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Ioffe-Type Criteria in Extended Quasi-Metric Spaces*

Radek Cibulka
NTIS and Department of Mathematics, Faculty of Applied Sciences
University of West Bohemia, 306 14 Pilsen, Czech Republic
cibi@kma.zcu.cz

Tomáš Roubal
NTIS and Department of Mathematics, Faculty of Applied Sciences
University of West Bohemia, 306 14 Pilsen, Czech Republic
roubalt@ntis.zcu.cz

Dedicated to Alexander Ioffe on the occasion of his 80th birthday.

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We study various regularity properties, including subregularity and semiregularity, of set-valued mappings acting in extended quasi-metric spaces. It turns out that this abstract framework allows to unify criteria for the usual (sub/semi) regularity as well as their directional and Hölder counterparts. Ioffe-type critera are obtained by applying a general version of the Ekeland's variational principle. We provide a self-contained material gathering and extending the existing theory on the topic. We (briefly) illustrate the importance of these criteria for applications. For example, we consider local convergence of a directional version of a Newton-type method for solving a generalized equation which may be applied when the usual (non-directional) regularity does not hold.

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1. Introduction

A fundamental concept in modern variational analysis and non-smooth optimization is *regularity* of a set-valued mapping F acting from a metric space (X, d) into (subsets of) another metric space (Y, ϱ), denoted by $F\colon X \rightrightarrows Y$, around a prescribed reference point $(\bar{x}, \bar{y})$ in its graph $\operatorname{gph} F$, see, for example, books [1, 7, 17, 18]. The term regularity means that the mapping under consideration obeys one of the three equivalent properties – metric regularity, openness with a linear rate around the reference point, and pseudo-Lipschitz property of the

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inverse. Let us focus on the second property. The mapping F is called *open with a linear rate* around $(\bar{x}, \bar{y})$ when $\bar{y} \in F(\bar{x})$ and there are positive constants c and ε along with a neighborhood $U \times V$ of $(\bar{x}, \bar{y})$ in $X \times Y$ such that

$$F(\mathbb{B}_X(x,t)) \supset \mathbb{B}_Y(y,ct) \text{ for each } (x,y) \in U \times V,\ y \in F(x), \text{ and } t \in (0,\varepsilon), \quad (1)$$

where $\mathbb{B}_Z(z,r)$ and $\mathbb{B}_Z[z,r]$ denote the open and the closed ball in the metric space Z centered at z with a radius $r > 0$, respectively. The supremum of $c > 0$ for which there exist a constant $\varepsilon > 0$ and a neighborhood $U \times V$ of $(\bar{x}, \bar{y})$ in $X \times Y$ such that (1) holds is called the *modulus of surjection* of F around $(\bar{x}, \bar{y})$ and is denoted by $\operatorname{sur} F(\bar{x}, \bar{y})$. There are two basic ways of weakening the regularity property by fixing one of the points involved in the definition (see [4] for a survey on various names of the properties as well as for the historical background).

The first resulting property, which has attracted a lot of attention during the last decades, is called *subregularity* meaning that the mapping under consideration obeys one of the three equivalent properties – metric subregularity, pseudo–openness with a linear rate at the reference point, and calmness of the inverse. Recall that F is said to be *pseudo-open with a linear rate at* $(\bar{x}, \bar{y})$ when $\bar{y} \in F(\bar{x})$ and there are positive constants c and ε along with a neighborhood U of $\bar{x}$ in X such that

$$F(\mathbb{B}_X(x,t)) \ni \bar{y} \ \text{ for each } \ x \in U, \text{ with } F(x) \cap \mathbb{B}_Y(\bar{y},ct) \neq \emptyset, \text{ and } t \in (0,\varepsilon). \quad (2)$$

The supremum of $c > 0$ for which there exist a constant $\varepsilon > 0$ and a neighborhood U of $\bar{x}$ in X such that (2) holds is called the *modulus of pseudo-openness* of F at $(\bar{x}, \bar{y})$ and is denoted by $\operatorname{popen} F(\bar{x}, \bar{y})$;

The latter resulting property called *semiregularity* means the mapping under consideration obeys one of the three equivalent properties – metric semiregularity, openness with a linear rate at the reference point, and pseudo-calmness of the inverse. Specifically, the mapping F is said to be *open with a linear rate at* $(\bar{x}, \bar{y})$ when $\bar{y} \in F(\bar{x})$ and there are positive constants c and ε such that

$$F(\mathbb{B}_X(\bar{x},t)) \supset \mathbb{B}_Y(\bar{y},ct) \quad \text{for each} \quad t \in (0,\varepsilon). \quad (3)$$

The supremum of $c > 0$ for which there exists a constant $\varepsilon > 0$ such that (3) holds is called the *modulus of openness* of F at $(\bar{x}, \bar{y})$ and is denoted by $\operatorname{lopen} F(\bar{x}, \bar{y})$.

In some applications, it is useful (and necessary) to consider Hölder versions of regularity, subregularity, and semiregularity [11] (cf. Definition 6.1). Recently, a directional version of regularity was introduced in [8] for set-valued mappings acting between Banach spaces, which covers common directional versions of regularity met in the literature such as directional metric regularity in a given direction from [15] and regularity along a subspace from [14]. The directional character is formulated by the use of a special type of the minimal time function (see Example 2.3). A directional version of metric subregularity in the spirit of [15] can be found in [13].

It is well-known [4, 17], that Ioffe-type criteria allow to obtain simpler proofs for (Hölder) regularity, subregularity, and semiregularity conditions in different instances. A common way of proving these criteria is the Ekeland's variational principle. In [3], a directional version of Ioffe's regularity criterion is proved by using the directional Ekeland's variational principle. Based on this adapted criterion, sufficient conditions for the directional regularity are proved easily and then applied to derive conditions for the directional regularities of a composition and a sum of set-valued mappings.

In this paper we continue the study of directional regularity concepts including subregularity and semiregularity in the spirit of [3]. It turns out that the extended quasi-metric spaces provide an appropriate framework for unifying criteria for the usual (sub/semi) regularity as well as their directional and Hölder counterparts.

The note is organized as follows. In Section 2, we state the Ekeland's variational principle in extended quasi-metric spaces and provide a (brief) comparison with various generalizations of this fundamental statement in the literature. In Section 3, Section 4, and Section 5, we study sufficient as well as necessary conditions for semiregularity, subregularity, and regularity, respectively. In Section 6, we illustrate possible applications of the unified theory. We also consider local convergence of the directional version of Newton-type methods for solving a generalized equation which may be applied when the usual (non-directional) regularity does not hold.

At the end of this introduction, we summarize common notions we use. Note that the specific ones are introduced at appropriate places later.

1.1. Notation and terminology

When we write $f\colon X \longrightarrow Y$, we mean that f is a (single-valued) mapping acting from X into Y while $F\colon X \rightrightarrows Y$ is a mapping from X into Y which may be set-valued. The set $\operatorname{dom} F := \{x \in X : \ F(x) \neq \emptyset\}$ is the *domain* of F, the *graph* of F is the set $\operatorname{gph} F := \{(x, y) \in X \times Y : \ y \in F(x)\}$ and the *inverse* of F is the mapping $Y \ni y \longmapsto \{x \in X : \ y \in F(x)\} =: F^{-1}(y) \subset X$; thus $F^{-1}\colon Y \rightrightarrows X$.

In any metric space, $\mathbb{B}_X[x, r]$ denotes the *closed ball* centered at x with a radius $r > 0$ and $\mathbb{B}_X(x, r)$ is the corresponding *open ball*. The sets $\mathbb{B}_X$ and $\mathbb{S}_X$ are respectively the *closed unit ball* and the *unit sphere* in a normed space X, and for a set $A \subset X$, the *cone generated by* A and the *linear span of* A is denoted by $\operatorname{cone} A$ and $\operatorname{span} A$, respectively. The *distance from a point* x to a subset C of a metric space (X, d) is $\operatorname{dist}(x, C) := \inf\{d(x, y) \ : \ y \in C\}$. We use the convention that $\inf \emptyset := \infty$ and as we work with non-negative quantities we set $\sup \emptyset := 0$. If a set is a singleton we identify it with its only element, that is, we write a instead of $\{a\}$.

The symbol $\mathcal{L}(X, Y)$ denotes the space of all linear bounded operators from a Banach space X into a Banach space Y. Then $\mathbb{R}^{m\times n} := \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ (we identify the linear mappings with the corresponding representation matrix with respect to the standard canonical bases) and $X^* := \mathcal{L}(X, \mathbb{R})$. Given $A \in \mathcal{L}(X, Y)$, the operator $A^* : Y^* \longrightarrow X^*$ denotes the adjoint (dual, transpose) operator to A.

The transpose of a matrix $A \in \mathbb{R}^{m\times n}$ is $A^T \in \mathbb{R}^{n\times m}$. We write $\operatorname{diag}\{a_1, \dots, a_n\}$ for a diagonal matrix whose diagonal entries starting in the upper left corner are

$a_1, \dots, a_n$. In $\mathbb{R}^n$, by $\langle \cdot, \cdot \rangle$ we denote the scalar product inducing the Euclidean norm $\| \cdot \|$. Given an extended real-valued function $\varphi \colon X \longrightarrow \mathbb{R} \cup \{\infty\}$ and a point $x \in X$, the *limit inferior* of φ at x is defined by

$$\liminf_{u \to x} \varphi(u) := \sup_{r>0} \inf_{u \in B_X(x,r)} \varphi(u);$$

φ is called *lower semicontinuous at* x if $\varphi(x) = \liminf_{u \to x} \varphi(u)$; finally the *domain* of φ is the set $\operatorname{dom} \varphi := \{u \in X : \ \varphi(u) < \infty\}$. Finally, the *calmness modulus* of a mapping g between metric spaces (X, d) and (Y, ϱ) at a point $x \in X$, denoted by $\operatorname{calm} g(x)$, is the infimum of $\ell > 0$ for which there is a neighborhood U of x such that $\varrho(g(u), g(x)) \le \ell \, d(u, x)$ for each $u \in U$.

2. Theoretical Background

In this section, we state the Ekeland's variational principle within the framework of extended quasi-metric spaces. Since the terminology used by various authors is not unified, we prefer to postulate the following set of properties without giving them particular names (cf. Remark 2.2).

Definition 2.1. Let a non-empty set X and a function $\varphi \colon X \times X \longrightarrow [0, \infty]$ be given. We say that φ *has property*

($\mathcal{P}_1$) if $\varphi(x, x) = 0$ for each $x \in X$;

($\mathcal{P}_2$) if $\varphi(x, y) \le \varphi(x, z) + \varphi(z, y)$ whenever $x, y, z \in X$;

($\mathcal{P}_3$) if $\varphi(x, y) > 0$ whenever $x, y \in X$ are distinct;

($\mathcal{P}_4$) provided that: if for each $\varepsilon > 0$ there is an index $k = k(\varepsilon)$ such that for each $n, m \in \mathbb{N}$ with $k \le n < m$ we have $\varphi(x_m, x_n) < \varepsilon$, then there is a point $u \in X$ such that $\varphi(u, x_n) \to 0$ as $n \to \infty$.

The *conjugate of* φ is the function $\overline{\varphi} \colon X \times X \longrightarrow [0, \infty]$ defined for $x, u \in X$ by $\overline{\varphi}(x, u) := \varphi(u, x)$. $\square$

Consider a non-empty set X and a function $\varphi \colon X \times X \longrightarrow [0, \infty]$ having property ($\mathcal{P}_1$). Given $x \in X$ and $r > 0$, we define sets

$$I\!B_X^{\varphi}(x, r) := \{u \in X : \varphi(x, u) < r\} \quad \text{and} \quad I\!B_X^{\varphi}[x, r] := \{u \in X : \varphi(x, u) \le r\}.$$

We define the topology τ_φ on X starting from the family $\mathcal{V}_\varphi(x)$ of neighborhoods of an arbitrary point $x \in X$:

$$V \in \mathcal{V}_\varphi(x) \quad \Longleftrightarrow \quad I\!B_X^{\varphi}(x, r) \subset V \quad \text{for some} \quad r > 0.$$

The convergence of a sequence (x_n) to x in (X, τ_φ) is characterized by

$$\lim_{\mathbb{N} \ni n \to \infty} x_n = x \ \text{ if and only if } \ \varphi(x, x_n) \to 0 \ \text{ as } \mathbb{N} \ni n \to \infty.$$

Note that, given $x \in X$ and $r > 0$, the set $I\!B_X^{\varphi}(x, r)$ is τ_φ-open but $I\!B_X^{\varphi}[x, r]$ has not to be τ_φ-closed (it is $\tau_{\overline{\varphi}}$-closed). Moreover, for each $x \in X$ the mapping $X \ni u \longmapsto \varphi(u, x)$ is τ_φ-lower semicontinuous while the mapping $X \ni u \longmapsto \varphi(x, u)$ is τ_φ-upper semicontinuous (cf. [6]).

Remark 2.2. (1) Let us comment on the terminology from [5]. Let X be a non-empty set and a function $\varphi\colon X \times X \longrightarrow [0,\infty)$ be given. Any function φ having properties $(\mathcal{P}_1) - (\mathcal{P}_2)$ is called a quasi-semimetric and the pair (X,φ) is called a quasi-semimetric space. If φ has properties $(\mathcal{P}_1) - (\mathcal{P}_3)$, then the topology τ_φ is T_1. Property $(\mathcal{P}_4)$ means that the space (X,φ) is right φ-K-complete, that is, every right φ-K-Cauchy sequence is convergent in (X,τ_φ). Note that there are several other names used in the literature (even by the same author).

(2) Let now a constant $\alpha \in (0,1)$, non-empty sets X and Y, and two functions $\varphi\colon (X \times X \longrightarrow [0,\infty]$ and $\varrho\colon Y \times Y \longrightarrow [0,\infty]$ be given. If both φ and ϱ have properties $(\mathcal{P}_1)$–$(\mathcal{P}_4)$ then so has the function $\omega\colon (X \times Y)^2 \longrightarrow [0,\infty]$ defined by

$$\omega((u,v),(u',v')) := \max\{\varphi(u,u'), \alpha\varrho(v,v')\}, \quad (u,v),(u',v') \in X \times Y. \tag{4}$$

Moreover, the topology τ_ω is the product topology on $X \times Y$.

Of course, any (positive multiple of) metric on a metric space X has properties $(\mathcal{P}_1) - (\mathcal{P}_3)$. Let us present another example (cf. [3, 8]).

Example 2.3. Let $(X, \|\cdot\|)$ be a Banach space. Given a non-empty set $L \subset \mathbb{S}_X$, the *directional minimal time function with respect to* L is the function

$$X \times X \ni (x,u) \longmapsto T_L(x,u) := \inf\{t \geq 0 : u - x \in tL\}. \tag{5}$$

Clearly, for each $x, u \in X$, if $T_L(x,u) < \infty$ (which is equivalent to $u - x \in \operatorname{cone} L$), then

$$T_{-L}(u,x) = T_L(x,u) = \|u - x\|.$$

If $\operatorname{cone} L$ is convex, then T_L has properties $(\mathcal{P}_1) - (\mathcal{P}_3)$. If L is closed, then so is $\operatorname{cone} L$, and the function T_L is lower semicontinuous and has property $(\mathcal{P}_4)$. $\square$

Now, we state the statement announced above and we equip it with a short direct proof for the reader's convenience.

Theorem 2.4. (Ekeland's variational principle) *Let a non-empty set X, a function $\varphi\colon X \times X \longrightarrow [0,\infty]$ having properties $(\mathcal{P}_1) - (\mathcal{P}_4)$, and a point $\bar{x} \in X$ be given. Consider a τ_φ-lower semicontinuous function $f\colon X \longrightarrow [0,\infty]$ such that $f(\bar{x}) < \infty$. Then there exists a point $u \in X$ such that $f(u) + \varphi(u,\bar{x}) \leq f(\bar{x})$ and*

$$f(u) < f(v) + \varphi(v,u) \quad \textit{whenever} \quad v \in X \setminus \{u\}. \tag{6}$$

Proof. We follow a standard pattern and construct inductively an appropriate sequence $x_1, x_2, \dots$ in $\operatorname{dom} f$. Let $x_1 := \bar{x}$. If $x_n \in \operatorname{dom} f$ is already defined for $n \in \mathbb{N}$, find $x_{n+1} \in X$ such that

$$f(x_{n+1}) + \varphi(x_{n+1}, x_n) \leq f(x_n) \quad \text{and} \quad f(x_{n+1}) < i_n + 1/n \tag{7}$$

where $\quad i_n := \inf\{f(x') : x' \in X \quad \text{and} \quad f(x') + \varphi(x', x_n) \leq f(x_n)\}.$

Note that $0 \leq i_n \leq f(x_n) < \infty$ hence a point x_{n+1} exists and lies necessarily in $\operatorname{dom} f$ because we have $f(x_{n+1}) \leq f(x_{n+1}) + \varphi(x_{n+1}, x_n) \leq f(x_n) < \infty$.

The first inequality in (7) implies that the sequence $\big(f(x_n)\big)$ is decreasing and bounded from below, so $\ell := \lim_{\mathbb{N}\ni n\to\infty} f(x_n)$ exists and is finite. Moreover, for all $1 \le n < m$, we have

$$\begin{aligned} 0 &\le \varphi(x_m, x_n) \le \varphi(x_m, x_{m-1}) + \cdots + \varphi(x_{n+1}, x_n) \\ &\le \big(f(x_{m-1}) - f(x_m)\big) + \cdots + \big(f(x_n) - f(x_{n+1})\big) \le f(x_n) - f(x_m). \end{aligned} \tag{8}$$

Hence for each $\varepsilon > 0$ there is an index $k = k(\varepsilon)$ such that for each $n, m \in \mathbb{N}$ with $k \le n < m$ we have $\varphi(x_m, x_n) < \varepsilon$. By $(\mathcal{P}_4)$, there is $u \in X$ such that $\varphi(u, x_n) \to 0$ as $n \to \infty$. Given $n \in \mathbb{N}$, as both f and $\varphi(\cdot, x_n)$ are τ_φ-lower semicontinuous, so is $f + \varphi(\cdot, x_n)$, and thus (8) implies that

$$f(u) + \varphi(u, x_n) \le \liminf_{p\to\infty} \big(\varphi(x_{n+p}, x_n) + f(x_{n+p})\big) \le f(x_n). \tag{9}$$

Taking $n = 1$ and remembering that $x_1 = \bar{x}$, we obtain the first inequality in the conclusion. Suppose that there is $v \in X$ such that

$$v \ne u \quad \text{and} \quad f(v) + \varphi(v, u) \le f(u).$$

This and (9) imply that, for each $n \in \mathbb{N}$, we have

$$f(v) + \varphi(v, x_n) \le f(v) + \varphi(v, u) + \varphi(u, x_n) \le f(u) + \varphi(u, x_n) \le f(x_n).$$

The definition of (i_n) implies that

$$\begin{aligned} f(v) &< f(v) + \varphi(v, u) \le f(u) \le \liminf_{n\to\infty} f(x_{n+1}) \le \liminf_{n\to\infty} \big(f(x_{n+1}) - 1/n\big) \\ &\le \liminf_{n\to\infty} i_n \le f(v), \end{aligned}$$

a contradiction. □

Clearly, we immediately get a basic form of the Ekeland's variational principle; cf. [7, Theorem 4B.5].

Corollary 2.5. *Let (X, d) be a complete metric space and let $\bar{x} \in X$ be fixed. Consider a lower semicontinuous function $f\colon X \longrightarrow [0, \infty]$ such that $f(\bar{x}) < \infty$. Then for every $c > 0$ there exists a point $u \in X$, depending on the choice of c, such that $f(u) + c\, d(u, \bar{x}) \le f(\bar{x})$ and*

$$f(u) < f(v) + c\, d(v, u) \quad \textit{whenever} \quad v \in X \setminus \{u\}.$$

Remark 2.6. Note that the above statement is equivalent to the most popular version of Corollary 2.5 involving two constants and three conditions in the conclusion; cf. [17, Theorem 2.6]. Indeed, consider a complete metric space (X, d), a point $\bar{x} \in X$, along with a proper lower semicontinuous and bounded from below function $f\colon X \longrightarrow \mathbb{R} \cup \{\infty\}$ such that $f(\bar{x}) \le \inf f(X) + \varepsilon$ for some $\varepsilon > 0$. Let $\lambda > 0$ be arbitrary. Applying Corollary 2.5 with $(f, c) := (f - \inf f(X), \varepsilon/\lambda)$, we conclude that there is a point $u := u(\varepsilon, \lambda) \in X$ such that $f(u) + \frac{\varepsilon}{\lambda}\, d(u, \bar{x}) \le f(\bar{x})$ and $f(u) < f(v) + \frac{\varepsilon}{\lambda}\, d(v, u)$ for each $v \in X \setminus \{u\}$. Then, necessarily,

$$\varepsilon\, d(u, \bar{x}) \le \lambda[f(\bar{x}) - f(u)] \le \lambda[f(\bar{x}) - \inf f(X)] \le \lambda\varepsilon,$$

that is, $d(u,\bar{x}) \le \lambda$. However, we would like to point out that Corollary 2.5 is enough in order to obtain regularity criteria.

Similarly, we get [5, Theorem 2.4. 1.] when (X,ϱ) is a T_1 quasi-metric space which is right ϱ-K-complete and f is τ_ϱ-lower semicontinuous. □

Second, we get the directional version of Corollary 2.5; cf. [8, Corollary 3.2].

Corollary 2.7. *Let $(X,\|\cdot\|)$ be a Banach space, $A \subset X$ be a non-empty closed set, $L \subset \mathbb{S}_X$ be a non-empty closed set such that* $\operatorname{cone} L$ *is convex, and $\bar{x} \in A$ be given. Consider a lower semicontinuous function $f\colon A \longrightarrow [0,\infty]$ such that $f(\bar{x}) < \infty$. Then for every $c > 0$, there exists a point $u \in A$, depending on the choice of c, such that $f(u) + c\,T_L(u,\bar{x}) \le f(\bar{x})$ and*

$$f(u) < f(v) + c\,T_L(v,u) \quad \textit{whenever} \quad v \in A \setminus \{u\}.$$

Proof. It suffices to apply Theorem 2.4 with $X := A$ and $\varphi := c\,T_L$ (see Example 2.3). □

In [2], the authors derived an extension of the Ekeland's variational principle in Euclidean space X involving a function $f\colon D \times D \longrightarrow \mathbb{R}$, with D being a closed subset of X, and $\varphi(x,u) := \|x-u\|$, x, $u \in X$. However, it seems that [2, Theorem 2.1] follows easily from the *usual version* of the principle (Corollary 2.5) similarly as the following slight extension of it.

Corollary 2.8. *Let a non-empty set X, a function $\varphi\colon X \times X \longrightarrow [0,\infty]$ having properties $(\mathcal{P}_1) - (\mathcal{P}_4)$, and a point $\bar{x} \in X$ be given.*
Consider a function $h\colon X \times X \longrightarrow (-\infty,\infty]$ such that

(i) *$h(\bar{x},\cdot)$ is bounded from below and τ_φ-lower semicontinuous;*

(ii) *$h(\bar{x},\bar{x}) = 0$;*

(iii) *$h(\bar{x},y) \le h(\bar{x},z) + h(z,y)$ for every $y, z \in X$.*

Then there exists a point $u \in X$ such that $h(\bar{x},u) + \varphi(u,\bar{x}) \le 0$ and

$$h(u,v) + \varphi(v,u) > 0 \quad \textit{whenever} \quad v \in X \setminus \{u\}.$$

Proof. Let $\ell := \inf_{x\in X} h(\bar{x},x)$, then $-\infty < \ell \le 0 < \infty$. By Theorem 2.4, with $f := h(\bar{x},\cdot) - \ell$, there is $u \in X$ such that

$$h(\bar{x},u) - \ell + \varphi(u,\bar{x}) = f(u) + \varphi(u,\bar{x}) \le f(\bar{x}) = -\ell$$

and for each $v \in X \setminus \{u\}$ we have

$$\begin{aligned} h(\bar{x},u) - \ell = f(u) < f(v) + \varphi(v,u) &= h(\bar{x},v) - \ell + \varphi(v,u) \\ &\overset{\text{(iii)}}{\le} h(\bar{x},u) + h(u,v) - \ell + \varphi(v,u); \end{aligned}$$

adding ℓ and $\ell - h(\bar{x},u)$, respectively, to the above displayed formulas, we finish the proof. □

3. Semiregularity

We start with a sufficient condition for semiregularity of a single-valued mapping.

Proposition 3.1. *Let non-empty sets X and Y, a function $\varphi\colon X\times X\longrightarrow[0,\infty]$ having properties $(\mathcal{P}_1)-(\mathcal{P}_4)$, a function $\psi\colon Y\times Y\longrightarrow[0,\infty]$ having properties $(\mathcal{P}_1)$ and $(\mathcal{P}_3)$, and a point $\bar{x}\in X$ be given. Consider a function $g\colon X\longrightarrow Y$, defined on whole X, such that for each $y\in Y$ the mapping $X\ni x\longmapsto\psi(g(x),y)$ is τ_φ-lower semicontinuous. Suppose that there is $r>0$ such that for any $x\in \mathbb{B}_X^{\overline{\varphi}}(\bar{x},r)$ and any $y\in\mathbb{B}_Y^{\psi}(g(\bar{x}),r)$ satisfying*

$$0<\psi(g(x),y)\le\psi(g(\bar{x}),y)-\varphi(x,\bar{x}), \tag{10}$$

there is a point $x'\in X$ satisfying

$$\varphi(x',x)<\psi(g(x),y)-\psi(g(x'),y). \tag{11}$$

Then $$g\big(\mathbb{B}_X^{\overline{\varphi}}(\bar{x},t)\big)\supset\mathbb{B}_Y^{\psi}(g(\bar{x}),t)\quad\text{for each}\quad t\in(0,r). \tag{12}$$

Proof. Fix any $t\in(0,r)$. Pick an arbitrary $y\in\mathbb{B}_Y^{\psi}(g(\bar{x}),t)$. If $y=g(\bar{x})$ then (12) holds trivially. Suppose that $y\neq g(\bar{x})$.

Since the mapping $X\ni x\longmapsto\psi(g(x),y)$ is τ_φ-lower semicontinuous, applying Theorem 2.4, with $f:=\psi(g(\cdot),y)$, we find $u\in X$ such that

$$\psi(g(u),y)+\varphi(u,\bar{x})\le\psi(g(\bar{x}),y) \tag{13}$$

and $$\psi(g(u),y)<\psi(g(v),y)+\varphi(v,u)\quad\text{whenever}\quad v\in X\setminus\{u\}. \tag{14}$$

By (13), we have

$$\overline{\varphi}(\bar{x},u)=\varphi(u,\bar{x})\le\psi(g(\bar{x}),y)-\psi(g(u),y)\le\psi(g(\bar{x}),y)<t.$$

Therefore $u\in\mathbb{B}_X^{\overline{\varphi}}(\bar{x},t)$ and $\psi(g(u),y)<t$. As $y\in\mathbb{B}_Y^{\psi}(g(\bar{x}),t)$ is arbitrary, (12) will follow once we show that $y=g(u)$.

Suppose on the contrary that $y\neq g(u)$. Therefore (10) holds, with $x:=u$. Thus find $x'\in X$ such that (11), with $x:=u$, holds. Clearly, $\psi(g(x'),y)<\infty$. Combining (14), with $v:=x'$, and (11), with $x:=u$, we get

$$\varphi(x',u)<\psi(g(u),y)-\psi(g(x'),y)\le\varphi(x',u),$$

a contradiction. Hence $y=g(u)$. □

The above result contains [4, Proposition 4.1 (i)] as well as its directional version which seems to be new.

Corollary 3.2. *Let (X,d) be a complete metric space, (Y,ϱ) be a metric space, and a point $\bar{x}\in X$ be fixed. Consider a continuous mapping $g\colon X\longrightarrow Y$, defined on whole of X, for which there are positive constants c and r such that for every $x\in\mathbb{B}_X(\bar{x},r)$ and every $y\in\mathbb{B}_Y(g(\bar{x}),cr)$ satisfying*

$$0<\varrho(g(x),y)\le\varrho(g(\bar{x}),y)-c\,d(x,\bar{x}) \tag{15}$$

there is a point $x'\in X$ such that $c\,d(x,x')<\varrho(g(x),y)-\varrho(g(x'),y)$.

Then $g\big(\mathbb{B}_X(\bar{x},t)\big)\supset\mathbb{B}_Y(g(\bar{x}),ct)$ for every $t\in(0,r)$.

Corollary 3.3. *Let $(X, \|\cdot\|)$ and $(Y, \|\cdot\|)$ be Banach spaces, non-empty closed sets $L \subset \mathbb{S}_X$ and $M \subset \mathbb{S}_Y$ be such that* $\operatorname{cone} L$ *is convex, and a point $\bar{x} \in X$ be given. Consider a continuous mapping $g\colon X \longrightarrow Y$ for which there are positive constants c and r such that the set $A := \operatorname{dom} g \cap I\!B_X[\bar{x}, r]$ is closed, and for every $x \in \big(I\!B_X(\bar{x}, r) \cap \operatorname{dom} g\big)$ and every $y \in I\!B_Y(g(\bar{x}), cr)$ satisfying*

$$0 < T_M(g(x), y) \leq T_M(g(\bar{x}), y) - c\, T_L(\bar{x}, x), \tag{16}$$

there is a point $x' \in \operatorname{dom} g$ satisfying

$$c\, T_L(x, x') < T_M(g(x), y) - T_M(g(x'), y). \tag{17}$$

Then $g\big(I\!B_X(\bar{x}, t) \cap (\bar{x} + \operatorname{cone} L) \cap \operatorname{dom} g\big) \supset I\!B_Y(g(\bar{x}), ct) \cap (g(\bar{x}) + \operatorname{cone} M)$ for each $t \in (0, r)$.

Proof. It suffices to apply Proposition 3.1, with $X := A$, $\varphi(x, u) := T_L(x, u)$, x, $u \in A$, and $\psi(y, v) := c^{-1}\, T_M(y, v)$, y, $v \in Y$. Indeed, in view of Example 2.3 it is enough to observe that the mapping $A \ni x \longmapsto T_M(g(x), y)$ is lower semicontinuous on A, since g is continuous on A and $T_M(\cdot, y)$ is lower semicontinuous because M is a closed subset of $\mathbb{S}_Y$. $\square$

When investigating Hölder semiregularity the following consequence of Proposition 3.1 is useful.

Corollary 3.4. *Let non-empty sets X and Y, a function $\varphi\colon X \times X \longrightarrow [0, \infty]$ having properties $(\mathcal{P}_1) - (\mathcal{P}_4)$, a function $\varrho\colon Y \times Y \longrightarrow [0, \infty]$ having properties $(\mathcal{P}_1)$ and $(\mathcal{P}_3)$, and a point $\bar{x} \in X$ be given. Consider a function $g\colon X \longrightarrow Y$, defined on whole X, such that for each $y \in Y$ the mapping $X \ni x \longmapsto \varrho(g(x), y)$ is τ_φ-lower semicontinuous, along with a continuous strictly increasing function $\phi\colon [0, \infty] \longrightarrow [0, \infty]$ such that $\phi(0) = 0$. Suppose that there is $r > 0$ such that for any $x \in I\!B^{\overline{\varphi}}_X(\bar{x}, r)$ and any $y \in I\!B^{\varrho}_Y(g(\bar{x}), r)$ satisfying*

$$0 < \phi(\varrho(g(x), y)) \leq \phi(\varrho(g(\bar{x}), y)) - \varphi(x, \bar{x}),$$

there is a point $x' \in X$ satisfying

$$\varphi(x', x) < \phi(\varrho(g(x), y)) - \phi(\varrho(g(x'), y)).$$

Then $g\big(I\!B^{\overline{\varphi}}_X(\bar{x}, t)\big) \supset I\!B^{\varrho}_Y(g(\bar{x}), \phi^{-1}(t))$ for each $t \in (0, \min\{r, \phi(r)\})$.

Proof. Let $\psi := \phi \circ \varrho$. Then ψ has properties $(\mathcal{P}_1)$ and $(\mathcal{P}_3)$. For each $y \in Y$ the mapping $X \ni x \longmapsto \psi(g(x), y)$ is τ_φ-lower semicontinuous. Corollary 3.1, with $r := \min\{r, \phi(r)\}$, implies that

$$g\big(I\!B^{\overline{\varphi}}_X(\bar{x}, t)\big) \supset I\!B^{\psi}_Y\big(g(\bar{x}), t\big) \quad \text{for each} \quad t \in (0, \min\{r, \phi(r)\}).$$

Fix any such t. Since ϕ is strictly increasing, we get

$$\begin{aligned} I\!B^{\psi}_Y(g(\bar{x}), t) &= \{u \in X : \phi\big(\varrho(g(\bar{x}), u)\big) < t\} = \{u \in X : \varrho(g(\bar{x}), u) < \phi^{-1}(t)\} \\ &= I\!B^{\varrho}_Y\big(g(\bar{x}), \phi^{-1}(t)\big). \end{aligned}$$

This implies that $g\big(I\!B^{\overline{\varphi}}_X(\bar{x}, t)\big) \supset I\!B^{\varrho}_Y\big(g(\bar{x}), \phi^{-1}(t)\big)$. $\square$

Note that [4, Proposition 4.1 (ii)] contains a necessary condition in the spirit of Corollary 3.2.

Proposition 3.5. *Let (X,d) and (Y,ϱ) be metric spaces, and a point $\bar{x} \in X$ be given. Consider a continuous mapping $g\colon X \longrightarrow Y$ for which there are positive constants c and r such that, for every $t \in (0,r)$,*

$$g\big(I\!B_X(\bar{x},t)\big) \supset I\!B_Y(g(\bar{x}),ct).$$

Then for every $c' \in (0,c)$, every $x \in I\!B_X(\bar{x},r)$, and every $y \in I\!B_Y(g(\bar{x}),c'r)$ satisfying $0 < \varrho(g(\bar{x}),y) \le \varrho(g(x),y) - c'\,d(x,\bar{x})$ there is a point $x' \in X$ such that $\varrho(g(x'),y) < \varrho(g(x),y) - c'\,d(x,x')$.

Similarly, we can derive a necessary condition corresponding to Corollary 3.3.

Proposition 3.6. *Let $(X,\|\cdot\|)$ and $(Y,\|\cdot\|)$ be normed spaces, non-empty sets $L \subset \mathbb{S}_X$ and $M \subset \mathbb{S}_Y$ be such that $\operatorname{cone} L$ is convex, and a point $\bar{x} \in X$ be given. Consider a continuous mapping $g\colon X \longrightarrow Y$ for which there are positive constants c and r such that, for each $t \in (0,r)$,*

$$g\big(I\!B_X(\bar{x},t) \cap (\bar{x} + \operatorname{cone} L) \cap \operatorname{dom} g\big) \supset I\!B_Y(g(\bar{x}),ct) \cap (g(\bar{x}) + \operatorname{cone} M).$$

Then, for every $c' \in (0,c)$, $x \in I\!B_X(\bar{x},r)$, and $y \in I\!B_Y(g(\bar{x}),c'r)$ satisfying

$$0 < T_M(g(\bar{x}),y) \le T_M(y,g(x)) - c'\,T_L(x,\bar{x}) < \infty, \tag{18}$$

there is a point $x' \in X$ such that $c'\,T_L(x,x') < T_M(y,g(x)) - T_M(y,g(x'))$.

Proof. Fix some $c' \in (0,c)$, $x \in I\!B_X(\bar{x},r)$, and $y \in I\!B_Y(g(\bar{x}),c'r)$ satisfying (18). Let $t := \|g(\bar{x})-y\|/c' = T_M(g(\bar{x}),y)/c'$. Then (18) implies that $0<t<(c'r)/c' = r$.

As $\quad y \in I\!B_Y[g(\bar{x}),c't] \cap (g(\bar{x}) + \operatorname{cone} M) \subset I\!B_Y(g(\bar{x}),ct) \cap (g(\bar{x}) + \operatorname{cone} M)$,

there is $x' \in I\!B(\bar{x},t) \cap (\bar{x} + \operatorname{cone} L)$ such that $g(x') = y$. Then

$$\begin{aligned} c'\,T_L(x,x') &\overset{(\mathcal{P}_2)}{\le} c'\,T_L(x,\bar{x}) + c'\,T_L(\bar{x},x') \\ &\overset{(18)}{<} T_M(y,g(x)) - T_M(g(\bar{x}),y) + c'\,T_L(\bar{x},x') \\ &= T_M(y,g(x)) - c't + c't = T_M(y,g(x)) \\ &\overset{(\mathcal{P}_1)}{=} T_M(y,g(x)) - T_M(y,g(x')). \end{aligned} \qquad \square$$

Although the above statements concern single-valued mappings, in view of Remark 2.2, using the restriction of the canonical projection to the graph of a given set-valued mapping we immediately get their set-valued versions (similarly to the classical case). Let us start with Proposition 3.1.

Theorem 3.7. *Let a constant $\alpha \in (0,1)$, non-empty sets X and Y, functions $\varphi\colon X \times X \longrightarrow [0,\infty]$ and ϱ, $\psi\colon Y \times Y \longrightarrow [0,\infty]$, and a point $(\bar{x},\bar{y}) \in X \times Y$ be given. Assume that φ and ϱ have properties $(\mathcal{P}_1)-(\mathcal{P}_4)$, that ψ has properties $(\mathcal{P}_1)$ and $(\mathcal{P}_3)$, that $\omega\colon (X \times Y)^2 \longrightarrow [0,\infty]$ is defined by (4), and that for each $y \in Y$ the mapping $Y \ni v \longmapsto \psi(v,y)$ is τ_ϱ-lower semicontinuous. Consider a set-valued mapping $F\colon X \rightrightarrows Y$, with $\bar{y} \in F(\bar{x})$, such that the set $\operatorname{gph} F$ is τ_ω-closed. Suppose that there is $r > 0$ such that for any $x \in I\!B^{\overline{\varphi}}_X(\bar{x},r)$, any $v \in I\!B^{\overline{\varrho}}_Y(\bar{y},r/\alpha) \cap F(x)$, and any $y \in I\!B^{\psi}_Y(\bar{y},r)$ satisfying*

$$0 < \psi(v,y) \le \psi(\bar{y},y) - \max\{\varphi(x,\bar{x}), \alpha\varrho(v,\bar{y})\}, \tag{19}$$

there is a pair $(x',v') \in \operatorname{gph} F$ such that

$$\max\{\varphi(x',x), \alpha\varrho(v',v)\} < \psi(v,y) - \psi(v',y). \tag{20}$$

Then $F\big(I\!B^{\overline{\varphi}}_X(\bar{x},t)\big) \supset I\!B^{\psi}_Y(\bar{y},t)$ for each $t \in (0,r)$.

Proof. Let $\widetilde{X} := \operatorname{gph} F$. We already know that ω has properties $(\mathcal{P}_1)-(\mathcal{P}_4)$. Let $g(x,y) := y$, $(x,y) \in \widetilde{X}$. Then g is τ_ω-to-τ_ϱ-continuous, hence for each $y \in Y$ the mapping $\widetilde{X} \ni (x,v) \longmapsto \psi(g(x,v),y)$ is τ_ω-lower semicontinuous.

Fix any $(x,v) \in I\!B^{\overline{\omega}}_{\widetilde{X}}\big((\bar{x},\bar{y}),r\big)$, that is, $x \in I\!B^{\overline{\varphi}}_X(\bar{x},r)$ and $v \in I\!B^{\overline{\varrho}}_Y(\bar{y},r/\alpha) \cap F(x)$, and any $y \in I\!B^{\psi}_Y(\bar{y},r)$ such that (19) holds. Find a pair $(x',v') \in \operatorname{gph} F = \widetilde{X}$ satisfying (20). Proposition 3.1, with $(X,\tau_\varphi) := (\widetilde{X},\tau_\omega)$, implies that for each $t \in (0,r)$ we have

$$I\!B^{\psi}_Y(\bar{y},t) \quad \subset \quad g\big(I\!B^{\overline{\omega}}_{\widetilde{X}}((\bar{x},\bar{y}),t)\big) = g\left(\big(I\!B^{\overline{\varphi}}_X(\bar{x},t) \times I\!B^{\overline{\varrho}}_Y(\bar{y},t)\big) \cap \operatorname{gph} F\right).$$

Fix an arbitrary $t \in (0,r)$. Pick any $y \in I\!B^{\psi}_Y(\bar{y},t)$. Then there are $x \in I\!B^{\overline{\varphi}}_X(\bar{x},t) \cap \operatorname{dom} F$ and $y' \in I\!B^{\overline{\varrho}}_Y(\bar{y},t) \cap F(x)$ such that $g(x,y') = y$. Hence $y' = y$. So $y \in F(x) \subset F\big(I\!B^{\overline{\varphi}}_X(\bar{x},t)\big)$. $\square$

Similarly, we get a set-valued version of Corollary 3.4.

Corollary 3.8. *Let a constant $\alpha \in (0,1)$, non-empty sets X and Y, functions $\varphi\colon X \times X \longrightarrow [0,\infty]$ and $\varrho\colon Y \times Y \longrightarrow [0,\infty]$, and a point $(\bar{x},\bar{y}) \in X \times Y$ be given. Assume that φ and ϱ have properties $(\mathcal{P}_1)$–$(\mathcal{P}_4)$ and that the function $\omega\colon (X \times Y)^2 \longrightarrow [0,\infty]$ is defined by (4). Consider a set-valued mapping $F\colon X \rightrightarrows Y$ with $\bar{y} \in F(\bar{x})$ and τ_ω-closed graph, along with a continuous strictly increasing function $\phi\colon [0,\infty] \longrightarrow [0,\infty]$ such that $\phi(0) = 0$. Suppose that there is $r > 0$ such that for any $x \in I\!B^{\overline{\varphi}}_X(\bar{x},r)$, any $v \in I\!B^{\overline{\varrho}}_Y(\bar{y},r/\alpha) \cap F(x)$, and any $y \in I\!B^{\varrho}_Y(\bar{y},r)$ satisfying*

$$0 < \phi(\varrho(v,y)) \le \phi(\varrho(\bar{y},v)) - \max\{\varphi(x,\bar{x}), \alpha\varrho(v,\bar{y})\}, \tag{21}$$

there is a pair $(x',v') \in \operatorname{gph} F$ such that

$$\max\{\varphi(x',x), \alpha\varrho(v',v)\} < \phi(\varrho(v,y)) - \phi(\varrho(v',y)). \tag{22}$$

Then $F\big(I\!B^{\overline{\varphi}}_X(\bar{x},t)\big) \supset I\!B^{\varrho}_Y\big(\bar{y},\phi^{-1}(t)\big)$ for each $t \in (0,\min\{r,\phi(r)\})$.

Proof. Let $\psi := \phi \circ \varrho$. It suffices to observe that for each $y \in Y$ the mapping $Y \ni v \longmapsto \psi(v, y)$ is τ_ϱ-lower semicontinuous, apply Theorem 3.7, and proceed as in the proof of Corollary 3.4. □

Of course, it is possible to deduce set-valued versions of Corollary 3.2 and Proposition 3.5 (cf. [4, Proposition 4.2]), as well as of Corollary 3.3 and Proposition 3.6 (which would be new). This is left to the interested reader (if any).

4. Subregularity

We start with a sufficient condition for subregularity of a single-valued mapping again.

Proposition 4.1. *Let non-empty sets X and Y, a function $\varphi\colon X \times X \longrightarrow [0,\infty]$ having properties $(\mathcal{P}_1) - (\mathcal{P}_4)$, a function $\psi\colon Y \times Y \longrightarrow [0,\infty]$ having properties $(\mathcal{P}_1)$ and $(\mathcal{P}_3)$, and a point $\bar{x} \in X$ be given. Consider a single-valued mapping $g\colon X \longrightarrow Y$, defined on whole X, such that the mapping $X \ni x \longmapsto \psi(g(x), g(\bar{x}))$ is τ_φ-lower semicontinuous. Suppose that there is $r > 0$ such that for any $u \in I\!B^{\overline{\varphi}}_X(\bar{x}, 2r)$, with $0 < \psi(g(u), g(\bar{x})) < \infty$, there is a point $x' \in X$ satisfying*

$$\varphi(x', u) < \psi(g(u), g(\bar{x})) - \psi(g(x'), g(\bar{x})). \tag{23}$$

Then $g\big(I\!B^{\overline{\varphi}}_X(x,t)\big) \ni g(\bar{x})$ for each $x \in I\!B^{\overline{\varphi}}_X(\bar{x}, r)$, with $g(x) \in I\!B^{\overline{\psi}}_Y(g(\bar{x}), t)$, and $t \in (0, r)$.

Proof. Fix any $t \in (0, r)$ and any $x \in I\!B^{\overline{\varphi}}_X(\bar{x}, r)$ with $g(x) \in I\!B^{\overline{\psi}}_Y(g(\bar{x}), t)$. Then $\psi(g(x), g(\bar{x})) < t < r$. In order to show that $g(\bar{x}) \in g(I\!B^{\overline{\varphi}}_X(x, t))$, we find $u \in I\!B^{\overline{\varphi}}_X(x, t)$ such that $g(u) = g(\bar{x})$.

Since the mapping $X \ni x \longmapsto \psi(g(x), g(\bar{x}))$ is τ_φ-lower semicontinuous, applying Theorem 2.4, with $f := \psi(g(\cdot), g(\bar{x}))$ and $\bar{x} := x$, we find $u \in X$ such that

$$\psi(g(u), g(\bar{x})) + \varphi(u, x) \le \psi(g(x), g(\bar{x}))$$

$$\text{and} \qquad \psi(g(u), g(\bar{x})) < \psi(g(v), g(\bar{x})) + \varphi(v, u) \quad \text{whenever } v \in X \setminus \{u\}. \tag{24}$$

$$\text{Then} \qquad \overline{\varphi}(x, u) = \varphi(u, x) \le \psi(g(x), g(\bar{x})) - \psi(g(u), g(\bar{x})) \le \psi(g(x), g(\bar{x})) < t$$

$$\text{and} \qquad \overline{\varphi}(\bar{x}, u) \le \overline{\varphi}(\bar{x}, x) + \overline{\varphi}(x, u) < r + t < 2r.$$

Therefore $u \in I\!B^{\overline{\varphi}}_X(x, t) \cap I\!B^{\overline{\varphi}}_X(\bar{x}, 2r)$. We claim that $g(u) = g(\bar{x})$. Suppose on the contrary that $g(u) \neq g(\bar{x})$. By the assumptions, we find $x' \in X$ such that (23) holds. Clearly, $\psi(g(x'), g(\bar{x})) < \infty$ and $x' \neq u$. Using (24), with $v := x'$, we get

$$\varphi(x', u) \overset{(23)}{<} \psi(g(u), g(\bar{x})) - \psi(g(x'), g(\bar{x})) < \varphi(x', u),$$

a contradiction. Hence $g(u) = g(\bar{x})$. □

It is elementary to write down sufficient conditions for subregularity and a directional version of it in the spirit of Corollary 3.2 and Corollary 3.3 (this is left to the reader). We mention necessary conditions for these properties only.

Proposition 4.2. *Let (X, d) and (Y, ϱ) be metric spaces, and a point $\bar{x} \in X$ be given. Consider a mapping $g\colon X \longrightarrow Y$ which is continuous at $\bar{x}$ and for which there are positive constants c and r such that $g\big(\mathbb{B}_X(x,t)\big) \ni g(\bar{x})$ for each $x \in \mathbb{B}_X(\bar{x}, r)$, with $g(x) \in \mathbb{B}_Y(g(\bar{x}), ct)$, and $t \in (0, r)$.*

Then for every $c' \in (0, c)$ there is some $r' > 0$ such that for every $x \in \mathbb{B}_X(\bar{x}, r')$ with $g(x) \neq g(\bar{x})$ there is a point $x' \in X$ satisfying

$$c'\, d(x, x') < \varrho(g(x), g(\bar{x})) - \varrho(g(x'), g(\bar{x})).$$

Proof. Fix any $c' \in (0, c)$. Since g is continuous at $\bar{x}$, we find $r' \in (0, r)$ such that $\varrho(g(x), g(\bar{x})) < c'r$ for each $x \in \mathbb{B}_X(\bar{x}, r')$. Pick an arbitrary $x \in \mathbb{B}_X(\bar{x}, r')$ with $g(x) \neq g(\bar{x})$. Let $t := \varrho(g(x), g(\bar{x}))/c'$. Then $t \in (0, r)$ and $\varrho(g(x), g(\bar{x})) = c't < ct$. Consequently, there is a point $x' \in \mathbb{B}_X(x, t)$ such that $g(x') = g(\bar{x})$. Then

$$c'\, d(x, x') < c't = \varrho(g(x), g(\bar{x})) = \varrho(g(x), g(\bar{x})) - \varrho(g(x'), g(\bar{x})). \qquad \square$$

Proposition 4.3. *Let normed spaces $(X, \|\cdot\|)$ and $(Y, \|\cdot\|)$, non-empty sets $L \subset \mathbb{S}_X$ and $M \subset \mathbb{S}_Y$, and a point $\bar{x} \in X$ be given. Consider a mapping $g\colon X \longrightarrow Y$ which is continuous at $\bar{x}$ and for which there are positive constants c and r such that for each $x \in \mathbb{B}_X(\bar{x}, r) \cap (\bar{x} + \operatorname{cone} L)$, with $g(x) \in \mathbb{B}_Y(g(\bar{x}), ct) \cap (g(\bar{x}) + \operatorname{cone} M)$, and each $t \in (0, r)$ we have*

$$g\big(\mathbb{B}_X(x, t) \cap (x + \operatorname{cone} L) \cap \operatorname{dom} g\big) \ni g(\bar{x}).$$

Then, for every $c' \in (0, c)$ there is some $r' > 0$ such that for every $x \in \mathbb{B}_X(\bar{x}, r') \cap (\bar{x} + \operatorname{cone} L)$ with $0 < T_M(g(\bar{x}), g(x)) < \infty$ there is a point $x' \in X$ satisfying $c'\, T_L(x', x) < T_M(g(\bar{x}), g(x)) - T_M(g(\bar{x}), g(x'))$.

Proof. Fix any $c' \in (0, c)$. Since g is continuous at $\bar{x}$, we find $r' \in (0, r)$ such that $\|g(x) - g(\bar{x})\| < c'r$ for each $x \in \mathbb{B}_X(\bar{x}, r')$. Pick an arbitrary $x \in \mathbb{B}_X(\bar{x}, r') \cap (\bar{x} + \operatorname{cone} L)$ with $0 < T_M(g(\bar{x}), g(x)) < \infty$. Let

$$t := \|g(x) - g(\bar{x})\|/c' = T_M(g(\bar{x}), g(x))/c'.$$

Then $t \in (0, r)$ and $\|g(x) - g(\bar{x})\| = c't < ct$, and moreover $g(x) \in g(\bar{x}) + \operatorname{cone} M$. Consequently, there is a point $x' \in \mathbb{B}_X(x, t) \cap (x + \operatorname{cone} L)$ such that $g(x') = g(\bar{x})$.

Then
$$\begin{aligned} c'\, T_L(x, x') &= c'\, \|x - x'\| < c't = T_M(g(\bar{x}), g(x)) \\ &= T_M(g(\bar{x}), g(x)) - T_M(g(\bar{x}), g(x')). \qquad \square \end{aligned}$$

As in Section 3, Proposition 4.1 has the following set-valued counterpart.

Theorem 4.4. *Let a constant $\alpha \in (0, 1)$, non-empty sets X and Y, functions $\varphi\colon X \times X \longrightarrow [0, \infty]$ and $\psi\colon Y \times Y \longrightarrow [0, \infty]$, and a point $(\bar{x}, \bar{y}) \in X \times Y$ be given. Assume that φ and ψ have properties $(\mathcal{P}_1) - (\mathcal{P}_4)$ and that $\omega\colon (X \times Y)^2 \longrightarrow [0, \infty]$ is defined by (4). Consider a set-valued mapping $F\colon X \rightrightarrows Y$, with $\bar{y} \in F(\bar{x})$, such that the set $\operatorname{gph} F$ is τ_ω-closed. Suppose that there is $r > 0$ such that for any*

$u \in I\!\!B_X^{\overline{\varphi}}(\bar{x}, 2r)$ and any $v \in I\!\!B_Y^{\overline{\psi}}(\bar{y}, 2r/\alpha) \cap F(u)$, with $0 < \psi(v, \bar{y}) \quad (< \infty)$, there is a point $(x', v') \in \operatorname{gph} F$ satisfying

$$\max\{\varphi(x', u), \alpha\psi(v', v)\} < \psi(v, \bar{y}) - \psi(v', \bar{y}). \tag{25}$$

Then $F\big(I\!\!B_X^{\overline{\varphi}}(x, t)\big) \ni \bar{y}$ for each $x \in I\!\!B_X^{\overline{\varphi}}(\bar{x}, r)$, with $F(x) \cap I\!\!B_Y^{\overline{\psi}}(\bar{y}, t) \neq \emptyset$, and $t \in (0, r)$.

Of course, the reader can easily write down also the set-valued versions of Proposition 4.2 and Proposition 4.3.

5. Regularity

In this section, we show that sufficient conditions for semiregularity can be used to obtain easily general versions of regularity criteria. Let a non-empty set Y along with a function $\varrho\colon Y \times Y \longrightarrow [0, \infty)$ having properties $(\mathcal{P}_1) - (\mathcal{P}_3)$ be given. Consider a set-valued mapping $G\colon Y \rightrightarrows Y$, with $y \in G(y)$ for each $y \in Y$. Define a function $\chi\colon Y \times Y \longrightarrow [0, \infty]$ by

$$\chi(y, v) := \begin{cases} \varrho(y, v) & \text{if } v \in G(y), \\ \infty & \text{otherwise.} \end{cases} \tag{26}$$

Clearly, χ has properties $(\mathcal{P}_1)$ and $(\mathcal{P}_3)$. Moreover, for each $r > 0$ and each $y \in Y$ we have

$$I\!\!B_Y^{\chi}(y, r) = I\!\!B_Y^{\varrho}(y, r) \cap G(y) \quad \text{and} \quad I\!\!B_Y^{\chi}[y, r] = I\!\!B_Y^{\varrho}[y, r] \cap G(y).$$

Example 5.1. Let a Banach space $(Y, \|\cdot\|)$ and a set $M \subset \mathbb{S}_Y$ be given. Define $G\colon Y \rightrightarrows Y$ by $G(y) := y + \operatorname{cone} M$, $y \in Y$. For each y, $v \in Y$ and each $r > 0$ we have

$$I\!\!B_Y^{\chi}(y, r) = I\!\!B_Y(y, r) \cap \big(y + \operatorname{cone} M\big) \quad \text{and} \quad \chi(y, v) = T_M(y, v). \qquad \square$$

The above example reveals that the following statements generalize criteria for directional openness around the reference point from [3], more precisely, their sufficiency parts, but the corresponding necessity parts follow easily in the similar way as in the case of semiregularity and subregularity.

Proposition 5.2. *Let non-empty sets X and Y, a function $\varphi\colon X \times X \longrightarrow [0, \infty]$ having properties $(\mathcal{P}_1)$–$(\mathcal{P}_4)$, a function $\varrho\colon Y \times Y \longrightarrow [0, \infty)$ having properties $(\mathcal{P}_1)$–$(\mathcal{P}_3)$, and a point $\bar{x} \in X$ be given. Let $G\colon Y \rightrightarrows Y$ be such that $y \in G(y)$ for each $y \in Y$ and let a function $\chi\colon Y \times Y \longrightarrow [0, \infty]$ be defined by (26).*

Consider a mapping $g\colon X \longrightarrow Y$, defined on whole X, such that for each $y \in Y$ the mapping $X \ni x \longmapsto \chi(g(x), y)$ is τ_φ-lower semicontinuous and the mapping $X \ni x \longmapsto \varrho(y, g(x))$ is $\tau_{\overline{\varphi}}$-upper semicontinuous.

Suppose that there is some $r > 0$ such that for any $x \in I\!B_X^{\overline{\varphi}}(\bar{x}, 2r)$ and any $y \in I\!B_Y^{\varrho}(g(\bar{x}), 2r)$, with $0 < \chi(g(x), y) < \infty$, there is a point $x' \in X$ satisfying

$$\varphi(x', x) < \chi(g(x), y) - \chi(g(x'), y). \tag{27}$$

Then there is $r' > 0$ such that

$$g\big(I\!B_X^{\overline{\varphi}}(x, t)\big) \supset I\!B_Y^{\chi}(g(x), t) \quad \text{for each} \quad x \in I\!B_X^{\overline{\varphi}}(\bar{x}, r') \quad \text{and} \quad t \in (0, r'). \tag{28}$$

Proof. Clearly, χ has properties $(\mathcal{P}_1)$ and $(\mathcal{P}_3)$. Find $r' \in (0, r)$ such that $\varrho(g(\bar{x}), g(x)) < r$ for each $x \in I\!B_X^{\overline{\varphi}}(\bar{x}, r')$. Pick an arbitrary $\tilde{x} \in I\!B_X^{\overline{\varphi}}(\bar{x}, r')$. We are going to apply Proposition 3.1 with $\bar{x} := \tilde{x}$, $r := r'$, and $\psi := \chi$. Fix any $x \in I\!B_X^{\overline{\varphi}}(\tilde{x}, r')$ and any $y \in I\!B_Y^{\chi}(g(\tilde{x}), r') \subset I\!B_Y^{\varrho}(g(\tilde{x}), r')$ such that

$$0 < \chi(g(x), y) \leq \chi(g(\tilde{x}), y) - \varphi(x, \tilde{x}).$$

Then $\chi(g(x), y) \leq \chi(g(\tilde{x}), y) < r' < \infty$. Also $x \in I\!B_X^{\overline{\varphi}}(\bar{x}, 2r)$ and $y \in I\!B_Y^{\varrho}(g(\bar{x}), 2r)$ because

$$\overline{\varphi}(\bar{x}, x) \leq \overline{\varphi}(\bar{x}, \tilde{x}) + \overline{\varphi}(\tilde{x}, x) < r' + r' = 2r' < 2r$$

and
$$\varrho(g(\bar{x}), y) \leq \varrho(g(\bar{x}), g(\tilde{x})) + \varrho(g(\tilde{x}), y) < r + r' < 2r.$$

Find $x' \in X$ such that (27) holds. Proposition 3.1 implies that

$$g\big(I\!B_X^{\overline{\varphi}}(\tilde{x}, t)\big) \supset I\!B_Y^{\chi}(g(\tilde{x}), t) \ \text{ for each } t \in (0, r'). \qquad \square$$

Finally, let us deduce a statement for set-valued mappings.

Theorem 5.3. *Let a constant $\alpha \in (0, 1)$, non-empty sets X and Y, functions $\varphi\colon X \times X \longrightarrow [0, \infty]$ and $\varrho\colon Y \times Y \longrightarrow [0, \infty)$, and a point $(\bar{x}, \bar{y}) \in X \times Y$ be given. Assume that φ and ϱ have properties $(\mathcal{P}_1)$–$(\mathcal{P}_4)$ and that $\omega\colon (X \times Y)^2 \longrightarrow [0, \infty]$ is defined by (4).*

Consider a set-valued mapping $F\colon X \rightrightarrows Y$ such that $\bar{y} \in F(\bar{x})$ and that $\operatorname{gph} F$ is both τ_ω-closed and $\tau_{\overline{\omega}}$-closed, a set-valued mapping $G\colon Y \rightrightarrows Y$ such that $y \in G(y)$ for each $y \in Y$, and a continuous strictly increasing function $\phi\colon [0, \infty] \longrightarrow [0, \infty]$ such that $\phi(0) = 0$. Let $\chi\colon Y \times Y \longrightarrow [0, \infty]$ be defined by (26). Assume that for each $y \in Y$ the mapping $Y \ni v \longmapsto \chi(v, y)$ is τ_ϱ-lower semicontinuous and the mapping $Y \ni v \longmapsto \varrho(y, v)$ is $\tau_{\overline{\varrho}}$-upper semicontinuous.

Suppose that there is some $r > 0$ such that for any $x \in I\!B_X^{\overline{\varphi}}(\bar{x}, 2r)$, any $v \in I\!B_Y^{\overline{\varrho}}(\bar{y}, 2r/\alpha) \cap F(x)$, and any $y \in I\!B_Y^{\varrho}(\bar{y}, 2r)$ with $0 < \chi(v, y) < \infty$, there is a pair $(x', v') \in \operatorname{gph} F$ satisfying

$$\max\{\varphi(x', x), \alpha\varrho(v', v)\} < \phi(\chi(v, y)) - \phi(\chi(v', y)). \tag{29}$$

Then there is $r' > 0$ such that for each $x \in I\!B_X^{\overline{\varphi}}(\bar{x}, r')$, each $v \in I\!B_Y^{\overline{\varrho}}(\bar{y}, r') \cap F(x)$, and each $t \in (0, r')$ we have

$$F\big(I\!B_X^{\overline{\varphi}}(x, t)\big) \supset I\!B_Y^{\chi}\big(v, \phi^{-1}(t)\big) \tag{30}$$

Proof. Let $\widetilde{X} := \operatorname{gph} F$. We already know that ω has properties $(\mathcal{P}_1)$–$(\mathcal{P}_4)$. Let $g(x,y) := y$, $(x,y) \in \widetilde{X}$. Then g is both τ_ω-to-τ_ϱ-continuous and $\tau_{\overline{\omega}}$-to-τ_ϱ-continuous, hence for each $y \in Y$ the mapping $\widetilde{X} \ni (x,v) \longmapsto \chi(g(x,v),y)$ is τ_ω-lower semicontinuous and the mapping $\widetilde{X} \ni (x,v) \longmapsto \varrho(y, g(x,v))$ is $\tau_{\overline{\omega}}$-upper semicontinuous.

Find $r' \in (0,r)$ such that $\varrho(g(\bar{x},\bar{y}), g(x,v)) < r$ for each $(x,v) \in I\!B^{\overline{\omega}}_{\widetilde{X}}((\bar{x},\bar{y}), r')$ and $\phi^{-1}(r') < r$. Fix any $t \in (0,r')$. Pick an arbitrary $(x,v) \in I\!B^{\overline{\omega}}_{\widetilde{X}}((\bar{x},\bar{y}), r')$ and $y \in I\!B^{\chi}_{Y}(g(x,v), \phi^{-1}(t))$. If $y = g(x,v)$, then (30) holds trivially. Suppose that $y \neq g(x,v)$. Then

$$\varrho(g(\bar{x},\bar{y}), y) \le \varrho(g(\bar{x},\bar{y}), g(x,v)) + \varrho(g(x,v), y) < r + \phi^{-1}(t) < 2r.$$

Since the mapping $\widetilde{X} \ni (u,z) \longmapsto \phi(\chi(g(u,z),y))$ is τ_ω-lower semicontinuous, applying Theorem 2.4, with $(X,\varphi) := (\widetilde{X}, \omega)$, $f := \phi(\chi(g(\cdot,\cdot),y))$, and $\bar{x} := (x,v)$, we find $(u,w) \in \widetilde{X}$ such that

$$\phi(\chi(g(u,w),y)) + \omega((u,w),(x,v)) \le \phi(\chi(g(x,v),y)) \tag{31}$$

and, whenever $(x',v') \in \widetilde{X} \setminus \{(u,w)\}$,

$$\phi(\chi(g(u,w),y)) < \phi(\chi(g(x',v'),y)) + \omega((x',v'),(u,w)). \tag{32}$$

By (31), we have

$$\overline{\omega}((x,v),(u,w)) = \omega((u,w),(x,v)) \le \phi(\chi(g(x,v),y)) < t$$

and $\quad \overline{\omega}((\bar{x},\bar{y}),(u,w)) \le \overline{\omega}((\bar{x},\bar{y}),(x,v)) + \overline{\omega}((x,v),(u,w)) < r' + t < 2r.$

Therefore $(u,w) \in I\!B^{\overline{\omega}}_{X}((x,v),t) \cap I\!B^{\overline{\omega}}_{X}((\bar{x},\bar{y}),2r)$.

We claim that $y = g(u,w)$. Suppose on the contrary that $y \neq g(u,w)$. Find $(x',v') \in \widetilde{X}$ such that (29), with $(x,v) := (u,w)$, holds. Clearly, $\chi(g(x',v'),y) < \infty$. Combining (29), with $(x,v) := (u,w)$, and (32), we get

$$\begin{aligned}\omega((x',v'),(u,w)) &< \phi(\chi(w,y)) - \phi(\chi(v',y)) \\ &= \phi(\chi(g(u,w),y)) - \phi(\chi(g(x',v'),y)) < \omega((x',v'),(u,w)),\end{aligned}$$

a contradiction. Hence $y = g(u,w)$.

We have shown that for each $t \in (0,r')$ and each $(x,v) \in I\!B^{\overline{\omega}}_{\widetilde{X}}((\bar{x},\bar{y}), r')$, meaning that $x \in I\!B^{\overline{\varphi}}_{X}(\bar{x}, r')$ and $v \in I\!B^{\overline{\varrho}}_{Y}(\bar{y}, r'/\alpha) \cap F(x)$, we have

$$I\!B^{\chi}_{Y}(v, \phi^{-1}(t)) \quad \subset \quad g\big(I\!B^{\overline{\omega}}_{\widetilde{X}}((x,v),t)\big) = g\left(\big(I\!B^{\overline{\varphi}}_{X}(x,t) \times I\!B^{\overline{\varrho}}_{Y}(v,t/\alpha)\big) \cap \operatorname{gph} F\right).$$

Fix any $t \in (0,r')$, $x \in I\!B^{\overline{\varphi}}_{X}(\bar{x}, r')$, and $v \in I\!B^{\overline{\varrho}}_{Y}(\bar{y}, r'/\alpha) \cap F(x)$. Pick an arbitrary $y \in I\!B^{\chi}_{Y}(v, \phi^{-1}(t))$. Find $x' \in I\!B^{\overline{\varphi}}_{X}(x,t) \cap \operatorname{dom} F$ and $v' \in I\!B^{\overline{\varrho}}_{Y}(v, t/\alpha) \cap F(x')$ such that $g(x',v') = y$. Hence $v' = y$ and consequently $y \in F(x') \subset F\big(I\!B^{\overline{\varphi}}_{X}(x,t)\big)$. $\square$

6. Applications

In this section, we mention several easy applications. First, recall the concept of Hölder regularity (i.e., openness around a reference point) from [11, 12].

Definition 6.1. Consider a set-valued mapping $F\colon X \rightrightarrows Y$ between metric spaces (X, d) and (Y, ϱ) and a point $(\bar{x}, \bar{y}) \in \operatorname{gph} F$. The mapping F is said to be *open with a rate* $k \geq 1$ around $(\bar{x}, \bar{y})$ when there are positive constants c and ε along with a neighborhood $U \times V$ of $(\bar{x}, \bar{y})$ in $X \times Y$ such that

$$F(I\!B_X(x,t)) \supset I\!B_Y\big(y, c^k t^k\big) \text{ whenever } (x,y) \in U \times V,\ y \in F(x) \text{ and } t \in (0, \varepsilon). \quad (33)$$

The supremum of $c > 0$ for which there exist a constant $\varepsilon > 0$ and a neighborhood $U \times V$ of $(\bar{x}, \bar{y})$ in $X \times Y$ such that (33) holds is called the *modulus of surjection with the rate* k of F around $(\bar{x}, \bar{y})$ and is denoted by $\operatorname{sur}^{(k)} F(\bar{x}, \bar{y})$.

Let us state a particular version of [11, Theorem 2.2] (see also [17, Corollary 2.99]) along with an elementary proof.

Theorem 6.2. *Let (X, d) and (Y, ϱ) be complete metric spaces and let a point $(\bar{x}, \bar{y}) \in X \times Y$ be given. Consider a set-valued mapping $F\colon X \rightrightarrows Y$, with a closed graph and $\bar{y} \in F(\bar{x})$, and a number $k \geq 1$. Suppose that there are $r > 0$, $\beta \in [0, 1)$, and $c > 0$ such that for each $x \in I\!B_X(\bar{x}, r)$, each $v \in I\!B_Y(\bar{y}, r) \cap F(x)$, and each $t \in (0, r)$ we have*

$$\sup_{y \in B_Y(v, c^k t^k)} \operatorname{dist}(y, F(I\!B_X(x,t))) \leq (\beta c t)^k. \quad (34)$$

Then $\operatorname{sur}^{(k)} F(\bar{x}, \bar{y}) \geq (1 - \beta)c$.

Proof. Fix any $\beta' \in (\beta, 1)$ and any $c' \in (\beta' c, c)$. Let $\alpha := 1/(1 + 2c^k)$, then $0 < \alpha < 1$. Let $r' := \min\{\alpha r/2, (c' \min\{r, 1\})^k/(1+1/\alpha)\}$. Fix any $x \in I\!B_X(\bar{x}, r')$, any $v \in I\!B_Y(\bar{y}, r'/\alpha) \cap F(x)$, and any $y \in I\!B_Y(\bar{y}, r')$ with $y \neq v$. Let $t := \varrho(v, y)^{1/k}/c'$. Then $0 < t < ((1 + 1/\alpha)r')^{1/k}/c' < \min\{r, 1\}$. By (34), there are $x' \in I\!B_X(x, t)$ and $v' \in F(x')$ such that $\varrho(v', y) < (\beta' c t)^k$. Then

$$\varrho(v', y)^{\frac{1}{k}} \ < \ \beta' c t = c' t - (c' - \beta' c)t = \varrho(v, y)^{\frac{1}{k}} - (c' - \beta' c)t.$$

As $d(x, x') < t$ and $\varrho(v, v') \leq \varrho(v, y) + \varrho(y, v') < c^k t^k + (\beta' c t)^k < 2c^k t^k < t/\alpha$, we get

$$\varrho(v', y)^{\frac{1}{k}}/(c' - \beta' c) < \varrho(v, y)^{\frac{1}{k}}/(c' - \beta' c) - \max\{d(x, x'), \alpha \varrho(v, v')\}.$$

Theorem 5.3, with $\phi := (\cdot)^{1/k}/(c' - \beta' c)$, $r := r'/2$, $G \equiv Y$, and $\chi = \varrho$, says that $\operatorname{sur}^{(k)} F(\bar{x}, \bar{y}) \geq c' - \beta' c$. Letting $\beta' \downarrow \beta$ and then $c' \uparrow c$ we conclude that $\operatorname{sur} F(\bar{x}, \bar{y}) \geq (1 - \beta)c$. □

In finite dimensions, Brouwer's fixed point theorem implies [4, Theorem 3.4]:

Theorem 6.3. *Consider a point $\bar{x} \in \mathbb{R}^n$ along with a mapping $g\colon \mathbb{R}^n \longrightarrow \mathbb{R}^m$ which is both defined and continuous in a vicinity of $\bar{x}$. Suppose that there is a surjective linear mapping $A\colon \mathbb{R}^n \longrightarrow \mathbb{R}^m$ such that* $\operatorname{calm}(g-A)(\bar{x}) < \operatorname{sur} A$. *Then $n \geq m$ and*

$$\operatorname{lopen} g(\bar{x}) \geq \operatorname{sur} A - \operatorname{calm}(g-A)(\bar{x}) > 0.$$

We have the following converse of the above statement.

Theorem 6.4. *Let a Banach space $(X, \|\cdot\|)$, a normed space $(Y, \|\cdot\|)$, and a point $\bar{x} \in X$ be given. Consider mappings $g\colon X \longrightarrow Y$ and $A \in \mathcal{L}(X,Y)$. Then*

$$\operatorname{sur} A \geq \operatorname{lopen} g(\bar{x}) - \operatorname{calm}(g-A)(\bar{x}).$$

Proof. If $\operatorname{lopen} g(\bar{x}) \leq \operatorname{calm}(g-A)(\bar{x})$, we are done.
Suppose now that we have $\operatorname{lopen} g(\bar{x}) > \operatorname{calm}(g-A)(\bar{x})$. Find c and ℓ such that $\operatorname{lopen} g(\bar{x}) > c > \ell > \operatorname{calm}(g-A)(\bar{x})$. There is $r > 0$ such that

$$g\big(I\!B_X(\bar{x},t)\big) \supset I\!B_Y(g(\bar{x}), ct) \quad \text{for each} \quad t \in (0, 2r) \tag{35}$$

and

$$\|g(x) - g(\bar{x}) - A(x-\bar{x})\| \leq \ell \|x - \bar{x}\| \quad \text{for each} \quad x \in I\!B_X(\bar{x}, 2r). \tag{36}$$

Further, fix any $t \in (0, 2r)$. Then

$$I\!B_Y(g(\bar{x}), ct) \subset g(\bar{x}) + A(I\!B_X(0,t)) + I\!B_Y(0, \ell t).$$

Indeed, given any fixed $w \in I\!B_Y(g(\bar{x}), ct)\big)$, by (35), there is $x \in I\!B_X(\bar{x}, t)$ such that $w = g(x)$, and, by (36),

$$w \in g(\bar{x}) + A(x - \bar{x}) + I\!B_Y(0, \ell t) \subset g(\bar{x}) + A(I\!B_X(0,t)) + I\!B_Y(0, \ell t).$$

Adding $Ax - g(\bar{x})$ we conclude that

$$I\!B_Y(Ax, ct) \subset A(I\!B_X(x,t)) + I\!B_Y(0, \ell t) \quad \text{for each} \quad x \in X. \tag{37}$$

Choose any $c' \in (\ell, c)$. Find $r' \in (0, \min\{r, c'r\})$ such that $\|Ax\| < c'r$ for each $x \in I\!B_X(0, r')$. Fix any $x \in I\!B_X(0,r')$ and any $y \in I\!B_Y(0, r')$ such that $Ax \neq y$. Let $t := \|Ax - y\|/c'$, then $0 < t < r + r = 2r$ and $y \in I\!B_Y(Ax, ct)$. Then, by (37), there is $x' \in I\!B_X(x,t)$ such that $\|y - Ax'\| < \ell t$. Thus

$$\|y - Ax'\| < \ell t = c't - (c' - \ell)t < \|Ax - y\| - (c' - \ell)\|x - x'\|.$$

Proposition 5.2, with $g := A$, $G \equiv Y$, $r := r'/2$ and $\bar{x} := 0$, implies that there is $r'' > 0$ such that

$$I\!B_Y(Ax, (c' - \ell)t) \subset A(I\!B_X(x,t)) \quad \text{for each} \quad x \in I\!B_X(0, r'') \quad \text{and} \quad t \in (0, r'').$$

Therefore $\operatorname{sur} A \geq c' - \ell$. Letting $c' \uparrow \operatorname{lopen} g(\bar{x})$ and $\ell \downarrow \operatorname{calm}(g-A)(\bar{x})$, we finish the proof. $\square$

Remark 6.5. Theorem 6.4 says that Y is necessarily complete. In particular, if g is both Fréchet differentiable and semiregular at $\bar{x}$ then the target space has to be complete. Of course, one can replace the last paragraph of the proof (following (37)) by the reference to Theorem 6.2 under the additional assumption that the target space is complete. $\square$

It is well-known that if $A \in \mathbb{R}^{m\times n}$ is surjective, then $\operatorname{sur} A$ equals to the least singular value of A. Let us present a directional version of this fact.

Theorem 6.6. *Let L be a non-trivial subspace of $\mathbb{R}^n$ and $P \in \mathbb{R}^{n\times n}$ be the orthogonal projection onto L. Consider a linear mapping $A\colon \mathbb{R}^n \longrightarrow \mathbb{R}^m$ such that the set $M := A(L)$ is a non-trivial subspace of $\mathbb{R}^m$. Then*

$$A(I\!B_{\mathbb{R}^n} \cap L) \supset I\!B_{\mathbb{R}^m}[0, c] \cap M, \tag{38}$$

where c is the least singular value of the matrix AP.

Proof. Let $j := \dim M$. The singular value decomposition of the matrix AP implies that there are numbers $\sigma_1, \sigma_2, \dots, \sigma_j$, with $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_j > 0$, and orthonormal vectors $v_1, v_2, \dots, v_n$ such that

$$(AP)^T APv_i = \sigma_i^2 v_i \quad \text{for each} \quad i = 1, 2, \dots, j, \tag{39}$$

and $$(AP)^T APv_i = 0 \quad \text{for each} \quad i = j+1, j+2, \dots, n.$$

Let $N := \operatorname{span}\{v_1, v_2, \dots, v_j\}$. Then N is a subspace of L. Indeed, since $P = P^T$ and $P(A^T APv_i) = \sigma_i^2 v_i$ for each $i = 1, 2, \dots, j$ by (39), we conclude that $v_1, v_2, \dots, v_j$ are elements of L.

Fix any $x \in \mathbb{S}_{\mathbb{R}^n} \cap N$. Find numbers $a_1, a_2, \dots, a_j$ such that $x = a_1 v_1 + a_2 v_2 + \dots + a_j v_j$ with $a_1^2 + a_2^2 + \dots + a_j^2 = 1$. Then

$$\begin{aligned}\|Ax\|^2 = \|APx\|^2 = \langle x, (AP)^T APx\rangle &= \left\langle \sum_{i=1}^{j} a_i v_i, \sum_{i=1}^{j} a_i (AP)^T APv_i \right\rangle \\ &= \left\langle \sum_{i=1}^{j} a_i v_i, \sum_{i=1}^{j} a_i \sigma_i^2 v_i \right\rangle = \sum_{i=1}^{j} a_i^2 \sigma_i^2 \|v_i\|^2 \geq \min_{1\leq i\leq j} \sigma_i^2 = \sigma_j^2.\end{aligned}$$

Letting $c := \sigma_j$, we conclude that (38) holds. $\square$

We also prove a directional version of Theorem 6.3.

Theorem 6.7. *Let L be a non-trivial subspace of $\mathbb{R}^n$, let $P \in \mathbb{R}^{n\times n}$ be the orthogonal projection onto L, and let $\bar{x} \in \mathbb{R}^n$ be given. Consider a linear mapping $A\colon \mathbb{R}^n \longrightarrow \mathbb{R}^m$ and a continuous mapping $g\colon \mathbb{R}^n \longrightarrow \mathbb{R}^m$. Suppose that the set $M := A(L)$ is a non-trivial subspace of $\mathbb{R}^m$, that there are positive constants r and μ such that*

$$T_M(g(\bar{x}) - A\bar{x}, g(x) - Ax) \leq \mu T_L(\bar{x}, x) \quad \textit{for each} \quad x \in I\!B_{\mathbb{R}^n}[\bar{x}, r], \tag{40}$$

and that $\mu < c$, where c is the least singular value of the matrix AP. Then $g\big(I\!B_{\mathbb{R}^n}[\bar{x}, t] \cap (\bar{x} + L)\big) \supset I\!B_{\mathbb{R}^m}[g(\bar{x}), (c - \mu)t] \cap (g(\bar{x}) + M)$ for each $t \in (0, r]$.

Proof. Without any loss of generality, suppose that $\bar{x} = 0$ and $g(\bar{x}) = 0$. Let $j := \dim M$. There are numbers $\sigma_1, \sigma_2, \dots, \sigma_j$, with $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_j > 0$, and orthogonal matrices $U \in \mathbb{R}^{m\times m}$ and $V \in \mathbb{R}^{n\times n}$, such that

$$U^T APV = \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{m\times n}, \quad \text{where} \quad \Sigma = \operatorname{diag}\{\sigma_1, \sigma_2, \dots, \sigma_j\}.$$

Let $B := V\widetilde{\Sigma}U^T \in \mathbb{R}^{n\times m}$, where $\widetilde{\Sigma} = \begin{bmatrix} \Sigma^{-1} & 0 \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{n\times m}$.

Then $\|B\| = 1/c$, the orthogonal projection onto M is APB, and $By \in L$ for each $y \in M$. Fix any $t \in (0, r]$ and any $y \in I\!B_{\mathbb{R}^m}[0, (c-\mu)t] \cap M$. Define the mapping $h_y \colon I\!B_{\mathbb{R}^n} \cap L \longrightarrow L$ by

$$h_y(u) := \frac{1}{t} B\big(A(tu) - g(tu) + y\big), \quad u \in I\!B_{\mathbb{R}^n} \cap L. \tag{41}$$

In particular, h_y is continuous on $I\!B_{\mathbb{R}^n} \cap L$. Further, for each $u \in I\!B_{\mathbb{R}^n} \cap L$, we have $tu \in I\!B_{\mathbb{R}^n}[0, r] \cap L$, so this and (40) imply that $A(tu) - g(tu) + y \in M$ and hence $B(A(tu) - g(tu) + y) \in L$. Therefore h_y is well-defined.

For any $u \in I\!B_{\mathbb{R}^n} \cap L$, we have

$$\begin{aligned} \|h_y(u)\| &\leq \frac{1}{t}\|B\|\|A(tu) - g(tu) + y\| \leq \frac{1}{ct}\big(\|A(tu) - g(tu)\| + \|y\|\big) \\ &\overset{(40)}{\leq} \frac{1}{ct}\big(\mu t\|u\| + (c-\mu)t\big) \leq 1. \end{aligned}$$

Therefore h_y maps $I\!B_{\mathbb{R}^n} \cap L$ into itself. By Brouwer's fixed point theorem, there is $u \in I\!B_{\mathbb{R}^n} \cap L$ such that $h_y(u) = u$. Hence $APh_y(u) = APu$, thus

$$APB\big(A(tu) - g(tu) + y\big) = A(tu) - g(tu) + y = tAP(u) = A(tu).$$

Then $x := tu$ is such that $g(x) = y$ and $x \in I\!B_{\mathbb{R}^n}[0, t] \cap L$. $\square$

Finally, we briefly consider local convergence of the directional version of Newton-type method for solving a generalized equation:

$$\text{Find} \quad x \in X \quad \text{such that} \quad f(x) + F(x) \ni 0, \tag{42}$$

where $(X, \|\cdot\|)$ and $(Y, \|\cdot\|)$ are Banach spaces, $f \colon X \longrightarrow Y$ is a single-valued mapping, and $F \colon X \rightrightarrows Y$ is a set-valued mapping. Suppose that we want to find a solution to (42) by the iteration in the form

$$f(x_k) + A_k(x_{k+1} - x_k) + F(x_{k+1}) \ni 0 \quad \text{for} \quad k \in \mathbb{N}_0, \tag{43}$$

where a sequence (A_k) in $\mathcal{L}(X, Y)$ and an initial point $x_0 \in X$ are given.

Theorem 6.8. *Let $(X, \|\cdot\|)$ and $(Y, \|\cdot\|)$ be Banach spaces. Consider a single-valued mapping $f\colon X \longrightarrow Y$, a set-valued mapping $F\colon X \rightrightarrows Y$, non-empty sets $L \subset \mathbb{S}_X$ and $M \subset \mathbb{S}_Y$, a point $\bar{x} \in X$ such that $f(\bar{x}) + F(\bar{x}) \ni 0$, and a sequence $(A_k)_{k=0}^{\infty}$ in $\mathcal{L}(X,Y)$. For each $k \in \mathbb{N}_0$, define mappings*

$$G_k := f(\bar{x}) + A_k(\cdot - \bar{x}) + F \quad \textit{and} \quad r_k := f(\bar{x}) - f + A_k(\cdot - \bar{x}).$$

Suppose that there are positive constants r, c, and ℓ, with $\ell < c$, such that

(i) *for every $t \in (0, r)$ and every $k \in \mathbb{N}_0$ we have*

$$\mathbb{B}_Y[0, ct] \cap \operatorname{cone} M \subset G_k\big(\mathbb{B}_X[\bar{x}, t] \cap (\bar{x} + \operatorname{cone} L)\big);$$

(ii) *for every $x \in \mathbb{B}_X[\bar{x}, r]$ and $k \in \mathbb{N}_0$ we have*

$$T_M(0, r_k(x)) \le \ell\, T_L(\bar{x}, x).$$

Then for every $x_0 \in \mathbb{B}_X[\bar{x}, r] \cap (\bar{x} + \operatorname{cone} L)$ there is a sequence (x_k) in the set $\mathbb{B}_X[\bar{x}, r] \cap (\bar{x} + \operatorname{cone} L)$ generated by (43) with the initial point x_0 and

$$\|x_{k+1} - \bar{x}\| \le \frac{\ell}{c}\, \|x_k - \bar{x}\| \quad \textit{for each} \quad k \in \mathbb{N}_0.$$

Proof. Choose any $x_0 \in \mathbb{B}_X[\bar{x}, r] \cap (\bar{x} + \operatorname{cone} L)$. Fix an arbitrary $k \in \mathbb{N}_0$. Suppose that $x_k \in \mathbb{B}_X[\bar{x}, r] \cap (\bar{x} + \operatorname{cone} L)$ is given. We shall find

$$x_{k+1} \in \mathbb{B}_X[\bar{x}, r] \cap (\bar{x} + \operatorname{cone} L)$$

such that (43) holds. This is equivalent to find $x_{k+1} \in X$ such that

$$G_k(x_{k+1}) \ni r_k(x_k). \tag{44}$$

Let $t := \ell\|x_k - \bar{x}\|/c$ $(< r)$. By (ii), we have $r_k(x_k) \in \operatorname{cone} M$ and

$$\|r_k(x_k)\| \le \ell\, \|x_k - \bar{x}\| = ct.$$

By (i), there is $x_{k+1} \in \mathbb{B}_X[\bar{x}, t] \cap (\bar{x} + \operatorname{cone} L)$ satisfying (44) and

$$\|x_{k+1} - \bar{x}\| \le t = \frac{\ell}{c}\, \|x_k - \bar{x}\|. \qquad \square$$

To establish the uniform versions of the directional regularity requested in the above statement one can use perturbation stability of this property established in [3] along with the assumption that the measure of non-compactness of the set of all A_k's is sufficiently small. This essentially follows the proof in the classical setting (cf. Step 1 of the proof of [4, Theorem 5.1]). One can also consider various inexact versions of the algorithm without any difficulty.

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