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On the Homotopical Stability of Isolated Critical Points of Continuous Functions

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Dedicated to Alexander Ioffe on the occasion of his 80th birthday.

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In the framework of the metric critical point theory, we prove a stability property of the critical groups, for a family of continuous functions which, in a common neighborhood of their respective unique critical point, satisfies a uniform Palais-Smale type condition and is continuous in the uniform topology.

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1. Introduction

Let (X, d) be a metric space and U be an open subset of X. For $\lambda \in [0, 1]$, let $f_\lambda \colon X \to \mathbb{R}$ be continuous and $u_\lambda \in U$ be such that u_λ is the only (lower) critical point of f_λ in $\overline{U}$ (see Definition 2.1). In [9, Theorem 5.2], we showed that if the family $\{f_\lambda : \lambda \in [0, 1]\}$ satisfies a uniform Palais-Smale type condition (see Definition 3.1 below), and if for each λ:

$$\max\{\|f_{\mu|U} - f_{\lambda|U}\|_\infty, \mathrm{Lip}(f_{\mu|U} - f_{\lambda|U})\} \to 0 \quad \text{as } \mu \to \lambda\,, \tag{1}$$

(where $f_{\lambda|U}$ denotes the restriction of f_λ to U), then the critical groups of f_λ at u_λ are independent of λ. This extends the classical result where the f_λ's are C^1 functions on a Banach space, satisfy the usual Palais-Smale condition, and the family $\{f_\lambda\}$ is continuous in the C^1-norm: see [2, Theorem I.5.6], [13, Theorem 8.8], see also [1, Corollary 1.4], dealing with restrictions on a closed convex set.

In [3, Theorem 3.1], Cingolani and Degiovanni showed that the result of [9] holds assuming only, instead of (1), that for each λ:

$$\|f_{\mu|U} - f_{\lambda|U}\|_\infty \to 0 \quad \text{as } \mu \to \lambda\,.$$

However, their proof is technically rather involved, so it is the purpose of this note to give a simpler proof of the result of [3], thanks to the use, as in [9], of the kind of “linearizing effect” provided by the so-called change-of-metric principle.

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In Section 2, we give some preliminary facts on singular homology and critical groups (we refer to [14] for the various properties of singular homology that are used). The main result is in Section 3. We recall in an Appendix (Section 4) our main tools from metric critical point theory, which are a deformation theorem and a change-of-metric principle.

For $u \in X$ and $\rho > 0$, we set:

$$B_\rho(u) := \{v \in X \ : \ d(v,u) < \rho\}\,, \quad B_\rho[u] := \{v \in X \ : \ d(v,u) \le \rho\}\,.$$

For $f\colon X \to \mathbb{R}$ and $a \in \mathbb{R}$ we set:

$$[f{<}a] := \{u \in X \ : \ f(u) < a\} \quad [f{\le}a] := \{u \in X \ : \ f(u) \le a\}$$

and for $a, b \in \mathbb{R}$ with $a < b$ we set $[a{<}f{<}b] := [f{<}b] \setminus [f{\le}a]$. We also use the notations $[f{\ge}a]$ and $[a{\le}f{\le}b]$, with similar meaning.

2. Preliminaries

In this section, (X, d) is a metric space and $f\colon X \to \mathbb{R}$ is continuous. We start by recalling the notion of weak slope from [10, 8].

Definition 2.1. For $u \in X$, let $|df|(u)$ denote the supremum of the σ's in $[0, +\infty[$ such that there exists $\delta > 0$ and $\mathcal{H} : B_\delta(u) \times [0, \delta] \to X$ continuous with

$$d(\mathcal{H}(v,t), v) \le t\,, \qquad f(\mathcal{H}(v,t)) \le f(v) - \sigma t$$

for every $(v, t) \in B_\delta(u) \times [0, \delta]$. The extended real number $|df|(u)$ is called the *weak slope* of f at u. If $|df|(u) = 0$, we say that u is a *(lower) critical point* of f.

This notion was introduced independently (without the terminology) in [12], after a variant was introduced in [11]. We recall that if X is a Finsler manifold of class C^1 and f is of class C^1 too, then $|df|(u) = \|f'(u)\|$, see [10, Corollary 2.12].

Definition 2.2. The *critical groups* of f at u are denoted and defined by:

$$C_q(f; u) := H_q([f{<}f(u)] \cup \{u\}, [f{<}f(u)])\,, \quad q \in \mathbb{Z}_+\,,$$

where H_q denotes the q-th relative singular homology group with real coefficients.

Using the graded group notation:

$$C_*(f; u) := (C_q(f; u))_{q\in\mathbb{Z}_+}\,,$$

as well as the excision property of singular homology, if W is a neighborhood of u we have:

$$C_*(f; u) = H_*(W \cap [f{<}f(u)] \cup \{u\}, W \cap [f{<}f(u)])\,. \tag{2}$$

According to [4, Proposition 3.4], we have:

$$C_*(f; u) \ne 0 \ \Rightarrow \ |df|(u) = 0\,.$$

According to [4, Proposition 3.7], if u is an *isolated* (lower) critical point of f, we also have:

$$C_*(f;u) = H_*(W \cap [f \le f(u)], W \cap [f \le f(u)] \setminus \{u\})$$

(W a neighborhood of u), the latter graded group being more commonly referred to in the literature as the *definition* of the critical groups of f at u, see, e.g., [2, 13].

In what follows, we shall repeatedly use the following basic property of (singular) homology: If $C \subset B \subset A \subset X$, considering the *long exact sequence of the triple* (A, B, C), then:

$$H_*(A,B) = \{0\}_* \;\Leftrightarrow\; H_*(B,C) = H_*(A,C)\,,$$

$$H_*(B,C) = \{0\}_* \;\Leftrightarrow\; H_*(A,C) = H_*(A,B)\,.$$

Proposition 2.3. *Let $V_2 \subset V_1 \subset W_2 \subset W_1 \subset X$.*

(a) *If $H_*(W_1,W_2) = H_*(V_1,V_2) = \{0\}_*$, then $H_*(W_2,V_1) = H_*(W_1,V_2)$.*

(b) *If $H_*(W_1,V_1) = H_*(W_2,V_2) = \{0\}_*$, then $H_*(W_1,W_2) = \{0\}_*$.*

Proof. (a) From the long exact sequence of the triple (W_1,W_2,V_1) we see that $H_*(W_2,V_1) = H_*(W_1,V_1)$ since $H_*(W_1,W_2) = \{0\}_*$. From the long exact sequence of the triple (W_1,V_1,V_2), since $H_*(V_1,V_2) = \{0\}_*$ we see that $H_*(W_1,V_2) = H_*(W_1,V_1)$.

(b) In the following commutative diagram the arrows are induced by inclusions:

$$\begin{tikzcd} H_*(W_1,V_2) \arrow[rr] \arrow[dr] & & H_*(W_1,W_2) \\ & H_*(W_1,V_1) \arrow[ur] & \end{tikzcd} .$$

From the long exact sequence of the triple (W_1,W_2,V_2), since $H_*(W_2,V_2) = \{0\}_*$ we see that the horizontal arrow is an isomorphism. Since also $H_*(W_1,V_1) = \{0\}_*$, the conclusion follows. □

Proposition 2.4. *Let $u_0 \in X$ with $f(u_0) = |df|(u_0) = 0$, let $\rho > 0$, and let $a \in [-\rho, 0]$. For $b \in]0,\rho]$, set:*

$$W_b := \{u \in X \;:\; f(u) + d(u,u_0) \le b\}\,.$$

Assume that $B_{2\rho}[u_0]$ is complete, and that $\inf\limits_{u \in B_{2\rho}[u_0]\setminus\{u_0\}} |df|(u) > 1\,.$ *Then:*

(a) $|df_{|W_b}|(u_0) = 0$ *and* $|df_{|W_b}|(u) \ge |df|(u)$ *for every* $u \in W_b \cap [f \ge a]$ *;*

(b) $C_*(f;u_0) = H_*(W_b, W_b \cap [f<a])\,.$

(c) $H_*(W_{b_1}, W_{b_2}) = \{0\}_*$ *for all* $0 < b_2 < b_1 < \rho\,.$

Proof. (a) Clearly, W_b is a closed neighborhood of u_0, and $W_b \cap [f \geq a] \subset B_{2\rho}[u_0]$. Thus, $|df_{|W_b}|(u_0) = |df|(u_0) = 0$, while for $u \in W_b \cap [f \geq a]$, $u \neq u_0$, we have $|df|(u) > 1$. Let such u be fixed, and let $1 \leq \sigma < |df|(u)$, $\delta > 0$, and $\eta\colon B_\delta(u) \times [0,\delta] \to X$ continuous be such that

$$d(\eta(v,t),v) \leq t\,, \quad \text{and} \quad f(\eta(v,t)) \leq f(v) - \sigma t\,.$$

If $v \in B_\delta(u) \cap W_b$ and $t \in [0,\delta]$, then

$$f(\eta(v,t)) + d(\eta(v,t),u_0) \leq f(\eta(v,t)) + d(\eta(v,t),v) + d(v,u_0) \leq f(v) + d(v,v_0) \leq b\,,$$

so that $\eta(v,t) \in W_b$, which shows that $|df_{|W_b}|(u) \geq \sigma$, and the conclusion follows.

(b) Thanks to (a), applying Theorem 4.1 to $f_{|W_b}$, we obtain that $W_b \cap [f<0] \cup \{u_0\}$ is a weak deformation retract of W_b, and that $W_b \cap [f<a]$ is a weak deformation retract of $W_b \cap [f<0]$. Thus, from the homotopy property of singular homology:

$$H_*(W_b, W_b \cap [f<0] \cup \{u_0\}) = H_*(W_b \cap [f<0], W_b \cap [f<a]) = \{0\}_*\,,$$

and the conclusion follows from Proposition 2.3(a) and (2).

(c) Set $\tilde{f}(u) := f(u) + d(u,u_0)$, so that $|d\tilde{f}|(u) \geq |df|(u) - 1$ for every $u \in X$ (arguing similarly as in the proof of (a)), while $W_{b_i} = [\tilde{f} \leq b_i]$, $i = 1,2$. Thus:

$$\inf_{u \in [0<\tilde{f}<\rho]} |d\tilde{f}|(u) > 0\,,$$

so that W_{b_2} is a strong deformation retract of W_{b_1}, according to Theorem 4.1(a), and the conclusion follows from the homotopy property of singular homology. □

3. Main result

In this section, (X,d) is a metric space, $\{f_\lambda\colon X \to \mathbb{R} \,:\, \lambda \in [0,1]\}$ is a family of continuous functions, and U is an open subset of X.

Definition 3.1. We say that $\{f_\lambda \,:\, \lambda \in [0,1]\}$ satisfies *the (uniform) Palais-Smale condition in* $\overline{U}$ (*condition* (PS) *in* $\overline{U}$, for short), if for every sequence $\lambda_h \to \lambda$ in $[0,1]$ and every sequence (u_h) in $\overline{U}$, with $(f_{\lambda_h}(u_h))$ bounded and $|df_{\lambda_h}|(u_h) \to 0$, there exists a subsequence (u_{h_k}) converging to u with $|df_\lambda|(u) = 0$.

Proposition 3.2. *Assume that* $\{f_\lambda(u) \,:\, \lambda \in [0,1]\,,\ u \in \overline{U}\}$ *is bounded, that* $\{f_\lambda \,:\, \lambda \in [0,1]\}$ *satisfies condition* (PS) *in* $\overline{U}$, *and that for every* $\lambda \in [0,1]$, f_λ *has a unique (lower) critical point* u_λ *in* $\overline{U}$. *Then:*

(a) $\lambda \mapsto u_\lambda$ *is continuous;*

(b) *There is a nondecreasing, continuous* $\beta\colon [0,+\infty[\to [0,+\infty[$, *with* $\beta(s) = 0$ *if and only if* $s = 0$, *and such that:*

$$|df_\lambda|(u) \geq 2\beta(d(u,u_\lambda)) \quad \textit{for every } u \in \overline{U} \setminus \{u_\lambda\} \textit{ and every } \lambda \in [0,1]\,.$$

Proof. See the proof of [9, Proposition 2.2]. □

Theorem 3.3. *Let (X,d) be a metric space, $\{f_\lambda\colon X \to \mathbb{R} \,:\, \lambda \in [0,1]\}$ be a family of continuous functions, U be an open subset of X, and $\{u_\lambda \,:\, \lambda \in [0,1]\} \subset U$. Assume that $\overline{U}$ is complete, that $\{f_\lambda \,:\, \lambda \in [0,1]\}$ satisfies condition (PS) in $\overline{U}$, and that for every $\lambda \in [0,1]$:*

(a) $(u \in \overline{U}$ *and* $|df_\lambda|(u) = 0) \;\Rightarrow\; u = u_\lambda$*;*

(b) $f_\mu \to f_\lambda$ *uniformly in* U*, as* $\mu \to \lambda$ *in* $[0,1]$.

Then, $C_*(f_0; u_0) = C_*(f_1; u_1)$.

Proof. Since we are going to use the change-of-metric procedure, we may assume that X is isometrically embedded in a Banach space. Thus, up to considering $\tilde{f}_\lambda : X + u_0 - u_\lambda \to \mathbb{R}$ defined by

$$\tilde{f}_\lambda(u) := f_\lambda(u + u_\lambda - u_0) - f_\lambda(u_\lambda)\,,$$

and taking Proposition 3.2(a) into account, we may assume that $u_\lambda = u_0$ and $f_\lambda(u_0) = 0$ for every $\lambda \in [0,1]$. Moreover, we may assume that U and $\{f_\lambda(u) \,:\, \lambda \in [0,1]\,,\; u \in \overline{U}\}$ are bounded.

Let $\lambda \in [0,1]$ be fixed. We need only show that for $\mu \in [0,1]$ sufficiently close to λ, we have $C_*(f_\mu; u_0) = C_*(f_\lambda; u_0)$. According to Proposition 3.2(b), let $\beta\colon [0,+\infty[\to [0,+\infty[$ be continuous and nondecreasing, with $\beta(s) = 0$ if and only if $s = 0$, and such that:

$$|df_\lambda|(u) \geq 2\beta(d(u,u_0)) \quad \text{for every } u \in \overline{U} \setminus \{u_0\} \text{ and every } \lambda \in [0,1]\,.$$

Let $\tilde{d} = \tilde{d}(\{u_0\}, \beta)$ be the metric given by Theorem 4.2, and let $\rho > 0$ be such that

$$\widetilde{B}_{2\rho}[u_0] := \{u \in X \,:\, \tilde{d}(u,u_0) \leq 2\rho\} \subset U\,.$$

Let $\varepsilon \in]0, \rho/4[$, and set:

$$W_\lambda := \{u \in X \,:\, f_\lambda(u) + \tilde{d}(u,u_0) \leq 4\varepsilon\}\,,$$

$$V_\lambda := \{u \in X \,:\, f_\lambda(u) + \tilde{d}(u,u_0) \leq 2\varepsilon\}\,.$$

Let $\delta > 0$ be such that for $\mu \in [0,1]$:

$$|\mu - \lambda| \leq \delta \quad \Rightarrow \quad \|(f_\mu - f_\lambda)_{|\overline{U}}\|_\infty \leq \varepsilon\,.$$

Let such μ be fixed, and set:

$$W_\mu := \{u \in X \,:\, f_\mu(u) + \tilde{d}(u,u_0) \leq 3\varepsilon\}\,,$$

$$V_\mu := \{u \in X \,:\, f_\mu(u) + \tilde{d}(u,u_0) \leq \varepsilon\}\,,$$

so that $V_\mu \subset V_\lambda \subset W_\mu \subset W_\lambda$. For $f := f_\lambda, f_\mu$, and for $u \in \overline{U} \setminus \{u_0\}$, we have:

$$|\tilde{d}f|(u) = \frac{|df|(u)}{\beta(d(u,u_0))} \geq 2\,.$$

From Proposition 2.4(c), we have:

$$H_*(W_\lambda, V_\lambda) = H_*(W_\mu, V_\mu) = \{0\}_* ,$$

and from Proposition 2.3(b), we obtain:

$$H_*(W_\lambda, W_\mu) = \{0\}_* . \tag{3}$$

Now, set:

$$\begin{aligned} W'_\lambda &:= \{u \in W_\lambda \; : \; f_\lambda(u) < 0\} , \\ V'_\lambda &:= \{u \in V_\lambda \; : \; f_\lambda(u) < -2\varepsilon\} , \\ W'_\mu &:= \{u \in W_\mu \; : \; f_\mu(u) < -\varepsilon\} , \\ V'_\mu &:= \{u \in V_\mu \; : \; f_\mu(u) < -3\varepsilon\} , \end{aligned}$$

so that $V'_\mu \subset V'_\lambda \subset W'_\mu \subset W'_\lambda$. We have:

$$C_*(f_\lambda; u_0) = H_*(W_\lambda, W'_\lambda) = H_*(V_\lambda, V'_\lambda) = H_*(W_\lambda, V'_\lambda) . \tag{4}$$

Indeed, the first two equalities follow from Proposition 2.4(b), the third one from the fact that $H_*(W_\lambda, V_\lambda) = \{0\}_*$, and considering the long exact sequence of $(W_\lambda, V_\lambda, V'_\lambda)$. Considering the long exact sequence of $(W_\lambda, W'_\lambda, V'_\lambda)$, we then obtain that $H_*(W'_\lambda, V'_\lambda) = \{0\}_*$. Similarly, we have:

$$C_*(f_\mu; u_0) = H_*(W_\mu, W'_\mu) = H_*(V_\mu, V'_\mu) = H_*(W_\mu, V'_\mu) , \tag{5}$$

so that $H_*(W'_\mu, V'_\mu) = \{0\}_*$. From Proposition 2.3(b), we thus have

$$H_*(W'_\lambda, W'_\mu) = \{0\}_* . \tag{6}$$

From (3) and (6), considering the long exact sequences of $(W_\lambda, W_\mu, W'_\mu)$ and of $(W_\lambda, W'_\lambda, W'_\mu)$, we obtain:

$$H_*(W_\mu, W'_\mu) = H_*(W_\lambda, W'_\mu) = H_*(W_\lambda, W'_\lambda) ,$$

so that $C_*(f_\mu; u_0) = C_*(f_\lambda; u_0)$ according to (4) and (5). □

4. Appendix

In this section, (X, d) is a metric space and $f \colon X \to \mathbb{R}$ is continuous. For $a \in \mathbb{R}$, we let:

$$K_a := \{u \in X \; : \; f(u) = a \, , \; |df|(u) = 0\}$$

denote the set of lower critical points of f at level a.

Theorem 4.1. *Let (X, d) be a metric space, let $f \colon X \to \mathbb{R}$ be continuous, and let $-\infty < a < b \le +\infty$. Assume that for every $c \in]a, b[$, $[a \le f \le c]$ is complete and*

$$\inf_{u \in [a<f<c]} |df|(u) > 0 . \tag{7}$$

Then:

(a) $[f \le a]$ *is a strong deformation retract of* $[f < b]$ *;*

(b) $[f < a] \cup K_a$ *is a weak deformation retract of* $[f < b]$ *.*

This result follows from the deformation results in [6], which build on the original ones from [8]. See also [7], which revisits the whole deformation technique in metric critical point theory[1]—in particular, see [7, Theorem 3.2] (which yields Theorem 4.1(a) for $b \in \mathbb{R}$) and the proofs of [7, Theorem 3.6, Theorem 5.2] (which show how to get the case $b = +\infty$ in (a), and (b)). Note that Theorem 4.1 is truly a nonsmooth result. Indeed, if f is (say) a C^1 function on a Banach space, then assumption (7) is not satisfied, in general, if a is a critical level of f.

Theorem 4.2. *Let (X, d) be a complete metric space, let A be a nonempty subset of X, and let $\beta\colon [0, +\infty[\to [0, +\infty[$ with $\beta(s) > 0$ for $s > 0$. Assume that either $\beta(0) > 0$, or A is compact and its connected components are single points. Assume further that:*

$$\int_0^{+\infty} \beta(s)\, ds = +\infty\,.$$

Then, there exists a metric $\tilde{d} = \tilde{d}(A, \beta)$ on X, which is topologically equivalent to d, and such that:

(a) *$(X, \tilde{d})$ is complete;*

(b) *If $f\colon X \to \mathbb{R}$ is continuous and $u \in X$ is such that $\beta(d(u, A)) \neq 0$, then:*

$$|\tilde{d}f|(u) = \frac{|df|(u)}{\beta(d(u, A))}\,,$$

where $|\tilde{d}f|$ denotes the weak slope of f with respect to the metric $\tilde{d}$.

The first version of this result was given in [5, Theorem 4.1], in the case $\beta(0) > 0$. The more general assumption was considered in [6, 9], in order to provide a metric extension of Chang's Second Deformation Lemma in a smooth setting, see [2, Theorem I.3.2]. Recall that, after embedding X isometrically into a Banach space Y (according to the Arens-Eells theorem), the metric $\tilde{d}$ is defined as follows: For $u, v \in Y$, let $\Gamma_{u,v}$ denote the set of piecewise C^1 paths $\gamma\colon [0, 1] \to Y$ with $\gamma(0) = u$ and $\gamma(1) = v$, and set

$$\tilde{d}(u, v) := \inf_{\gamma \in \Gamma_{u,v}} \int_0^1 \beta(d(\gamma(t), A)) \|\gamma'(t)\|\, dt\,.$$

A complete proof of the theorem can be found in [7, Theorem 4.1].

[1] I take this opportunity to correct a bad typo at the end of the proof of [7, Theorem 2.6], where the final deformation should read:

$$\eta(u, t) := \begin{cases} \hat{\eta}(u, t) & \text{if } f(u) > a \\ u & \text{if } f(u) \leq a \end{cases}.$$

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