Journal of Convex Analysis
Volume 27 (2020), No. 1, 237–276

Various Lipschitz-Like Properties for Functions and Sets. II: Subdifferential and Normal Characterizations

Rafael Correa
Centro de Modelamiento Matemático and Departamento de Ingeniería Matemática, Universidad de Chile, Santiago, Chile; and: Universidad de O'Higgins, Rancagua, Chile
rcorrea@dim.uchile.cl

Pedro Gajardo
Departamento de Matemática, Universidad Técnica Federico Santa María, Valparaíso, Chile
pedro.gajardo@usm.cl

Lionel Thibault
Institut Montpelliérain Alexander Grothendieck, Université de Montpellier, France
lionel.thibault@umontpellier.fr

Dedicated to Alexander Ioffe on the occasion of his 80th birthday.

Received: February 22, 2019
Revised manuscript received: June 27, 2019
Accepted: July 12, 2019

The present paper is a continuation of our previous article: "Various Lipschitz-like properties of functions and sets. I: Directional derivative and tangential characterizations" [SIAM J. Optim. 20(4) (2010) 1766–1785]. Here we provide diverse subdifferential and normal characterizations of K directionally Lipschitzian functions and sets for bounded sets K of a Banach space.

Keywords: K directionally Lipschitzian function, epi-Lipschitzian set, compactly epi-Lipschitzian set, subdifferential, normal cone, multidirectional mean value inequality.

2010 Mathematics Subject Classification: Primary 26A24, 49J52; secondary 28B20, 47L07.

1. Introduction

In order to establish in a unified way several old and new characterizations of diverse Lipschitz-like properties, we defined in [11] the concepts of K directionally Lipschitzian sets and functions. A closed set C of a Banach space is known to be compactly epi-Lipschitzian at a point $\bar{x} \in C$ if there exist two reals $r > 0$ and $\gamma > 0$ and a compact set Q in the Banach space such that

$$C \cap B(\bar{x}, r) + tB(0, \gamma) \subset C - tQ \quad \text{for all } t \in]0, r[,$$

where $B(\bar{x}, r)$ denotes the open ball centered at $\bar{x}$ with radius r. When Q is a singleton set $\{\bar{v}\}$, the set C is said to epi-Lipschitzian at $\bar{x}$ (in the direction $\bar{v}$). Given any nonempty bounded set K of the Banach space, noticing that $C - tK$

ISSN 0944-6532 / \$ 2.50

fails to be closed in general, the above inclusion led us in [11] to declare that the closed set C is K directionally Lipschitzian at $\bar{x} \in C$ provided that there exist two reals $r > 0$ and $\gamma > 0$ such that

$$C \cap B(\bar{x}, r) + tB(0, \gamma) \subset \mathrm{cl}\,(C - tK) \quad \text{for all } t \in]0, r[.$$

The use of the closure of $C - tK$ is crucial and this has been highlighted in [11]. This closure operation will also be clear through the development in the diverse sections. Using epigraphs of lower semicontinuous functions, one defines the K directionally Lipschitzian property for such functions in terms of suitable behavior of differential quotient. This can also be achieved in a direct way as done in Definition 2.6.

In our paper [11] we proved several characterizations of K directionally Lipschitzian functions and sets. Our characterizations in [11] for functions are in terms of the Hadamard lower directional derivatives and for sets they are in terms of the Bouligand tangent cones. Our approach in [11] is mainly based on a multidirectional mean value inequality with Hadamard lower directional derivatives.

The present paper is a continuation of [11] in the sense that we provide here other characterizations of K directionally Lipschitzian functions and sets, but in terms of subdifferentials and normal cones. The main subdifferentials and normal cones which work in general Banach spaces for our analysis are the approximate and geometric subdifferentials and normal cones introduced and developed by Ioffe in a series of seminal works [13, 14, 15].

The paper is organized as follows. Section 2 recalls the concepts of Clarke subdifferential and Ioffe approximate subdifferential as well as other basic subdifferentials. In the same section the concepts of K directionally Lipschitzian functions and sets are given and we define certain constants related to these concepts. Diverse examples are furnished as illustrations. In Section 3 we utilize an abstract notion of normal cone to establish a general condition for the K directional Lipschitzian property of sets. When K is compact in the Banach space, we show that the condition, with either the Ioffe approximate normal cone or the Ioffe geometric normal, is a complete characterization of K directionally Lipschitzian sets. In Asplund spaces, the same characterization is obtained with Fréchet and Mordukhovich limiting normal cones. Conditions/characterizations with Fréchet and limiting subdifferentials in Asplund spaces are also derived for the K directional Lipschizian property for lower semicontinuous functions. Similar characterizations with the Ioffe approximate subdifferential in general Banach spaces are established in Section 4 for lower semicontinuous functions.

2. Preliminaries

Throughout all the paper, unless otherwise stated, $(X, \|\cdot\|)$ will be a *real Banach space*, X^* its topological dual, and $\langle \cdot, \cdot \rangle$ the duality pairing. For $x \in X$ and $r > 0$, we denote by $B(x, r)$ the *open ball* of X centered at x with radius r. In a similar way, B_X (resp. $\overline{B}_X$) will denote the *open (resp. closed) unit ball* centered at zero.

In this work we will deal with various notions of subdifferentials. Let us start with the Clarke subdifferential. One of the best ways to introduce this concept (see [6]) is to consider first the case of a locally Lipschitz continuous function $f\colon X \longrightarrow \mathbb{R}$ and define its *Clarke subdifferential* at $\bar{x} \in X$ as

$$\partial_C f(\bar{x}) := \{x^* \in X^* \ : \ \langle x^*, v\rangle \leq d^0 f(\bar{x}; v) \quad \text{for all} \ \ v \in X\}, \tag{1}$$

where $d^0 f(\bar{x}; v)$ is the *Clarke directional derivative* given by

$$d^0 f(\bar{x}; v) := \limsup_{\substack{x \to \bar{x} \\ t \to 0^+}} t^{-1}[f(x+tv) - f(x)]. \tag{2}$$

Through (1), the *Clarke normal cone* to a subset S of X at $\bar{x} \in S$ is the weak star closed cone generated by the Clarke subdifferential of the distance function to S at $\bar{x}$, that is,

$$N_C(S; \bar{x}) := \mathrm{cl}_{\,w^*}\left(\mathbb{R}_+ \partial_C d_S(\bar{x})\right),$$

where $\mathrm{cl}_{\,w^*}$ stands for the weak star closure in X^*. For an extended real-valued function $f\colon X \longrightarrow \mathbb{R} \cup \{+\infty\}$, its *Clarke subdifferential* at $\bar{x} \in \operatorname{dom} f := \{x \in X \ : \ f(x) < +\infty\}$, can then be defined in a geometric manner as

$$x^* \in \partial_C f(\bar{x}) \Leftrightarrow (x^*, -1) \in N_C(\operatorname{epi} f; \bar{x}, f(\bar{x})). \tag{3}$$

Here, $\operatorname{epi} f$ denotes the *epigraph of* f, i.e.,

$$\operatorname{epi} f := \{(x, r) \in X \times \mathbb{R} \ : \ f(x) \leq r\},$$

and we write $N_C(\operatorname{epi} f; \bar{x}, f(\bar{x}))$ in place of $N_C(\operatorname{epi} f; (\bar{x}, f(\bar{x})))$. Any $x^* \in \partial_C f(\bar{x})$ is called a *Clarke subgradient* of f at $\bar{x}$.

We mention (see [26]) that we also have the equality

$$N_C(S, \bar{x}) = \partial_C \Psi_S(\bar{x}), \tag{4}$$

where Ψ_S is the *indicator function* of the set $S \subset X$ defined for all x in the space X by

$$\Psi_S(x) = \begin{cases} 0 & \text{if } \ x \in S \\ +\infty & \text{if } \ x \notin S. \end{cases}$$

Fix now an extended real-valued function $f\colon X \longrightarrow \mathbb{R} \cup \{+\infty\}$. Analogously to (1), the *Hadamard subdifferential* of f at $\bar{x} \in \operatorname{dom} f$ is given (see, e.g., [14, 30]) by

$$\partial_H f(\bar{x}) := \{x^* \in X^* \ : \ \langle x^*, v\rangle \leq d_H^- f(\bar{x}; v) \quad \text{for all} \ \ v \in X\}, \tag{5}$$

where the *Hadamard lower directional derivative* $d_H^- f(\bar{x}; v)$ at $\bar{x} \in X$ in a direction $v \in X$ is defined by

$$d_H^- f(\bar{x}; v) := \liminf_{\substack{v' \to v \\ t \to 0^+}} t^{-1}[f(\bar{x}+tv') - f(\bar{x})]. \tag{6}$$

Similar to equality (4), with the indicator function Ψ_S, one defines the *Hadamard normal cone* to S at $\bar{x} \in S$, as

$$N_H(S, \bar{x}) := \partial_H \Psi_S(\bar{x}).$$

The Hadamard subdifferential allows one to define the *Ioffe approximate subdifferential* ([14]) of f at $\bar{x} \in \operatorname{dom} f$, first considered in finite dimensions in [13] with notation $\partial_a f(\bar{x})$ and extended in [13] to Banach spaces with notation $\partial_A f(\bar{x})$. For convenience, we keep notation $\partial_a f(\bar{x})$ in the Banach space setting. The approximate subdifferential $\partial_a f(\bar{x})$ is then defined by

$$\partial_a f(\bar{x}) := \bigcap_{L \in \mathcal{F}} \operatorname*{Lim\,sup}_{x \to_f \bar{x}} \partial_H f_{x+L}(x), \tag{7}$$

where $f_{x+L} = f + \Psi_{x+L}$. The notation $x \to_f \bar{x}$ means $x \to \bar{x}$, with respect to the norm topology in X, together with $f(x) \to f(\bar{x})$, and the topological limit superior (or outer limit) $\operatorname{Lim\,sup}_{x \to_f \bar{x}}$ is taken with respect to the weak-star topology in X^*. The intersection in (7) is taken over $\mathcal{F}$, the collection of all finite-dimensional vector subspaces of X.

With the Ioffe approximate subdifferential, it is naturally associated, for any set S and $x \in S$, the cone $\partial_a \Psi_S(x)$ called the *Ioffe approximate normal cone* of S at x denoted by $N_a(S; x)$, that is,

$$N_a(S; x) := \partial_a \Psi_S(x).$$

Viceversa, this cone reproduces the Ioffe approximate subdifferential of a function f at $\bar{x} \in \operatorname{dom} f$, through the equality (see [14])

$$\partial_a f(\bar{x}) = \{x^* \in X^* \ : \ (x^*, -1) \in N_a(\operatorname{epi} f; \bar{x}, f(\bar{x}))\}.$$

Remark 2.1. Using the distance function d_S instead of the indicator function Ψ_S, leads to consider the cone $\mathbb{R}_+ \partial_a d_S(x)$ for $x \in S$. This cone (as well its closure) is generally different (in the context of infinite dimensional spaces) from $N_a(S; x)$. It does not depend on the norm in the sense that for any other equivalent norm and the associated distance function $\tilde{d}_S$, one has (see [15, 30]) $\mathbb{R}_+ \partial_a d_S(x) = \mathbb{R}_+ \partial_a \tilde{d}_S(x)$. $\square$

Because of the geometric nature of the cone $\mathbb{R}_+ \partial_a d_S(x)$, through the distance function d_S, Ioffe in [15] (see also [19, 21]) called it the *geometric normal cone* of S at x. We will denote it as $N_G(S; x)$, that is,

$$N_G(S; x) := \mathbb{R}_+ \partial_a d_S(x). \tag{8}$$

Reverse to the way of the above approximate concepts and like (3), we associate to the function f its *Ioffe geometric subdifferential* $\partial_G f(\bar{x})$ at $\bar{x} \in \operatorname{dom} f$ by

$$x^* \in \partial_G f(\bar{x}) \Leftrightarrow (x^*, -1) \in N_G(\operatorname{epi} f; \bar{x}, f(\bar{x})). \tag{9}$$

If the function f is locally Lipschitz continuous at $\bar{x}$, it is known (see [15]) that $\partial_a f(\bar{x}) = \partial_G f(\bar{x})$. Applying this to (8), we see that one also has

$$N_G(S; x) = \mathbb{R}_+ \partial_G d_S(x). \tag{10}$$

For all above subdifferentials we take the convention that they are empty at points outside $\operatorname{dom} f$. Comparisons between these subdifferentials are given by the inclusions

$$\partial_G f(\bar{x}) \subset \partial_a f(\bar{x}) \quad \text{and} \quad \partial_G f(\bar{x}) \subset \partial_C f(\bar{x}).$$

No such inclusion holds, in infinite dimensions, between $\partial_a f(\bar{x})$ and $\partial_C f(\bar{x})$, for general functions f.

Several results in the paper will require other important notions of subdifferential. It is then convenient to work with an abstract general concept of subdifferential, allowing us to state many results in a general unified framework.

Definition 2.2. Following [12, 15, 18, 28, 31, 30] we call *subdifferential* on X any operator ∂ which associates with any function $f\colon X \to \mathbb{R} \cup \{+\infty\}$ (resp. with any function in a class of functions containing the convex ones and stable for the operations involved in the properties below) and with any $x \in X$ a subset $\partial f(x)$ of X^* which satisfies the following properties:

(P1) $\partial f(x) = \emptyset$ if $x \notin \operatorname{dom} f$;

(P2) $\partial f(x) = \partial g(x)$ whenever f and g coincide on a neighborhood of x;

(P3) $\partial f(x)$ is equal to the subdifferential in the sense of convex analysis whenever f is convex and continuous;

(P4) If f is lower semicontinuous near x, g is convex continuous, and x is a local minimum point of $f + g$, then for any $\varepsilon > 0$ there exists $u,\ v \in B(x, \varepsilon)$ with $|f(x) - f(u)| < \varepsilon$ such that

$$0 \in \partial f(u) + \partial g(v) + \varepsilon B_{X^*};$$

(P5) For any $\alpha \in \mathbb{R}$ and for $g\colon X \times \mathbb{R} \longrightarrow \mathbb{R} \cup \{+\infty\}$ given by $g(x, r) := f(x) + \alpha r$ one has the inclusion

$$\partial g(x, r) \subset \partial f(x) \times \{\alpha\}.$$

Remark 2.3. Taking Property (P4) into account, a subdifferential as in the above definition can be seen as a subdifferential in the strong sense. When the following weaker property (P4') holds in place of (P4), we will say that ∂ is a *subdifferential* in the weak sense (see [12, 28, 31]):

(P4') If f is lower semicontinuous near x, g is convex continuous, and x is a local minimum point of $f + g$, then one has

$$0 \in ({}^{\mathrm{seq}}\operatorname*{Lim\,sup}_{y \to_f x} \partial f(y)) + \partial g(x),$$

where ${}^{\mathrm{seq}}\operatorname{Lim\,sup}$ denotes here the weak-star sequential limit superior, that is, $x^* \in {}^{\mathrm{seq}}\operatorname{Lim\,sup}_{y \to_f x} \partial f(y)$ if and only if there are a sequence $(x_n)_{n \in \mathbb{N}}$ in X with $(x_n, f(x_n)) \to (x, f(x))$ and a sequence $(x_n^*)_{n \in \mathbb{N}}$ in X^* with $x_n^* \in \partial f(x_n)$ and $x_n^* \to x^*$ with respect to the weak star topology.

We point out that for any lower semicontinuous function f with $\operatorname{dom} f \neq \emptyset$, and any such subdifferential ∂ in the weak sense, Zagrodny's mean value theorem

[28, 30, 34] assures us that the set $\operatorname{Dom}\partial f := \{x \in X : \partial f(x) \neq \emptyset\}$ is graphically dense in $\operatorname{dom} f$ in the sense that for any $x \in \operatorname{dom} f$ there exists a sequence $(x_n)_{n\in\mathbb{N}}$ in X with $x_n \in \operatorname{Dom}\partial f$ such that $(x_n, f(x_n)) \to (x, f(x))$ as $n \to \infty$. □

In a Banach space the Ioffe geometric subdifferential ∂_G, the Ioffe approximate subdifferential ∂_a, and the Clarke subdifferential ∂_C are subdifferentials in the strong sense fulfilling all properties of Definition 2.2 (see the properties of such subdifferentials in [8, 14, 15, 30]). In Banach spaces with Gâteaux differentiable (outside of zero) equivalent renorm, the Hadamard subdifferential is a subdifferential in the weak sense (i.e., satisfying (P4') instead of (P4) in Definition 2.2: see Theorem 1 in [18][1]). Other important examples of subdifferentials, in the context of *Asplund spaces* (see [25, 30]), are the *Fréchet subdifferential* $\partial_F f(\bar{x})$ defined by

$$\partial_F f(\bar{x}) := \left\{ x^* \in X^* \,:\, \liminf_{x\to\bar{x}} \frac{f(x) - f(\bar{x}) - \langle x^*, x-\bar{x}\rangle}{\|x-\bar{x}\|} \geq 0 \right\}, \tag{11}$$

and the *Mordukhovich limiting subdifferential* $\partial_L f(\bar{x})$ given by

$$\partial_L f(\bar{x}) := {}^{\text{seq}}\operatorname*{Lim\,sup}_{x\to_f \bar{x}} \partial_F f(x).$$

Recall that a Banach space is an Asplund space when the topological dual of any separable vector subspace is itself separable.

The *proximal subdifferential* $\partial_P f(\bar{x})$ of f at $\bar{x} \in \operatorname{dom} f$ is defined (see [8, 27, 30]) as the set of all $x^* \in X^*$ such that for some $r > 0$ the inequality

$$f(x) \geq f(\bar{x}) + \langle x^*, x-\bar{x}\rangle - r\|x-\bar{x}\|^2$$

holds for all x in a neighborhood of $\bar{x}$. In *Hilbert spaces*, the proximal subdifferential is also a subdifferential in the sense of Definition 2.2. It is known that, for a lower semicontinuous function f on a Hilbert space, the Mordukhovich limiting subdifferential of f can also be described with the proximal subdifferential through the equality (see [16, 30])

$$\partial_L f(\bar{x}) = {}^{\text{seq}}\operatorname*{Lim\,sup}_{x\to_f \bar{x}} \partial_P f(x).$$

The Fréchet, Mordukhovich limiting, and proximal normal cones of a set of X, denoted by N_F, N_L, and N_P respectively, are defined through the corresponding subdifferentials of the indicator function of S, that is,

$$N_F(S;\bar{x}) := \partial_F \Psi_S(\bar{x}), \quad N_L(S;\bar{x}) := \partial_L \Psi_S(\bar{x}), \quad N_P(S;\bar{x}) := \partial_P \Psi_S(\bar{x}).$$

Finally, the viscosity β-subdifferentials (see, e.g., [5, 18] for the definitions) are also subdifferentials (in the strong sense of Definition 2.2) in certain Banach spaces, for example, when there exists an equivalent β-differentiable norm.

In addition to the above concepts of subdifferentials, the notion of limit of infima over enlargements of a set will be crucial in the development of the paper (as it

[1]We thank A. Ioffe for providing the reference to us.

was the case in our previous paper [11]). For an extended real-valued function ϕ on X and a nonempty set $S \subset X$, the limit of infima of ϕ over enlargements of S, denoted by linf (as in [11]), is given by

$$\operatorname*{linf}_{v\in S}\phi(v) := \sup_{\delta>0}\inf_{v\in S+\delta B_X}\phi(v) = \lim_{\delta\to 0^+}\inf_{v\in S+\delta B_X}\phi(v). \tag{12}$$

The second equality being due to the nonincreasing property of $\delta \mapsto \inf\limits_{v\in S+\delta B_X}\phi(v)$.
In a similar way we define $\operatorname*{lsup}\limits_{v\in S}\phi(v)$.
Some simple and important properties of linf are provided in the following lemma.

Lemma 2.4. *Let $\phi, \varphi\colon X \to \mathbb{R}\cup\{+\infty\}$ be two functions, $S\subset X$ a nonempty set.*

(a) *The following inequalities hold:*

$$\operatorname*{linf}_{v\in S}\phi(v) + \operatorname*{linf}_{v\in S}\varphi(v) \le \operatorname*{linf}_{v\in S}(\phi(v)+\varphi(v)) \le \operatorname*{linf}_{v\in S}\phi(v) + \operatorname*{lsup}_{v\in S}\varphi(v).$$

(b) *For any constant $r\in\mathbb{R}$ and $t>0$:* $\operatorname*{linf}\limits_{v\in S}(t\phi(v)+r) = t\operatorname*{linf}\limits_{v\in S}\phi(v)+r.$

(c) *If the set S is compact and ϕ is lower semicontinuous, then*

$$\operatorname*{linf}_{v\in S}\phi(v) = \inf_{v\in S}\phi(v).$$

Proof. Assertion (a) is a direct consequence of the second equality in (12) and (b) follows directly from (a). Finally, (c) is proved in [11] (but it was previously stated in [1]). □

Our main variational tool in the next section is Theorem 2.5 below. Six years after Zagrodny's fundamental mean value theorem [34], Clarke and Ledyaev in [7] provided a deep multidirectional mean value inequality.

For a lower semicontinuous function $f\colon X \longrightarrow \mathbb{R}\cup\{+\infty\}$, two fixed points $a, b\in X$ with $a\in \operatorname{dom} f$, and a real number $\rho\le f(b)$, Zagrodny's theorem concerns the estimation of $\rho - f(a)$ in terms of subgradients of f at points near the segment $[a,b]$. Considering a closed convex set C in place of the point b, Clarke and Ledyaev proved for $\rho \le \operatorname{linf}_{z\in C} f(z)$, in the context of Hilbert spaces, a similar estimation in terms of proximal subgradients at points close to the convex hull of $\{a\}\cup C$.

The extension of this result, to various classes of subdifferentials in general Banach spaces, is due to Aussel, Corvellec and Lassonde [1]. Independently, in the context of Banach spaces with $\beta-$ differentiable renorm, the multidirectional mean value inequality has been also extended by Zhu in [35].

Given a point $a\in X$ and a convex set C of X, it will be convenient to denote the *convex hull* of $\{a\}\cup C$ by

$$[a, C] := \operatorname{co}(\{a\}\cup C).$$

This set is usually called the *drop* formed by the point a and the convex set C.

Theorem 2.5. *Let $C \subset X$ be a nonempty closed convex set, $f\colon X \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous function, and $a \in \operatorname{dom} f$. Assume that for the drop*

$$D = [a, C] := \{\lambda a + (1-\lambda)c \,:\, \lambda \in [0,1],\ c \in C\},$$

there exists some $s > 0$ such that the function f is bounded from below over the enlargement $D + sB_X$ of D. Then, for every real number $\rho \leq \operatorname{linf}_{z \in C} f(z)$ and for any subdifferential ∂ (see Definition 2.2), there exist sequences $(x_n)_{n \in \mathbb{N}}$ in $\operatorname{Dom} \partial f := \{x \in X : \partial f(x) \neq \emptyset\}$ *and $x_n^* \in \partial f(x_n)$ such that $d_D(x_n) \to 0$ and*

$$\langle x_n^*, c - a\rangle \geq \rho - f(a) - (1/n)\|c - a\| - 1/n \quad \forall\, c \in C,\ \forall\, n \in \mathbb{N}. \tag{13}$$

Proof. The proof follows the arguments of [1]. □

We introduce now the main extended Lipschitzian concept, for extended real-valued functions and sets, with which we will be concerned in this paper.

Definition 2.6. (see Definition 3.1 in [11]) Let K be a nonempty set of X. We say that a lower semicontinuous function $f\colon X \longrightarrow \mathbb{R} \cup \{+\infty\}$ is K *directionally Lipschitzian* at $\bar{x} \in \operatorname{dom} f$ if

$$\beta_K(f; \bar{x}) := \limsup_{\substack{x \to_f \bar{x} \\ t \to 0^+,\ b \to 0}} \ \operatorname{linf}_{v \in K} t^{-1}[f(x + tv + tb) - f(x)] < +\infty, \tag{14}$$

or equivalently, if there exist $\beta \in \mathbb{R}$, $\gamma > 0$, and $r > 0$ such that

$$\operatorname{linf}_{v \in K} t^{-1}[f(x + tv + tb) - f(x)] < \beta \quad \forall\, t \in]0, r],\ \forall\, b \in \gamma B_X,\ \forall\, x \in B_f(\bar{x}, r), \tag{15}$$

where $B_f(\bar{x}, r) := \{x \in B(\bar{x}, r) \,:\, |f(x) - f(\bar{x})| < r\}$.

Observe that the value $\beta_K(f; \bar{x})$ is equal to the infimum of $\beta \in \mathbb{R}$ for which there exists $\gamma > 0$ and $r > 0$ such that (15) is fulfilled. This quantity will be called the *K directionally Lipschitzian index of f at $\bar{x}$*. When $\beta_K(f; \bar{x})$ is less than $+\infty$, for every $\beta > \beta_K(f; \bar{x})$, we define the set

$$\Gamma_K(f; \bar{x}; \beta) := \left\{ \gamma > 0 \;\middle|\; \begin{array}{l} \exists\, r > 0 \quad \text{such that} \\ \displaystyle\sup_{\substack{x \in B_f(\bar{x}, r) \\ t \in]0, r[,\, b \in \gamma B_X}} \ \operatorname{linf}_{v \in K} t^{-1}[f(x + tv + tb) - f(x)] < \beta \end{array} \right\}. \tag{16}$$

Notice that the set $\Gamma_K(f; \bar{x}; \beta)$ is nonempty whenever $\beta > \beta_K(f; \bar{x})$. The supremum of this set, will be denoted by $\gamma_K(f; \bar{x}; \beta)$, that is,

$$\gamma_K(f; \bar{x}; \beta) := \sup \Gamma_K(f; \bar{x}; \beta) \quad \text{for} \ \ \beta > \beta_K(f; \bar{x}), \tag{17}$$

and, for $\beta = \beta_K(f; \bar{x})$ we put

$$\gamma_K(f; \bar{x}; \beta) = \inf_{\beta' > \beta_K(f; \bar{x})} \gamma_K(f; \bar{x}; \beta').$$

Clearly, we note that the expression whose linf is taken in (15) is well-defined because $f(x)$ is finite for $x \in B_f(\bar{x}, r)$.

It is also worth pointing out that f is K directionally Lipschitzian at $\bar{x}$ if and only if it is ρK directionally Lipschitzian for any real $\rho > 0$.

Definition 2.6 could be considered for any extended real-valued function, not necessary lower semicontinous, if we replace (14) by

$$\limsup_{\substack{(x,\alpha)\to_{\mathrm{epi}\, f}(\bar{x},f(\bar{x}))\\ t\to 0^+,\, b\to 0}} \ \operatorname*{linf}_{v\in K} t^{-1}[f(x+tv+tb)-\alpha] < +\infty.$$

This modification would allow us to recover Rockafellar's definition in [26] of directionally Lipschitzian functions which are not necessarily lower semicontinuous. For the sake of simplicity, we restrict our work to lower semicontinuous functions.

Example 2.7. As we pointed out in [11], one can verify that the notion of K directionally Lipschitzian behavior recovers some well-known concepts in variational analysis.

- A lower semicontinuous function is locally Lipschitzian at a point $\bar{x}$ of its effective domain if and only if it is K directionally Lipschitzian at $\bar{x}$, with $K = \{0\}$.
- A lower semicontinuous function is directionally Lipschitzian at a point $\bar{x}$ of its effective domain (see [26]) if and only if it is K directionally Lipschitzian at $\bar{x}$, with $K = \{\bar{v}\}$, for some $\bar{v} \in X$.
- A lower semicontinuous function is compactly epi-Lipschitzian at a point of its effective domain (see [3, 24]) if and only if it is K directionally Lipschitzian at that point for some compact set $K \subset X$.

Recall (see [3, 24, 26]) that a lower semicontinuous function $f\colon X \to \mathbb{R}\cup\{+\infty\}$ is directionally Lipschitzian (resp. compactly epi-Lipschitzian) at $\bar{x} \in \mathrm{dom}\, f$ if and only if there exists a vector $\bar{v}$ (resp. a compact set K) of X, a real number β, and positive numbers r and γ such that

$$t^{-1}[f(x+t\bar{v}+tb)-f(x)] < \beta \ (\text{resp. } \inf_{v\in K} t^{-1}[f(x+tv+tb)-f(x)] < \beta) \tag{18}$$

for all $x \in B_f(\bar{x}, r)$, $b \in \gamma B_X$, and $t \in]0, r[$.

Finally, observe that every lower semicontinuous function is $\overline{B}_X$ (the closed unit ball) directionally Lipschitzian at any point of its effective domain, and hence every lower semicontinuous function on X is compactly epi-Lipschitzian at any point of its effective domain whenever X is a finite-dimensional space. Therefore, the compactly epi-Lipschitzian property of any lower semicontinuous function f on X characterizes the fact that X is finite dimensional since, taking as f the indicator function of the singleton $\{0\}$, the second inequality of (18) entails that for the positive numbers r, γ and the compact set K in (18) one has $\gamma B_X \subset K$, and hence the compact set K is a neighborhood of the origin. □

The idea for introducing the values $\beta_K(f;\bar{x})$ and $\gamma_K(f;\bar{x},\beta)$ for $\beta > \beta_K(f;\bar{x})$, is that, if the function f is Lipschitzian around $\bar{x}$ (i.e., $K = \{0\}$), then the mentioned quantities are related to the exact Lipschitzian constant of f at $\bar{x}$ defined by

$$\operatorname{lip}(f;\bar{x}) := \limsup_{\substack{x,\, y \to \bar{x} \\ x \neq y}} \frac{|f(x) - f(y)|}{\|x - y\|}.$$

The precise relationship is established in the following proposition.

Proposition 2.8. *Let $f\colon X \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a Lipschitzian function around $\bar{x} \in \operatorname{dom} f$. Then, with $K = \{0\}$, one has the equality*

$$\operatorname{lip}(f;\bar{x}) = \limsup_{\beta \searrow \beta_K(f;\bar{x})} \frac{\beta}{\gamma_K(f;\bar{x};\beta)}.$$

Proof. Let us define $K = \{0\}$. First, observe that $\beta_K(f;\bar{x}) = 0$. Indeed, if $L \geq 0$ is a Lipschitz constant in a neighborhood of $\bar{x}$, one has

$$\beta_K(f;\bar{x}) = \limsup_{\substack{x \to_f \bar{x} \\ t \to 0^+,\, b \to 0}} t^{-1}[f(x + tb) - f(x)] \leq \limsup_{b \to 0} L\|b\| = 0.$$

On the other hand, the converse inequality $\beta_K(f;\bar{x}) \geq 0$ follows by taking $b = 0$ in the first upper limit. Now, take L and L' satisfying

$$L > L' > \limsup_{\substack{x,\ y \to \bar{x} \\ x \neq y}} \frac{|f(x) - f(y)|}{\|x - y\|} =: \operatorname{lip}(f;\bar{x}).$$

Then, there exists $\varepsilon > 0$ such that

$$|f(x) - f(y)| \leq L'\|x - y\| \qquad \forall\ x,\ y \in B(\bar{x}, \varepsilon).$$

Thus, for each $\alpha > 0$, there exists $r > 0$ (e.g., $r = (\varepsilon/3)\min\{1, 1/\alpha\}$) such that

$$\sup_{\substack{x \in B_f(\bar{x}, r) \\ t \in\,]0,r[,\, b \in \alpha B_X}} t^{-1}[f(x + tb) - f(x)] \leq L'\alpha < L\alpha.$$

From (16) and (17), we deduce that $\alpha \leq \gamma_K(f;\bar{x};L\alpha)$ for all $\alpha > 0$, and hence, for all $\beta > 0$ we obtain (taking $\alpha = \beta/L$)

$$\frac{\beta}{L} \leq \gamma_K(f;\bar{x};\beta),$$

which implies $\displaystyle\limsup_{\beta \searrow \beta_K(f;\bar{x})} \frac{\beta}{\gamma_K(f;\bar{x},\beta)} \leq L \quad \forall\ L > \operatorname{lip}(f;\bar{x}),$

and thus $\displaystyle\limsup_{\beta \searrow \beta_K(f;\bar{x})} \frac{\beta}{\gamma_K(f;\bar{x},\beta)} \leq \operatorname{lip}(f;\bar{x}).$

To prove the converse inequality, choose sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ in X converging to $\bar{x}$ with $x_n \neq y_n$ for all $n \in \mathbb{N}$ and such that

$$\operatorname{lip}(f;\bar{x}) = \lim_{n \to \infty} t_n^{-1}[f(x_n + t_n b_n) - f(x_n)],$$

where $t_n := \|x_n - y_n\|$ and $b_n := (y_n - x_n)/\|x_n - y_n\|$. Take any $\beta > \beta_K(f;\bar{x})$ and any $\gamma \in \Gamma_K(f;\bar{x};\beta)$. The definition of $\beta_K(f;\bar{x})$ and $\Gamma_K(f;\bar{x};\beta)$ along with the equality $K = \{0\}$ furnish (see (16)) some real $r > 0$ such that

$$t^{-1}[f(x+tb) - f(x)] < \beta$$

for all $t \in]0,r[$, all $b \in \gamma\overline{B}_X$ and all $x \in B(\bar{x},r)$. Putting $s_n := t_n/\gamma$ for every $n \in \mathbb{N}$, it ensues that

$$\begin{aligned} \operatorname{lip}(f;\bar{x}) &= \lim_{n\to\infty} t_n^{-1}[f(x_n + t_n b_n) - f(x_n)] \\ &= \gamma^{-1} \lim_{n\to\infty} s_n^{-1}[f(x_n + s_n(\gamma b_n)) - f(x_n)] \le \beta/\gamma. \end{aligned}$$

This being true for every $\gamma \in \Gamma_K(f;\bar{x};\beta)$, we see that $\operatorname{lip}(f;\bar{x}) \le \beta/\gamma_K(f;\bar{x};\beta)$. Taking the limit as $\beta \searrow \beta_K(f;\bar{x})$ yields the desired inequality

$$\operatorname{lip}(f;\bar{x}) \le \limsup_{\beta\searrow\beta_K(f;\bar{x})} \frac{\beta}{\gamma_K(f;\bar{x},\beta)}.$$

The proof is then complete. □

In a natural way, through indicator functions, we can introduce the K directionally Lipschitzian property for sets.

Definition 2.9. (See Definition 4.1 in [11]) We say that a closed set $S \subset X$ is K *directionally Lipschitzian at* $\bar{x} \in S$ if its indicator function Ψ_S is K directionally Lipschitzian at $\bar{x}$. This means that there exist $r > 0$ and $\gamma > 0$ such that for all $t \in\ ]0,r[$ one has

$$S \cap B(\bar{x},r) + t\gamma B_X \subset \bigcap_{\delta>0}(S - t(K + \delta B_X)) = \operatorname{cl}(S - tK). \tag{19}$$

For the function Ψ_S, we can observe that the value $\gamma_K(\Psi_S;\bar{x};\beta)$ is constant for all $\beta \ge \beta_K(\Psi_S;\bar{x})$. Therefore, we will use the notation $\gamma_K(S;\bar{x})$ instead of $\gamma_K(\Psi_S;\bar{x};\beta)$ with $\beta = \beta_K(\Psi_S;\bar{x})$ in the context of sets. Moreover, one can directly check that $\gamma_K(S;\bar{x})$ corresponds in fact to

$$\gamma_K(S;\bar{x}) = \sup\left\{\gamma > 0 \;\middle|\; \begin{array}{l} \exists\, r>0 \quad \text{such that} \\ S\cap B(\bar{x},r) + t\gamma B_X \subset \operatorname{cl}(S - tK)\ \ \forall t \in]0,r[\end{array}\right\}. \tag{20}$$

Remark 2.10. It is worth pointing out the fact that when K is compact, the set $\operatorname{cl}(S - tK)$ in (19) is equal to $S - tK$. This allows us to see that the K directionally Lipschitzian notion recovers two well-known concepts in variational analysis. Recall that a closed set S is epi-Lipschitzian in the sense of [26] (resp. compactly epi-Lipschitzian in the sense of [3]) at $\bar{x} \in S$ when there exist a vector $\bar{v} \in X$ (resp. a compact set K of X) and real numbers $r,\ \gamma > 0$ such that

$$S \cap B(\bar{x},r) +]0,r[B(\bar{v},\gamma) \subset S$$

$$(\text{resp. } S \cap B(\bar{x},r) + t\gamma B_X \subset S - tK \quad \text{for all } t \in\ ]0,r[).$$

Therefore, we have the following equivalences:

- A closed set is epi-Lipschitzian at one of its points if and only if it is $\{v\}$ directionally Lipschitzian at that point for some $v \in X$;
- A closed set is compactly epi-Lipschitzian at one of its points if and only if it is K directionally Lipschitzian at that point for some compact set $K \subset X$.

Finally, we observe that every closed set is a $\overline{B}_X$ directionally Lipschitzian set, where we recall that $\overline{B}_X$ is the closed unit ball in X. Therefore, if X is a finite-dimensional space, every closed set of X is compactly epi-Lipschitzian. In fact, arguing like in Example 2.7 for lower semicontinuous functions, we see that the latter property characterizes the finite-dimensional vector spaces in the sense that a Banach space is finite-dimensional if and only if everyone of its closed sets is compactly epi-Lipschitzian. Indeed, X is finite dimensional whenever $\{0\}$, and a fortiori every singleton, is a compactly epi-Lipschitzian set. $\square$

3. Various subdifferential characterizations

This section is devoted to characterizations of K directionally Lipschitzian functions in terms of subdifferentials. For a bounded set K of X, in what follows we will denote $|K| := \sup_{u \in K} \|u\|$.

Proposition 3.1. *Let $f\colon X \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous function which is continuous with respect to its domain with the induced topology. Let $\bar{x} \in \operatorname{dom} f$ and ∂ be a subdifferential (see Definition 2.2). If for a closed bounded set $K \subset X$ there exist $r > 0$, $\gamma > 0$ and $\beta \in \mathbb{R}$ such that*

$$\inf_{v \in K} \langle x^*, v\rangle + \gamma \|x^*\| \leq \beta \qquad \forall\, x \in B_f(\bar{x}, r), \quad \forall\, x^* \in \partial f(x), \tag{21}$$

then f is K' directionally Lipschitzian at $\bar{x}$, where $K' := \overline{\mathrm{co}}\, K$. In fact (see Definition 2.6), one has $\beta \geq \beta_{K'}(f; \bar{x})$ and $\gamma \leq \inf_{\beta' > \beta} \gamma_{K'}(f; \bar{x}; \beta')$ (more precisely $\gamma \in \Gamma_{K'}(f; \bar{x}; \beta')$ for all $\beta' > \beta$).

Proof. Let $K' = \overline{\mathrm{co}}\, K$ and $\bar{\nu} \in\ ]0, +\infty]$ be such that f is bounded from below on $B(\bar{x}, \bar{\nu})$ (from the lower semicontinuity of f). For any positive η, the relative continuity of f on its domain allows us to choose some $\xi(r) \in\ ]0, r]$ such that

$$B(\bar{x}, \xi(r)) \cap \operatorname{dom} f \subset B_f(\bar{x}, r). \tag{22}$$

Without loss of generality we assume $r < \bar{\nu}/2$. Take a positive r' such that

$$0 < r' \leq \frac{\xi(r)}{2(1 + \gamma + |K|)} \tag{23}$$

(here $|K| = \sup\limits_{u \in K} \|u\| = \sup\limits_{u \in K'} \|u\|$) and

$$B(\bar{x}, r') + \gamma r' B_X + \ [0, r'[\ K' \subset B(\bar{x}, \xi(r)/2). \tag{24}$$

Fix $x \in B_f(\bar{x}, r') \subset \operatorname{dom} f$, $t \in\ ]0, r'[$, and $b \in \gamma B_X$. Define $C := x + tb + tK'$ and $D := [x, C] := \{\lambda x + (1 - \lambda y : y \in C\}$. From (24) we see on one hand that

$$D \subset B(\bar{x}, \xi(r)/2) \subset B(\bar{x}, \bar{\nu}/2). \tag{25}$$

On the other hand, for $\delta_x := r' - \|x - \bar{x}\| > 0$ we have from (23)

$$C + \delta_x B_X \subset B(\bar{x}, \xi(r)) \subset B(\bar{x}, \bar{\nu}),$$

and hence $$\operatorname*{linf}_{z \in C} f(z) \geq \inf_{z \in C + \delta_x B_X} f(z) \geq \inf_{z \in B(\bar{x}, \bar{\nu})} f(z) > -\infty.$$

The last inequality is due to the boundedness from below of f on $B(\bar{x}, \bar{\nu})$. Since $\operatorname{linf}_{z \in C} f(z) > -\infty$, we can fix any real number $\rho < \operatorname*{linf}_{z \in C} f(z)$. Taking into account the boundedness from below of f over $D + (\bar{\nu}/2)\overline{B}_X$ (because of (25)) and that $x \in \operatorname{dom} f$, from Theorem 2.5 we know that there exist a sequence $(x_n)_{n \in \mathbb{N}}$ in $\operatorname{Dom} \partial f$, and $x_n^* \in \partial f(x_n)$ (x_n, x_n^* depending on ρ) with $d_D(x_n) \to 0$ such that

$$\langle x_n^*, tb + tv \rangle \geq \rho - f(x) - t\|v + b\|/n - 1/n$$

for all $v \in K'$ and $n \in \mathbb{N}$. This yields, for all $v \in K'$ and all integers n

$$\langle x_n^*, b + v \rangle \geq t^{-1}[\rho - f(x)] - \|v + b\|/n - 1/tn,$$

and hence $$\langle x_n^*, b + v \rangle \geq t^{-1}[\rho - f(x)] - \mu/n - 1/tn, \tag{26}$$

where $\mu := \gamma + |K'| = \gamma + |K|$ is finite due to the boundedness of K'.

Since $d_D(x_n) < \xi(r)/2$ for n large enough, we have by (22) and the first inclusion in (25),

$$x_n \in B(\bar{x}, \xi(r)) \cap \operatorname{Dom} \partial f \subset B(\bar{x}, \xi(r)) \cap \operatorname{dom} f \subset B_f(\bar{x}, r),$$

and hence taking the supremum over $v \in \gamma B_X$ in the inequality (26) using the assumption (21) we obtain

$$\beta \geq t^{-1}[\rho - f(x)] - \mu/n - 1/tn.$$

Passing to the limit as $n \to +\infty$, we see that $\beta \geq t^{-1}[\rho - f(x)]$. This being true for all real numbers $\rho < \operatorname*{linf}_{z \in C} f(z)$, we obtain

$$\beta \geq t^{-1} \left[\operatorname*{linf}_{z \in C} f(z) - f(x) \right].$$

Observe now that by Lemma 2.4 (part (b)), one has

$$t^{-1} \left[\operatorname*{linf}_{z \in C} f(z) - f(x) \right] = \operatorname*{linf}_{z \in C} t^{-1} \left[f(z) - f(x) \right] = \operatorname*{linf}_{v \in K'} t^{-1} \left[f(x + tb + tv) - f(x) \right].$$

Then, for all $x \in B_f(\bar{x}, r')$, $t \in \ ]0, r'[$, and $b \in \gamma B_X$ we get

$$\beta \geq \operatorname*{linf}_{v \in K'} t^{-1} \left[f(x + tb + tv) - f(x) \right]. \tag{27}$$

On one hand, for every $\beta' > \beta$ one has $\gamma \in \Gamma_{K'}(f; \bar{x}; \beta')$, and in consequence $\gamma \leq \inf_{\beta' > \beta} \gamma_{K'}(f; \bar{x}; \beta')$ (see (16) and (17)). On the other hand, taking the limit superior as $x \to_f \bar{x}$, $t \to 0^+$, and $b \to 0$, in (27), we obtain $\beta_{K'}(f; \bar{x}) \leq \beta$ (see (14)), and then f is K' directionally Lipschitzian at the point $\bar{x}$. $\square$

Remark 3.2. If f is the indicator function of a closed set of X, we can take $\xi(r) = r$ in (22) and hence with $r' := r/2(1 + \gamma + |K|)$, the inequality (27) holds for all $x \in B_f(\bar{x}, r')$, $t \in\ ]0, r'[$, and $b \in \gamma B_X$. □

Remark 3.3. We can observe that the above proposition still holds if we consider any operator ∂' containing a subdifferential ∂. In fact, it is easy to check that the mean value inequality (13) is also satisfied for ∂'. As example of an important subdifferential operator which is not a subdifferential in the sense of Definition 2.2, we mention the Hadamard subdifferential ∂_H. Nevertheless, it contains the Fréchet subdifferential which, in Asplund spaces, fulfills properties (P1)–(P5) of Definition 2.2. □

Our next step is to apply Proposition 3.1, to the indicator function of a closed set S, in order to obtain sufficient conditions for the K directional Lipschitzian property of the set S.

For this purpose, the formula $N(S; x) = \partial\Psi_S(x)$, for $x \in S$, will be used for all normal cones (and the respective subdifferentials) involved in Corollary 3.5 below. These formulas, recalled in Section 2, appear in the literature for all normal cones mentioned in that section. Concerning the Ioffe geometric normal cone N_G, for completeness in the following lemma we provide a proof.

Lemma 3.4. *For a nonempty set $S \subset X$ and $x \in S$, one has*

$$\partial_G\Psi_S(x) = N_G(S; x).$$

Proof. Recall, by definition, that $N_G(S; x) = \mathbb{R}_+\partial_a d_S(x)$. Considering in the product space $X \times \mathbb{R}$ the norm $\|(y, s)\|_1 := \|y\| + |s|$, one has $d_{S\times\mathbb{R}_+}(y, s) = d_S(y) + \zeta(s)$, where $\mathbb{R}_+ := [0, +\infty[$ and $\zeta(s) := \max\{0, -s\}$. From [14, Proposition 4.4], we then have that

$$\partial_a d_{S\times\mathbb{R}_+}(u, s) = \partial_a d_S(u) \times \partial_a\zeta(s). \tag{28}$$

Since $\zeta(\cdot)$ is convex, we have $\quad \partial_a\zeta(0) = [-1, 0]. \quad (29)$

From (9), the Ioffe geometric subdifferential of the indicator function, at $x \in S$, is given by

$$\partial_G\Psi_S(x) = \{x^* \in X^* \ :\ (x^*, -1) \in N_G(\operatorname{epi}\Psi_S; x, 0)\}. \tag{30}$$

Further $\quad N_G(\operatorname{epi}\Psi_S; x, 0) = N_G(S \times \mathbb{R}_+; x, 0) = \mathbb{R}_+\partial_a d_{S\times\mathbb{R}_+}(x, 0).$

Then, according to (28) and (30), we have

$$\partial_G\Psi_S(x) = \{x^* \in X^* \ :\ (x^*, -1) \in \mathbb{R}_+(\partial_a d_S(x) \times [-1, 0])\}. \tag{31}$$

This expression implies directly $\partial_G\Psi_S(x) \subset N_G(S; x)$.

In order to prove the reverse inclusion, fix a non null $x^* \in N_G(S; x)$. Then, by definition of N_G, there exist $\lambda_{x^*} > 0$ and $u^* \in \partial_a d_S(x)$ such that $x^* = \lambda_{x^*}u^*$. Now, consider the equivalent norm $|||\cdot||| := \lambda_{x^*}\|\cdot\|$ (depending on x^*) and denote by

$\tilde{d}_S$ the distance function to the set S, associated to the norm $|||\cdot|||$. In view of the obvious equality $\tilde{d}_S(\cdot) = \lambda_{x^*} d_S(\cdot)$, we have

$$\partial_a \tilde{d}_S(x) = \lambda_{x^*} \partial_a d_S(x),$$

and hence $x^* \in \partial_a \tilde{d}_S(x)$. If in the product space $X \times \mathbb{R}$ we consider the norm

$$|||(y,s)|||_1 := |||y||| + |s|,$$

and if for a set $T \subset X \times \mathbb{R}$ we denote by $\tilde{d}_T$ the distance function corresponding to $|||\cdot|||_1$, we have as above,

$$\tilde{d}_{S\times\mathbb{R}_+}(y,s) = \tilde{d}_S(y) + \zeta(s),$$

which gives

$$(x^*, -1) \in \partial_a \tilde{d}_S(x) \times [-1, 0] = \partial_a \tilde{d}_S(x) \times \partial_a \zeta(0) = \partial_a \tilde{d}_{S\times\mathbb{R}_+}(x, 0).$$

On the other hand, the cone generated by $\partial_a d_{S\times\mathbb{R}_+}(x,0)$ is the same if we consider any equivalent norm (see Remark 2.1), we have

$$\mathbb{R}_+ \partial_a d_{S\times\mathbb{R}_+}(x,0) = \mathbb{R}_+ \partial_a \tilde{d}_{S\times\mathbb{R}_+}(x,0).$$

Thus, we obtain

$$(x^*, -1) \in \mathbb{R}_+ \partial_a d_{S\times\mathbb{R}_+}(x,0) = \mathbb{R}_+ \left(\partial_a d_S(x) \times [-1,0]\right),$$

the last equality being due to (28) and (29). Therefore, from (31) we deduce that $x^* \in \partial_G \Psi_S(x)$, which ensures that $N_G(S;x) \subset \partial_G \Psi_S(x)$. □

We can now establish sufficient conditions, for the K directional Lipschitzian behavior of sets, in terms of any normal cone considered in Section 2.

Corollary 3.5. *Let S be a nonempty closed set of X and $\bar{x} \in S$. Assume that for some $r > 0$, $\gamma > 0$, and for a bounded set $K \subset X$ the inequality*

$$\inf_{v\in K} \langle x^*, v\rangle + \gamma \|x^*\| \leq 0$$

holds for all $x \in S \cap B(\bar{x}, r)$ and $x^* \in N(S;x)$. In each of the following cases, the set S is $\overline{\mathrm{co}}\, K$ directionally Lipschitzian at $\bar{x} \in S$:

- *the normal cone N is either N_C, or N_a, or N_G;*
- *X is an Asplund space and N is either N_H, or N_F, or N_L;*
- *X is a Hilbert space and N is N_P.*

Moreover, one has that $\gamma \leq \gamma_{K'}(S;\bar{x})$ where $K' = \overline{\mathrm{co}}\, K$.

Proof. The indicator function Ψ_S of the set S is obviously continuous with respect to its domain with the induced topology and the mentioned normal cones are the corresponding subdifferentials of Ψ_S (see Lemma 3.4 for N_G). So, we may apply Proposition 3.1, because in each case, the considered subdifferential satisfies properties of Definition 2.2 or it contains a subdifferential in the sense of that definition. This result gives $\gamma \le \inf_{\beta'>0} \gamma_{K'}(\Psi_S;\bar{x};\beta')$, but as it was pointed out in Definition 2.9, one has (see (20))

$$\gamma_{K'}(S;\bar{x}) = \inf_{\beta'>0} \gamma_{K'}(\Psi_S;\bar{x};\beta'),$$

which proves the desired result. □

In order to study the converse implication in Corollary 3.5, let us consider now the following result, in Proposition 3.6 below, established by Jourani and Thibault in [23]. For completeness, we provide a simple proof.

First, we know by (7) that for a subset S of X and $x \in S$ we have

$$\partial_a d_S(x) = \bigcap_{L\in\mathcal{F}} \operatorname*{Lim\,sup}_{u\to x} \partial_H (d_S)_{u+L}(u). \tag{32}$$

For the proof of Proposition 3.6 below, we need a formula along the line of (32), with $u \to_S x$, that is, $u \to x$ with $u \in S$. Such a formula was provided in [15, Proposition 2.4] as

$$\partial_a d_S(x) = \bigcap_{L\in\mathcal{F}} \operatorname*{Lim\,sup}_{\substack{S\ni u\to x\\ \varepsilon\to 0^+}} \partial_H^{\varepsilon} (d_S)_{u+L}(u), \tag{33}$$

where for a function $f\colon X \to \mathbb{R}\cup\{+\infty\}$ and $x \in \operatorname{dom} f$

$$\partial_H^{\varepsilon} f(x) := \{x^* \in X^* \ : \ \langle x^*, v\rangle \le d_H^- f(x;v) + \varepsilon\|v\| \ \forall v \in X\}$$

is the *Hadamard ε-subdifferential* of f at x.

Proposition 3.6. ([23]) *Let S be a closed nonempty subset of X which is K directionally Lipschitzian at $\bar{x} \in S$ for some compact set $K \subset X$ and let a positive $\gamma < \gamma_K(S;\bar{x})$. Then, there exists some real number $r > 0$ such that for all $x \in S \cap B(\bar{x}, r)$ and $x^* \in N_G(S;x)$ one has*

$$\inf_{v\in K} \langle x^*, v\rangle + \gamma\|x^*\| \le 0.$$

Proof. Consider a real number $r > 0$ such that for all $t \in \ ]0, r[$ the inclusion

$$S \cap B(\bar{x}, 2r) + t\gamma B_X \subset S - tK \tag{34}$$

holds. Let $x \in S \cap B(\bar{x}, r)$ and $x^* \in \partial_a d_S(x)$. Fix any real $\delta > 0$ and choose some finite subset $K_\delta \subset K$ such that $K \subset K_\delta + \delta B_X$. Take any $u \in S \cap B(x, r)$ and $b \in \gamma B_X$. Take also a sequence of positive numbers $t_n \to 0^+$ with $t_n < r$. By (34) there exists for each integer n some $v_n \in K$ (depending on u and b) with $u + t_n(b + v_n) \in S$. Choose some $v_n^\delta \in K_\delta$ such that $\|v_n - v_n^\delta\| < \delta$. Taking a subsequence if necessary, we may suppose that $(v_n^\delta)_{n\in\mathbb{N}}$ converges to some $v^\delta \in K_\delta$. Put $L_b := \operatorname{span}[\{b\} \cup K_\delta]$ (the vector space generated by $\{b\} \cup K_\delta$) and for any $\varepsilon > 0$, consider any $u^* \in \partial_H^{\varepsilon}(d_S)_{u+L_b}(u)$. On one hand, we have

$$\begin{aligned} \langle u^*, b + v^\delta\rangle &\le d_H^-(d_S)(u; b + v^\delta) + \varepsilon\|b + v^\delta\| \\ &\le d_H^-(d_S)(u; b + v^\delta) + \varepsilon\zeta, \end{aligned} \tag{35}$$

where $\zeta := \gamma + \max_{y \in K} \|y\|$. On the other hand, choosing $b_n^\delta \in B_X$ such that $v_n = v_n^\delta + \delta b_n^\delta$, we have

$$t_n^{-1}[d_S(u + t_n(b + v_n^\delta)) - d_S(u)] \leq$$
$$\leq t_n^{-1}[d_S(u + t_n(b + v_n^\delta + \delta b_n^\delta)) - d_S(u)] + t_n^{-1}\|t_n \delta b_n^\delta\| = \delta \|b_n^\delta\| \leq \delta,$$

and hence
$$d_H^-(d_S)(u; b + v^\delta) \leq \delta.$$

Using (35) we obtain
$$\langle u^*, b + v^\delta \rangle \leq \delta + \varepsilon\zeta. \tag{36}$$

By (33) we know that $x^* \in \operatorname*{Lim\,sup}_{\substack{S \ni u \to x \\ \varepsilon \to 0^+}} \partial_H^\varepsilon (d_S)_{u+L_b}(u)$. Therefore there exist nets $u_{j,\varepsilon} \to x$ with $u_{j,\varepsilon} \in S \cap B(x, r)$ and $u_{j,\varepsilon}^* \xrightarrow{w^*} x^*$ with $u_{j,\varepsilon}^* \in \partial_H^\varepsilon (d_S)_{u_{j,\varepsilon}+L_b}(u_{j,\varepsilon})$, the nets being indexed by both j and ε, where $\xrightarrow{w^*}$ denotes the weak-star convergence in X^*. Since $x \in B(\bar{x}, r)$, we may suppose that $u_{j,\varepsilon} \in S \cap B(\bar{x}, r)$. By (36) there exists some $v_{j,\varepsilon}^\delta \in K_\delta$ such that $\langle u_{j,\varepsilon}^*, b + v_{j,\varepsilon}^\delta \rangle \leq \delta + \varepsilon\zeta$. Since K_δ is a finite set, we easily obtain some $w \in K_\delta$ for which $\langle x^*, b + w \rangle \leq \delta$, and hence

$$\langle x^*, b \rangle + \inf_{v \in K} \langle x^*, v \rangle \leq \delta.$$

Taking the supremum over $b \in \gamma B_X$ and making $\delta \to 0^+$ we get for any $x^* \in \partial_a d_S(x)$

$$\gamma \|x^*\| + \inf_{v \in K} \langle x^*, v \rangle \leq 0,$$

and this inequality still obviously holds for all $x^* \in \mathbb{R}_+ \partial d_S(x) = N_G(S; x)$. The proof is then complete. □

In [11] several conditions for the compactly epi-Lipschitzian property of sets have been provided in terms of certain tangent cones. Obviously, those conditions can be translated with any normal cone which enjoys the property to be the polar of such a tangent cone. It is, for example, the case of the Clarke tangent cone. However, all the normal cones considered in Section 2, except the Clarke one, cannot be seen as polars of tangent cones and more generally, some of them are not even convex. One of the interests in such normal cones, is that very often we can provide necessary and sufficient conditions (for the compactly epi-Lipschitzian property, or more generally for the K directionally Lipschitzian behavior) in terms of these cones. Indeed, from Corollary 3.5 and Propostion 3.6, we obtain characterizations as it is established in Theorem 3.7 and Corollary 3.8 below.

As the definition of the K directionally Lipschitzian property highlights the set K, the novelty of Theorem 3.7 is to involve the same set K in the characterizations that the theorem provides.

Theorem 3.7. *For a closed subset S of X, $\bar{x} \in S$, and a nonempty compact convex set $K \subset X$, the following are equivalent:*

(a) *S is K directionally Lipschitzian at $\bar{x} \in S$;*

(b) *For some $r > 0$ and $\gamma > 0$ the inequality*

$$\inf_{v \in K} \langle x^*, v\rangle + \gamma \|x^*\| \leq 0 \tag{37}$$

holds for all $x \in S \cap B(\bar{x}, r)$ and $x^* \in N_G(S; x)$;

(c) *The same as in* (b) *but with $N_a(S; x)$ in place of $N_G(S; x)$.*

If X is an Asplund space (resp. a Hilbert space), then anyone of the above properties (a), (b), (c) *is equivalent to:*

(d) *The same as in* (b) *but using in place of $N_G(S; x)$, one of the normal cones $N_H(S; x)$, $N_F(S; x)$, or $N_L(S; x)$ (resp. the normal cone $N_P(S; x)$).*

In fact, the equivalences between (b), (c), *and* (d) *hold with the same constants r and $\gamma > 0$.*

Moreover, considering any of the above normal cones one has $\gamma \leq \gamma_K(S; \bar{x})$ for all $\gamma > 0$ such that (37) *holds. Conversely, if S is K directionally Lipschitzian at $\bar{x}$ then, for every $\gamma < \gamma_K(S; \bar{x})$ we can find $r > 0$ such that* (37) *is fulfilled.*

Proof. The first implication (a) $\Rightarrow$ (b) follows from Proposition 3.6. If one has property (b), by Corollary 3.5 the set S is K directionally Lipshitzian at $\bar{x} \in S$, that is, (b) $\Rightarrow$ (a). So, the equivalence (a) $\Leftrightarrow$ (b) holds.

Corollary 3.5 also ensures that (c) entails (a), and the converse implication (a) $\Rightarrow$ (c) follows from the equality $N_a(S; \cdot) = N_G(S; \cdot)$, established by Jourani in [22], whenever S is compactly epi-Lipschitzian (which is the case because K is compact), and from the above equivalence (a) $\Leftrightarrow$ (b).

The implication (a) $\Rightarrow$ (d) is a consequence of the above implication (a) $\Rightarrow$ (c), since anyone of the normal cones involved in (d) is contained in $N_a(S; \cdot)$. Further, the converse implication (d) $\Rightarrow$ (a) follows from Corollary 3.5, implying (a) $\Leftrightarrow$ (d).

According to Corollary 3.5, considering any of the normal cones (in (b), (c) and (d)) one has $\gamma \leq \gamma_K(S; \bar{x})$ for all $\gamma > 0$ such that (37) holds. Conversely, if S is K directionally Lipschitzian at $\bar{x}$, by Proposition 3.6 we have that for every positive $\gamma < \gamma_K(S; \bar{x})$ we can find $r > 0$ such that (37) is fulfilled with N_G and then, with all other normal cones (by the equivalence between (b) with (c) and (d)). □

The following corollary, concerning the characterization of compactly epi-Lipschitzian property, is a direct consequence of Theorem 3.7. The implication $(a) \Rightarrow (b)$ (as we already pointed out) has been first proved by Jourani and Thibault in [23] (see also Proposition 3.6). The reverse implication $(b) \Rightarrow (a)$ has been established by Ioffe in [17].

Corollary 3.8. *For a closed subset S of X and $\bar{x} \in S$, the following are equivalent.*

(a) *The set S is compactly epi-Lipschitzian at $\bar{x} \in S$;*

(b) *For some $r > 0$, $\gamma > 0$, and for some compact subset $K \subset X$ the inequality*

$$\inf_{v \in K} \langle x^*, v\rangle + \gamma \|x^*\| \leq 0 \tag{38}$$

holds for all $x \in S \cap B(\bar{x}, r)$ and $x^* \in N_G(S; x)$;

(c) *The same as in* (b) *but with* $N_a(S;x)$ *in place of* $N_G(S;x)$*.*

If X *is an Asplund space (resp. a Hilbert space), then anyone of the above properties* (a), (b), (c) *is equivalent to:*

(d) *The same as in* (b) *but using in place of* $N_G(S;x)$*, one of the normal cones* $N_H(S;x)$*,* $N_F(S;x)$*, or* $N_L(S;x)$ *(resp. the normal cone* $N_P(S;x)$*).*

Characterizations in the line of Corollary 3.8 or, more generally, in the line of Theorem 3.7, cannot be obtained with the Clarke normal cone. Indeed, Example 4.1 in [2] shows an example of a compactly epi-Lipschitzian set S that does not satisfy the inequality (38) with $N_C(S;\cdot)$.

In order to derive characterizations for the K Lipschitzian property of lower semicontinuous functions, we need to link this behavior of a lower semicontinuous function with the same property for the indicator function of its epigraph. This link, provided in the lemma below, is in the line of Lemma 3.5 in [11], but here we emphasize the estimation of constants β and γ involved in the definition of K directionally Lipschitzian property. The interest of both constants has been pointed out in Proposition 2.8.

Lemma 3.9. *Let* $f\colon X \longrightarrow \mathbb{R}\cup\{+\infty\}$ *be a lower semicontinuous function and* K *be a subset of* X*.*

(a) *If for some* $\beta\in\mathbb{R}$ *and* $\gamma>0$ *there exist* $r>0$ *and* $\eta>0$ *such that*

$$\operatorname*{linf}_{(v,\beta)\in K\times\{\beta\}} \Psi_{\mathrm{epi}\, f}((x,s)+t(b,b')+t(v,\beta))=0 \tag{39}$$

holds for all $(x,s)\in B((\bar{x},f(\bar{x})),r)\cap\ \mathrm{epi}\, f$*,* $t\in]0,r[$*,* $(b,b')\in\gamma\,(B_X\times]-\eta,\eta[)$*, then* f *is* K *directionally Lipschitzian at* $\bar{x}$*, with*

$$\operatorname*{linf}_{v\in K} t^{-1}[f(x+tb+tv)-f(x)]<\beta$$

for all $x\in B_f(\bar{x},r)$*,* $t\in]0,r[$ *and* $b\in B_X$*. Moreover,* $\beta\geq\beta_K(f;\bar{x})$ *and* $\gamma\leq\gamma_K(f;\bar{x};\beta)$*.*

(b) *If* f *is* K *directionally Lipschitzian at* $\bar{x}\in\mathrm{dom}\, f$*, then for any real numbers* $\beta>\beta_K(f;\bar{x})$ *and* $\gamma\in]0,\gamma_K(f;\bar{x};\beta)[$ *there exist* $r>0$ *and* $\eta>0$ *such that* (39) *holds.*

Proof. The proofs of (a) and (b) are very similar, so we will only present that of (a). If for some $\beta\in\mathbb{R}$ and $\gamma>0$ there exist $r>0$ and $\eta>0$ such that (39) holds, then for all $\delta\in\ ]0,\gamma\eta/2[$, $(x,s)\in B((\bar{x},f(\bar{x})),r)\cap\ \mathrm{epi}\, f$, $t\in\ ]0,r[$, and $(b,b')\in\gamma\,(B_X\times]-\eta,\eta[)$, there exists $v_\delta\in K+\delta B_X$ and $\beta_\delta<\beta+\delta<\beta+\gamma\eta/2$ such that

$$\Psi_{\mathrm{epi}\, f}((x,s)+t(b,b')+t(v_\delta,\beta_\delta))=0. \tag{40}$$

The above equality implies (taking $x\in B_f(\bar{x},r)$, $s=f(x)$, and $b'=-\gamma\eta/2$) that

$$t^{-1}\left[f(x+tb+tv_\delta)-f(x)\right]\leq\beta_\delta-\frac{\gamma\eta}{2}<\beta,$$

and hence
$$\operatorname*{linf}_{v\in K} t^{-1}\left[f(x+tb+tv)-f(x)\right]<\beta$$

for all $x \in B_f(\bar{x}, r)$, $t \in \ ]0, r[$, $\ b \in \gamma B_X$, which means that f is K directionally Lipschitzian at $\bar{x}$, together with $\beta \geq \beta_K(f; \bar{x})$ and $\gamma \leq \gamma_K(f; \bar{x}; \beta)$. □

In the same line than Lemma 3.9, the following two lemmas link the K directionally Lipschitzian property of a function f to the same property (with an appropriate set) of its epigraph epi f.

Lemma 3.10. *Let $f\colon X \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous function and K be a subset of X. If f is K directionally Lipschitzian at $\bar{x}$, then for any $\beta > \beta_K(f; \bar{x})$ and for any $0 < \bar{\gamma} < \gamma_K(f; \bar{x}; \beta)$ the set* $\operatorname{epi} f$ *is $K \times [\beta - \bar{\gamma}, \beta + \bar{\gamma}]$ directionally Lipschitzian at $(\bar{x}, f(\bar{x}))$ and $\bar{\gamma} \leq \gamma_{K \times [\beta - \bar{\gamma}, \beta + \bar{\gamma}]}(\operatorname{epi} f; \bar{x}, f(\bar{x}))$ (see definition in* (20)*).*

Proof. Take $\beta > \beta_K(f; \bar{x})$ and $0 < \bar{\gamma} < \gamma_K(f; \bar{x}; \beta)$. Then, by lower semicontinuity of f and by Definition of $\beta_K(f; \bar{x})$ (see Definition 2.9) there exists $r > 0$ such that $f(\bar{x}) - r < f(x)$ for all $x \in B(\bar{x}, r)$ and such that

$$\operatorname*{linf}_{v \in K} t^{-1}[f(x + tv + tb) - f(x)] < \beta \tag{41}$$

for all $t \in \ ]0, r[$, $b \in \bar{\gamma} B_X$, and $x \in B_f(\bar{x}, r)$. Take $t \in \ ]0, r[$ and consider any $(\tilde{x}, \tilde{s}) \in (\operatorname{epi} f) \cap B_{X \times \mathbb{R}}((\bar{x}, f(\bar{x})), r) + t\bar{\gamma}(B_X \times [-1, 1])$. We observe that $(\tilde{x}, \tilde{s})$ can be written as $\tilde{x} = x + tb$ and $\tilde{s} = s + t\bar{\gamma}\alpha$ where $b \in \bar{\gamma} B_X$, $\alpha \in [-1, 1]$ and $f(x) \leq s$ along with $(x, s) \in B_{X \times \mathbb{R}}((\bar{x}, f(\bar{x})), r)$, implying $x \in B_f(\bar{x}, r)$. Therefore, from (41) one obtains that for every $\delta > 0$ there exists $v_\delta \in K + \delta B_X$ such that

$$f(\tilde{x} + tv_\delta) < f(x) + t\beta \leq s + t\beta.$$

The latter inequality means that $(\tilde{x}, \tilde{s})$ can be written as $\tilde{x} = x' - tv_\delta$ and $\tilde{s} = s' - t(\beta - \bar{\gamma}\alpha)$, with $(x', s') \in \operatorname{epi} f$. Therefore, keeping in mind that $\alpha \in [-1, 1]$, we see that

$$(\tilde{x}, \tilde{s}) \in \operatorname{epi} f - t\left((K + \delta B_X) \times [\beta - \bar{\gamma} - \delta, \beta + \bar{\gamma} + \delta]\right) \qquad \forall\ \delta > 0,$$

which implies the inclusion

$$\begin{aligned} &(\operatorname{epi} f) \cap B_{X \times \mathbb{R}}((\bar{x}, f(\bar{x})), r) + t\bar{\gamma}(B_X \times [-1, 1]) \\ \subset\ &\bigcap_{\delta > 0} \left(\operatorname{epi} f - t\left((K + \delta B_X) \times [\beta - \bar{\gamma} - \delta, \beta + \bar{\gamma} + \delta]\right)\right). \end{aligned}$$

This allows us to conclude, according to Definition 2.9, that epi f is $K \times [\beta - \bar{\gamma}, \beta + \bar{\gamma}]$ directionally Lipschitzian at $(\bar{x}, f(\bar{x}))$ and $\bar{\gamma} \leq \gamma_{K \times [\beta - \bar{\gamma}, \beta + \bar{\gamma}]}(\operatorname{epi} f; \bar{x}, f(\bar{x}))$. □

The next lemma shows a certain reverse conclusion of Lemma 3.10.

Lemma 3.11. *Let $f\colon X \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous function and K be a subset of X. If for some $\beta \in \mathbb{R}$ and $\bar{\gamma} > 0$ the set* $\operatorname{epi} f$ *is $K \times [\beta - \bar{\gamma}, \beta + \bar{\gamma}]$ directionally Lipschitzian at $(\bar{x}, f(\bar{x}))$ with $\bar{\gamma} < \gamma_{K \times [\beta - \bar{\gamma}, \beta + \bar{\gamma}]}(\operatorname{epi} f; \bar{x}, f(\bar{x}))$, then f is K directionally Lipschitzian at $\bar{x}$. Moreover, $\beta \geq \beta_K(f; \bar{x})$ and $\bar{\gamma} \leq \gamma_K(f; \bar{x}; \beta)$.*

Proof. As a consequence of the assumptions, there exists $r > 0$ such that for every $t \in \ ]0, r[$ we have the inclusion

$$
\begin{aligned}
&(\operatorname{epi} f) \cap B_{X\times\mathbb{R}}((\bar{x}, f(\bar{x})), r) + t\bar{\gamma}(B_X \times [-1,1]) \\
\subset &\bigcap_{\delta>0} (\operatorname{epi} f - t\left((K+\delta B_X) \times [\beta - \bar{\gamma} - \delta, \beta + \bar{\gamma} + \delta]\right)). \qquad (42)
\end{aligned}
$$

Take $x \in B_f(\bar{x}, r)$, $b \in \bar{\gamma} B_X$, $t \in]0, r[$. Since $(x, f(x)) \in (\operatorname{epi} f) \cap B_{X\times\mathbb{R}}((\bar{x}, f(\bar{x})), r)$, one has that $(x + tb, f(x) - t\bar{\gamma})$ belongs to the left-hand side of (42). Therefore, for any $\delta > 0$, there exists $(\tilde{x}, \tilde{s}) \in \operatorname{epi} f$ and $v_\delta \in K + \delta B_X$ such that $x + tb = \tilde{x} - tv_\delta$ and $f(x) - t\bar{\gamma} = \tilde{s} - t\alpha$ for some $\alpha \in [\beta - \bar{\gamma} - \delta, \beta + \bar{\gamma} + \delta]$.

Hence, $\quad f(x + tb + tv_\delta) \le \tilde{s} = f(x) - t\bar{\gamma} + t\alpha \le f(x) + t(\beta + \delta),$

which implies $\quad \liminf_{v\in K} t^{-1}[f(x + tv + tb) - f(x)] \le \beta$

for all $t \in\ ]0, r[$, $b \in \bar{\gamma} B_X$, and $x \in B_f(\bar{x}, r)$, proving thus that f is K directionally Lipschitzian at $\bar{x}$, and directly we deduce $\beta \ge \beta_K(f; \bar{x})$ and $\bar{\gamma} \le \gamma_K(f; \bar{x}; \beta)$ from the definitions of these values. $\square$

Given a subdifferential ∂ and a function f we define the *horizon ∂-subdifferential* $\partial^\infty f(x)$ of f at $x \in \operatorname{dom} f$ by

$$\partial^\infty f(x) := \{x^* \in X^* : (x^*, 0) \in \partial \Psi_{\operatorname{epi} f}(x, f(x))\}. \qquad (43)$$

For the general case of lower semicontinuous functions, we will need additional properties for the subdifferential ∂. Indeed, for any lower semicontinuous function f and $x \in \operatorname{dom} f$ we will assume that the subdifferential ∂ satisfies the following properties (used in Theorem 3.13 below):

(P6) the subset $\partial \Psi_{\operatorname{epi} f}(x, f(x)) \subset X^* \times \mathbb{R}$ is a cone;

(P7) for any $(x^*, -r^*) \in \partial \Psi_{\operatorname{epi} f}(x, r)$ one has $r^* \ge 0$;

(P8) $\partial \Psi_{\operatorname{epi} f}(x, r) \subset \partial \Psi_{\operatorname{epi} f}(x, f(x))$ for any $r \in \mathbb{R}$;

(P9) the equality $\partial f(x) = \{x^* \in X^* : (x^*, -1) \in \partial \Psi_{\operatorname{epi} f}(x, f(x)\}$ holds;

(P10) for any continuous linear functional $x^* \in \partial^\infty f(x)$ (the horizon ∂-subdifferential of f at x) there exist sequences $(\alpha_n)_{n\in\mathbb{N}}$ in $]0, +\infty[$ tending to 0, $(x_n)_{n\in\mathbb{N}}$ in X with $(x_n, f(x_n)) \to (x, f(x))$, and $(x_n^*)_{n\in\mathbb{N}}$ in X^* with $x_n^* \in \partial f(x_n)$ and such that $\alpha_n x_n^* \xrightarrow{w^*} x^*$.

The fact that $\operatorname{Dom} \partial f$ is graphically dense in $\operatorname{dom} f$, ensures that the null linear functional on X is always the weak-star limit of such a sequence $(\alpha_n x_n^*)_{n\in\mathbb{N}}$.

Some of these fundamental additional properties have already been used in [10] for other purposes.

All the properties (P1)–(P10) hold in appropriate spaces, as stated in Section 2, for the proximal, Fréchet, and viscosity subdifferentials.

Proposition 3.12. *Let $f : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous function, $\bar{x} \in \operatorname{dom} f$, and ∂ be a subdifferential satisfying in addition* (P6)–(P9)*. If for a compact set $K \subset X$ there exist $r > 0$, $\gamma > 0$, $\omega \ge 0$ and $\beta \in \mathbb{R}$ such that for every $x \in B_f(\bar{x}, r)$*

$$\inf_{v\in K} \langle x^*, v\rangle + \gamma \|x^*\| \le \begin{cases} \beta & \forall\ x^* \in \partial f(x) \\ \omega & \forall\ x^* \in \partial^\infty f(x), \end{cases} \qquad (44)$$

then f is K' directionally Lipschitzian at $\bar{x}$, where $K' := \overline{\mathrm{co}}\, K$. Moreover we have $\beta \geq \beta_{K'}(f;\bar{x})$ and $\gamma \leq \inf_{\beta'>\beta} \gamma_{K'}(f;\bar{x};\beta')$. More precisely, $\gamma \in \Gamma_{K'}(f;\bar{x};\beta')$ for all $\beta' > \beta_{K'}(f;\bar{x})$.

Proof. Put $K' = \overline{\mathrm{co}}\, K$ and fix $\varepsilon > 0$. For $\beta' := \beta + \varepsilon$ let us show that (44) implies, for some $r' > 0$ and $\eta > 0$, the inequality

$$\inf_{(v,\beta')\in K'\times\{\beta'\}} \{\langle y^*, v\rangle + s^*\beta'\} + \gamma(\|y^*\| + \eta|s^*|) \leq 0 \tag{45}$$

for all $(x,s) \in \operatorname{epi} f \cap B((\bar{x}, f(\bar{x})), r')$ and for all $(y^*, s^*) \in \partial\Psi_{\operatorname{epi} f}(x,s)$. Observe that (45) is an expression in the line of the first inequality in (44) written with $\Psi_{\operatorname{epi} f}$ in place of the function f.

Now take any $x \in B_f(\bar{x}, r)$ and any $x^* \in \partial^\infty f(x)$. For any integer $n \in \mathbb{N}$ we see from (P6) and the definition of $\partial^\infty f$ that $nx^* \in \partial^\infty f(x)$, so by (44)

$$n \inf_{v\in K'} \langle x^*, v\rangle + \gamma n\|x^*\| \leq \omega.$$

This being true for any $n \in \mathbb{N}$, it ensues that

$$\inf_{v\in K'} \langle x^*, v\rangle + \gamma\|x^*\| \leq 0 \qquad \forall\, x \in B_f(\bar{x}, r), \;\; \forall\, x^* \in \partial^\infty f(x). \tag{46}$$

On one hand, from (44), we have that for each $x \in B_f(\bar{x}, r) \cap \operatorname{Dom} \partial f$ and each $x^* \in \partial f(x)$ there exists $v \in K'$ such that $\langle x^*, v\rangle + \gamma\|x^*\| \leq \beta$, and hence, for $\eta = \varepsilon/\gamma$

$$\langle x^*, v\rangle - \beta' + \gamma(\|x^*\| + \eta) \leq 0. \tag{47}$$

On the other hand, from properties (P6)–(P7)–(P9) we have

$$(y^*, s^*) \in \partial\Psi_{\operatorname{epi} f}(x, f(x)) \Rightarrow \begin{cases} -\frac{y^*}{s^*} \in \partial f(x) & \text{if } s^* < 0 \\ \\ y^* \in \partial^\infty f(x) & \text{if } s^* = 0. \end{cases} \tag{48}$$

By the lower semicontinuity of f, choose for each $\theta > 0$ some $\sigma(\theta) \in]0, \theta]$ such that

$$f(x) > f(\bar{x}) - \theta \qquad \forall\, x \in B(\bar{x}, \sigma(\theta)).$$

Then, for any $(x,s) \in \operatorname{epi} f \cap B((\bar{x}, f(\bar{x})), \sigma(r))$ and

$$(y^*, s^*) \in \partial\Psi_{\operatorname{epi} f}(x,s) \subset \partial\Psi_{\operatorname{epi} f}(x, f(x))$$

(the latter inclusion being due to property (P8) of ∂) we claim that we can find $(v, \beta') \in K' \times \{\beta'\}$ such that

$$\langle y^*, v\rangle + s^*\beta' + \gamma(\|y^*\| + \eta|s^*|) \leq 0. \tag{49}$$

Indeed, if $s^* < 0$ then $-y^*/s^* \in \partial f(x)$ and therefore (because $x \in B_f(\bar{x}, r)$) we can use (47) in order to obtain (49). Finally, if $s^* = 0$ we conclude (49) from (46). This means that (45) is satisfied with $r' = \sigma(r)$.

In the product space $X \times \mathbb{R}$ consider the norm $|||(x,s)||| := \max\left\{\|x\|, \frac{1}{\eta}|s|\right\}$, and so the dual norm in $X^* \times \mathbb{R}$ is $|||(x^*, s^*)|||_* = \|x^*\| + \eta|s^*|$. Since $\Psi_{\operatorname{epi} f}$ is continuous with respect to its domain, from (45), Proposition 3.1, and Remark 3.2, we deduce for $r'' := \sigma(r)/2(1+\gamma+\max\{|K|, \beta'\})$ that

$$\inf_{(v,\beta')\in K'\times\{\beta'\}} t^{-1}[\Psi_{\operatorname{epi} f}((x,s)+t(v,\beta')+t(b,b')) - \Psi_{\operatorname{epi} f}(x,s)] \le 0$$

is valid for all $(x,s) \in B(\bar{x}, r'')\times]f(\bar{x}) - \eta r'', f(\bar{x}) + \eta r''[\cap \operatorname{epi} f$, $t \in]0, r''[$, and $(b, b') \in \gamma B_X \times]-\gamma\eta, \gamma\eta[$. Lemma 3.9 and part (c) of Lemma 2.4 tell us that f is K' directionally Lipschitzian at $\bar{x}$ with

$$\inf_{v\in K'} t^{-1}[f(x+tv+tb) - f(x)] < \beta'$$

for all $x \in B_f(\bar{x}, r'')$, $t \in]0, r''[$, and $b \in \gamma B_X$, as well as with $\beta_{K'}(f; \bar{x}) \le \beta'$ and $\gamma \le \gamma_K(f; \bar{x}; \beta')$. These inequalities being fulfilled for all $\beta' > \beta$, we conclude $\beta_{K'}(f;\bar{x}) \le \beta$ and $\gamma \le \inf\limits_{\beta'>\beta} \gamma_K(f;\bar{x};\beta')$. □

The next theorem considers the situation when the subdifferential satisfies the additional basic property (P10).

Theorem 3.13. *Let $f\colon X \longrightarrow \mathbb{R}\cup\{+\infty\}$ be a lower semicontinuous function, $\bar{x} \in \operatorname{dom} f$, and ∂ be a subdifferential satisfying in addition* (P6)–(P10)*. If for a compact set $K \subset X$ there exist $r > 0$, $\gamma > 0$, and $\beta \in \mathbb{R}$ such that*

$$\inf_{v\in K}\langle x^*, v\rangle + \gamma\|x^*\| \le \beta \qquad \forall\, x \in B_f(\bar{x}, r), \quad \forall\, x^* \in \partial f(x), \tag{50}$$

then f is K' directionally Lipschitzian at $\bar{x}$, where $K' := \overline{\operatorname{co}}\, K$. Moreover, we have $\beta \ge \beta_{K'}(f;\bar{x})$ and $\gamma \le \inf_{\beta'>\beta} \gamma_{K'}(f;\bar{x};\beta')$. More precisely, $\gamma \in \Gamma_{K'}(f;\bar{x};\beta')$ for all $\beta' > \beta_{K'}(f;\bar{x})$.

Proof. Since the subset K is compact, the function $x^* \mapsto \inf_{v\in K}\langle x^*, v\rangle + \gamma\|x^*\|$ is sequentially lower semicontinuous with respect to the weak-star topology on X^*. Then, considering the set

$$\Lambda_f(x) := \{x^* \in X^* : \exists x_n \to_f x, \exists \alpha_n \to 0^+, \exists x_n^* \in \partial f(x_n) \text{ with } \alpha_n x_n^* \xrightarrow{w^*} x^*\},$$

we see that (50) implies that for every $x \in B_f(\bar{x}, r)$ we have

$$\inf_{v\in K}\langle x^*, v\rangle + \gamma\|x^*\| \le 0$$

for every $x^* \in \Lambda_f(x)$. Given any $x \in B_f(\bar{x}, r)$, the latter inequality holds true in particular for every $x^* \in \partial^\infty f(x)$ according to the inclusion $\partial^\infty f(x) \subset \Lambda_f(x)$ due to (P10). Consequently, the theorem follows from Proposition 3.12. □

Remark 3.14. The proof of Proposition 3.12 shows that for the constants β, γ, r of assumption (44) in this proposition, or of assumption (50) in Theorem 3.13, and for $\beta' > \beta$, there exists $r'' := \sigma(r)/2(1+\gamma+\max\{|K|, \beta'\})$ such that

$$\inf_{v\in\overline{\operatorname{co}} K} t^{-1}[f(x+tv+tb) - f(x)] < \beta'$$

for all $x \in B_f(\bar{x}, r'')$, $t \in \,]0, r''[$, and $b \in B_X$. Above $\sigma(\cdot)$ is required to be any function associated with the lower semicontinuity of f in such a way that $\sigma(\theta) \in]0, \theta]$ and $f(x) > f(\bar{x}) - \theta$ for all $x \in B(\bar{x}, \sigma(\theta))$. □

Theorem 3.13 holds in Asplund Banach spaces for the Fréchet subdifferential, and in Hilbert spaces the theorem holds for the proximal subdifferential, because the mentioned subdifferentials are, in these spaces, subdifferentials which satisfy (P6)-(P7)-(P8)-(P9)-(P10). Since the Mordukhovich limiting and Hadamard subdifferentials contain the Fréchet one, Theorem 3.13 is also satisfied for the Mordukhovich limiting and Hadamard subdifferentials in Asplund spaces. Theorem 3.13 also holds for viscosity subdifferentials in appropriate spaces.

Unfortunately, we cannot apply directly Theorem 3.13 to the Ioffe approximate subdifferential because we do not know if this subdifferential satisfies property (P10). The study with this subdifferential will be treated separately afterwards.

In the following, we establish the converse version of Theorem 3.13 for some subdifferentials, starting with the Fréchet, proximal, Hadamard, and Mordukhovich limiting subdifferentials.

Proposition 3.15. *Let K be a closed bounded subset of X and $f\colon X \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous function which is K directionally Lipschitzian at $\bar{x} \in \operatorname{dom} f$. Then, for any $\beta > \beta_K(f;\bar{x})$ and $\gamma \in \,]0, \gamma_K(f;\bar{x};\beta)[$ there exists $r > 0$ such that the relation*

$$\inf_{v \in K} \langle x^*, v\rangle + \gamma\|x^*\| \leq \beta \qquad \forall\, x \in B_f(\bar{x}, r), \quad \forall\, x^* \in \partial f(x): \tag{51}$$

(a) *holds with $\partial = \partial_F$ (the Fréchet subdifferential) and $\partial = \partial_P$ (the proximal subdifferential);*

(b) *holds with $\partial = \partial_H$ (the Hadamard subdifferential) and $\partial = \partial_L$ (the Mordukhovich limiting subdifferential) whenever K is compact.*

Proof. For $\beta > \beta_K(f;\bar{x})$ and $\gamma \in \,]0, \gamma_K(f;\bar{x};\beta)[$, from Definition 2.6 (and that of $\beta_K(f;\bar{x})$ and $\gamma_K(f;\bar{x};\beta)$) take $r > 0$ such that

$$\sup_{\substack{x \in B_f(\bar{x}, r) \\ b \in \gamma B_X \\ t \in]0, r[}} \operatorname*{linf}_{v \in K} t^{-1}[f(x + tv + tb) - f(x)] < \beta. \tag{52}$$

(a) Fix $\beta' \in \mathbb{R}$ such that $\operatorname{linf}_{v \in K} t^{-1}[f(x + tv + tb) - f(x)] < \beta' < \beta$, for all $t \in]0, r]$, $b \in \gamma B_X$, and $x \in B_f(\bar{x}, r)$. Consider $\varepsilon > 0$ satisfying

$$(|K| + 1 + \gamma)\varepsilon + \beta' < \beta. \tag{53}$$

Fix $x \in B_f(\bar{x}, r) \cap \operatorname{Dom} \partial_F f$, $x^* \in \partial_F f(x)$, and $b \in \gamma B_X$. By (12) choose a positive real $\delta_0 < 1$ (depending on x, b and β') such that

$$\inf_{v \in K + \delta B_X} t^{-1}[f(x + tv + tb) - f(x)] < \beta' \qquad \forall \delta \in]0, \delta_0]. \tag{54}$$

From the definition of Fréchet subdifferential choose also a real $\eta \in]0, r[$ (depending on x, x^* and ε) such that for all $x' \in x + \eta B_X$

$$\langle x^*, x' - x\rangle \leq f(x') - f(x) + \varepsilon\|x' - x\|.$$

Take $\eta' \in]0, 1[$ satisfying $\eta'(|K| + 1 + \gamma) < \eta$. Then, fixing any $\delta \in]0, \delta_0[$, $v \in K$, $u \in \delta B_X$, and $t \in]0, \eta']$ we have

$$\langle x^*, x + tv + tu + tb - x\rangle \leq f(x + tv + tu + tb) - f(x) + \varepsilon\|tv + tu + tb\|,$$

and hence

$$\langle x^*, v + b\rangle \leq t^{-1}[f(x + tv + tu + tb) - f(x)] + \varepsilon(\|v\| + \|u\| + \|b\|) - \langle x^*, u\rangle,$$

implying (since $\delta_0 < 1$ and $b \in \gamma B_X$)

$$\langle x^*, v\rangle + \langle x^*, b\rangle \leq t^{-1}[f(x + tv + tu + tb) - f(x)] + \varepsilon(|K| + 1 + \gamma) + \delta\|x^*\|.$$

If we take the infimum over all $v \in K$ and $u \in \delta B_X$ we obtain by (54)

$$\begin{aligned} &\langle x^*, b\rangle + \inf_{v\in K}\langle x^*, v\rangle \leq \\ &\leq \inf_{v'\in K+\delta B_X} t^{-1}[f(x + tv' + tb) - f(x)] + \varepsilon(|K| + 1 + \gamma) + \delta\|x^*\| \\ &\leq \beta' + \varepsilon(|K| + 1 + \gamma) + \delta\|x^*\|, \end{aligned}$$

and hence it ensures, from (53), that

$$\langle x^*, b\rangle + \inf_{v\in K}\langle x^*, v\rangle \leq \beta + \delta\|x^*\|.$$

This being true for all $\delta \in]0, \delta_0[$, we obtain for all $b \in \gamma B_X$

$$\langle x^*, b\rangle + \inf_{v\in K}\langle x^*, v\rangle \leq \beta,$$

which gives

$$\gamma\|x^*\| + \inf_{v\in K}\langle x^*, v\rangle \leq \beta.$$

This translates the desired property with ∂_F.

The property for the proximal subdifferential ∂_P follows directly from the inclusion $\partial_P \subset \partial_F$.

(b) Fix $x \in B_f(\bar{x}, r)$, $b \in \gamma B_X$, and $(t_n)_{n\in\mathbb{N}}$ in $]0, +\infty[$ with $t_n \to 0^+$. Since K is compact, from (c) in Lemma 2.4 and inequality (52), for n large enough there exists $v_n \in K$ such that

$$t_n^{-1}[f(x + t_n v_n + t_n b) - f(x)] < \beta.$$

Taking a subsequence of $(v_n)_{n\in\mathbb{N}}$ converging to some $v \in K$ (depending on x, b and $(t_n)_{n\in\mathbb{N}}$) one obtains $d_H^- f(x; v + b) \leq \beta$, which entails that, for any $x^* \in \partial_H f(x)$ and any $b \in \gamma B_X$, there exists $v \in K$ such that

$$\langle x^*, v + b\rangle \leq \beta.$$

This proves the desired result for the Hadamard subdifferential ∂_H. The result for the Mordukhovich limiting subdifferential ∂_L follows from the previous part (a) with ∂_F, and the fact that the function $x^* \mapsto \inf_{v\in K}\langle x^*, v\rangle + \gamma\|x^*\|$ is sequentially lower semicontinuous for the weak-star topology when K is compact. □

Remark 3.16. The proof of (b), for the Hadamard subdifferential, in the above theorem can also be obtained by applying Proposition 3.3 in [11], because this proposition provides a necessary condition for the K directionally Lipschitzian behavior in terms of the Hadamard directional derivative. □

The inequalities (50) (in Theorem 3.13) and (51) (in Proposition 3.15) lead us to introduce, in addition to $\beta_K(f,\bar{x})$, another index $\beta_K^{\partial}(f;\bar{x})$ associated with a subdifferential ∂ as follows.

Definition 3.17. Let $f\colon X \longrightarrow \mathbb{R}\cup\{+\infty\}$ be a lower semicontinuous function, $\bar{x}\in \operatorname{dom} f$, and $K\subset X$. For a subdifferential ∂ we define the value $\beta_K^{\partial}(f;\bar{x})$ as the infimum of $\beta\in\mathbb{R}$ for which there exist $r>0$ and $\gamma>0$ such that

$$\inf_{v\in K}\langle x^*, v\rangle + \gamma\|x^*\| \le \beta$$

for all $x\in B_f(\bar{x}, r)$ and $x^*\in\partial f(x)$.

The next theorem combines Theorem 3.13 and Proposition 3.15 to establish characterizations of K directionally Lipschitzian functions in Asplund and Hilbert spaces, in terms of Hadamard, Fréchet, Mordukhovich limiting, and proximal subdifferentials. Adaptations to viscosity subdifferentials in appropriate spaces is left to the reader. The case of general Banach spaces will require the Ioffe approximate subdifferential and it will be considered afterwards.

Theorem 3.18. *For any lower semicontinuous function $f\colon X \longrightarrow \mathbb{R}\cup\{+\infty\}$, $\bar{x}\in\operatorname{dom} f$, and a convex compact subset $K\subset X$, the following assertions hold.*

(a) *If X is an Asplund space, then f is K directionally Lipschitzian at $\bar{x}$ if and only if there exist $r>0$, $\gamma>0$, and $\beta\in\mathbb{R}$ such that*

$$\inf_{v\in K}\langle x^*, v\rangle + \gamma\|x^*\| \le \beta \qquad \forall\, x\in B_f(\bar{x}, r),\ \ \forall\, x^*\in\partial f(x), \tag{55}$$

where ∂ is either ∂_H, or ∂_F, or ∂_L (Hadamard, or Fréchet, or Mordukhovich limiting subdifferential respectively).

(b) *If X is a Hilbert space, then f is K directionally Lipschitzian at $\bar{x}$ if and only if there exist $r>0$, $\gamma>0$, and $\beta\in\mathbb{R}$ such that* (55) *holds for $\partial=\partial_P$ (the proximal subdifferential).*

Moreover, for all the involved subdifferentials ∂, in the above appropriate spaces, one has the equality

$$\beta_K^{\partial}(f;\bar{x}) = \beta_K(f;\bar{x}). \tag{56}$$

Proof. Let us start by proving the equality (56), for the involved subdifferentials. Characterizations (a) and (b) will follow directly from this equality.

For the inequality $\beta_K^\partial(f;\bar{x}) \leq \beta_K(f;\bar{x})$, assume $\beta_K(f;\bar{x}) < +\infty$. In such a case, f is K directionally Lipschitzian at $\bar{x}$ and, from Proposition 3.15, one has that for each $\beta > \beta_K(f;\bar{x})$, there exists $r > 0$ and $\gamma > 0$ such that

$$\inf_{v \in K} \langle x^*, v\rangle + \gamma\|x^*\| \leq \beta \qquad \forall\, x \in B_f(\bar{x}, r), \ \ \forall\, x^* \in \partial f(x) \tag{57}$$

for ∂_H, ∂_F, ∂_L, and ∂_P. From Definition 3.17 of $\beta_K^\partial(f;\bar{x})$ (for these subdifferentials), we then get

$$\beta \geq \beta_K^\partial(f;\bar{x}) \qquad \forall\, \beta > \beta_K(f;\bar{x}),$$

which justifies the desired inequality.

To prove the reverse inequality $\beta(f;\bar{x}) \leq \beta_K^\partial(f;\bar{x})$, assume that $\beta_K^\partial(f;\bar{x}) < +\infty$ for ∂_H, ∂_F, and ∂_L on an Asplund space or for ∂_P if X is a Hilbert space. Take any real $\beta > \beta_K^\partial(f;\bar{x})$. This latter inequality yields that there exist reals $r > 0$ and $\gamma > 0$ such that (57) holds.

Since the Fréchet subdifferential ∂_F in Asplund spaces and the proximal subdifferential ∂_P in Hilbert spaces, are subdifferentials satisfying properties (P6)–(P7)–(P8)–(P9)–(P10), by Theorem 3.13 we get that f is directionally Lipschitzian at $\bar{x}$ and $\beta \geq \beta_K(f;\bar{x})$. Hence,

$$\beta \geq \beta_K(f;\bar{x}) \qquad \forall\, \beta > \beta_K^\partial(f;\bar{x}),$$

proving the inequality $\beta_K^\partial(f;\bar{x}) \geq \beta_K(f;\bar{x})$ for $\partial = \partial_F$ and $\partial = \partial_P$. The inequality with $\partial = \partial_H$ and $\partial = \partial_L$ follows from the fact that these subdifferentials contain the Fréchet subdifferential and from the observation that the inclusion $\partial' \subset \partial$ implies that $\beta_K^{\partial'}(f;\bar{x}) \leq \beta_K^\partial(f;\bar{x})$. □

As a first corollary, we deduce some characterizations of compactly epi-Lipschitzian behavior for functions. The corollary directly follows from the fact that we already noticed in Example 2.7 that a function is compactly epi-Lipschitzian if and only if it is K directionally Lipschitzian for some compact set $K \subset X$.

Corollary 3.19. *For any lower semicontinuous function $f\colon X \longrightarrow \mathbb{R} \cup \{+\infty\}$ and $\bar{x} \in \operatorname{dom} f$, the following assertions hold.*

(a) *If X is an Asplund space, then f is compactly epi-Lipschitzian at $\bar{x}$ if and only if there exist a compact set $K \subset X$, $r > 0$, $\gamma > 0$, and $\beta \in \mathbb{R}$ such that*

$$\inf_{v \in K} \langle x^*, v\rangle + \gamma\|x^*\| \leq \beta \qquad \forall\, x \in B_f(\bar{x}, r), \ \ \forall\, x^* \in \partial_H f(x). \tag{58}$$

(b) *If X is an Asplund space, then f is compactly epi-Lipschitzian at $\bar{x}$ if and only if there exist a compact set $K \subset X$, $r > 0$, $\gamma > 0$, and $\beta \in \mathbb{R}$ such that condition* (58) *is satisfied with $\partial = \partial_F$.*

(c) *If X is an Asplund space, then f is compactly epi-Lipschitzian at $\bar{x}$ if and only if there exist a compact set $K \subset X$, $r > 0$, $\gamma > 0$, and $\beta \in \mathbb{R}$ such that condition* (58) *is satisfied with $\partial = \partial_L$.*

(d) *If X is a Hilbert space, then f is compactly epi-Lipschitzian at $\bar{x}$ if and only if there exist a compact set $K \subset X$, $r > 0$, $\gamma > 0$, and $\beta \in \mathbb{R}$ such that condition* (58) *is satisfied with $\partial = \partial_P$.*

The next corollary shows that Theorem 3.18 also recovers some known characterizations of Lipschitzian and directionally Lipschitzian behavior (see [33, 32, 9]).

Corollary 3.20. *Let $f\colon X \longrightarrow \mathbb{R}\cup\{+\infty\}$ be a lower semicontinuous function and $\bar{x} \in \operatorname{dom} f$.*

(a) *If X is an Asplund space, then f is locally Lipschitzian at $\bar{x}$ if and only if the Hadamard subdifferential is locally bounded around $\bar{x}$.*

(b) *If X is an Asplund space, then f is locally Lipschitzian at $\bar{x}$ if and only if the Fréchet (resp. the Mordukhovich limiting) subdifferential of f is locally bounded around $\bar{x}$.*

(c) *If X is a Hilbert space, then f is locally Lipschitzian at $\bar{x}$ if and only if the proximal subdifferential of f is locally bounded around $\bar{x}$.*

(d) *If X is an Asplund space, then f is directionally Lipschitzian at $\bar{x}$ if and only if there exist $\bar{v}\in X$, $r>0$, $\gamma>0$ and $\beta\in\mathbb{R}$ such that*

$$\langle x^*, \bar{v}\rangle + \gamma\|x^*\| \le \beta \qquad \forall\, x \in B_f(\bar{x}, r),\ \ \forall\, x^* \in \partial_H f(x). \tag{59}$$

(e) *If X is an Asplund space, then f is directionally Lipschitzian at $\bar{x}$ if and only if there exists $\bar{v}\in X$, $r>0$, $\gamma>0$, and $\beta\in\mathbb{R}$ such that* (59) *holds for the Fréchet subdifferential ∂_F (resp. the Mordukhovich limiting subdifferential ∂_L).*

(f) *If X is a Hilbert space, then f is directionally Lipschitzian at $\bar{x}$ if and only if there exists $\bar{v}\in X$, $r>0$, $\gamma>0$, and $\beta\in\mathbb{R}$ such that* (59) *is satisfied for the proximal subdifferential ∂_P.*

4. Characterization with the Ioffe approximate subdifferential

The characterizations in Theorem 3.18 (and its corollaries) with the Hadamard, Fréchet, Mordukhovich limiting or proximal subdifferentials require that the Banach space X be Asplund or Hilbert. In the present section, our objective is to obtain similar characterizations on any Banach space with the Ioffe approximate subdifferential ∂_a.

Firstly, let us show that property (50) for an operator ∂ and a compact set $K\subset X$ implies the same property for a finite subset $K'\subset K$.

Lemma 4.1. *Let $f\colon X \longrightarrow \mathbb{R}\cup\{+\infty\}$ be a lower semicontinuous function, $\bar{x}\in$ $\operatorname{dom} f$, and ∂ be a subdifferential. If for a subset $K\subset X$ there exist $r>0$, $\gamma>0$, and $\beta\in\mathbb{R}$ such that*

$$\inf_{v\in K}\langle x^*, v\rangle + \gamma\|x^*\| \le \beta \qquad \forall\, x\in B_f(\bar{x}, r),\ \ \forall\, x^*\in\partial f(x), \tag{60}$$

then for any $\gamma'\in\ ]0,\gamma[$ and any set K' for which $K\subset K' + (\gamma-\gamma')B_X$, one has

$$\inf_{v\in K'}\langle x^*, v\rangle + \gamma'\|x^*\| \le \beta \qquad \forall\, x\in B_f(\bar{x}, r),\ \ \forall\, x^*\in\partial f(x). \tag{61}$$

Proof. Let γ' be in $]0,\gamma[$ and any set K' such that $K\subset K'+(\gamma-\gamma')B_X$. For $x\in B_f(\bar{x}, r)$ and $x^*\in\partial f(x)$ we have

$$\inf_{v\in K'}\langle x^*,v\rangle+\gamma'\|x^*\|=\inf_{v\in K'}\langle x^*,v\rangle+(\gamma'-\gamma)\|x^*\|+\gamma\|x^*\|=$$
$$=\inf_{v\in K'}\langle x^*,v\rangle+\inf_{w\in(\gamma-\gamma')B_X}\langle x^*,w\rangle+\gamma\|x^*\|\le\inf_{v\in K'+(\gamma-\gamma')B_X}\langle x^*,v\rangle+\gamma\|x^*\|\le$$
$$\le\inf_{v\in K}\langle x^*,v\rangle+\gamma\|x^*\|\le\beta\,,$$

the last inequality being due to (60). □

Our objective now is to show that if we have the property (61) for the Ioffe approximate subdifferential ∂_a with a finite set K', then we can obtain the same property with the Hadamard subdifferential restricted to finite-dimensional vector subspaces containing K'. Before that, we present two necessary lemmas. The first one was established by Borwein and Zhu in [4] and the second one is a slight adaptation of a result given by Thibault in [29] for convex functions, the proof of which is in Appendix. Both results can be established for a subdifferential ∂ which satisfies the properties (P4) in an exact way, and (P5) in a general form, that is,

(P4") If f is lower semicontinuous near x, g is convex continuous, and x is a local minimum point of $f+g$, one has $0\in\partial f(x)+\partial g(x)$.

(P5") For a function $f\colon X_1\times X_2\longrightarrow\mathbb{R}\cup\{+\infty\}$, $f(x_1,x_2):=f_1(x_1)+f_2(x_2)$, one has the inclusion $\partial f(x_1,x_2)\subset\partial f_1(x_1)\times\partial f_2(x_2)$.

Lemma 4.2. ([4]) *Let X be a Banach space and $f_i\colon X\to\mathbb{R}\cup\{+\infty\}$ with $i=1,2,3$ be lower semicontinuous functions which are finite at some point in a closed subset $S\subset X$. Assume that $f_3(x)=+\infty$ outside some finite-dimensional vector subspace of X and put for $(x_1,x_2,x_3)\in X^3$*

$$\Delta(x_1,x_2,x_3):=\tfrac{1}{2}(\|x_1-x_2\|^2+\|x_2-x_3\|^2+\|x_3-x_1\|^2)$$

and $$w_t(x_1,x_2,x_3):=f_1(x_1)+f_2(x_2)+f_3(x_3)+t\Delta(x_1,x_2,x_3).$$

Then $$\inf_{(x_1,x_2,x_3)\in S^3}w_t(x_1,x_2,x_3)\underset{t\to\infty}{\longrightarrow}\inf_{x\in S}(f_1(x)+f_2(x)+f_3(x)).$$

Furthermore, for any $(x_{1,t},x_{2,t},x_{3,t})\in S^3$ satisfying

$$w_t(x_{1,t},x_{2,t},x_{3,t})-\inf_{(x_1,x_2,x_3)\in S^3}w_t(x_1,x_2,x_3)\underset{t\to+\infty}{\longrightarrow}0$$

one has $$t\Delta(x_{1,t},x_{2,t},x_{3,t})\underset{t\to+\infty}{\longrightarrow}0.$$

The next lemma follows the proof (given in Appendix) of Theorem 2.9 in Borwein and Zhu [4] and Lemma 2 in Thibault [29]. The above lemma is required for the proof of this next lemma.

Lemma 4.3. *Let X be a Banach space and $f_i\colon X\to\mathbb{R}\cup\{+\infty\}$, with $i=1,2,3$, be lower semicontinuous functions which are finite at $\bar{x}\in X$ and ∂ be a subdifferential that satisfies* (P1)–(P2)–(P3)–(P4")–(P5"). *Assume that $f_3(x)=+\infty$ outside a finite-dimensional vector subspace of X and assume that $\bar{x}$ is a local minimum* *point for $f_1 + f_2 + f_3$ over X. Then, for any real number $\varepsilon > 0$, there exist $x_i^* \in \partial f_i(x_i)$ such that*

$$\|x_i - \bar{x}\| < \varepsilon, |f_i(x_i) - f_i(\bar{x})| < \varepsilon, \|x_1^* + x_2^* + x_3^*\| < \varepsilon.$$

Proof. See the proof in the Appendix. □

Since the Ioffe approximate subdifferential ∂_a satisfies (P1)–(P2)–(P3)–(P4")–(P5"), we will use Lemma 4.3 in order to prove the following proposition.

Proposition 4.4. *Let $f\colon X \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous function and $\bar{x} \in \operatorname{dom} f$. Let K be a finite subset of X, $\beta \in \mathbb{R}$, $r > 0$, and $\gamma > 0$, such that*

$$\inf_{v\in K} \langle x^*, v\rangle + \gamma\|x^*\| \leq \beta \qquad \forall\ x \in B_f(\bar{x}, r), \quad \forall\ x^* \in \partial_a f(x). \tag{62}$$

Then, for any finite-dimensional vector subspace $L \subset X$ containing K one has

$$\inf_{v\in K} \langle x^*, v\rangle + \gamma \sup_{b\in B_X\cap L} \langle x^*, b\rangle \leq \beta \qquad \forall\ x \in B_f(\bar{x}, r), \quad \forall\ x^* \in \partial_H f_{x+L}(x), \tag{63}$$

where $f_{x+L} = f + \Psi_{x+L}$ and ∂_H is the Hadamard subdifferential.

Proof. Let $L \subset X$ be a finite-dimensional vector space containing the finite subset K. Take $x \in B_f(\bar{x}, r)$ and $x^* \in \partial_H f_{x+L}(x)$. Fix any $\gamma' \in \ ]0, \gamma[$. For $\varepsilon \in]0, \gamma - \gamma'[$ it is straightforward to show (see [18]) that the function $y \mapsto f(y) - \langle x^*, y - x\rangle + \varepsilon\|y - x\| + \Psi_L(y - x)$ attains a local minimum at x. If we apply Lemma 4.3 to $f_1 = f$, $f_2 = -\langle x^*, \cdot - x\rangle + \varepsilon\| \cdot - x\|$, and $f_3 = \Psi_L(\cdot - x)$ with $0 < \varepsilon' < \min\{\varepsilon, r - \|x - \bar{x}\|, r - |f(x) - f(\bar{x})|\}$ one obtains that there exist $x_1 \in B_f(x, \varepsilon') \subset B_f(\bar{x}, r)$, $x_2 \in B(x, \varepsilon')$, $x_3 \in B(x, \varepsilon')$ and

$$x_1^* \in \partial_a f(x_1), \quad x_2^* \in \partial_a f_2(x_2) \subset -x^* + \varepsilon B_{X^*}, \quad x_3^* \in \partial_a f_3(x_3) = L^\perp,$$

such that $x_1^* + x_2^* + x_3^* \in \varepsilon' B_{X^*}$, which implies

$$x^* \in x_1^* + 2\varepsilon B_{X^*} + L^\perp. \tag{64}$$

Using property (62) we also have $\inf_{v\in K} \langle x_1^*, v\rangle + \gamma\|x_1^*\| \leq \beta$.

Combining this inequality with (64), we deduce for $\mu := \max_{v\in K} \|v\|$, that

$$\inf_{v\in K} \langle x^*, v\rangle + \gamma' \sup_{b\in B_X\cap L} \langle x^*, b\rangle \leq \beta + 2\varepsilon(\mu + \gamma') < \beta + 2(\gamma - \gamma')(\mu + \gamma').$$

Taking the limit as $\gamma' \uparrow \gamma$, we conclude

$$\inf_{v\in K} \langle x^*, v\rangle + \gamma \sup_{b\in B_X\cap L} \langle x^*, b\rangle \leq \beta. \qquad \square$$

For a closed vector subspace $E \subset X$ the following lemma expresses the link between $\partial_H f_{x+L}(x)$ and $\partial_H (f\,|_E)_{x+L}(x)$ where $f\,|_E$ is the restriction to the vector subspace E of the function f and where L is a finite-dimensional vector subspace of E. This result will allow us to obtain property (63) for the restriction of the function f to a suitable vector subspace.

Lemma 4.5. *Consider a lower semicontinuous function $f\colon X \to \mathbb{R} \cup \{+\infty\}$, a closed vector subspace E of X, the injection $j\colon E \to X$, i.e., $j(u) = u$ for all $u \in E$, and $g := f \circ j\colon E \to \mathbb{R} \cup \{+\infty\}$ the restriction of f to E (i.e., $g = f\,|_E$). Let L be any finite-dimensional vector subspace of E. Then for any $u \in E \cap \operatorname{dom} f$, $u^* \in \partial_H g_{x+L}(u)$, and $\varepsilon > 0$, there exist some $y \in u + \varepsilon B_X$ with $|f(y) - f(u)| < \varepsilon$ and some $y^* \in \partial^- f_{y+L}(y)$ such that*

$$u^* \in y^* \circ j + \varepsilon B_{E^*} + L^{\perp_{E^*}},$$

where $L^{\perp_{E^*}}$ denotes the orthogonal space of L in E^*, i.e., the space of all elements of E^* which are null on L.

Proof. See the Appendix. □

If property (63) holds for a lower semicontinuous function f, we show, in the following proposition, that it is also true for the restriction of f to any closed vector subspace $E \subset X$ containing the finite dimensional vector space L involved in (63).

Proposition 4.6. *Let $f\colon X \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous function, K be a finite subset of X, E be a closed vector subspace of X containing K, and $\bar{x} \in \operatorname{dom} f \cap E$. If for $r > 0$, $\gamma > 0$, $\beta \in \mathbb{R}$, and a finite-dimensional vector space $L \subset E$ containing K one has*

$$\inf_{v\in K}\langle x^*, v\rangle + \gamma \sup_{b\in B_X\cap L}\langle x^*, b\rangle \le \beta \quad \forall\, x \in B_f(\bar{x}, r), \quad \forall\, x^* \in \partial_H f_{x+L}(x), \tag{65}$$

then one has

$$\inf_{v\in K}\langle u^*, v\rangle + \gamma \sup_{b\in B_E\cap L}\langle u^*, b\rangle \le \beta \;\; \forall\, u \in B_{f|_E}(\bar{x}, r), \;\; \forall\, u^* \in \partial_H (f\,|_E)_{u+L}(u). \tag{66}$$

Proof. Let $u \in B_{f|_E}(\bar{x}, r)$, and $u^* \in \partial_H (f\,|_E)_{u+L}(u)$. Take any $\varepsilon > 0$ such that

$$\varepsilon < \min\{r - \|x - \bar{x}\|, r - |f(x) - f(\bar{x})|\}.$$

From Lemma 4.5 one obtains that there exist $y \in B_f(u, \varepsilon) \subset B_f(\bar{x}, r)$ and

$$y^* \in \partial_H f_{y+L}(y), \quad w^* \in \varepsilon B_{E^*}, \quad z^* \in L^{\perp_{E^*}},$$

such that

$$u^* = y^*\,|_E + w^* + z^*.$$

For $v \in K \subset L \subset E$ and $b \in \gamma B_E \cap L$ we have

$$\begin{aligned}
\langle u^*, v\rangle &= \langle y^*, v\rangle + \langle w^*, v\rangle \le \langle y^*, v\rangle + \varepsilon \max_{v'\in K}\|v'\| \\
\langle u^*, b\rangle &= \langle y^*, b\rangle + \langle w^*, b\rangle \le \langle y^*, b\rangle + \varepsilon\gamma,
\end{aligned}$$

which implies

$$\begin{aligned}
\inf_{v\in K}\langle u^*, v\rangle &\le \inf_{v\in K}\langle y^*, v\rangle + \varepsilon \max_{v\in K}\|v\| \\
\gamma \sup_{b\in B_E\cap L}\langle u^*, b\rangle &\le \gamma \sup_{b\in B_E\cap L}\langle y^*, b\rangle + \varepsilon\gamma \le \gamma \sup_{b\in B_X\cap L}\langle y^*, b\rangle + \varepsilon\gamma.
\end{aligned}$$

By (65) it follows that

$$\inf_{v\in K}\langle u^*,v\rangle+\gamma\sup_{b\in B_E\cap L}\langle u^*,b\rangle\le\beta+\varepsilon\max_{v\in K}\|v\|+\varepsilon\gamma,$$

and taking the limit as $\varepsilon\downarrow 0$ we obtain the desired inequality. □

The above propositions and lemmas have been proved in order to obtain the K directionally Lipschitzian behavior for a lower semicontinuous function f whenever it satisfies the property

$$\inf_{v\in K}\langle x^*,v\rangle+\gamma\|x^*\|\le\beta\qquad\forall\ x\in B_f(\bar{x},r),\ \ \forall\ x^*\in\partial_a f(x).\tag{67}$$

The following lemma provides us another result which will be one of the keys of the proof of the next theorem establishing the K directionally Lipschitzian property of f under condition (67).

Lemma 4.7. *Let $f\colon X\longrightarrow\mathbb{R}\cup\{+\infty\}$ be a lower semicontinuous function, $\bar{x}\in$ $\operatorname{dom} f$, and $K\subset X$ be a compact subset. If there exist $r>0$, $\gamma>0$, and $\beta\in\mathbb{R}$ such that*

$$\inf_{v\in K}\langle x^*,v\rangle+\gamma\|x^*\|\le\beta\qquad\forall\ x\in B_f(\bar{x},r),\ \ \forall\ x^*\in\partial_a f(x),\tag{68}$$

then, for any closed vector subspace E containing $K\cup\{\bar{x}\}$ we have

$$\inf_{v\in K}\langle u^*,v\rangle+\gamma\|u^*\|_{E^*}\le\beta\qquad\forall\ u\in B_{f|_E}(\bar{x},r),\ \ \forall\ u^*\in\partial_H(f\mid_E)(u).\tag{69}$$

Proof. Fix any positive number $\gamma'<\gamma$, take any finite subset K' of K such that $K\subset K'+(\gamma-\gamma')B_X$. By Lemma 4.1 we have

$$\inf_{v\in K'}\langle x^*,v\rangle+\gamma'\|x^*\|\le\beta\qquad\forall\ x\in B_f(\bar{x},r),\ \ \forall\ x^*\in\partial_a f(x),$$

and by Proposition 4.4, for any finite-dimensional vector space L of X containing K', we have

$$\inf_{v\in K'}\langle x^*,v\rangle+\gamma'\sup_{b\in B_X\cap L}\langle x^*,b\rangle\le\beta\qquad\forall\ x\in B_f(\bar{x},r),\ \ \forall\ x^*\in\partial_H f_{x+L}f(x).$$

Fix any closed vector subspace E of X containing $K\cup\{\bar{x}\}$. Proposition 4.6 ensures

$$\inf_{v\in K'}\langle u^*,v\rangle+\gamma'\sup_{b\in B_E\cap L}\langle u^*,b\rangle\le\beta\qquad\forall\ u\in B_{f|_E}(\bar{x},r),\ \ \forall\ u^*\in\partial_H(f\mid_E)_{u+L}(u).$$

Since $\partial_H(f\mid_E)(u)\subset\partial_H(f\mid_E)_{u+V}(u)$ for any closed vector subspace V of E we deduce that

$$\inf_{v\in K'}\langle u^*,v\rangle+\gamma'\sup_{b\in B_E\cap L}\langle u^*,b\rangle\le\beta\qquad\forall\ u\in B_{f|_E}(\bar{x},r),\ \ \forall\ u^*\in\partial_H(f\mid_E)(u),$$

for all finite-dimensional vector subspaces $L \subset E$ containing K'. Therefore, for all $u \in B_{f|_E}(\bar{x}, r)$ and $u^* \in \partial_H(f\;|_E)(u)$ we have

$$\inf_{v \in K} \langle u^*, v\rangle + \gamma' \sup_{b \in B_E \cap L} \langle u^*, b\rangle \leq \inf_{v \in K'} \langle u^*, v\rangle + \gamma' \sup_{b \in B_E \cap L} \langle u^*, b\rangle \leq \beta,$$

and hence, this being true for any finite-dimensional subspace L of E containing K', we obtain

$$\inf_{v \in K} \langle u^*, v\rangle + \gamma' \|u^*\|_{E^*} \leq \beta.$$

Taking the limit as $\gamma' \uparrow \gamma$ finally yields, for all $u \in B_{f|_E}(\bar{x}, r)$ and $u^* \in \partial_H(f\;|_E)(u)$,

$$\inf_{v \in K} \langle u^*, v\rangle + \gamma \|u^*\|_{E^*} \leq \beta. \qquad \square$$

The idea now is to obtain property (69) for appropriate vector subspaces E where the Hadamard subdifferential is a subdifferential and satisfies (P6)–(P7)–(P8)–(P9)–(P10). Thus, using Theorem 3.13, we will be able to conclude that $f\;|_E$ is K directionally Lipschitzian, and then to show that f is K directionally Lipschitzian. This is realized in the following theorem.

Theorem 4.8. *Let $f\colon X \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous function and $\bar{x} \in \operatorname{dom} f$. If for a compact set $K \subset X$ there exist $r > 0$, $\gamma > 0$, and $\beta \in \mathbb{R}$ such that*

$$\inf_{v \in K} \langle x^*, v\rangle + \gamma \|x^*\| \leq \beta \qquad \forall\, x \in B_f(\bar{x}, r), \;\; \forall\, x^* \in \partial_a f(x), \tag{70}$$

then f is K' directionally Lipschitzian at $\bar{x}$, where $K' := \overline{\operatorname{co}}\, K$. Moreover $\beta \geq \beta_{K'}(f; \bar{x})$ and $\gamma \leq \inf_{\beta' > \beta} \gamma_{K'}(f; \bar{x}; \beta')$.

Proof. By lower semicontinuity of f at $\bar{x}$, fix a function σ on $]0, +\infty[$ such that $\sigma(\theta) \in]0, \theta]$ and $f(x) > f(\bar{x}) - \theta$ for all $x \in B(\bar{x}, \sigma(\theta))$. Fix any positive $\gamma' < \gamma$ and choose a finite subset K' of K such that $K \subset K' + (\gamma - \gamma')B_X$. Lemma 4.1 gives

$$\inf_{v \in K'} \langle x^*, v\rangle + \gamma' \|x^*\| \leq \beta \qquad \forall\, x \in B_f(\bar{x}, r), \;\; \forall\, x^* \in \partial_a f(x).$$

Let $E \subset X$ be any finite-dimensional vector space containing $K' \cup \{\bar{x}\}$. Then, Lemma 4.7 says that

$$\inf_{v \in K'} \langle u^*, v\rangle + \gamma' \|u^*\|_{E^*} \leq \beta \qquad \forall\, u \in B_{f|_E}(\bar{x}, r), \;\; \forall\, u^* \in \partial_H\left(f\;|_E\right)(u).$$

Take $\beta' > \beta$. By Theorem 3.13 and Remark 3.14 for $r' = \sigma(r)/2(1 + \gamma + \max\{|K| + \beta'\})$ and $K'' = \operatorname{co} K'$ we have

$$\inf_{v \in K''} t^{-1}[f(u + tv + tb) - f(u)] \leq \beta'$$

for all $u \in E \cap B_f(\bar{x}, r')$, $t \in\;]0, r'[$ and $v \in \gamma' B_X \cap E$. This being true for every finite-dimensional subspace $E \subset X$ containing $K' \cup \{\bar{x}\}$, it results that

$$\inf_{v \in \overline{\operatorname{co}} K} t^{-1}[f(x + tv + tb) - f(x)] \leq \inf_{v \in K''} t^{-1}[f(x + tv + tb) - f(x)] \leq \beta'$$

for all $x \in B_f(\bar{x}, r')$, $t \in\;]0, r'[$ and $v \in \gamma' B_X$. This tells us that f is $\overline{\operatorname{co}}\, K$ directionally Lipschitzian at $\bar{x}$ with $\beta \geq \beta_{K'}(f; \bar{x})$ and $\gamma \leq \inf_{\beta' > \beta} \gamma_{K'}(f; \bar{x}; \beta')$, which concludes the proof. $\square$

We can now give a characterization of K directionally Lipzchitzian behavior of f in terms of the Ioffe approximate subdifferential $\partial_a f$.

Theorem 4.9. *Let $f\colon X \longrightarrow \mathbb{R}\cup\{+\infty\}$ be a lower semicontinuous function, $\bar{x}\in \operatorname{dom} f$, and K be a convex compact subset of X. Then the following properties are equivalent:*

(a) *f is K directionally Lipschitzian at $\bar{x}$;*

(b) *There exist $r>0$, $\gamma>0$, and $\beta\in\mathbb{R}$ such that*

$$\inf_{v\in K}\langle x^*,v\rangle+\gamma\|x^*\|\le\beta \qquad \forall\ x\in B_f(\bar{x},r),\ \ \forall\ x^*\in\partial_a f(x); \tag{71}$$

(c) *There exist $r>0$, $\gamma>0$, and $\beta\in\mathbb{R}$ such that for every $x\in B_f(\bar{x},r)$*

$$\inf_{v\in K}\langle x^*,v\rangle+\gamma\|x^*\|\le\begin{cases}\beta & \forall\ x^*\in\partial_G f(x)\\ 0 & \forall\ x^*\in\partial_G^\infty f(x).\end{cases} \tag{72}$$

Moreover, one has the equality

$$\beta_K(f;\bar{x})=\beta_K^{\partial_a}(f;\bar{x}), \tag{73}$$

and these quantities are also equal to $\beta_K^{\partial_G}(f;\bar{x})$ under either (a), or (b), or (c).

Proof. (I) Assume first that $\beta_K(f;\bar{x})<+\infty$. In such a case, f is K directionally Lipschitzian at $\bar{x}$ and, by Lemma 3.9 part (b), for any $\beta>\beta_K(f;\bar{x})$ and $\gamma'\in\ ]0,\gamma_K(f;\bar{x};\beta)[$ there exist $r'>0$ and $\eta>0$ such that the equality

$$\operatorname*{linf}_{(v,\beta)\in K\times\{\beta\}}\Psi_{\operatorname{epi} f}((x,s)+t(b,b')+t(v,\beta))=0$$

holds for all $(x,s)\in B((\bar{x},f(\bar{x})),r')\cap\operatorname{epi} f$, $\ t\in\ ]0,r'[$, $\ (b,b')\in\gamma'\,(B_X\times]-\eta,\eta[)$. This implies that $\operatorname{epi} f$ is a $K\times\{\beta\}$ directionally Lipschitzian set at $(\bar{x},f(\bar{x}))$ and, from Theorem 3.7, there exists $r>0$ and $\gamma>0$ such that

$$\inf_{v\in K}\langle x^*,v\rangle-s^*\beta+\gamma(\|x^*\|+|s^*|)\le 0 \tag{74}$$

for all $(x,s)\in B((\bar{x},f(\bar{x})),r)\cap\operatorname{epi} f$ and for all $(x^*,-s^*)\in N_a(\operatorname{epi} f;(x,s))$ (resp. for all $(x^*,-s^*)\in N_G(\operatorname{epi} f;(x,s))$.

(II) Let us prove that (a) $\Leftrightarrow$ (b). Since the value $\beta_K^{\partial_a}(f;\bar{x})$ is the infimum of $\beta\in\mathbb{R}$ for which there exist $r>0$ and $\gamma>0$ such that (71) holds, and $\beta_K(f;\bar{x})<+\infty$ if and only if f is K directionally Lipschitzian at $\bar{x}$, in order to prove this equivalence (a) $\Leftrightarrow$ (b) it is enough to prove the equality $\beta_K^{\partial_a}(f;\bar{x})=\beta_K(f;\bar{x})$ of (73). For the inequality $\beta_K^{\partial_a}(f;\bar{x})\le\beta_K(f;\bar{x})$ we may suppose that $\beta_K(f;\bar{x})<+\infty$. By (I) take $r>0$ and $\gamma>0$ such that (74) holds. Taking any $x\in B_f(\bar{x},r)$ and $x^*\in\partial_a f(x)$ and noting that $(x,f(x))\in B((\bar{x},f(\bar{x})),r)$ along with $(x^*,-1)\in N_a(\operatorname{epi} f;(x,f(x)))$, that inequality (74) assures us that

$$\inf_{v\in K}\langle x^*,v\rangle+\gamma\|x^*\|\le\beta.$$

Consequently, $\beta \geq \beta_K^{\partial_a}(f;\bar{x})$ for all $\beta > \beta_K(f;\bar{x})$, which entails the desired inequality $\beta_K^{\partial_a}(f;\bar{x}) \leq \beta_K(f;\bar{x})$.

To prove the converse inequality $\beta_K^{\partial_a}(f;\bar{x}) \geq \beta_K(f;\bar{x})$, assume $\beta_K^{\partial_a}(f;\bar{x}) < +\infty$ and take any $\beta > \beta_K^{\partial_a}(f;\bar{x})$. Then there exist reals $r > 0$ and $\gamma > 0$ such that (71) holds. From Theorem 4.8 we deduce that f is K directionally Lipschitzian at $\bar{x}$ and $\beta \geq \beta_K(f;\bar{x})$. This being true for all $\beta > \beta_K^{\partial_a}(f;\bar{x})$, we get the inequality $\beta_K^{\partial_a}(f;\bar{x}) \geq \beta_K(f;\bar{x})$ as desired. Consequently, the equality $\beta_K^{\partial_a}(f;\bar{x}) = \beta_K(f;\bar{x})$ is justified.

(III) We proceed now to show the equivalence (a) $\Leftrightarrow$ (c). Asssuming (a), by (I) let $r > 0$ and $\gamma > 0$ such that (74) holds with $N_G(\operatorname{epi} f;\cdot,\cdot)$. Fix any $x \in B_f(\bar{x}, r)$. For any $x^* \in \partial_G f(x)$ (if any), we have $(x^*, -1) \in N_G(\operatorname{epi} f; x, f(x))$, so the inequality in (74) implies with $s^* = 1$ that $\inf_{v\in K}\langle x^*, v\rangle - \beta + \gamma(\|x^*\| + 1) \leq 0$, then in particular $\inf_{v\in K}\langle x^*, v\rangle + \gamma\|x^*\| \leq \beta$. On the other hand, taking any $x^* \in \partial_G^\infty f(x)$ (and keeping mind that this means $(x^*, 0) \in N_G(\operatorname{epi} f; x, f(x))$) we see with $s^* = 0$ in (74) that $\inf_{v\in K}\langle x^*, v\rangle + \gamma\|x^*\| \leq 0$. Taking the facts together they justify the implication (a) $\Rightarrow$ (c).

On the other hand, the implication (c) $\Rightarrow$ (a) is a consequence of Proposition 3.12. Therefore, properties (a), (b) and (c) are pairwise equivalent.

(IV) Note that one always has $\beta_K^{\partial_a}(f;\bar{x}) \geq \beta_K^{\partial_G}(f;\bar{x})$ according to the inclusion $\partial_G f(x) \subset \partial_a f(x)$ (see Definition 3.17). Now, assume (c) and take any $\beta' \in \mathbb{R}$ such that there exist reals $r' > 0$ and $\gamma' > 0$ such that

$$\inf_{v\in K}\langle x^*, v\rangle + \gamma'\|x^*\| \leq \beta' \qquad \forall x \in B_f(\bar{x}, r'),\ \forall x^* \in \partial_G f(x).$$

By the property (c) for some reals $r'' > 0$ and $\gamma'' > 0$ we also have

$$\inf_{v\in K}\langle x^*, v\rangle + \gamma''\|x^*\| \leq 0 \qquad \forall x \in B_f(\bar{x}, r''),\ \forall x^* \in \partial_G^\infty f(x).$$

Putting $r := \min\{r', r''\} > 0$, $\gamma := \min\{\gamma', \gamma''\} > 0$, we see for every $x \in B_f(\bar{x}, r)$:

$$\inf_{v\in K}\langle x^*, v\rangle + \gamma\|x^*\| \leq \begin{cases} \beta' & \forall x^* \in \partial_G f(x) \\ 0 & \forall x^* \in \partial_G^\infty f(x). \end{cases}$$

Proposition 3.12 again ensures that $\beta' \geq \beta_K(f;\bar{x})$, which in turn entails that $\beta_K^{\partial_G}(f;\bar{x}) \geq \beta_K(f;\bar{x})$. From this, the above equality $\beta_K(f;\bar{x}) = \beta_K^{\partial_a}(f;\bar{x})$, and the above inequality $\beta_K^{\partial_a}(f;\bar{x}) \geq \beta_K^{\partial_G}(f;\bar{x})$ we deduce that

$$\beta_K^{\partial_G}(f;\bar{x}) \geq \beta_K(f;\bar{x}) = \beta_K^{\partial_a}(f;\bar{x}) \geq \beta_K^{\partial_G}(f;\bar{x}),$$

so the desired final equality $\beta_K(f;\bar{x}) = \beta_K^{\partial_G}(f;\bar{x})$ holds true under (c). This finishes the proof of the theorem. □

Remark 4.10. The implication (a) $\Rightarrow$ (c) in the previous theorem is obtained in [24] without the links between the constants $\beta_K^{\partial_G}(f;\bar{x})$ and $\beta_K(f;\bar{x})$. □

As corollary we obtain the following characterization of compactly epi-Lipschitzian functions.

Corollary 4.11. *A lower semicontinuous function $f\colon X \longrightarrow \mathbb{R} \cup \{+\infty\}$ is compactly epi-Lipschitzian at $\bar{x} \in \operatorname{dom} f$ if and only if there exist a compact set $K \subset X$, $r > 0$, $\gamma > 0$, and $\beta \in \mathbb{R}$ such that the inequality*

$$\inf_{v \in K} \langle x^*, v\rangle + \gamma \|x^*\| \le \beta \qquad \forall\ x \in B_f(\bar{x}, r), \quad \forall\ x^* \in \partial_a f(x),$$

is satisfied.

5. Conclusion

This is the second of a series of our works devoted to K directionally Lipschitzian property. The first work [11] introduced the concept for functions and sets and established diverse characterizations via the Hadamard lower directional derivatives and Bouligand tangent cones. The present work provides characterizations using subdifferentials and normal cones, mainly with the Ioffe approximate subdifferential and normal cone in the general Banach setting. The corresponding concept for set-valued mappings and its study will be the object of future works.

Acknowledgments. R. Correa and P. Gajardo would like to thank the "Institut Montpelliérain Alexander Grothendieck" at the Université de Montpellier for supporting their work and the hospitality during their visits in France. L. Thibault would also like to thank the support and hospitality provided by authorities of Universidad de Chile and Universidad Técnica Federico Santa María (Chile) during his stays in Chile.

This research was partially supported by FONDECYT grants (Chile) N 1120239 and N 1160204 and of Basal Program CMM-AFB 170001 from CONICYT.

6. Appendix

Proof of Lemma 4.3

Fix $\varepsilon > 0$ and take $0 < \varepsilon' < \varepsilon/3$ such that $\bar{x}$ is a minimum point for $f_1 + f_2 + f_3$ over $B(\bar{x}, \varepsilon')$. Applying Lemma 4.2 with the closed subset $S := \bar{x} + \varepsilon' \overline{B}_X$ we obtain

$$\inf_{(x_1,x_2,x_3)\in S^3} w_t(x_1, x_2, x_3) \to f_1(\bar{x}) + f_2(\bar{x}) + f_3(\bar{x})$$

when $t \to +\infty$ where

$$w_t(x_1, x_2, x_3) := f_1(x_1) + f_2(x_2) + f_3(x_3) + t\Delta(x_1, x_2, x_3),$$

and $$\Delta(x_1, x_2, x_3) := \frac{1}{2}(\|x_1 - x_2\|^2 + \|x_2 - x_3\|^2 + \|x_3 - x_1\|^2).$$

Hence, for each integer $n \ge 1$ there exists some real number $t_n > n$ such that

$$f_1(\bar{x}) + f_2(\bar{x}) + f_3(\bar{x}) < \inf_{(x_1,x_2,x_3)\in S^3} w_{t_n}(x_1, x_2, x_3) + \frac{\varepsilon'^2}{n}. \tag{75}$$

As $\Delta(\bar{x}, \bar{x}, \bar{x}) = 0$, this inequality can be rewritten in the form

$$w_{t_n}(\bar{x}, \bar{x}, \bar{x}) < \inf\{w_{t_n}(x_1, x_2, x_3) : x_i \in \bar{x} + \varepsilon' \overline{B}_X\} + \frac{\varepsilon'^2}{n}. \tag{76}$$

By the Ekeland variational principle applied to the complete metric space S^3, there exist

$$x_{i,n} \in \bar{x} + \frac{\varepsilon'}{n} \overline{B}_X \tag{77}$$

such that
$$w_{t_n}(x_{1,n}, x_{2,n}, x_{3,n}) \leq w_{t_n}(\bar{x}, \bar{x}, \bar{x}) \tag{78}$$

and $(x_{1,n}, x_{2,n}, x_{3,n})$ is a minimum point on S^3 for the function

$$(x_1, x_2, x_3) \mapsto f_1(x_1) + f_2(x_2) + f_3(x_3) + t_n \Delta(x_1, x_2, x_3) + \\ + \varepsilon'(\|x_1 - \bar{x}\| + \|x_2 - \bar{x}\| + \|x_3 - \bar{x}\|).$$

From (P5'), for $f\colon X^3 \longrightarrow \mathbb{R} \cup \{+\infty\}$ with

$$f(x_1, x_2, x_3) = f_1(x_1) + f_2(x_2) + f_3(x_3)$$

one has
$$\partial f(x_1, x_2, x_3) = \partial f_1(x_1) \times \partial f_2(x_2) \times \partial f_3(x_3).$$

Since Δ is a convex continuous function, from (P4'), we obtain that there exist $x_i^* \in \partial f_i(x_{i,n})$ such that

$$-(x_{1,n}^*, x_{2,n}^*, x_{3,n}^*) \in t_n \partial \Delta(x_{1,n}, x_{2,n}, x_{3,n}) + 3\varepsilon' B.$$

Observing that for any $(u_1^*, u_2^*, u_3^*) \in \partial \Delta(u_1, u_2, u_3)$ (from (P3) it is the subdifferential in the sense of convex analysis) one has

$$u_1^* + u_2^* + u_3^* = 0,$$

and one gets
$$\|x_{1,n}^* + x_{2,n}^* + x_{3,n}^*\| \leq 3'\varepsilon < \varepsilon. \tag{79}$$

By (75) and (78) one has

$$w_{t_n}(x_{1,n}, x_{2,n}, x_{3,n}) \underset{n\to\infty}{\longrightarrow} f_1(\bar{x}) + f_2(\bar{x}) + f_3(\bar{x}) \tag{80}$$

and by definition of w_t one has

$$f_1(x_{1,n}) + f_2(x_{2,n}) + f_3(x_{3,n}) \leq w_{t_n}(x_{1,n}, x_{2,n}, x_{3,n}).$$

As $x_{i,n} \to \bar{x}$ and each function f_i is lower semicontinuous, it follows from the last inequality and from (80) that

$$f_1(x_{1,n}) + f_2(x_{2,n}) + f_3(x_{3,n}) \underset{n\to\infty}{\longrightarrow} f_1(\bar{x}) + f_2(\bar{x}) + f_3(\bar{x}).$$

The latter and the lower semicontinuity of f_i once again imply that

$$f_i(x_{i,n}) \underset{n\to\infty}{\longrightarrow} f_i(\bar{x}). \tag{81}$$

Thus, for n large enough, we obtain, from (77), (79), and (81) that

$$\|x_{i,n} - \bar{x}\| < \varepsilon, |f_i(x_{i,n}) - f_i(\bar{x})| < \varepsilon, \|x_{1,n}^* + x_{2,n}^* + x_{3,n}^*\| < \varepsilon.$$

Proof of Lemma 4.5

Fix $\bar{u} \in E \cap \mathrm{dom}\, f$ and $u^* \in \partial_H g_{\bar{u}+L}(\bar{u})$ (if any). Define $h^1, h^2 \colon E \times X \to \mathbb{R} \cup \{+\infty\}$ by $h^1(u,x) = f(x)$ and $h^2 = \psi_{\mathrm{gph}\, j}$. For $\varphi := h^1 + h^2$ we have

$$\varphi_{(\bar{u},\bar{u})+L\times L} = h^1_{(\bar{u},\bar{u})+L\times L} + h^2_{(\bar{u},\bar{u})+L\times L}.$$

Observe first that for all $(v,w) \in E \times X$ and all $t > 0$ we have

$$t^{-1}[g_{\bar{u}+L}(\bar{u}+tv) - g_{\bar{u}+L} \leq t^{-1}[\varphi_{(\bar{u},\bar{u})+L\times L}(\bar{u}+tv, \bar{u}+tw) - \varphi_{(\bar{u},\bar{u})+L\times L}(\bar{u},\bar{u})]$$

and hence

$$d_H^- g_{\bar{u}+L}(\bar{u}; v) \leq d_H^- \varphi_{(\bar{u},\bar{u})+L\times L}(\bar{u},\bar{u}; v, w) = d_H^-(h^1+h^2)_{(\bar{u},\bar{u})+L\times L}(\bar{u},\bar{u}; v, w).$$

According to the inclusion $u^* \in \partial_H g_{\bar{u}+L}(\bar{u})$, the next inequality yields for all $(v,w) \in E \times X$

$$\langle u^*, v\rangle_{E^*,E} + \langle 0, w\rangle_{X^*,X} \leq d_H^-(h^1+h^2)_{(\bar{u},\bar{u})+L\times L}(\bar{u},\bar{u}; v, w),$$

that is, $(u^*, 0) \in \partial_H(h^1+h^2)_{(\bar{u},\bar{u})+L\times L}(\bar{u},\bar{u})$. Since $L \times L$ is a finite dimensional subspace of $E \times X$, the set $\varepsilon B_{E^*\times X^*} + (L\times L)^{\perp_{E^*\times X^*}}$ is a w^*-neighborhood of zero in $E^* \times X^*$ and from Lemma 3 in [13] there are $(\zeta_i, \xi_i) \in (\bar{u},\bar{u}) + \varepsilon B_{E\times X}$ with $|h^i(\zeta_i,\xi_i) - h^i(\bar{u},\bar{u})| \leq \varepsilon$ and $(\zeta_i^*, \xi_i^*) \in \partial h^i_{(\zeta_i,\xi_i)+L\times L}(\zeta_i,\xi_i)$, i=1,2, such that

$$(u^*, 0) \in (\zeta_1^*, \xi_1^*) + (\zeta_2^*, \xi_2^*) + \frac{\varepsilon}{2} B_{E^*\times X^*} + (L\times L)^{\perp_{E^*\times X^*}}. \tag{82}$$

By definition of h^1 we have $\zeta_1^* = 0$, $|f(\zeta_1) - f(\bar{u})| \leq \varepsilon$, and $\xi_1^* \in \partial_H f_{\xi_1+L}(\xi_1)$. Now observe that $h^2_{(\zeta_2,\xi_2)+L\times L}(u,x) = 0$ if $x - \xi_2 = u - \zeta_2 \in L$ and $h^2_{(\zeta_2,\xi_2)+L\times L}(u,x) = +\infty$ otherwise. Then, for every $(u,x) \in E \times X$ with $x - \xi_2 = u - \zeta_2 \in L$

$$\langle \zeta_2^*, u - \zeta_2\rangle_{E^*,E} + \langle \xi_2^*, x - \xi_2\rangle_{X^*,X} \leq 0,$$

that is,

$$\langle \zeta_2^*, u'\rangle_{E^*,E} + \langle \xi_2^* \circ j, u'\rangle_{E^*,E} \leq 0 \; \forall u' \in L,$$

and hence

$$\zeta_2^* + \xi_2^* \circ j \in L^{\perp_{E^*}}.$$

Putting this into (82) we obtain

$$u^* \in -\xi_2^* \circ j + \tfrac{\varepsilon}{2} B_{E^*} + L^{\perp_{E^*}}. \tag{83}$$

Further, (82) also ensures that

$$-\xi_2^* = \xi_1^* + \tfrac{\varepsilon}{2} b^* + l^* \text{ with } b^* \in B_{X^*} \text{ and } l^* \in L^{\perp_{X^*}},$$

and hence

$$-\xi_2^* \circ j = \xi_1^* \circ j + \frac{\varepsilon}{2} b^* \circ j + l^* \circ j.$$

Using this into (83) and noting that $\|b^* \circ j\|_{E^*} \leq 1$ and $l^* \circ j \in L^{\perp_{E^*}}$, we see that

$$u^* \in \xi_1^* \circ j + \varepsilon B_{E^*} + L^{\perp_{E^*}},$$

and hence the choice of $y := \zeta_1$ and $y^* := \zeta_1^*$ concludes the proof of the lemma.

References

[1] D. Aussel, J.-N. Corvellec, M. Lassonde: *Nonsmooth constrained optimization and multidirectional mean value inequalities*, SIAM J. Optim. 9(3) (1999) 690–706 (electronic).

[2] J. M. Borwein: *Epi-Lipschitz-like sets in Banach space: theorems and examples*, Nonlinear Analysis 11(10) (1987) 1207–1217.

[3] J. M. Borwein, H. M. Strójwas: *Tangential approximations*, Nonlinear Analysis 9(12) (1985) 1347–1366.

[4] J. M. Borwein, Q. J. Zhu: *Viscosity solutions and viscosity subderivatives in smooth Banach spaces with applications to metric regularity*, SIAM J. Control Optim. 34(5) (1996) 1568–1591.

[5] J. M. Borwein, Q. J. Zhu: *Techniques of Variational Analysis*, CMS Books in Mathematics 20, Springer, New York (2005).

[6] F. H. Clarke: *Generalized gradients and applications*, Trans. Amer. Math. Soc. 205 (1975) 247–262.

[7] F. H. Clarke, Yu. S. Ledyaev: *Mean value inequalities in Hilbert space*, Trans. Amer. Math. Soc. 344(1) (1994) 307–324.

[8] F. H. Clarke, Yu. S. Ledyaev, R. J. Stern, P. R. Wolenski: *Nonsmooth Analysis and Control Theory*, Graduate Texts in Mathematics 178, Springer, New York (1998).

[9] R. Correa, P. Gajardo, L. Thibault: *Subdifferential representation formula and subdifferential criteria for the behavior of nonsmooth functions*, Nonlinear Analysis 65(4) (2006) 864–891.

[10] R. Correa, P. Gajardo, L. Thibault: *Links between directional derivatives through multidirectional mean value inequalities*, Math. Programming, Ser. B 116(1-2) (2009) 57–77.

[11] R. Correa, P. Gajardo, L. Thibault: *Various Lipschitz-like properties for functions and sets. I: Directional derivative and tangential characterizations*, SIAM J. Optim. 20(4) (2010) 1766–1785.

[12] R. Correa, A. Jofré, L. Thibault: *Subdifferential characterization of convexity*, in: *Recent Advances in Nonsmooth Optimization*, World Scientific, River Edge (1995) 18–23.

[13] A. D. Ioffe: *Approximate subdifferentials and applications. I: The finite-dimensional theory*, Trans. Amer. Math. Soc. 281(1) (1984) 389–416.

[14] A. D. Ioffe: *Approximate subdifferentials and applications. II*, Mathematika 33(1) (1986) 111–128.

[15] A. D. Ioffe: *Approximate subdifferentials and applications. III: The metric theory*, Mathematika 36(1) (1989) 1–38.

[16] A. D. Ioffe: *Proximal analysis and approximate subdifferentials*, J. London Math. Soc. (2) 41(1) (1990) 175–192.

[17] A. D. Ioffe: *Codirectional compactness, metric regularity and subdifferential calculus*, in: *Constructive, Experimental, and Nonlinear Analysis*, Limoges 1999), CMS Conf. Proceedings 27, American Mathematical Society, Providence (2000) 123–163.

[18] A. D. Ioffe: *Metric regularity and subdifferential calculus*, Uspekhi Mat. Nauk 55(3(333)) (2000) 103–162.

[19] A. D. Ioffe: *Metric regularity – a survey. Part 1: Theory*, J. Australian Math. Soc. 101(2) (2016) 188–243.

[20] A. D. Ioffe: *Metric regularity – a survey. Part 2: Applications*, J. Australian Math. Soc. 101(3) (2016) 376–417.

[21] A. D. Ioffe: *Variational Analysis of Regular Mappings. Theory and Applications*, Springer Monographs in Mathematis, Springer, New York (2017).

[22] A. Jourani: *The role of locally compact cones in nonsmooth analysis*, Comm. Appl. Nonlinear Analysis 5(2) (1998) 1–35.

[23] A. Jourani, L. Thibault: *Metric regularity for strongly compactly Lipschitzian mappings*, Nonlinear Analysis 24(2) (1995) 229–240.

[24] A. Jourani, L. Thibault: *Extensions of subdifferential calculus rules in Banach spaces*, Canad. J. Math. 48(4) (1996) 834–848.

[25] B. S. Mordukhovich: *Variational Analysis and Generalized Differentiation. I: Basic Theory*, Grundlehren der Mathematischen Wissenschaften 330, Springer, Berlin (2006).

[26] R. T. Rockafellar: *Directionally Lipschitzian functions and subdifferential calculus*, Proc. London Math. Soc. (3) 39(2) (1979) 331–355.

[27] R. T. Rockafellar, R. J.-B. Wets: *Variational Analysis*, Grundlehren der Mathematischen Wissenschaften 317, Springer, Berlin (1998).

[28] L. Thibault: *A note on the Zagrodny mean value theorem*, Optimization 35(2) (1995) 127–130.

[29] L. Thibault: *Limiting convex subdifferential calculus with applications to integration and maximal monotonicity of subdifferential*, in: *Constructive, Experimental, and Nonlinear Analysis*, Limoges 1999, CMS Conf. Proceedings 27, American Mathematical Society, Providence (2000) 279–289.

[30] L. Thibault: *Unilateral Variational Analysis in Banach Spaces*, to appear.

[31] L. Thibault, D. Zagrodny: *Integration of subdifferentials of lower semicontinuous functions on Banach spaces*, J. Math. Anal. Appl. 189(1) (1995) 33–58.

[32] L. Thibault, N. Zlateva: *Integrability of subdifferentials of directionally Lipschitz functions*, Proc. Amer. Math. Soc. 133(10) (2005) 2939–2948 (electronic).

[33] J. S. Treiman: *Generalized gradients, Lipschitz behavior and directional derivatives*, Canad. J. Math. 37(6) (1985) 1074–1084.

[34] D. Zagrodny: *Approximate mean value theorem for upper subderivatives*, Nonlinear Analysis 12(12) (1988) 1413–1428.

[35] Q. Zhu: *Clarke-Ledyaev mean value inequalities in smooth Banach spaces*, Nonlinear Analysis 32(3) (1998) 315–324.