Journal of Convex Analysis
Volume 27 (2020), No. 1, 307–312

Non Approximate Derivability of the Takagi Function

Juan Ferrera
IMI, Dep. de Análisis Matemático y Matemática Applicada, Facultad Ciencias Matemáticas, Universidad Complutense, 28040 Madrid, Spain
ferrera@mat.ucm.es

Javier Gómez-Gil
Dep. de Análisis Matemático y Matemática Applicada, Facultad Ciencias Matemáticas, Universidad Complutense, 28040 Madrid, Spain
gomezgil@mat.ucm.es

Dedicated to Alexander Ioffe on the occasion of his 80th birthday.

Received: February 12, 2019
Revised manuscript received: April 14, 2019
Accepted: April 16, 2019

The Takagi function is a classical example of a continuous nowhere differentiable function. In this paper we prove that it is nowhere approximately derivable.

Keywords: Approximate differentiability, Takagi function.

2010 Mathematics Subject Classification: 26A27, 26A24.

1. Introduction

The Takagi function is probably the easiest example of a continuous nowhere derivable function, it was introduced in 1903 (see [12]), and ever since, it has caught the interest of mathematicians, as a matter of fact it was often rediscovered, for instance, in 1930, using base ten instead of base two by Van der Waerden (see [13]). For an extensive information about this function, see the surveys [1] and [8].

We say that a function f satisfies the C^1 *Lusin property* if for every $\varepsilon > 0$ there exists a C^1 function g such that the set $\{x : f(x) \neq g(x)\}$ has Lebesgue measure less than ε. It is known that the Takagi's function does not satisfy the C^1 Lusin property, more precisely, that it agrees with no C^1 function on any set of positive measure (see [3]). In particular, this implies that the Takagi function does not satisfy the Lipschitz property (see [10]), however it is "almost" Lipschitz in the sense that it is α-Hölder continuous for every $0 < \alpha < 1$ (see [11]).

It is known that every function with non empty Fréchet subdifferential a.e. satisfies the C^1 Lusin property (see [2]), hence the Takagi function cannot be Fréchet

ISSN 0944-6532 / $ 2.50

subdifferentiable a.e., as a matter of fact it is known that it has non empty subdifferential at x if and only if x is a dyadic number (see [4] or [5]).

The Lusin properties are closely related with the approximate differentiability, for instance a one dimensional function has the C^1 Lusin property if and only if it is approximately derivable a.e., for this result see [14] or [9] for similar results involving funcions on $\mathbb{R}^n$.

From all these results we may deduce that the Takagi function is not approximately derivable a.e., in other words: there exists a positive measure set such that the Takagi function is not approximately derivable at any point of that set. The aim of this paper is to give a direct, and relatively elementary, proof of a stronger result, namely that the Takagi function is nowhere approximately derivable. Although it is known that almost every, in the sense of category, continuous function $f\colon \mathbb{R} \to \mathbb{R}$ is nowhere approximately derivable, see [6] , it is not easy to provide examples of such functions, see for instance the examples that appear in [6] or [7], clearly the functions that they define are not as elementary as the Takagi function. On the other hand the relevance of the Takagi function worth the study of its approximate derivability.

2. Results

We introduce some definitions and notation. $\mathcal{L}$ will denote the Lebesgue measure on $\mathbb{R}$. For a function $f\colon \mathbb{R} \to \mathbb{R}$, l is the *approximate limit* of f at x, $\operatorname{ap}\lim_{y\to x} f(y) = l$, if for every $\varepsilon > 0$,

$$\lim_{r\to 0^+} \frac{\mathcal{L}\big((x-r,x+r)\cap\{y\in\mathbb{R} : |f(y)-l|\geq\varepsilon\}\big)}{2r} = 0.$$

We say that a function $f\colon \mathbb{R}\to\mathbb{R}$ is *approximately derivable at x* if there exists a real number $f'_{ap}(x)$, named approximate derivative of f at x, such that

$$\operatorname{ap}\lim_{y\to x} \frac{f(y)-f(x)-f'_{ap}(x)(y-x)}{y-x} = 0.$$

The following result, see [7], page 139, is immediate:

Proposition 2.1. *Let $f\colon \mathbb{R}\to\mathbb{R}$ be a function, let x be a point of $\mathbb{R}$ and suppose that f is approximately derivable at x with approximate derivative $f'_{ap}(x)$. Then, for any real number $\alpha > f'_{ap}(x)$, we have*

$$\lim_{r\to 0^+} \frac{\mathcal{L}\left\{y\in(x-r,x+r)\setminus\{x\} : \frac{f(y)-f(x)}{y-x}\geq\alpha\right\}}{2r} = 0.$$

Similarly, for any real number $\beta < f'_{ap}(x)$, we have

$$\lim_{r\to 0^+} \frac{\mathcal{L}\left\{y\in(x-r,x+r)\setminus\{x\} : \frac{f(y)-f(x)}{y-x}\leq\beta\right\}}{2r} = 0.$$

We define the *Takagi function*: For $n \in \mathbb{N}$, $n \geq 0$, let $D_n = \{k2^{-n} : k \in \mathbb{Z}\} \subset \mathbb{R}$ and let $D = \cup D_n$ be the set of all dyadic numbers. The Takagi function $T \colon \mathbb{R} \to \mathbb{R}$ is defined as

$$T(x) = \sum_{k=1}^{\infty} g_k(x) = \lim_n G_n(x) \tag{1}$$

where $g_k(x) = \min(|x-y| : y \in D_k)$, and $G_n = g_1 + \cdots + g_n$. The aim of this paper is to prove the following Theorem:

Theorem 2.2. *The Takagi function is nowhere approximately derivable.*

We split the proof in several Propositions. First, we assume that $x \notin D$, we have that for all n there exist $x_n, y_n \in D_n$ such that $x \in (x_n, y_n)$ and $(x_n, y_n) \cap D_n = \varnothing$. It is clear that for $k < n$ we have that $(x_n, y_n) \subset (x_k, y_k)$ and

$$g_k(x') - g_k(x) = g'_k(x)(x' - x) \tag{2}$$

for all $x' \in [x_n, y_n]$. Note that g_k is derivable at x because $x \notin D$, moreover $g'_k(x) = \pm 1$. We start with the following estimation:

Lemma 2.3. *Let $x \notin D$. If $g'_n(x) = 1$ then*

$$\mathcal{L}\left\{y :\ 0 < |x-y| < 2^{-n},\ \frac{T(y)-T(x)}{y-x} \leq G'_{n-1}(x) + \frac{2}{5}\right\} \geq \frac{1}{2^{n+5}}. \tag{3}$$

Analogously, if $g'_n(x) = -1$ then

$$\mathcal{L}\left\{y :\ 0 < |x-y| < 2^{-n},\ \frac{T(y)-T(x)}{y-x} \geq G'_{n-1}(x) - \frac{2}{5}\right\} \geq \frac{1}{2^{n+5}}. \tag{4}$$

Proof. Let x_n, y_n be as above, if $g'_n(x) = 1$ then x is closer to x_n than to y_n, hence $y_n - x > 2^{-n-1}$. For $0 < t < 2^{-n-5}$ if $k = 0, 1, 2, \ldots$, as $y_n \in D_n \subset D_{n+k}$ and $\sup |g_{n+k}| \leq 2^{-n-k-1}$, we have that

$$g_{n+k}(y_n - t) \leq \min\left\{2^{-n-5}, 2^{-n-k-1}\right\}$$

and then, by (2),

$$\begin{aligned} T(y_n - t) - T(x) &\leq G_{n-1}(y_n - t) - G_{n-1}(x) + \frac{5}{2^{n+5}} + \sum_{k>4} \frac{1}{2^{n+k+1}} \\ &\leq G'_{n-1}(x)(y_n - t - x) + \frac{6}{2^{n+5}}. \end{aligned}$$

Therefore,

$$\frac{T(y_n - t) - T(x)}{y_n - t - x} \leq G'_{n-1}(x) + \frac{2}{5}.$$

since $y_n - t - x \geq 2^{-n-1} - 2^{-n-5} = 15 \cdot 2^{-n-5}$. The case $g'_n(x) = -1$ is similar. $\square$

Proposition 2.4. *Let $x \notin D$, if either $I = \liminf_n G'_n(x)$ or $S = \limsup_n G'_n(x)$, is finite then T is not approximately derivable at x.*

Proof. Let us suppose that I is finite and T is approximately derivable at x. As I is finite and $G'_n(x) \in \mathbb{Z}$ for all n, then there exists n_0 such that $G'_n(x) \geq I$ for all $n \geq n_0$ and there exists a strictly increasing subsequence $(n_k)_{k\geq 0}$ such that $G'_{n_k-1}(x) = I$ for all k. It is clear that $g'_{n_k-1}(x) = -1$ and $g'_{n_k}(x) = +1$ for all $k > 0$ and hence, by Lemma 2.3,

$$\limsup_{r\to 0^+} \frac{\mathcal{L}\left\{y:\, 0<|y-x|<r,\ \frac{T(y)-T(x)}{y-x} \leq I+\frac{2}{5}\right\}}{2r}$$

$$\geq \limsup_k \frac{\mathcal{L}\left\{y:\ 0<|y-x|<2^{-n_k},\ \frac{T(y)-T(x)}{y-x} \leq G'_{n_k-1}(x)+\frac{2}{5}\right\}}{2^{-n_k+1}} \geq \frac{1}{2^6} > 0$$

and

$$\limsup_{r\to 0^+} \frac{\mathcal{L}\left\{y:\, 0<|y-x|<r,\ \frac{T(y)-T(x)}{y-x} \geq I+1-\frac{2}{5}\right\}}{2r}$$

$$\geq \limsup_k \frac{\mathcal{L}\left\{y:\ 0<|y-x|<2^{-n_k+1},\ \frac{T(y)-T(x)}{y-x} \geq G'_{n_k-2}(x)-\frac{2}{5}\right\}}{2^{-n_k+2}} \geq \frac{1}{2^6} > 0$$

since $I+1 = G'_{n_k-2}(x)$. We deduce, by Proposition 2.1, the following contradiction.

$$T'_{ap}(x) \leq I + \frac{2}{5} < I + 1 - \frac{2}{5} \leq T'_{ap}(x).$$

The case S finite is similar. □

The alternative occurs when both limits are infinite. Then we also have that T is not approximately derivable.

Proposition 2.5. *Let $x \notin D$, if $I = \liminf_n G'_n(x)$ and $S = \limsup_n G'_n(x)$ are infinite, then T is not approximately derivable at x.*

Proof. As the result is obvious if $I = S$, we will suppose that $I = -\infty$ and $S = +\infty$. Then, there exists a subsequence $(n_k)_k$ such that for all k

$$G'_{n_k}(x) < G'_{n_{k+1}}(x), \quad g'_{n_k+1}(x) = -1 \ \text{ and } \ G'_{n_k}(x) \underset{k}{\to} +\infty.$$

By Proposition 2.1, if T is approximately derivable at x and $\alpha > T'_{ap}(x)$ there exists $\delta > 0$ such that

$$\mathcal{L}\left\{y \in (x-r, x+r)\setminus\{x\} :\ \frac{T(y)-T(x)}{y-x} \geq \alpha\right\} < \frac{1}{32}r \tag{5}$$

for $0 < r < \delta$. But if k is such that $G'_{n_k-1}(x) > \alpha + 1$ and $2^{-n_k} < \delta$ we have, by Lemma 2.3, that

$$\mathcal{L}\left\{y : 0 < |y - x| < \frac{1}{2^{n_k}}, \frac{T(y) - T(x)}{y - x} \geq G'_{n_k-1}(x) - \frac{2}{5}\right\}$$
$$\geq \frac{1}{2^{n_k+5}} = \frac{1}{32}\frac{1}{2^{n_k}},$$

contradicting (5). □

It only remains to prove that T is not approximately derivable at any dyadic number.

Proposition 2.6. *If $x \in D$, then T is not approximately derivable at x.*

Proof. If $x \in D$, let n_0 be the smaller integer such that $x \in D_{n_0+1}$. Let n be such that $n > 2n_0$. If $2|h| < 2^{-n}$ then $g_k(x + h) = |h|$ for every k, $n_0 < k \leq n$. Thus

$$T(x + h) - T(x) \geq \sum_{k=1}^{n_0}\big(g_k(x + h) - g_k(x)\big) + \sum_{k=n_0+1}^{n} g_k(x + h) \geq$$

$$\geq -n_0|h| + (n - n_0)|h| = (n - 2n_0)\,|h|.$$

This implies that

$$\mathcal{L}\left\{y : 0 < |y - x| < \frac{1}{2^{n+1}}, \frac{T(y) - T(x)}{y - x} \geq n - 2n_0\right\} \geq \frac{1}{4}\,2^{-n}.$$

It follows from Proposition 2.1 that T is not approximately derivable at x. □

Combining Propositions 2.4, 2.5, and 2.6, we have the proof of Theorem 2.2.

References

[1] P. C. Alaart, K. Kawamura: *The Takagi function: a survey*, Real Anal. Exchange 37 (2012) 1–54.

[2] D. Azagra, J. Ferrera, M. García-Bravo J. Gómez-Gil: *Subdifferentiable functions satisfy Lusin properties of class C^1 or C^2*, J. Approximation Theory 230 (2018) 1–12.

[3] J. B. Brown, G. Kozlowski: *Smooth interpolation, Hölder continuity, and the Takagi-Van der Waerden function*, Amer. Math. Monthly 110(2) (2003) 142–147.

[4] J. Ferrera, J. Gómez-Gil: *Generalized Takagi-Van der Waerden functions and their subdifferentials*, J. Convex Analysis 25(4) (2018) 1355–1369.

[5] P. Gora, R. J. Stern: *Subdifferential analysis of the Van der Waerden function*, J. Convex Analysis 18(3) (2011) 699–705.

[6] V. Jarník: *Sur la derivabilité des fonctions continues*, Spisy Privodov, Fak. Univ. Karlovy 129 (1934) 3–9.

[7] A. Kharazishvili: *Strange Functions in Real Analysis*, 2nd ed., Monographs and Textbooks in Pure and Applied Mathematics, CRC Press, Boca Raton (2005).

[8] J. C. Lagarias: *The Takagi function and its properties*, RIMS Kôkyûroku Besatsu B34 (2012) 153–189.

[9] F.-C. Liu, W.-S. Tai: *Approximate Taylor polynomials and differentiation of functions*, Topol. Methods Nonlinear Anal. 3(1) (1994) 189–196.

[10] J. Marcinkiewicz: *Sur les séries de Fourier*, Fund. Math. 27 (1936) 38–69.

[11] A. Shidfar, K. Sabetfatkhri: *On the continuity of Van der Waerden's Function in the Hölder sense*, Amer. Math. Monthly 93(5) (1986) 375–376.

[12] T. Takagi: *A simple example of the coninuous function without derivative*, Proc. Phys. Math. Soc. Tokio Ser. II, 1 (1903) 176–177.

[13] B. L. Van der Waerden: *Ein einfaches Beispiel einer nichdifferenzierbaren stetigen Funktion*, Math. Zeitschrift 32 (1930) 474–475.

[14] H. Whitney: *On totally differentiable and smooth functions*, Pacific J. Math. 1 (1951) 143–159.