|a_1 - b\| \le \rho(A_1, A_2),$$

and by [5, Corollary 2.1]: $$\|b - a_2\| \le \frac{R}{R + d}\|x_1 - x_2\|.$$ □

Note that taking the limit as $R \to +\infty$ we get the inequality

$$\|a_1 - a_2\| \le \|x_1 - x_2\| + \rho(A_1, A_2)$$

for any subsets $A_1, A_2 \in \mathcal{F}$. The last estimate was proved in [4].

Lemma 2.2. *Let $B \in \mathcal{F}$. Then $\rho(\{0\}, B) = h(\{0\}, B)$.*

Proof. The inequality $\rho(\{0\}, B) \ge h(\{0\}, B)$ is obvious. We prove the converse inequality. Put $h = h(\{0\}, B)$. Hence for any unit vector p we have

$$h(\{0\}, B(p)) \le h(\{0\}, B),$$

and hence $$\rho(\{0\}, B) \le \sup_{\|p\|=1} h(\{0\}, B(p)) \le h.$$ □

Lemma 2.3. *Let $A, B \in \mathcal{F}$. Then $\rho(A, A + B) = h(A, A + B)$.*

Proof. The proof follows from Lemma 2.2 and the equalities $\rho(A, A + B) = \rho(\{0\}, B)$, $h(A, A + B) = h(\{0\}, B)$. □

Suppose that E is a separable Banach space and (T,τ,μ) is a compact topological space T with σ-algebra of measurable sets τ and Lebesgue's measure μ. Definition of measurable set-valued mapping and criteria for measurability can be found in [9]. In particular, in a separable Banach space for any measurable set-valued mapping $F\colon T\to Cl(E)$ ($Cl(E)$ is the set of all closed nonempty subsets from E) there exists countable family of measurable selectors f_m with

$$F(t)=\overline{\bigcup_{m=1}^{\infty} f_m(t)},\quad \text{a.e. } t\in T.$$

Put $$S_T(F)=\{f\in L_1(T,E)\mid f(t)\in F(t)\ \text{ a.e. } t\in T\}.$$

The Aumann integral of F [2] is defined as follows

$$\int\limits_T F(t)\,dt=\Big\{x\in E\ \Big|\ x=\int\limits_T f(t)\,dt,\quad f\in S_T(F).\Big\}$$

The value $\int_T F(t)\,dt$ is closed and convex.

In the case $T=T_1\cup T_2$ (T_i is measurable) it is easy to see by definition that

$$\int\limits_T F(t)\,dt=\int\limits_{T_1} F(t)\,dt+\int\limits_{T_2} F(t)\,dt.$$

In the case $E=\mathbb{R}^n$ the results can be found in [14, §8.1, 8.2].

Lemma 2.4. *Let $\mathcal{H}$ be a separable real Hilbert space. Let $G\colon[t_0,T]\to\mathcal{F}$ be a measurable mapping such that* $\operatorname{ess\,sup}_{t\in[t_0,T]} h(\{0\},G(t))<+\infty$. *For $F_0\in\mathcal{F}$ put*

$$F(t)=F_0+\int\limits_{t_0}^{t} G(\tau)\,d\tau,\qquad t\in[t_0,T].$$

Then for any $t_1,t_2\in[t_0,T]:\quad h(F(t_1),F(t_2))=\rho(F(t_1),F(t_2))$.

Proof. Let $t_1\le t_2$. The proof follows from the equality

$$F(t_2)=F(t_1)+\int\limits_{t_1}^{t_2} G(\tau)\,d\tau,$$

the fact that the integral is a set from $\mathcal{F}$ and Lemma 2.3. □

Lemma 2.5. *Let $\mathcal{H}$ be a separable real Hilbert space. Let $G\colon[t_0,T]\to\mathcal{F}$ be a measurable mapping such that* $\operatorname{ess\,sup}_{t\in[t_0,T]} h(\{0\},G(t))<+\infty$. *For $F_0\in\mathcal{F}$ put*

$$F(t)=F_0+\int\limits_{t_0}^{t} G(\tau)\,d\tau,\qquad t\in[t_0,T].$$

If the set $F(T)$ is strongly convex of radius $R>0$, then for any $t\in[t_0,T)$ the set $F(t)$ is also strongly convex of radius R.

Proof. From the equality

$$F(T) = F_0 + \int_{t_0}^{t} G(\tau)\,d\tau + \int_{t}^{T} G(\tau)\,d\tau$$

and the fact that any above integral belongs to $\mathcal{F}$ we get $F(t) = F(T) \stackrel{*}{-} H$, where $H = \int_t^T G(\tau)\,d\tau$. Thus the set $F(t)$ is intersection of balls of radius R. □

Another sufficient condition for the set-valued integral to be a strongly convex set can be found in [12].

Theorem 2.6. *Suppose that $\mathcal{H}$ is a separable Hilbert space, $f\colon \mathcal{H} \to \mathbb{R}$ is a strongly convex function with a constant of strong convexity $\varkappa > 0$ and with a Lipschitz continuous gradient with Lipschitz constant $L_1 > 0$. If the sets A_1 and A_2 are values of some set-valued integral, i.e. $A_i = F_0 + \int_{t_0}^{t_i} G(\tau)\,d\tau$, where F_0 and $G(t)$ are from Lemma 2.4, then the solution a_i of the problem $\min_{x\in A_i} f(x)$, $i = 1, 2$, satisfies the estimate*

$$\|a_1 - a_2\| \le \frac{1}{1 - \sqrt{1 - \frac{\varkappa^2}{L_1^2}}}\, h(A_1, A_2).$$

Proof. The proof follows from Lemma 2.4 and formula (1). □

3. Stability of minimization problems for strongly convex sets with small radius of strong convexity

Theorem 3.1. *Let $A_1, A_2 \in \mathcal{F}$ be strongly convex sets of radius $R > 0$. Let $f\colon \mathcal{H} \to \mathbb{R}$ be a function such that the problem*

$$\min_{x\in A_i} f(x), \quad i = 1, 2,$$

has a unique solution $a_i \in A_i$, and let $f(a_1) \le f(a_2)$. Assume that there exist numbers $m > 0$, $L_1 > 0$ such that for any $x, y \in \mathcal{L}(f(a_2))\backslash \operatorname{int}\mathcal{L}(f(a_1))$ we have $\|f'(x) - f'(y)\| \le L_1\|x - y\|$ and $\|f'(x)\| \ge m$. Assume also that $R < m/L_1$. Then

$$\|a_1 - a_2\| \le \frac{1}{1 - \frac{RL_1}{m}}\, \rho(A_1, A_2).$$

Proof. Take $b \in \partial A_2$ such that $b \in A_2(-f'(a_1))$. By the definition of the Pliś metric $\|a_1 - b\| \le \rho(A_1, A_2)$. For any $\|p_i\| = 1$, $\lambda_i \ge 1$, $i = 1, 2$, we have the inequality $\|p_1 - p_2\| \le \|\lambda_1 p_1 - \lambda_2 p_2\|$, it can be verified by squared. Using the Lipschitz property of the supporting elements of strongly convex sets [7, Theorem 4.1], we have for unit vectors $p_i = -f'(a_i)/\|f'(a_i)\|$, $i = 1, 2$, the estimate

$$\|a_2 - b\| \le R\|p_1 - p_2\| \le R\left\|\frac{f'(a_1)}{m} - \frac{f'(a_2)}{m}\right\| \le \frac{RL_1}{m}\|a_1 - a_2\|$$

and

$$\|a_1 - a_2\| \le \|a_1 - b\| + \|a_2 - b\| \le \rho(A_1, A_2) + \frac{RL_1}{m}\|a_1 - a_2\|.$$ □

In the same way the next theorem can be proved.

Theorem 3.2. *Let $A_1, A_2 \in \mathcal{F}$ be strongly convex sets of radius $R > 0$. Let $f\colon \mathcal{H} \to \mathbb{R}$ be a function such that the problem*

$$\max_{x\in A_i} f(x), \quad i = 1, 2,$$

has a unique solution $a_i \in A_i$, $f(a_1) \le f(a_2)$. Assume that there exist numbers $m > 0$, $L_1 > 0$ that for any $x, y \in \mathcal{L}(f(a_2))\backslash \operatorname{int} \mathcal{L}(f(a_1))$ we have $\|f'(x)-f'(y)\| \le L_1\|x-y\|$ and $\|f'(x)\| \ge m$. Assume also that $R < m/L_1$. Then

$$\|a_1 - a_2\| \le \frac{1}{1-\frac{RL_1}{m}}\ \rho(A_1, A_2).$$

Note that by [20, Proposition 4.13], [3, Theorem 2.1] under some mild additional assumptions the level sets $\mathcal{L}_i = \mathcal{L}(f(a_i))$, $i = 1, 2$, are proximally smooth with constant $r = m/L_1$ and the solutions a_i are unique due to $R < m/L_1$ [3].

4. Stability of minimization problems for functions with weakly convex level sets and for quasiconvex functions

Theorem 4.1. *Let $A_1, A_2 \in \mathcal{F}$ be strongly convex sets of radius $R > 0$. Let $f\colon \mathcal{H} \to \mathbb{R}$ be a function such that the problem*

$$\min_{x\in A_i} f(x), \quad i = 1, 2,$$

has unique solution $a_i \in A_i$, $f(a_1) \le f(a_2)$. Put $\mathcal{L}_i = \mathcal{L}(f(a_i))$, $i=1,2$, and suppose that the set $\mathcal{L}_1$ and its complement $\operatorname{cl}(\mathcal{H}\backslash\mathcal{L}_1)$ are weakly convex with constant $r > 0$ and $R < r$. Suppose that there exist positive constants m, L_1, L such that

(1) *$\|f'(x)\| \ge m$ for all $x \in \mathcal{H}\backslash \operatorname{int} \mathcal{L}_1$,*

(2) *$\|f'(x) - f'(y)\| \le L_1\|x-y\|$, for all $x, y \in \mathcal{H}\backslash \operatorname{int} \mathcal{L}_1$,*

(3) *$\|f(x) - f(y)\| \le L\|x-y\|$, for all $x, y \in \mathcal{H}\backslash \operatorname{int} \mathcal{L}_1$,*

(4) *$\frac{L}{m} h(A_1, A_2) < r$,*

(5) *for all $x \in \partial\mathcal{L}_2$ and $y = P_{\mathcal{L}_1}x$ (exists and unique) $f(x) - f(y) \ge m\|x-y\|$, where m is from (1).*

Then

$$\|a_1 - a_2\| \le \frac{1}{1-\frac{R}{r}}\left(\rho(A_1,A_2) + \frac{LL_1R}{m^2}h(A_1,A_2) + \frac{Lh(A_1,A_2)}{m}\right) + \frac{Lh(A_1,A_2)}{m}.$$

Proof. We have by (4) and (5) $\varrho(a_2, \mathcal{L}_1) \le \frac{f(a_2)-f(a_1)}{m} \le \frac{Lh(A_1,A_2)}{m} < r$, hence $P_{\mathcal{L}_1}a_2$ is a singleton. Putting $b = P_{\mathcal{L}_1}a_2$, we have $\|a_2 - b\| \le \frac{Lh(A_1,A_2)}{m}$.

Let $-f'(b) \in N(A_2, w)$ for some $w \in \partial A_2$. From the strong convexity of the set A_2 we obtain that [7, Theorem 4.1]

$$\|w - a_2\| \le R\left\|\frac{f'(b)}{m} - \frac{f'(a_2)}{m}\right\| \le \frac{L_1R}{m}\|a_2 - b\| \le \frac{LL_1R}{m^2}h(A_1,A_2). \tag{5}$$

Denote $A_2' := A_2 - w + b$. From the weak convexity of the set $\mathcal{L}_1$ with constant $r > R$ we get $A_2' \cap \operatorname{int} \mathcal{L}_1 = \emptyset$ and the point b is a minimizer for f on the set A_2'. Hence

$$\begin{aligned} \|w-b\| &\le \|w-a_2\| + \|a_2 - b\| = \|w-a_2\| + \varrho(a_2, \mathcal{L}_1) \\ &\le \frac{LL_1R}{m^2} h(A_1,A_2) + \frac{Lh(A_1,A_2)}{m}, \end{aligned} \tag{6}$$

$$\rho(A_1, A_2') \le \rho(A_1,A_2) + \rho(A_2', A_2) = \rho(A_1,A_2) + \|w-b\|. \tag{7}$$

Define $b_1 = a_1$, $b_2 = b$, $p_i = f'(b_i)/\|f'(b_i)\|$, $x_i = b_i - tp_i$, $t \ge 0$, $i = 1,2$. By the definition we get

$$\|x_1 - x_2\|^2 = \|b_1 - b_2\|^2 - 2t(b_1 - b_2, p_1 - p_2) + t^2\|p_1 - p_2\|^2. \tag{8}$$

Since $p_i \in N(\mathcal{L}_i, b_i)$, from [20, Proposition 3.7], [13, Theorem 3.1 (d)] we obtain $r(b_1 - b_2, p_1 - p_2) \ge -\|b_1 - b_2\|^2$. By [20, Proposition 3.6], [13, Theorem 3.6]

$$\|p_1 - p_2\| \le \frac{1}{r}\|b_1 - b_2\|,$$

and finally from Formula (8) we obtain

$$\|x_1 - x_2\|^2 \le \|b_1 - b_2\|^2 \left(1 + \frac{2t}{r} + \frac{t^2}{r^2}\right),$$

and hence
$$\|x_1 - x_2\| \le \|b_1 - b_2\| \left(1 + \frac{t}{r}\right). \tag{9}$$

For any $\alpha \in (0,1)$ we can find $t_\alpha > 0$ such that $h(A_1, A_2') \le t(1-\alpha)$ for all $t > t_\alpha$, and thus $\varrho(x_1, A_1) = \varrho(x_2, A_2') = t$ and

$$\varrho(x_1, A_2') \ge \varrho(x_1, A_1) - h(A_1, A_2') \ge t\alpha.$$

From this and Lemma 2.1 we have

$$\|b_1 - b_2\| \le \frac{R}{R + t\alpha}\|x_1 - x_2\| + \rho(A_1, A_2') \le \frac{1 + \frac{t}{r}}{1 + \frac{t\alpha}{R}}\|b_1 - b_2\| + \rho(A_1, A_2'),$$

where the second inequality is due to (9). Passing to the limit as $t \to +\infty$ in the previous formula we find

$$\|b_1 - b_2\| \le \frac{R}{r\alpha}\|b_1 - b_2\| + \rho(A_1, A_2').$$

Taking the limit as $\alpha \to 1 - 0$ we obtain

$$\|a_1 - b\| = \|b_1 - b_2\| \le \frac{1}{1 - \frac{R}{r}} \rho(A_1, A_2') \tag{10}$$

and, taking in mind (7), $\|a_1 - a_2\| \le \|a_1 - b\| + \|b - a_2\| \le$

$$\le \frac{1}{1-\frac{R}{r}} \left(\rho(A_1,A_2) + \frac{LL_1R}{m^2}h(A_1,A_2) + \frac{Lh(A_1,A_2)}{m}\right) + \frac{Lh(A_1,A_2)}{m}. \qquad \square$$

Note that weak convexity of level sets for functions is considered in [20], [1].

Remark 4.2. In case $f(a_1) = f(a_2)$ we get by Theorem 3.2 from (10) ($b = a_2$)

$$\|a_1 - a_2\| \le \frac{1}{1 - \frac{R}{r}} \rho(A_1, A_2). \tag{11}$$

Example 4.3. Suppose that $r > R > 0$ and $\mathcal{L}_1 = \mathcal{L}_2 = \mathbb{R}^2 \backslash B_r(0)$. The sets $\mathcal{L}_1$ and $\mathbb{R}^2 \backslash \operatorname{int} \mathcal{L}_1$ are weakly convex with constant r. Consider the balls $A_1, A_2 \subset B_r(0)$ of radius R which touch each other and both balls A_1 and A_2 touch $\partial\mathcal{L}_1 = \partial B_r(0)$. Let $a_i = A_i \cap \partial\mathcal{L}_1$, $i = 1, 2$. Then, from trivial similarity, we have

$$\frac{\|a_1 - a_2\|}{2R} = \frac{r}{r - R}, \quad \rho(A_1, A_2) = 2R, \quad \text{and}$$

$$\|a_1 - a_2\| = \frac{2Rr}{r - R} = \frac{1}{1 - \frac{R}{r}} \rho(A_1, A_2).$$

Thus the estimate (11) is exact. □

We want to admit that in Theorem 4.1 we separate "radial" direction (formula (7)), where we need some sort of Lipschitz condition for the derivative in "radial" direction (2), and "axial" direction (formula (10)), where we use weak convexity of the level sets $\mathcal{L}_1$ and $\operatorname{cl}(\mathcal{H} \backslash \mathcal{L}_1)$. We cannot completely refuse the Lipschitz condition of f' in the "radial" direction (at least in the present proof), or replace it with some geometrical property.

Theorem 4.4. *Let $A_1, A_2 \in \mathcal{F}$ be strongly convex sets of radius $R > 0$. Let $f\colon \mathcal{H} \to \mathbb{R}$ be a quasiconvex function and $a_i \in A_i$, $i = 1, 2$, be solution of the problem*

$$\min_{x \in A_i} f(x), \quad i = 1, 2,$$

and let $f(a_1) \le f(a_2)$. Put $\mathcal{L}_i = \mathcal{L}(f(a_i))$, $i = 1, 2$. Suppose that there exist positive constants m, L_1, L such that

(1) $\|f'(x)\| \ge m$ *for all* $x \in \mathcal{H} \backslash \operatorname{int} \mathcal{L}_1$,

(2) $\|f'(x) - f'(y)\| \le L_1 \|x - y\|$, *for all* $x, y \in \mathcal{H} \backslash \operatorname{int} \mathcal{L}_1$,

(3) $\|f(x) - f(y)\| \le L \|x - y\|$, *for all* $x, y \in \mathcal{H} \backslash \operatorname{int} \mathcal{L}_1$.

Suppose that, for $t_0 = m/(2RL_1^2)$,

$$h(A_1, A_2) \left(1 + \frac{LL_1R}{m^2} + \frac{L}{m}\right) < \frac{m}{2} t_0, \tag{12}$$

Then $\|a_1 - a_2\| \le$

$$\le \frac{1}{1 - q} \left(\rho(A_1, A_2) + \frac{LL_1R}{m^2} h(A_1, A_2) + \frac{Lh(A_1, A_2)}{m}\right) + \frac{Lh(A_1, A_2)}{m},$$

where $q = \left(1 + \frac{m^2}{4R^2L_1^2}\right)^{-\frac{1}{2}}$.

Proof. The proof is similar to the proof of Theorem 4.1, except for the step below. Assume b, w and A_2' as defined in the proof of Theorem 4.1.

Define $b_1 = a_1$, $b_2 = b$, $x_i = b_i - tmf'(b_i)$, $t \geq 0$, $i = 1, 2$. By definition we get

$$\|x_1 - x_2\|^2 = \|b_1 - b_2\|^2 - 2t(b_1 - b_2, f'(b_1) - f'(b_2)) + t^2\|f'(b_1) - f'(b_2)\|^2. \quad (13)$$

From the convexity of the set $\mathcal{L}_1$ we have $(b_1 - b_2, f'(b_1) - f'(b_2)) \geq 0$. By the Lipschitz condition for f' we have $\|f'(b_1)-f'(b_2)\| \leq L_1\|b_1-b_2\|$, and from formula (13) we obtain that

$$\|x_1 - x_2\| \leq \|b_1 - b_2\|\sqrt{1 + t^2 L_1^2}.$$

On the other hand, $h(A_1, A_2') \leq h(A_1, A_2) + h(A_2', A_2) \leq h(A_1, A_2) + \|w - b\|$, and $\varrho(x_1, A_1) = \varrho(x_2, A_2') = tm$. Then from (12) with $t = t_0$ and (6) we have

$$\varrho(x_1, A_2') \geq \varrho(x_1, A_1) - h(A_1, A_2') \geq \frac{1}{2}tm.$$

From this and Lemma 2.1 we get

$$\|b_1 - b_2\| \leq \frac{R}{R + \frac{1}{2}tm}\|x_1 - x_2\| + \rho(A_1, A_2') \leq \frac{\sqrt{1 + t^2 L_1^2}}{1 + \frac{tm}{2R}}\|b_1 - b_2\| + \rho(A_1, A_2').$$

Taking $t = t_0$, we obtain that

$$\|a_1 - b\| = \|b_1 - b_2\| \leq \frac{1}{1 - q}\varrho(A_1, A_2'), \quad \text{where} \quad q = \frac{1}{\sqrt{1 + \frac{m^2}{4R^2 L_1^2}}}.$$

Thus, by (6) and (7): $\|a_1 - a_2\| \leq \|a_1 - b\| + \|b - a_2\| \leq$

$$\leq \frac{1}{1 - q}\left(\rho(A_1, A_2) + \frac{LL_1R}{m^2}h(A_1, A_2) + \frac{Lh(A_1, A_2)}{m}\right) + \frac{Lh(A_1, A_2)}{m}. \qquad \square$$

Remark 4.5. In the case $f(a_1) = f(a_2)$ we have by Theorem 4.4 ($b = a_2$)

$$\|a_1 - a_2\| \leq \frac{1}{1 - q}\rho(A_1, A_2), \qquad q = \frac{1}{\sqrt{1 + \frac{m^2}{4R^2 L_1^2}}}.$$

Remark 4.6. Note that it's sufficient to demand Lipschitz and other conditions for functions f in Theorems 4.1, 4.4 on the set $\mathcal{L}(\mu)\backslash \operatorname{int} \mathcal{L}(f(a_1))$, where $\mu = \sup\limits_{x \in A_1 \cup A_2} f(x)$.

Remark 4.7. If in Theorems 3.1, 3.2, 4.1, 4.4 the space $\mathcal{H}$ is separable and the sets A_1 and A_2 are values of some set-valued integral with strongly convex images, i.e. $A_i = F_0 + \int_{t_0}^{t_i} G(\tau)\, d\tau$, then in statements of all these theorems one can replace $\rho(A_1, A_2)$ with $h(A_1, A_2)$.

Indeed, in this case by Lemma 2.4 $\rho(A_1, A_2) = h(A_1, A_2)$. Strong convexity of the integral takes place under conditions of Lemma 2.5, see also [12].

5. Appendix: metrics with uniform continuity of the metric projection

Let d be some metric on the space $\mathcal{F}$.

We shall discuss the question: when for any $A, B \in \mathcal{F}$, and for any $x \in \mathcal{H}$, the following inequality holds

$$\|P_A x - P_B x\| \leq w(d(A, B)), \tag{14}$$

where $w\colon [0, +\infty) \to [0, +\infty)$ is an increasing function such that $0 = w(0) = \lim_{t \to +0} w(t)$. Thus we want to describe some metrics on $\mathcal{F}$ with uniformly continuous metric projection with respect to the set (uniformly on x).

Lemma 5.1. *Let $A_1, A_2 \in \mathcal{F}$. Then for any $\varepsilon > 0$ there exists $x \in \mathcal{H}$ with*

$$(1 - \varepsilon)\rho(A_1, A_2) \leq \|P_{A_1} x - P_{A_2} x\|.$$

Proof. Suppose that $\rho(A_1, A_2) > 0$. Fix $\varepsilon \in (0, 1)$. Take a vector $p \in \mathcal{H}$, $\|p\| = 1$, satisfies the inequality

$$h(A_1(p), A_2(p)) > \Big(1 - \frac{1}{8}\varepsilon\Big)\rho(A_1, A_2).$$

From the definition of the Hausdorff metric without loss of generality we can find a point $a_1 \in A_1(p)$ such that its metric projection $a_2 = P_{A_2(p)} a_1$ gives

$$\begin{aligned}\|a_1 - a_2\| &> \Big(1 - \frac{1}{8}\varepsilon\Big) h(A_1(p), A_2(p)) > \Big(1 - \frac{1}{8}\varepsilon\Big)^2 \rho(A_1, A_2)\\ &> \Big(1 - \frac{1}{2}\varepsilon\Big)\rho(A_1, A_2).\end{aligned}$$

For any natural number k define the point $x_k = a_1 + kp$ and the point $y_k = P_{A_2} x_k$. By definition $a_1 = P_{A_1} x_k$ for any k. Since the sequence (y_k) is bounded, we can suppose that it weakly converges to a point w. The sequence of vectors $p_k = (x_k - y_k)/(\|x_k - y_k\|)$ converge to the vector p in the strong topology and $y_k \in A_2(p_k)$. Using the norm$\times$weak closedness of graphs of subdifferentials of lower semicontinuous convex functions, we have $w \in A_2(p)$ (recall that for a set $A \subset \mathcal{F}$ and $p \in \mathcal{H}$ $A(p) = \partial s(p, A)$). From the lower weak semicontinuity of the norm in $\mathcal{H}$ and the equality $a_2 = P_{A_2(p)} a_1$ we obtain

$$\liminf \|a_1 - y_k\| \geq \|a_1 - w\| \geq \|a_1 - a_2\| \geq \left(1 - \frac{1}{2}\varepsilon\right)\rho(A_1, A_2).$$

Thus we can find a number k, for which

$$\begin{aligned}\|a_1 - y_k\| &\geq \Big(1 - \frac{1}{4}\varepsilon\Big)\|a_1 - a_2\| \geq \Big(1 - \frac{1}{4}\varepsilon\Big)\Big(1 - \frac{1}{2}\varepsilon\Big)\rho(A_1, A_2)\\ &\geq (1 - \varepsilon)\rho(A_1, A_2).\end{aligned}$$

□

We shall consider two more axioms in addition to the three standard axioms of the metric d:

(A1) $\forall \lambda \geq 0\ \forall A, B \in \mathcal{F}$: $d(\lambda A, \lambda B) = \lambda d(A, B)$,

(A2) $\forall A, B, C \in \mathcal{F}$: $d(A + C, B + C) \leq d(A, B)$.

Note that (A2) is equivalent to property (A3):

(A3) $\forall A, B, C, D \in \mathcal{F}$: $d(A + C, B + D) \leq d(A, B) + d(C, D)$.

Suppose that the metric d satisfies axiom (A1) and condition (14). Then by Lemma 5.1 for any $\varepsilon > 0$, $A, B \in \mathcal{F}$ there exists $x \in \mathcal{H}$ with

$$(1 - \varepsilon)\rho(A, B) \leq \|P_A x - P_B x\| \leq w(d(A, B)).$$

Taking the limit as $\varepsilon \to +0$, we obtain that $\rho(A, B) \leq w(d(A, B))$.

Put $\lambda = \alpha/d(A, B)$ for some $\alpha > 0$. Then by (A1) $\lambda\rho(A, B) \leq w(\alpha)$, i.e.

$$\rho(A, B) \leq \frac{w(\alpha)}{\alpha} d(A, B) \qquad \forall \alpha > 0.$$

Define $K = \inf\limits_{\alpha > 0} \frac{w(\alpha)}{\alpha}$. Then $$\rho(A, B) \leq K d(A, B) \qquad \forall A, B \in \mathcal{F}. \tag{15}$$

Moreover, taking $\lambda = \alpha/d(A, B)$ ($\alpha > 0$) and using (A1), we have

$$\lambda\|P_A x - P_B x\| = \|P_{\lambda A}\lambda x - P_{\lambda B}\lambda x\| \leq w(\lambda d(A, B)) = w(\alpha),$$

i.e. $\|P_A x - P_B x\| \leq \frac{w(\alpha)}{\alpha} d(A, B)$ for any $\alpha > 0$. Hence

$$\|P_A x - P_B x\| \leq K d(A, B).$$

We shall say that the metrics ρ and d are *uniformly equivalent*, if for any sequence $\{(A_k, B_k)\}_{k=1}^{\infty}$, where $A_k, B_k \in \mathcal{F}$ for all k, the following two implications hold

(T1) $\rho(A_k, B_k) \to 0$ implies $d(A_k, B_k) \to 0$, and

(T2) $d(A_k, B_k) \to 0$ implies $\rho(A_k, B_k) \to 0$.

From inequality (15) we see that (T2) holds. Hence, the uniform equivalence of ρ and d means (T1) and vice versa.

Theorem 5.2. *Suppose that metric d satisfies* (A1), (A2) *and* (15). *If condition* (T1) *holds, then there exists $Q > 0$ such that*

$$\forall A, B \in \mathcal{F}: \quad \rho(A, B) \geq Q d(A, B).$$

Proof. Enclose $\mathcal{F}$ into the linear space $L(\mathcal{F})$ of pairs (A, B), $A, B \in \mathcal{F}$ [16]. Endow the space $L(\mathcal{F})$ with norms $\|(A, B)\|_\rho = \rho(A, B)$ and $\|(A, B)\|_d = d(A, B)$ (due to axioms (A1) and (A3)). We have $\|(A, B)\|_\rho \leq K\|(A, B)\|_d$ for all $(A, B) \in L(\mathcal{F})$, and the completions of the normed spaces $(L(\mathcal{F}), \|\cdot\|_d)$ and $(L(\mathcal{F}), \|\cdot\|_\rho)$

coincide because of property (T1). The norms $\|\cdot\|_\rho$ and $\|\cdot\|_d$ are equivalent by the bounded inverse theorem. Thus there exists $Q > 0$ such that for all $A, B \in \mathcal{F}$

$$\rho(A, B) = \|(A, B)\|_\rho \geq Q\|(A, B)\|_d = Qd(A, B). \qquad \square$$

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