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A Uniform Approach to Hölder Calmness of Subdifferentials*

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For finite-valued convex functions f defined on the n-dimensional Euclidean space, we are interested in the set-valued mapping assigning to each pair (f, x) the subdifferential of f at x. Our approach is uniform with respect to f in the sense that it involves pairs of functions close enough to each other, but not necessarily around a nominal function. More precisely, we provide lower and upper estimates, in terms of Hausdorff excesses, of the subdifferential of one of such functions at a nominal point in terms of the subdifferential of nearby functions in a ball centered in such a point. In particular, we obtain the $(1/2)$-Hölder calmness of our mapping at a nominal pair (f, x) under the assumption that the subdifferential mapping viewed as a set-valued mapping from $\mathbb{R}^n$ to $\mathbb{R}^n$ with f fixed is calm at each point of $\{x\} \times \partial f(x)$.

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1. Introduction

Let Γ represent the set of all finite-valued convex functions $f\colon \mathbb{R}^n \longrightarrow \mathbb{R}$ (hence, functions in Γ are continuous), and consider $\mathcal{S}\colon \Gamma \times \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ given by

$$\mathcal{S}(f,x) := \partial f(x) := \{u \in \mathbb{R}^n : f(y) - f(x) \geq u'(y-x) \text{ for all } y \in \mathbb{R}^n\},$$

where elements in $\mathbb{R}^n$ are regarded as column vectors and the prime denotes transposition. We are concerned with quantifying the stability of $\mathcal{S}$ around a nominal point $x_0 \in \mathbb{R}^n$ and uniformly with respect to f. More precisely, we provide lower and upper estimates of $\partial f_1(x_0)$ by means of $\partial f_2(x_0 + \eta\mathbb{B})$, for $\eta > 0$ small enough, with $\mathbb{B}$ standing for the closed unit ball for the Euclidean norm in $\mathbb{R}^n$, and where functions f_1 and f_2 in Γ are close enough to each other with respect to the standard uniformity for the topology of uniform convergence on bounded subsets. In particular, in the last section we obtain the $(1/2)$ - Hölder calmness (definitions are given later) of $\mathcal{S}$ at (f_0, x_0) under the assumption that ∂f_0 is calm at (x_0, u) for all $u \in \partial f_0(x_0)$.

Indeed, the finite-valuedness of such functions is not essential, and we could consider convex functions $f\colon \mathbb{R}^n \longrightarrow \mathbb{R}\cup\{+\infty\}$ if we work in a ball $x_0 + \alpha\mathbb{B}$ contained in the interior (assumed non-empty) of the effective domain of f, since in such a ball f coincides with the finite-valued convex function given by

$$g(y) := \max\{f(x) + u'(y-x) : x \in x_0 + \alpha\mathbb{B},\ u \in \partial f(x)\},$$

where the maximum is attained because the set $\{(x,u) : x \in x_0 + \alpha\mathbb{B},\ u \in \partial f(x)\}$ is compact (see [15, Theorems 24.4 and 24.7]).

Throughout the paper, $\mathbb{R}^n$ is assumed to be endowed with the Euclidean norm, whereas $\mathbb{R}^{n+1}$ is equipped with the norm given by

$$\left\| \begin{pmatrix} u \\ w \end{pmatrix} \right\| = \max\{\|u\|, |w|\},\ u \in \mathbb{R}^n,\ w \in \mathbb{R}.$$

Given $0 < q \leq 1$, we say that a set-valued mapping $\mathcal{M}\colon Y \rightrightarrows X$ between metric spaces (with both distances denoted by d) is *q-Hölder calm* at $(y_0, x_0) \in \operatorname{gph}\mathcal{M}$ (the graph of $\mathcal{M}$) if there exist some neighborhoods V of y_0 and U of x_0 along with a constant $\kappa \geq 0$ such that

$$d(x, \mathcal{M}(y_0)) \leq \kappa d(y, y_0)^q \text{ for all } y \in V \text{ and all } x \in \mathcal{M}(y) \cap U. \tag{1}$$

$\mathcal{M}$ is said to be *calm* at (y_0, x_0) when it is 1 - Hölder calm at this point. In this paper we are confined to the case $X = \mathbb{R}^n$. For additional information about calmness and other related variational concepts, the reader is referred to [7, 10, 13, 16]. An alternative formulation of (1) can be given in terms of the Hausdorff excess as

$$e(\mathcal{M}(y) \cap U, \mathcal{M}(y_0)) \leq \kappa d(y, y_0)^q \text{ for all } y \in V,$$

where the *excess* of a subset A of $\mathbb{R}^n$ over another subset B is defined as

$$e(A,B) := \inf\{\varepsilon > 0 : A \subset B + \varepsilon\mathbb{B}\} = \sup\{d(x,B) : x \in A\},$$

under the convention $\inf\emptyset = +\infty$. The terminology 'Hölder calmness' is used in [1] in a more restrictive sense (with Hausdorff distance instead of excess). Our current use of the concept q - Hölder calmness in this paper is called *calmness* $[q]$ in [12] and can be checked to be equivalent to the concept of *q-Hölder metric subregularity* used in [11, 14] for the inverse mapping $\mathcal{M}^{-1}$ at (x_0, y_0), which corresponds to metric subregularity of order $1/q$ if we adopt the terminology of [8]. Recall that, for $x \in X$, $\mathcal{M}^{-1}(x) := \{y \in Y : x \in \mathcal{M}(y)\}$.

The structure of the paper is as follows: After a section of preliminaries, Section 3 provides lower Hölder estimates for the subdifferential at a given point in terms of subdifferentials of nearby functions at nearby points. These estimates are indeed uniform with respect to functions. Section 4 tackles upper estimates under calmness assumptions on the subdifferential of the nominal function f_0 at each point of $\{x_0\} \times \partial f_0(x_0)$. We finish the paper with a section of conclusions (Section 5), leading to the $(1/2)$ - Hölder calmness of $\mathcal{S}$ at the pair (f_0, x_0).

2. Preliminaries and first results

This section provides the notation and preliminary results needed later on. Recall that $\mathbb{B}$ stands for the closed unit ball with respect to the Euclidean norm in $\mathbb{R}^n$. The corresponding unit sphere is denoted by $\mathbb{S}$ and 0_n represents the zero vector in $\mathbb{R}^n$. From the topological side, $\text{int}A$, $\text{cl}A$, and $\text{bd}A$, stand, respectively, for the interior, the closure, and the boundary of $A \subset \mathbb{R}^n$; whereas $\text{cone}A$ represents the convex cone generated by A (with the convention $\text{cone}\,\emptyset := \{0_n\}$). At the beginning of the introduction we alluded to the *subdifferential* of $f \in \Gamma$ at $x \in \mathbb{R}^n$, denoted by $\partial f(x)$, and for any $A \subset \mathbb{R}^n$ we adopt the usual notation $\partial f(A) := \bigcup_{x\in A} \partial f(x)$. The *epigraph* of $f \in \Gamma$ is given by $\text{epi}f := \{(x,y) \in \mathbb{R}^n \times \mathbb{R} : y \geq f(x)\}$. Given two functions f and g, their *closed convex hull* is the lower semicontinuous convex function, denoted by $\text{cl}\,\text{conv}\{f,g\}$, whose epigraph is $\text{cl}\,\text{conv}\{\text{epi}f \cup \text{epi}g\}$. Note that the closed convex hull of two finite-valued functions can assume values of $-\infty$. By δ_A we represent the *indicator function* of the non-empy set A; i.e., $\delta_A(x) = 0$ if $x \in A$, and $\delta_A(x) = +\infty$ otherwise.

Throughout the paper we use the following specific notation: $x_0 \in \mathbb{R}^n$ is a given (fixed) point and $f_0 \in \Gamma$ is a given finite-valued convex function; for any given $\alpha > 0$, d_α denotes the supremum (pseudo-)distance on $x_0 + \alpha\mathbb{B}$; i.e., for any two functions $f_1, f_2 \in \Gamma$

$$d_\alpha(f_1, f_2) = \max_{z \in x_0 + \alpha\mathbb{B}} |f_1(z) - f_2(z)|.$$

This section is organized in four subsections. Subsection 2.3 appeals to the *Fenchel conjugate* of $f \in \Gamma$, which is the convex function $f^* : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ defined as

$$f^*(u) := \sup_{x \in \mathbb{R}^n} (u'x - f(x)).$$

In relation to Subsection 2.4, a *bornology* on a metric space X is a family $\mathcal{B}$ of subsets of X that contains the singletons, that is stable under taking subsets, and that is stable under finite unions.

2.1. On continuity properties of the subdifferential

Recall that the set-valued mapping $\mathcal{M}\colon Y \rightrightarrows X$ is said to be *upper semicontinuous* at $y_0 \in Y$ [3] if for each open set W containing $\mathcal{M}(y_0)$ there exists a neighborhood V of y_0 such that $\mathcal{M}(y) \subset W$ whenever $y \in V$. When $\mathcal{M}(y_0)$ is compact, this concept is equivalent to *Hausdorff upper semicontinuity*, which considers open sets of the form $W = \mathcal{M}(y_0) + \varepsilon\, \mathrm{int}\mathbb{B}$ for $\varepsilon > 0$.

The following proposition is a direct consequence of [15, Theorem 24.5]. For convenience we state it with closed balls and nonstrict inequalities.

Proposition 2.1. (Upper semicontinuity of $\mathcal{S}$) *Given* $\varepsilon > 0$, *there exists* $\delta > 0$

such that $$\partial f(x) \subset \partial f_0(x_0) + \varepsilon\mathbb{B};$$

provided that $f \in \Gamma$ *satisfies* $d_\alpha(f, f_0) \leq \delta$ *and* $\|x - x_0\| \leq \delta$.

Proof. Reasoning by contradiction, assume the existence of $\varepsilon > 0$, and sequences of convex functions $\{f_r\}$ and points $\{x_r\}$ verifying $d_\alpha(f_r, f_0) \leq \frac{1}{r}$ and $\|x_r - x_0\| \leq \frac{1}{r}$ for all r, such that

$$\partial f_r(x_r) \not\subset \partial f_0(x_0) + \varepsilon\mathbb{B}, \text{ for all } r.$$

Since $\{f_r\}$ converges to f_0 on $x_0 + \alpha\, \mathrm{int}\mathbb{B}$, this violates [15, Theorem 24.5]. □

Remark 2.2. (On the lower semicontinuity counterpart) One can easily check that $\partial f(x)$ and $\partial f_0(x_0)$ cannot be switched in Proposition 2.1, not even in the case $f = f_0$. Just consider $f_0(x) = |x|$, $x_0 = 0$, and approach x_0 by $x_r = \frac{1}{r}$, $r = 1, 2, \ldots$ Then $\partial f_0(x_0) = [-1, 1]$, while $\partial f(x_r) = \{1\}$.

2.2. On calmness of the subdifferential

Given any $f \in \Gamma$, the so-called *Fenchel equality* characterizes the fact that $u_0 \in \partial f(x_0)$ by the equality $f^*(u_0) + f(x_0) = u_0' x_0$. The next characterization of the calmness property of ∂f at $(x_0, u_0) \in \mathrm{gph}\partial f$ follows directly from a recent result of Aragón and Geoffroy, valid in a general reflexive Banach space [2, Corollary 4.3] together with Fenchel's equality.

Proposition 2.3. *Let* $f \in \Gamma$, $x_0 \in \mathbb{R}^n$, *and* $u_0 \in \partial f(x_0)$ *be given. Then* ∂f *is calm at* (x_0, u_0) *if and only if there exist a neighborhood* U *of* u_0 *and a positive constant* c *such that*

$$f^*(u) + f(x_0) \geq u' x_0 + c d(u, \partial f(x_0))^2 \text{ for all } u \in U. \tag{2}$$

Specifically, if ∂f *is calm at* (x_0, u_0) *with constant* κ, *then* (2) *holds for all* $c < 1/(4\kappa)$; *conversely, if* (2) *holds with constant* c, *then* ∂f *is calm at* (x_0, u_0) *with constant* $1/c$.

Example 2.4. Let $f(x) = |x|$ as a convex function on $\mathbb{R}$. Since $\partial f(0) = [-1, 1]$ contains $\partial f(x)$ for each $x \in \mathbb{R}$, it is immediate that ∂f is calm at $(0, u_0)$ for each $u_0 \in \partial f(0)$. We now use Proposition 2.2 to independently check this.

As is well-known, f^* is the indicator function of $[-1, 1]$, that is, of the closed unit ball. We claim that whenever $u_0 \in \partial f(0)$, then

$$f^*(u) + f(0) \geq u \cdot 0 + cd(u, [-1, 1])^2$$

holds for all u close to u_0 no matter what $c > 0$ may be. If $u_0 \in\ ]-1, 1[$, take $U =]-1, 1[$ itself; then for each $u \in U$, both sides of the inequality are zero. The remaining cases $u_0 = 1$ and $u_0 = -1$ are handled similarly, so we just look at the first. Let $U =]\frac{1}{2}, \frac{3}{2}[$; if $u \in]1, \frac{3}{2}[$ the left-hand side of the inequality is $+\infty$, thus exceeding the right-hand side. On the other hand, if $u \in]\frac{1}{2}, 1]$, both sides of the inequality are again zero.

Example 2.5. As a second illustration, consider $f \colon \mathbb{R} \to \mathbb{R}$ given by $f(x) = |x|^p$, with $p > 1$. Then, by appealing to Example 1 in [15, p. 106] (see also [15, Theorem 16.1]), we have $f^*(u) = (p-1)\, p^{-q} |u|^q$, with $1/p + 1/q = 1$, and conclude that ∂f is calm at $(0, 0)$ if and only if $p \geq 2$. In such a case, for $U = \mathbb{B}$ *we* can take $c = (p-1)\, p^{-q}$.

2.3. A linear semi-infinite approach to the subdifferential

It is easy to check that, for any $f \in \Gamma$ and any $x_0 \in \mathbb{R}^n$, $\partial f(x_0)$ may be seen as the feasible set of the linear semi-infinite inequality system (LSIS for short), in the variable $u \in \mathbb{R}^n$,

$$\sigma(x_0) := \left\{ (t - x_0)' u \leq f(t) - f(x_0),\ t \in x_0 + \alpha\mathbb{B} \right\},$$

with $\alpha > 0$ being arbitrary. LSIS have been extensively studied in [9]. In particular, [9, Corollary 6.2.1] provides an alternative way of establishing Proposition 2.1, due to the boundedness $\partial f(x_0)$.

The calmness of the feasible set mapping of *continuous* LSIS of the form $\sigma = \{a_t' u \leq b_t,\ t \in T\}$, where 'continuous' means that T is a compact Hausdorff space and $t \mapsto \binom{a_t}{b_t}$ is continuous on T (see [9, Section 6.4]), has been characterized in [5, Theorem 3 and Corollary 2]. Observe that our system $\sigma(x_0)$ is continuous in the referred sense. However, the kind of perturbations considered in [5, Section 5] –where the parameter space is the set $C(T, \mathbb{R}^n \times \mathbb{R})$ of all continuous functions $t \mapsto \binom{a_t}{b_t}$ endowed with the supremum norm– is much more general than those perturbations providing systems of the form $\sigma(x)$ for x close to x_0. Accordingly, the sufficient condition for the calmness of ∂f at $(x_0, u_0) \in \operatorname{gph} \partial f$ derived from applying this LSIS approach is far from being necessary.

Specifically, writing $(a, b) \in C(T, \mathbb{R}^n \times \mathbb{R})$ instead of $\{\binom{a_t}{b_t}\}_{t \in T}$ for simplicity, and considering the feasible set mapping $\mathcal{F} \colon C(T, \mathbb{R}^n \times \mathbb{R}) \rightrightarrows \mathbb{R}^n$ given by

$$\mathcal{F}(a, b) := \{u \in \mathbb{R}^n : a_t' u \leq b_t,\ t \in T\},$$

the aforementioned [5, Theorem 3] (inspired by [18, Theorem 2.2]), together with [5, Corollary 2], characterizes the calmness of $\mathcal{F}$ at $((a_0, b_0), u_0) \in \operatorname{gph}\mathcal{F}$ in terms of the fulfillment by parameter (a_0, b_0) of the Abadie constraint qualification (ACQ) and the so-called 'uniform dual boundedness' (see [5] for details), both for u *around* u_0. Moreover, a direct application of [5, Corollary 1] to our current $\sigma(x_0)$ provides the following characterization of ACQ *at* any given $u \in \partial f(x_0)$:

$$(\operatorname{cl}\mathcal{L}(x_0)) \cap \left\{\begin{pmatrix} u \\ -1 \end{pmatrix}\right\}^{\perp} = \operatorname{cl}\left(\mathcal{L}(x_0) \cap \left\{\begin{pmatrix} u \\ -1 \end{pmatrix}\right\}^{\perp}\right), \tag{3}$$

where $\mathcal{L}(x_0)$ is the so-called *second-moment cone*, defined by (see, e.g., [9, p. 81])

$$\mathcal{L}(x_0) := \operatorname{cone}\left\{\begin{pmatrix} t - x_0 \\ f(t) - f(x_0) \end{pmatrix} : t \in x_0 + \alpha\mathbb{B}\right\},$$

and $\perp$ means orthogonal with respect to the usual inner product in $\mathbb{R}^{n+1}$. Observe that condition (3) holds easily for polyhedral functions, but fails for general finite-valued convex functions, as the following example suggests.

Example 2.6. Let $f\colon \mathbb{R} \longrightarrow \mathbb{R}$ be given by $f(x) := \max\{x, \frac{1}{2}x^2\}$ and take $x_0 = 0$. We have $\partial f(0) = [0, 1]$ and, if $g(x) := \frac{1}{2}x^2$, [15, Theorem 16.5] yields

$$f = \operatorname{cl}\operatorname{conv}\left\{\delta_{\{1\}}, g^*\right\},$$

with $g^*(u) := \frac{1}{2}u^2$, i.e.,
$$f(u) = \begin{cases} u^2/2, & \textit{if } u < 0 \textit{ or } u > 2, \\ 0, & \textit{if } 0 \le u \le 1, \\ 2(u-1), & \textit{if } 1 < u \le 2. \end{cases}$$

Accordingly, Proposition 2.3 shows that ∂f is calm at $(0, u_0)$ for any $u_0 \in [0, 1]$. More specifically, (2) holds for $u_0 = 1$ and any $c > 0$ on $U =]0, 1 + 2/c[$, and for $u_0 = 0$ and any $c \le 1/2$ on $U =]-\infty, 1[$. On the other hand, (3) holds for all $u \in]0, 1]$ but fails for $u = 0$. More in detail, both sides of (3) are $\{0_2\}$ for $u \in]0, 1[$, both are cone $\left\{\binom{1}{1}\right\}$ for $u = 1$, whereas for $u = 0$ the left-hand side of (3) is cone $\left\{\binom{-1}{0}\right\}$ and the right-hand side is $\{0_2\}$.

2.4. A uniform approach with respect to functions

Since the pointwise convergence of sequences of functions in Γ entails their uniform convergence on bounded subsets [15, Theorem 10.8], it is appropriate to use the standard uniformity of uniform convergence on bounded subsets in the sequel to obtain estimates. We develop the relevant terminology and structure in a deliberately leisurely fashion.

Let $\mathbf{V}$ be a family of real functions on $\mathbb{R}^n$. Given a cover $\mathcal{A}$ of $\mathbb{R}^n$, if a net of functions in $\mathbf{V}$ converges uniformly on members of $\mathcal{A}$ to a function $f \in \mathbf{V}$, then it obviously converges pointwise to f and the convergence is uniform on each member of the smallest bornology containing $\mathcal{A}$. Thus, pointwise convergence - that is, uniform convergence on the singleton subsets - amounts to uniform convergence on finite subsets, and uniform convergence on all balls with a common center amounts to uniform convergence on bounded subsets. Similarly, uniform convergence on compact subsets of $\mathbb{R}^n$ amounts to uniform convergence on bounded subsets without any assumptions on $\mathbf{V}$ at all.

Given a fixed bornology $\mathcal{B}$, the *uniformity of uniform convergence* on members of $\mathcal{B}$ for the functions in $\mathbf{V}$ has for its standard base all entourages of the form

$$\{(f,g) \in \mathbf{V} \times \mathbf{V} : \forall x \in B,\ |f(x) - g(x)| < \delta\}$$

where B runs over $\mathcal{B}$ and δ runs over the positive reals. In fact, we can replace the bornology by any subfamily $\mathcal{B}_0$ that is cofinal with respect to inclusion in it (i.e., if for every $B \in \mathcal{B}$ there is a $B_0 \in \mathcal{B}_0$ such that $B \subset B_0$) to adequately describe the uniformity.

Here, we initially look at the family of real-valued functions on $\mathbb{R}^n$ that are bounded on bounded subsets (this forms a vector space) and where $\mathcal{B}$ is the family of bounded subsets. Of course, to describe the uniformity, $\mathcal{B}$ can be replaced by any subfamily that is cofinal in $\mathcal{B}$ with respect to inclusion. Since we can find a countable such cofinal subfamily, then we have a countable base for the uniformity itself. Since the uniformity is separated, it follows from basic uniform space theory that there is a metric ρ on the function space whose induced uniformity is the given one [17, Theorem 38.3]. Actually, it is easy to write down a formula for such a metric. Here is one way to do it:

$$\rho(f,g) := \sum_{k=1}^{\infty} \frac{1}{2^k} \min\ \{d_k(f,g), 1\}, \tag{4}$$

where for $\alpha > 0,\ d_\alpha(f,g) = \sup\{|f(x) - g(x)| : ||x - x_0|| \leq \alpha\}$ and x_0 is a fixed point. Note that the functions are assumed bounded on bounded subsets and that is why we get finite suprema to be used in our formula. While classical, such a formula is not really helpful for our purposes, and it is easier to work with a base of the uniformity directly. Moreover, it is well-known (see, e.g., [6, Lemma 3.2]) that, for each $\varepsilon > 0$, there exist $k \in \mathbb{N}$ and $\delta > 0$ such that $\rho(f,g) < \varepsilon$ for each pair of functions (f,g) satisfying $d_k(f,g) < \delta$.

3. Lower Hölder type estimates for subdifferentials

As is well-known, finite-valued convex functions on $\mathbb{R}^n$ are locally Lipschitz, and locally Lipschitz functions on a general metric space are precisely those functions that are Lipschitz when restricted to relatively compact subsets (see, e.g., Theorem 4.2 of [4]). Thus, elements of Γ are Lipschitz and bounded when restricted to bounded sets of $\mathbb{R}^n$.

Given $x_0 \in \mathbb{R}^n,\ \alpha > 0$ and $\delta > 0$, put

$$\mathcal{U}(x_0, \alpha, \delta) := \{(f,g) \in \Gamma \times \Gamma : d_\alpha(f,g) < \delta\}.$$

Varying both δ and α over $]0, +\infty[$, such sets form a base for the uniformity of uniform convergence on bounded subsets on the space Γ of finite convex functions.

Of course, given a fixed finite-valued convex function f_0, all sets of the form

$$\mathcal{U}(x_0, \alpha, \delta)(f_0) := \{f \in \Gamma : d_\alpha(f, f_0) < \delta\} = \{f \in \Gamma : (f, f_0) \in \mathcal{U}(x_0, \alpha, \delta)\}$$

where α and δ run over the positive reals form a local base at f_0 for the induced topology of uniform convergence on bounded subsets for Γ.

In the next lemma we appeal to the concept of ε-subdifferential of $f \in \Gamma$, defined for any $\varepsilon > 0$ and $x \in \mathbb{R}^n$ as

$$\partial_\varepsilon f(x) := \{u \in \mathbb{R}^n : f(y) - f(x) \geq u'(y - x) - \varepsilon \text{ for all } y \in \mathbb{R}^n\}.$$

Lemma 3.1. *Let f_0 be a finite-valued convex function. Let $x_0 \in \mathbb{R}^n$ and let $\delta > 0$ and $\lambda > 0$ be arbitrary. Then $\bigcup\{\partial_\delta f(x_0) : f \in \mathcal{U}(x_0, 1, \lambda)(f_0)\}$ is a bounded subset of $\mathbb{R}^n$.*

Proof. Suppose boundedness fails. Then for each $r \in \mathbb{N}$ there exists $f_r \in \mathcal{U}(x_0, 1, \lambda)(f_0)$ and $u_r \in \partial_\delta f_r(x_0)$ with $||u_r|| > r$. By definition, for each r the function f_r majorizes $x \mapsto f_r(x_0) + u_r'(x - x_0) - \delta$. For each r, put $x_r = x_0 + \frac{u_r}{||u_r||} \in x_0 + \mathbb{B}$. Since $f_r(x_0) > f_0(x_0) - \lambda$, the function f_r majorizes

$$x \mapsto f_0(x_0) + u_r'(x - x_0) - \delta - \lambda,$$

but $f(x_0) + u_r'(x_r - x_0) = f(x_0) + ||u_r||$ can be made arbitrarily large, violating the uniform boundedness of $\mathcal{U}(x_0, 1, \lambda)(f_0)$ on $x_0 + \mathbb{B}$ guaranteed by the compactness of closed balls, a contradiction. □

Remark 2.2 speaks to the necessity of enlarging $\partial f(x)$ in some way to obtain a valid lower semicontinuity type condition. We offer the following result.

Theorem 3.2. *Let f_0 be a finite-valued convex function, $x_0 \in \mathbb{R}^n$, and let $\delta_0 > 0$. Then there exists $\alpha > 1$ such that whenever $\delta \in]0, \delta_0[$ and $f \in \mathcal{U}(x_0, \alpha, \delta/2)(f_0)$,*

we have
$$\partial f_0(x_0) \subset \partial f(x_0 + \sqrt{\delta}\mathbb{B}) + 2\sqrt{\delta}\mathbb{B}.$$

Proof. Since the vector difference of two bounded sets is bounded, by the previous lemma $\partial_\delta f_0(x_0) - \partial_\delta f(x_0)$ where f runs over $\mathcal{U}(x_0, 1, \delta_0)(f_0)$ is bounded and thus there exists $\alpha > 1$ such that $\alpha\mathbb{B}$ contains all such vectors. Let $\delta \in]0, \delta_0[$ be arbitrary. Since

$$\mathcal{U}(x_0, \alpha, \delta/2)(f_0) \subset \mathcal{U}(x_0, 1, \delta_0)(f_0),$$

by the choice of α, whenever $f \in \mathcal{U}(x_0, \alpha, \delta/2)(f_0)$ and $u \in \partial_\delta f(x_0)$ and $u_0 \in \partial f_0(x_0)$, we have

$$u_0 - u \in \partial_\delta f_0(x_0) - \partial_\delta f(x_0) \subset \alpha\mathbb{B}.$$

Next, let $f \in \mathcal{U}(x_0, \alpha, \delta/2)(f_0)$ and $u_0 \in \partial f_0(x_0)$ be arbitrary. Since $d_\alpha(f, f_0) < \frac{\delta}{2}$, it is clear that $f|_{x_0+\alpha\mathbb{B}}$ majorizes the restricted affine functional h defined by

$$h(x) = f_0(x_0) + u_0'(x - x_0) - \frac{\delta}{2}, \quad x \in x_0 + \alpha\mathbb{B}.$$

By the separation theorem, we can separate epi$f$ from the compact convex set $\{(x, h(x)) : x \in x_0 + \alpha\mathbb{B}\}$ by a nonvertical hyperplane that can be viewed as the graph of an affine functional g of the form $g(x) = u'(x - x_0) + \beta$ where $f_0(x_0) - \frac{\delta}{2} \leq \beta \leq f(x_0) < f_0(x_0) + \frac{\delta}{2}$.

From these estimates it follows that $f(x_0) - \beta < \delta$ and so $u \in \partial_\delta f(x_0)$. We next intend to show that $||u_0 - u|| \leq \sqrt{\delta}$. If this fails, we still have by the choice of α that $x := x_0 + u_0 - u \in x_0 + \alpha\mathbb{B}$ (take into account that $u_0 \in \partial f_0(x_0) \subset \partial_\delta f_0(x_0)$). We compute

$$\begin{aligned} g(x) &= u'(x - x_0) + \beta = u'(u_0 - u) + \beta \\ &= (u - u_0)'(u_0 - u) + u_0'(u_0 - u) + \beta \\ &< -\delta + u_0'(u_0 - u) + f_0(x_0) + \frac{\delta}{2} \\ &= f_0(x_0) + u_0'(x - x_0) - \frac{\delta}{2} = h(x). \end{aligned}$$

This contradicts the fact that g majorizes h on $x_0 + \alpha\mathbb{B}$, and establishes the claim.

Since $u \in \partial_\delta f\,(x_0)$, we now invoke the Brønsted-Rockafellar Theorem (see, e.g., [3, Theorem 8.3.3]) to find x_1 and $u_1 \in \partial f(x_1)$ with $||x_0 - x_1|| \leq \sqrt{\delta}$ and also $||u - u_1|| \leq \sqrt{\delta}$. By the triangle inequality $||u_0 - u_1|| \leq 2\sqrt{\delta}$. Consequently $u_0 \in \partial f(x_1) + 2\sqrt{\delta}\mathbb{B}$ as required. □

One would like to have uniform estimates applicable to all (f, g) in some basic entourage rather than to all f sufficiently close to some nominal function f_0. This could be obtained from our proof technique if we knew that, for all (f, g) in some basic entourage $\mathcal{U}(x_0, \alpha, \lambda)$,

$$\{u_f - u_g : u_f \in \partial f(x_0),\ u_g \in \partial_\delta g(x_0)\}$$

were a uniformly bounded family of sets. Unfortunately this fails even in the case $n = 1$, as the following example shows.

Example 3.3. For each $r \in \mathbb{N}$ we give two functions $f, g \in \Gamma$ whose global uniform distance between them is $1/r$ yet there exist $u_f \in \partial f(0)$ and $u_g \in \partial g(0)$ with $|u_f - u_g| > r$. For a fixed r, let f be defined by $f(x) = 2r|x|$ and let g be piecewise defined by

$$g(x) = \begin{cases} r^3x^2 & \text{if } |x| \leq \frac{1}{r^2} \\ \frac{1}{r} + 2r\left(x - \frac{1}{r^2}\right) & \text{if } x > \frac{1}{r^2} \\ \frac{1}{r} - 2r\left(x + \frac{1}{r^2}\right) & \text{if } x < -\frac{1}{r^2}. \end{cases}$$

Note that both f, g are convex and that $f \geq g$. The uniform distance of $\frac{1}{r}$ between f and g is achieved at each point of $]-\infty, -\frac{1}{r^2}] \cup [\frac{1}{r^2}, \infty[$ while $\partial f(0) = [-2r, 2r]$ and $\partial g(0) = \{0\}$.

3.1. Uniform lower Hölder type estimates

We can obtain indeed a uniform estimate for pairs of functions in any basic uniform entourage $\mathcal{U}(x_0, \alpha, \delta)$ with $\delta \leq \alpha^2$. The price to pay is a larger constant than in Theorem 3.2.

Theorem 3.4. *Consider* $0 < \delta \leq \alpha^2$. *If* $f_1, f_2 \in \Gamma$ *are such that* $d_\alpha(f_1, f_2) \leq \delta$,

then
$$\partial f_1(x_0) \subset \partial f_2\left(x_0 + \sqrt{\delta}\,\mathbb{B}\right) + 4\sqrt{\delta}\,\mathbb{B}\,.$$

In other words, $e\left(\partial f_1(x_0), \partial f_2\left(x_0 + \sqrt{\delta}\,\mathbb{B}\right)\right) \leq 4\sqrt{\delta}\,.$

Proof. Consider any pair of finite-valued convex functions f_1 and f_2 such that $d_\alpha(f_1, f_2) \leq \delta$. Take any $u \in \partial f_1(x_0)$ and let us see that

$$u \in \partial \widetilde{f_2}(x_1) \quad \text{for some } x_1 \in x_0 + \sqrt{\delta}\,\mathbb{B},$$

where $\widetilde{f_2} \colon \mathbb{R}^n \to \mathbb{R}$ is defined by

$$\widetilde{f_2}(z) := f_2(z) + 2\|z - x_0\|^2, \ z \in \mathbb{R}^n.$$

Remember that $\mathbb{S}$ denotes the unit sphere of $\mathbb{R}^n$ for the Euclidean norm. For any $z \in x_0 + \sqrt{\delta}\,\mathbb{S}$,

$$\begin{aligned}
\widetilde{f_2}(z) - \widetilde{f_2}(x_0) &= f_1(z) - f_1(x_0) + \widetilde{f_2}(z) - f_1(z) + f_1(x_0) - \widetilde{f_2}(x_0) \\
&\geq u'(z - x_0) + f_2(z) - f_1(z) + 2\|z - x_0\|^2 + f_1(x_0) - f_2(x_0) \\
&\geq u'(z - x_0) + 2\delta - 2d_\alpha(f_1, f_2) \geq u'(z - x_0) + 2\delta - 2\delta = u'(z - x_0)\,.
\end{aligned}$$

Now consider the convex function $g \colon \mathbb{R}^n \to \mathbb{R}$, given by

$$g(z) := \widetilde{f_2}(z) - \widetilde{f_2}(x_0) - u'(z - x_0), \ z \in \mathbb{R}^n.$$

Observe that
$$g(x_0) = 0, \quad \text{and} \quad g(z) \geq 0 \text{ for all } z \in x_0 + \sqrt{\delta}\,\mathbb{S}.$$

Take a minimizer of g on the compact set $x_0 + \sqrt{\delta}\,\mathbb{B}$, say y. If y lies in the interior of $x_0 + \sqrt{\delta}\,\mathbb{B}$, then $x_1 := y$ is a global minimizer of g (by convexity). Otherwise, if $y \in x_0 + \sqrt{\delta}\,\mathbb{S}$, then the minimum of g on $x_0 + \sqrt{\delta}\,\mathbb{B}$ is zero and $x_1 := x_0$ is also a global minimizer of g. In both cases

$$0_n \in \partial g(x_1) = \partial \widetilde{f_2}(x_1) - u, \quad \text{with } x_1 \in x_0 + \sqrt{\delta}\,\mathbb{B}.$$

So, $u \in \partial \widetilde{f_2}(x_1)$. Finally, we have

$$u \in \partial \widetilde{f_2}(x_1) = \partial f_2(x_1) + 4(x_1 - x_0) \in \partial f_2\left(x_0 + \sqrt{\delta}\mathbb{B}\right) + 4\sqrt{\delta}\,\mathbb{B}.$$

Since u has been arbitrarily chosen, we have

$$\partial f_1(x_0) \subset \partial f_2\left(x_0 + \sqrt{\delta}\,\mathbb{B}\right) + 4\sqrt{\delta}\,\mathbb{B}. \qquad \square$$

The following corollary will be useful in the next section.

Corollary 3.5. *Consider* $0 < \delta \leq \frac{\alpha^2}{4}$. *If* $f_1, f_2 \in \Gamma$ *are such that* $d_\alpha(f_1, f_2) \leq \delta$,

then
$$\partial f_1\left(x_0 + \sqrt{\delta}\,\mathbb{B}\right) \subset \partial f_2\left(x_0 + 2\sqrt{\delta}\,\mathbb{B}\right) + 4\sqrt{\delta}\,\mathbb{B}.$$

Proof. Fix any positive δ such that $\delta \leq \frac{\alpha^2}{4}$. Take two finite convex functions, f_1 and f_2, such that $d_\alpha(f_1, f_2) \leq \delta$. Consider any $x_1 \in x_0 + \sqrt{\delta}\,\mathbb{B}$ ($\subset x_0 + \frac{\alpha}{2}\mathbb{B}$), and let us see that

$$\partial f_1(x_1) \subset \partial f_2\left(x_0 + 2\sqrt{\delta}\,\mathbb{B}\right) + 4\sqrt{\delta}\,\mathbb{B}.$$

Observe that $x_1 + \frac{\alpha}{2}\mathbb{B} \subset x_0 + \alpha\mathbb{B}$, and then

$$\sup_{z \in x_1 + \frac{\alpha}{2}\mathbb{B}} |f_1(z) - f_2(z)| \leq d_\alpha(f_1, f_2) \leq \delta \leq \left(\frac{\alpha}{2}\right)^2.$$

Now, apply the previous theorem with $\frac{\alpha}{2}$ instead of α and x_1 instead of x_0 to conclude

$$\partial f_1(x_1) \subset \partial f_2\left(x_1 + \sqrt{\delta}\,\mathbb{B}\right) + 4\sqrt{\delta}\,\mathbb{B} \subset \partial f_2\left(x_0 + 2\sqrt{\delta}\,\mathbb{B}\right) + 4\sqrt{\delta}\,\mathbb{B}. \qquad \square$$

4. Upper Hölder type estimates for subdifferentials

In the following theorem X and Y are metric spaces, where we write for simplicity d for the metric of both. Suppose that $\mathcal{M}\colon Y \rightrightarrows X$ is a set-valued mapping. Let $y_0 \in Y$ be a point at which $\mathcal{M}$ is nonempty compact-valued and upper semicontinuous. We will show that calmness of $\mathcal{M}$ at each point (y_0, x) where x runs over $\mathcal{M}(y_0)$ can be interpreted as a local upper Lipschitz condition for $\mathcal{M}$ at y_0.

Theorem 4.1. *Let* $\mathcal{M}\colon Y \rightrightarrows X$ *be such that at a particular* y_0 *we have* $\mathcal{M}(y_0) \neq \emptyset$ *is compact and* $\mathcal{M}$ *is upper semicontinuous at* y_0. *The following conditions are equivalent:*

(i) *for all* $x \in \mathcal{M}(y_0), \mathcal{M}$ *is calm at* (y_0, x);

(ii) *there exists* $\kappa > 0$ *and a neighborhood* V *of* y_0 *such that for all* $y \in V$,

$$e(\mathcal{M}(y), \mathcal{M}(y_0)) \leq \kappa d(y, y_0).$$

Proof. (i) ⇒ (ii). For each $x \in \mathcal{M}(y_0)$, choose neighborhoods $W(x)$ and $V_x(y_0)$ so that $\mathcal{M}$ is calm at (y_0, x) on $V_x(y_0) \times W(x)$ with constant $\kappa(x)$. By compactness of $\mathcal{M}(y_0)$, we can find $x_1, \ldots, x_n$ such that $\mathcal{M}(y_0) \subset W := \cup_{j=1}^n W(x_j)$. By upper semicontinuity, we can find a neighborhood V of y_0 contained in $\cap_{j=1}^n V_{x_j}(y_0)$ such that $\mathcal{M}(V) \subset W$. Put $\kappa = \max_{1\leq j\leq n} \kappa(x_j)$. We claim that these choices for V and κ do the job.

Let $y \in V$ and $x \in \mathcal{M}(y)$ be arbitrary. Since $x \in W$, there exists x_j with $x \in W(x_j)$. By calmness at (y_0, x_j) and since $y \in V_{x_j}(y_0)$, we compute

$$d(x, \mathcal{M}(y_0)) \leq \kappa(x_j)d(y, y_0) \leq \kappa d(y, y_0).$$

This shows $e(\mathcal{M}(y), \mathcal{M}(y_0)) \leq \kappa d(y, y_0)$ as required.

(ii) $\Rightarrow$ (i). Let V and κ be as given in condition (ii). We claim that for each $x \in \mathcal{M}(y_0)$, calmness of $\mathcal{M}$ occurs at (y_0, x) with constant κ on $V \times X$. Let $z \in \mathcal{M}(y) = X \cap \mathcal{M}(y)$ where $y \in V$. Then, by (ii),

$$d(z, \mathcal{M}(y_0)) \leq e(\mathcal{M}(y), \mathcal{M}(y_0)) \leq \kappa d(y, y_0). \qquad \square$$

At this point we recall Proposition 2.3 in relation to the next results.

Corollary 4.2. *Given $f_0 \in \Gamma$ and $x_0 \in \mathbb{R}^n$, the following conditions are equivalent:*

(i) *∂f_0 is calm at (x_0, u) for every $u \in \partial f_0(x_0)$;*

(ii) *there exists $\kappa_0 > 0$ and $\delta_0 > 0$ such that $0 < \delta < \delta_0$ implies*

$$\partial f_0(x_0 + \delta\mathbb{B}) \subset \partial f_0(x_0) + \kappa_0\delta\mathbb{B}.$$

Theorem 4.3. *Let $f_0 \in \Gamma$ and $\alpha > 0$. Assume that ∂f_0 is calm at (x_0, u) for every $u \in \partial f_0(x_0)$. Then, there exist $\kappa_1 > 0$ and $\delta_1 > 0$ such that for any $0 < \delta < \delta_1$ and any convex function $f : \mathbb{R}^n \to \mathbb{R}$ verifying $d_\alpha(f, f_0) \leq \delta$ we have*

$$\partial f\left(x_0 + \sqrt{\delta}\,\mathbb{B}\right) \subset \partial f_0(x_0) + \kappa_1\sqrt{\delta}\,\mathbb{B}.$$

In other words, recalling the mapping $\mathcal{S}$ in Section 1,

$$\left.\begin{array}{r} d_\alpha(f, f_0) \leq \delta \\ \|x - x_0\| \leq \sqrt{\delta} \end{array}\right\} \Rightarrow e\left(\mathcal{S}(f, x), \mathcal{S}(f_0, x_0)\right) \leq \kappa_1\sqrt{\delta}.$$

Proof. Take $\kappa_0, \delta_0 > 0$ as in Corollary 4.2(ii). Let $\delta_1 := \min\{\frac{\delta_0^2}{4}, \frac{\alpha^2}{4}\}$. Consider f and δ such that $d_\alpha(f, f_0) \leq \delta < \delta_1$. Then, by combining Corollaries 3.5 and 4.2(ii), one has

$$\partial f\left(x_0 + \sqrt{\delta}\,\mathbb{B}\right) \subset \partial f_0\left(x_0 + 2\sqrt{\delta}\,\mathbb{B}\right) + 4\sqrt{\delta}\,\mathbb{B} \subset \partial f_0(x_0) + (2\kappa_0 + 4)\sqrt{\delta}\,\mathbb{B}.$$

(Observe that $2\sqrt{\delta} < 2\sqrt{\delta_1} < \delta_0$). Finally, just take $\kappa_1 := 2\kappa_0 + 4$. $\square$

Combining the previous theorem with Theorem 3.4, and recalling the definition of Hausdorff distance [3] between nonempty closed subsets of $\mathbb{R}^n$ (in our case these subsets are indeed compact and convex), given as

$$d_H(A, B) := \max\{e(A, B), e(B, A)\},$$

we obtain the following corollary.

Corollary 4.4. *Let* $\alpha > 0$ *and assume that* ∂f_0 *is calm at* (x_0, u) *for any* $u \in \partial f_0(x_0)$. *Then, there exist* $\kappa_1 > 0$ *and* $\delta_1 > 0$ *such that for any* $0 < \delta < \delta_1$ *and any function* $f \in \Gamma$ *verifying* $d_\alpha(f, f_0) \leq \delta$, *we have*

$$d_H\left(\partial f\left(x_0 + \sqrt{\delta}\,\mathbb{B}\right), \partial f_0(x_0)\right) \leq \kappa_1\sqrt{\delta}.$$

5. Conclusions

Keeping the notation of the previous sections, the main results of this paper are Theorems 3.2, providing a lower Hölder type estimation for $\partial f_0(x_0)$ by means of subdifferentials of nearby functions in a ball around x_0; Theorem 3.4, of uniform nature with respect to functions (i.e., valid for pair of functions close enough to each other but not necessarily around a nominal function f_0); and Theorem 4.3 providing an upper Hölder type estimation. From the last one we can directly derive the $(1/2)$ - Hölder calmness – recall (1) – of the mapping $\mathcal{S}$ defined at the beginning of Section 1 at (f_0, x_0). To do this we can consider $\Gamma \times \mathbb{R}^n$ endowed with the pseudometric given by

$$d((f,x),(f_0,x_0)) := \max\{d_\alpha(f,f_0), \|x - x_0\|\}, \text{ for a certain } \alpha > 0.$$

Note that the notion of q - Hölder calmness, although defined in Section 1 for metric spaces, also makes sense for pseudometric ones (where distances between different elements may be zero). An alternative to this consists working directly with the metric ρ given in (4) instead of d_α. In summary, we have the following result:

Theorem 5.1. *Let* $(f_0, x_0) \in \Gamma \times \mathbb{R}^n$. *Assume that* ∂f_0 *is calm at* (x_0, u) *for any* $u \in \partial f_0(x_0)$. *Then, there exist* $\kappa_1 > 0$ *and* $0 < \delta_1 \leq 1$ *such that for any* $(f, x) \in \Gamma \times \mathbb{R}^n$ *verifying* $d((f,x),(f_0,x_0)) \leq \delta_1$ *we have*

$$e(\mathcal{S}(f,x), \mathcal{S}(f_0,x_0)) \leq \kappa_1\sqrt{d((f,x),(f_0,x_0))}.$$

In other words, $\mathcal{S}$ *is* $(1/2)$ *- Hölder calm at* (f_0, x_0).

Proof. Just apply the last assertion in the statement of Theorem 4.3 with $\delta = d((f,x),(f_0,x_0))$, by observing that $0 < \delta \leq 1$ entails $\delta \leq \sqrt{\delta}$. □

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